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SPECTRAL ANALYSIS OF A Q-DIFFERENCE OPERATOR WHICH ARISES FROM THE QUANTUM $SU(1,1)$ GROUP

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Dedicated to Professor Huzihiro Araki on his 60th anniversary

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ABSTRACT. This paper is devoted to the study of an explicitly given second order difference operator which appears in the "representation theory" of the quantum $SU(1,1)$ group of non-compact type. We set up a situation in which the operator is shown to be self-adjoint, and the spectral analysis of the operator is developed. The "eigenfunctions" are perfectly given in terms of the basic hypergeometric functions. We then prove an explicit spectral expansion theorem which corresponds to the Fok-Mehler formula in the classical situation.

KEYWORDS: *Non-compact quantum groups, Zonal spherical functions, spectral theory of a self-adjoint operator.*

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0. INTRODUCTION AND RESULTS

The purpose of this paper is to give a detailed spectral analysis of a difference operator of a very special type. The operator which we study in this paper comes from the q-analogue with its background based on the "unitary representations" of the quantum group $SU_q(1,1)$ of non-compact type.

In order to study the structure of the quantum group $SU_q(1,1)$, of non-compact type, we are obliged to pass to the infinite dimensional representations as in the case of the classical Lie group $SU(1,1)$. However, almost all of the researches on the quantum groups are mainly concerned with the finite dimensional representations based on the purely algebraic techniques.

Following to the present approach to the quantum groups, one of our basic tool is the coordinate ring of the quantum group. Due to the non-commutativity of the coordinate ring, we are not allowed to think of the "underlying space" in a usual sense. Then, in the case of the "compact real form" $SU_q(2)$, the elements of the coordinate ring are realized as all bounded operators and we have no analytical difficulties. (See [14]). However, the "non-compactness" of our quantum group $SU_q(1,1)$ causes enormous difficulties from the point of view of functional analysis. As a result, rigorous and natural frameworks which deal with the unitary representations of the quantum groups of "non-compact type" have not yet been carefully established.

At the present situation, the technically possible direction for our research is to work on the spectral analysis of the operator which comes from the Casimir element. This is certainly supposed to be the first step to build up the theory of the unitary representations of the quantum group $SU_q(1,1)$.

In the publications [8], we also discussed about some classifications of the formal unitary representations in terms of the "complex spin". Then, the amazing thing is that, only in the quantized situation, there appears a new continuous family of "infinite dimensional unitary representations". We also have a formal observation that this new family disappears if we take the "classical limit $q \to 1$". Also at the same time, the parameter space corresponding to the usual "principal series representations" becomes compact set whereas it is a non-compact half line in the classical situation.

Then after, one of the author announced in [12] that if we take in account of a suitable boundary condition for the operator, the parameter space of the new family is not any more continuous but discrete. This new discrete family was also discovered by [13] and now called by the name "strange series".

From the point of view of spectral theory, our problem looks as follows. The classification of the unitary representations corresponds to the spectral decomposition of the operator. In the classical situation, the operator corresponds to the usual regular-singular differential operator of second order which is the radial part of the Casimir operator on the classical Lie group $SU(1,1)$. Then the spectral theory was well known that the operator has only continuous spectrum over which the spectral expansion theorem was established. But, when we turn to the quantized situation, the spectrum does not look like that. The discovery of the strange series yields the existence of the point spectrum for the operator which never happens in the classical situation. These eigenfunctions are expressed in terms of the basic hypergeometric functions which is known as the q-analogue of the hypergeometric functions of Gauss.

However, rigorous treatments of this operator from the point of view of spectral analysis rather seem to be missing at the moment. Therefore, the main purpose of this paper is to take a full advantage of the functional analysis for this difference operator. The operator is shown to be unbounded self-adjoint operator having both continuous and discrete spectrum corresponding to the "principal series" and the "strange series", respectively. Here, it should be remarked that the operator is semi-bounded and the continuous spectrum is a bounded set lying in the low energy region. Then the unboundedness of the operator is due to the point spectrum which lies in the high energy region. This type of phenomena are quite rare in the case of classical differential operators.

In the present publication, we are not able to go into the detailed discussions of the theory of "unitary representations of $SU_q(1,1)$". As we have already pointed out, we still have some subtle problems even for the definition of the natural coordinate ring defined over $SU_q(1,1)$. This matter was started to be discussed in [9] with lots more delicate problems being untouched. This paper will be just a beginning part and the further discussions will be given in our forthcoming publications. In particular, this paper concerns only with the case of zonal part. Then the non-zonal part is fully discussed in the publication [4].

One of our main emphasis in this paper is the rigorous appearence of the point spectrum which suggests the existence of the new series of representations which we call the strange series. This is certainly a subtle object and in the "quantized situation", we also have observed some strange behaviour of the heat kernel associated with the quantum Laplacian even in the case of compact quantum spaces. This is discussed in [5].

This paper is organized as follows. After this introductory section, we give a brief survey on the calculus over the quantized half open interval in Section 1. The reduction of our problem to the q-difference operator which we study in this paper is also given in the same section. Then in Section 2, the self-adjointness of the operator is discussed. In the proof, we present a function space on which the operator is shown to be essentially self-adjoint.

In Section 3, the eigenfunctions and the generalized eigenfunctions are given and shown to be expressed in terms of the basic hypergeometric functions. This will be a brief survey of the results essentially obtained in [8] from the different point of view. Then the expression given in Section 3 is used to give a construction of the Green kernel. To prove that the given expression is actually a Green kernel of the operator, the connection formula for the basic hypergeometric functions obtained in [10] plays a quite important role.

The full description of the spectrum of the operator is given in Section 4. Then the explicit eigenfunction expansion as our main theorem (Theorem 5.6) is given in Section 5. The Plancherel formula is explicitly given in Section 6.

Finally, in Section 7, we present some discussions concerning the classical limit.

1. REDUCTION TO THE Q-DIFFERENCE OPERATOR

Throughout this paper, we assume that the quantization parameter q satisfies $0 < q < 1$.

Here we describe very roughly the way to get the operator which we deal with in this paper. When we discuss about the quantum groups, the object is usually meant to be the quantum universal enveloping algebra which is $U_q(\mathfrak{sl}(2))$ in our situation. (For the quantum universal enveloping algebra, we refer [3] for the details.) In the classical situation, there are three different real forms $SU(2)$, $SU(1,1)$ and $SL(2,\mathbf{R})$ for the classical $U(\mathfrak{sl}(2))$. Accordingly, we are also able to discuss about the three different types of formal real forms on the quantum universal enveloping algebra $U_q(\mathfrak{sl}(2))$. The formal real form which we regard to correspond to the virtual object $SU_q(1,1)$ is described by the involutive structure on the Hopf algebra $U_q(\mathfrak{sl}(2))$ determined on the generators by $e^* = -f$, $f^* = -e$ and $k^* = k$ with the condition that the quantization parameter q is a non-zero real number. Then the "infinitesimal unitary representations" which we discussed in [8] are given by some families of involutive homomorphisms of the forms

$$U_q(\mathfrak{sl}(2)) \longrightarrow \mathrm{Mat}_{\infty}(\mathbf{C})$$

obtained by suitable modifications of the homomorphisms

$$\pi_\ell : U_q(\mathfrak{sl}(2)) \longrightarrow \mathrm{Mat}\,(I, \mathbf{C})$$

parametrized by the so-called "complex spin $\ell \in \mathbf{C}$". Here, the set I is an infinite set indexing the basis of the infinite matrix algebra $\mathrm{Mat}\,(I,\mathbf{C})$ and the homomorphisms π_ℓ are determined on the generators in terms of the matrix units $E_{i,j}$, $i, j \in I$ of $\mathrm{Mat}\,(I,\mathbf{C})$ as follows:

$$\pi_\ell(e) = \sum_{j\in I}[\ell - j + 1]_q E_{j-1,j}$$

$$\pi_\ell(f) = \sum_{j\in I}[\ell + j + 1]_q E_{j+1,j}$$

$$\pi_\ell(k) = \sum_{j\in I} q^{-j} E_{j,j}.$$

In these expressions, we also have to mention that the symbol $[n]_q$ for $n \in \mathbb{Z}$ is the (homogeneous) q-integer of n defined by

$$[n]_q := \frac{q^n - q^{-n}}{q^1 - q^{-1}} .$$

Then, following the classical ideas of Bargmann ([1]), we are lead to look at the action of the "Casimir element":

$$C := ef + \frac{q^{-1}k^2 + qk^{-2} - 2}{(q - q^{-1})^2} \in U_q(\mathfrak{sl}(2))$$

which generates the center of the quantum universal enveloping algebra. In the process of looking this object as an operator, there are some subtle problems to see the Hilbert space on which the operator is acting. Once a natural coordinate ring defined over $SU_q(1,1)$ is given, then we are able to define a suitable Hilbert space on which the operator in question is regarded to be acting. In other words, the operator which we deal with in this paper corresponds to the radial part of the Casimir operator C acting on $SU_q(1,1)$.

To reduce the argument to the spectral analysis of the q-difference operator which we deal with in this paper, we start with the coordinate ring $A(SU_q(1,1))$ of the quantum group $SU_q(1,1)$.

Due to [8], the coordinate ring $A(SU_q(1,1))$ is generated by the four elements x, u, v, and y satisfying the following relations.

$$xu = qux,\ xv = qvx,\ uy = qyu,\ vy = qyv,\ uv = vu,$$
$$xy - q^{-1}uv = yx - quv = 1.$$

Moreover, $A(SU_q(1,1))$ has the structure of the involutive Hopf algebra. (For the details, see [8].)

The element k of the quantum universal enveloping algebra $U_q(\mathfrak{sl}(2))$ acts on $A(SU_q(1,1))$ from the left and from the right. Let A_{00} be the subalgebra of $A(SU_q(1,1))$ of all elements which are bi-invariant under the action of k.

Then according to [8], the following proposition holds.

PROPOSITION 1.1. (i) *A_{00} is a polynomial ring $\mathbb{C}[\zeta]$ generated by $\zeta = -q^{-1}uv$.*

(ii) *We denote by C_{00} the action of the Casimir element C on A_{00}. Then, C_{00} is given by*

$$C_{00}\varphi(\zeta) = -q^{-1}T_{q^2}^{-1}D_{q^2}\zeta(1 - q^2\zeta)D_{q^2}\varphi(\zeta) + \frac{1}{q + 2 + q^{-1}}\varphi(\zeta), \tag{1.1}$$

where T_{q^2} and D_{q^2} denote a q-shift operator and a q-difference operator defined by

$$[T_{q^2}\varphi](\zeta) := \varphi(q^2\zeta), \qquad [D_{q^2}\varphi](\zeta) := \frac{\varphi(\zeta)-\varphi(q^2\zeta)}{(1-q^2)\zeta},$$

respectively.

To make our L^2-theoretical approach "compatible" with the algebraic framework discussed in [8], we introduce a q-interval $(-\infty,0]_{q^2}$, a function space $\mathcal{S}_0 = \mathcal{S}_0(-\infty,0]_{q^2}$, and a pairing $\langle\cdot,\cdot\rangle : A_{00}\times\mathcal{S}_0 \to \mathbf{C}$ in the following way.

Under the usual condition $0<q<1$, the definition of the q-interval $(-\infty,0]_{q^2}$ is given by

$$(-\infty,0]_{q^2} := \{-q^{2n} : n\in\mathbf{Z}\}.$$

Next, we define a function space $\mathcal{S}_0 = \mathcal{S}_0(-\infty,0]_{q^2}$ on the q-interval $(-\infty,0]_{q^2}$ as follows.

$$\mathcal{S}_0 \ni f \iff \begin{cases} (1) & \sup_{\zeta\in(-\infty,0]_{q^2}} |D_{q^2}{}^k f(\zeta)| < \infty \quad \text{for } k=0,1,2,\ldots, \\ (2) & \exists N,\ n<N \Rightarrow f(-q^{2n})=0. \end{cases}$$

To define the pairing, let us recall the definition of the Jackson integral. The Jackson integral of a finite interval is defined by the formula:

$$\int_{\zeta_1}^{\zeta_2} \varphi(\zeta)\,\mathrm{d}_{q^2}\zeta := \sum_{n=N_1}^{n=N_2-1} \varphi(-q^{2n})q^{2n}(1-q^2),$$

for a function φ on $(-\infty,0]_{q^2}$ and $\zeta_j = -q^{2N_j}\in(-\infty,0]_{q^2}, j=1,2$. Similarly, the Jackson integral over the whole q-interval $(-\infty,0]_{q^2}$ is as follows:

$$\int_{-\infty}^{0} \varphi(\zeta)\,\mathrm{d}_{q^2}\zeta := \sum_{n\in\mathbf{Z}} \varphi(-q^{2n})q^{2n}(1-q^2).$$

Due to the above notations, the pairing $\langle\cdot,\cdot\rangle : A_{00}\times\mathcal{S}_0\to\mathbf{C}$ is defined by

$$\langle\varphi,f\rangle := \int_{-\infty}^{0} \varphi(\zeta)f(\zeta)\,\mathrm{d}_{q^2}\zeta,$$

for $\varphi\in A_{00}$, and $f\in\mathcal{S}_0$. Here we remark that, in the definition above, we identify an element $\varphi\in A_{00}$ with a polynomial function on $\mathbf{R}$ (and, as a result, with a polynomial function on the q-interval $(-\infty,0]_{q^2}$).

The following lemma is essential in our reduction.

LEMMA 1.2. (i) *The bilinear form* $\langle\cdot,\cdot\rangle : A_{00}\times\mathcal{S}_0 \to \mathbf{C}$ *is non-degenerate.*

(ii) *We define a q-difference operator* $\widetilde{C_{00}}$ *acting on* $\mathcal{S}_0$ *by*

$$\widetilde{C_{00}}f(\zeta) = -q^{-1}[T_{q^2}^{-1}D_{q^2}\zeta(1-q^2\zeta)D_{q^2}]f(\zeta) + \frac{1}{q+2+q^{-1}}f(\zeta). \tag{1.2}$$

Then we have

$$\langle C_{00}\varphi, f\rangle = \langle \varphi, \widetilde{C_{00}}f\rangle, \tag{1.3}$$

for $\varphi \in A_{00}$, *and* $f \in \mathcal{S}_0$.

(iii) *Conversely, an operator* $\widetilde{C_{00}}$ *satisfying* (1.3) *is uniquely determined and given by* (1.2).

Proof. Since (ii) and (iii) follow imediately from (i) by direct calculations, we only have to prove (i). Namely, it is sufficient to prove the following:

(a) $\varphi \in A_{00}$ satisfies $\langle\varphi, f\rangle = 0$ for $\forall f \in \mathcal{S}_0 \Longrightarrow \varphi = 0$.

(b) $f \in \mathcal{S}_0$ satisfies $\langle\varphi, f\rangle = 0$ for $\forall\varphi \in A_{00} \Longrightarrow f = 0$.

For any $\eta \in (-\infty, 0]_{q^2}$, we define a function $\delta_\eta \in \mathcal{S}_0$ by $\delta_\eta(\eta) = 1$ and $\delta_\eta(\zeta) = 0$ for $\zeta \neq \eta$. Then, $\langle\varphi, \delta_\eta\rangle = 0$ means $\varphi(\eta) = 0$, which proves (a). Then the proof of (b) goes as follows. Take $f \in \mathcal{S}_0$ satisfying the assumption in (b). We define a function F on $\mathbf{R}$ by $F(x) = 0$ for $x \geqslant 0$ and $F(x) = f(-q^{2n})$ for $-q^{2n} \leqslant x < -q^{2(n+1)}$. Then F is a bounded function with a compact support. Thus, the Fourier transform $\widehat{F}(t)$ of F is extended to give an entire function of $t \in \mathbf{C}$. By the assumtion of f, we have

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \widehat{F}(0) = \int_{-\infty}^{\infty} x^m F(x)\,\mathrm{d}x = \sum_{n\in\mathbf{Z}} f(-q^{2n}) \int_{-q^{2n}}^{-q^{2(n+1)}} x^m\,\mathrm{d}x$$
$$= \frac{1-q^{2(m+1)}}{(m+1)(1-q^2)} \int_{-\infty}^{0} f(\zeta)\zeta^m\,\mathrm{d}_{q^2}\zeta = 0, \quad \text{for } m = 0,1,2,\dots .$$

Therefore, we obtain $\widehat{F} = 0$, which shows $f = 0$ by the Fourier inversion formula. ∎

The lemma above allows us to think of the action of the Casimir element C on $\mathcal{S}_0$ which essentially agree with the framework discussed in [8]. Then the lemma above enables us to set up the problem of the spectral analysis of the Casimir operator C on the zonal Hilbert space.

Now we go into the definition of $L^2(-\infty, 0]_{q^2}$. We define

$$L^2(-\infty, 0]_{q^2} := \{\varphi \text{ on } (-\infty, 0]_{q^2} : (\varphi, \varphi) < \infty\}.$$

where the inner product is defined by

$$(\varphi, \psi) := \int_{-\infty}^{0} \varphi(\zeta)\overline{\psi(\zeta)}\, \mathrm{d}_{q^2}\zeta$$

and the corresponding L^2-norm is denoted by $\|\cdot\| = \|\cdot\|_{L^2(-\infty,0]_{q^2}}$.

As a technical tool, we also introduce the space of all compactly supported functions by

$$\begin{aligned} &C_c(-\infty, 0]_{q^2} \\ &\quad := \{\varphi \text{ on } (-\infty, 0]_{q^2} : \varphi(-q^{2n}) = 0 \quad \text{except for finitely many } n \in \mathbf{Z}\}. \end{aligned}$$

It is then easily observed that we have the natural inclusions

$$C_c(-\infty, 0]_{q^2} \subset \mathcal{S}_0(-\infty, 0]_{q^2} \subset L^2(-\infty, 0]_{q^2}.$$

Let P be the operator defined by

$$P := T_{q^2}^{-1} D_{q^2} \zeta(1 - q^2\zeta) D_{q^2}. \tag{1.4}$$

For simplicity, we deal with the above q-difference operator P instead of $\widetilde{C_{0\,0}}$ in the following discussions.

REMARK 1.3. In case of the classical Lie group $SU(1,1)$, the q-difference operator $C_{0\,0}$ (or $\widetilde{C_{0\,0}}$) corresponds to the hypergeometric differential operator

$$-\frac{\mathrm{d}}{\mathrm{d}\zeta}\zeta(1-\zeta)\frac{\mathrm{d}}{\mathrm{d}\zeta} + \frac{1}{4},$$

on the half interval $(-\infty, 0]$, which is realized as a self-adjoint operator on the zonal Hilbert space $L^2(-\infty, 0]$ associated with the Casimir operator on $SU(1,1)$. Moreover, it is known that the spectrum of the differential operator above consists only of the absolutely continuous spectrum, which means that only the principal continuous series representation of $SU(1,1)$ appears in the zonal Hilbert space.

2. THE SELF-ADJOINTNESS OF THE OPERATOR

In this section, we go into the proof of the self-adjointness of the operator. Here we have to remark that all these operators can be regarded to act on the spaces of sequences over $\mathbf{Z}$, and then these operators are well defined in this sense. It is also seen by the definition that the space $C_c(-\infty,0]_{q^2}$ is invariant by the action of these operators. In either sense, the formal adjoint of the operator D_{q^2} is equal to $-\frac{1}{q^2}T_{q^2}^{-1}D_{q^2}$ and hence the operator is symmetric. In particular, the operator P acting on $\mathcal{S}_0(-\infty,0]_{q^2}$ which we denote by $(P,\mathcal{S}_0(-\infty,0]_{q^2})$ is a densely defined symmetric operator on $L^2(-\infty,0]_{q^2}$. The main theorem of this section is the following.

THEOREM 2.1. (i) *The operator* $(P,\mathcal{S}_0(-\infty,0]_{q^2})$ *is essentially self-adjoint.* (ii) *The self-adjoint extension of* $(P,\mathcal{S}_0(-\infty,0]_{q^2})$ *is given by* $(P,\mathcal{D})$, *where*

$$\mathcal{D} := \{\varphi \in L^2(-\infty,0]_{q^2} : P\varphi \in L^2(-\infty,0]_{q^2}\,, \lim_{\zeta\to 0}\sqrt{-\zeta}[D_{q^2}\varphi](\zeta)=0\}.$$

Here we remark that our notational convention of taking the limits as $\zeta \to 0$ and $\zeta \to -\infty$ are given by

$$\lim_{\zeta\to 0}\varphi(\zeta) := \lim_{n\to\infty}\varphi(-q^{2n}), \qquad \lim_{\zeta\to-\infty}\varphi(\zeta) := \lim_{n\to-\infty}\varphi(-q^{2n}).$$

The remaining part of this section is devoted to the proof of this theorem. The proof is devided into several steps as follows.

LEMMA 2.2. *For* $\varphi \in L^2(-\infty,0]_{q^2}$,

$$\lim_{\zeta\to 0}\sqrt{-\zeta}\varphi(\zeta) = \lim_{\zeta\to-\infty}\sqrt{-\zeta}\varphi(\zeta) = 0.$$

Proof. This follows from the definition of the Hilbert space $L^2(-\infty,0]_{q^2}$. ∎

LEMMA 2.3. *Suppose* $\varphi \in L^2(-\infty,0]_{q^2}$ *satisfies*

$$P\varphi \in L^2(-\infty,0]_{q^2},$$

$$\lim_{\zeta\to 0}\zeta[D_{q^2}\varphi](\zeta) = 0. \tag{2.1}$$

Then we have $\lim_{\zeta\to 0}\sqrt{-\zeta}[D_{q^2}\varphi](\zeta) = 0.$

Proof. We put $T_{q^2}P\varphi = f$. Then, by (2.1),

$$(1-q^2\zeta)\zeta[D_{q^2}\varphi](\zeta) = -\int_\zeta^0 f(\eta)\,\mathrm{d}_{q^2}\eta.$$

Therefore we have

$$\begin{aligned}|(1-q^2\zeta)\zeta[D_{q^2}\varphi](\zeta)| &\leqslant \int_\zeta^0 |f(\eta)|\,\mathrm{d}_{q^2}\eta \\ &\leqslant \sqrt{-\zeta}\sqrt{\int_\zeta^0 |f(\eta)|^2\,\mathrm{d}_{q^2}\eta}\,,\end{aligned}$$

where we used the Schwarz inequality to obtain the second inequality. Then,

$$\left|\sqrt{-\zeta}[D_{q^2}\varphi](\zeta)\right| \leqslant \frac{1}{|1-q^2\zeta|}\sqrt{\int_\zeta^0 |f(\eta)|^2\,\mathrm{d}_{q^2}\eta}\,.$$

Due to the assumption that the element $T_{q^2}P\varphi = f$ is in $L^2(-\infty,0]_{q^2}$, the left hand side of the above goes to zero when we take the limit $\zeta \to 0$. This proves the assertion. ∎

LEMMA 2.4. *The operator $(P,\mathcal{D})$ is symmetric.*

Proof. We see easily that the equality

$$\int_{\zeta_1}^{\zeta_2} \{[P\varphi](\zeta)\overline{\psi(\zeta)} - \varphi(\zeta)\overline{[P\psi](\zeta)}\}\,\mathrm{d}_{q^2}\zeta = F(\varphi,\psi)(\zeta_2) - F(\varphi,\psi)(\zeta_1)$$

holds for $\varphi,\psi \in \mathcal{D}$, where $F(\varphi,\psi)$ is defined by

$$\begin{aligned}F(\varphi,\psi)(\zeta) &= \frac{q^2}{1-q^2}(1-\zeta)\{\varphi(q^{-2}\zeta)\overline{\psi(\zeta)} - \varphi(\zeta)\overline{\psi(q^{-2}\zeta)}\} \\ &= \frac{q}{1-q^2}(1-\zeta)\times\Big\{T^{-1}\left(\sqrt{-\zeta}\,[D_{q^2}\varphi](\zeta)\right)\sqrt{-\zeta}\,\overline{\psi(\zeta)} \\ &\quad - \sqrt{-\zeta}\varphi(\zeta)T^{-1}\left(\sqrt{-\zeta}\,\overline{[D_{q^2}\psi](\zeta)}\right)\Big\}.\end{aligned} \tag{2.2}$$

Therefore, Lemma 2.2 combined with $\varphi, \psi \in \mathcal{D}$, we have $F(\varphi,\psi)(\zeta_2) \to 0$ as $\zeta_2 \to 0$. Furthermore, by Lemma 2.2, we also have

$$F(\varphi,\psi)(\zeta_1) = \frac{q^2}{1-q^2}\frac{1-\zeta_1}{-\zeta_1} \times \Big\{\sqrt{-\zeta_1}\varphi(q^{-2}\zeta_1)\sqrt{-\zeta_1}\,\overline{\psi(\zeta_1)} \\ - \sqrt{-\zeta_1}\,\varphi(\zeta_1)\left(\sqrt{-\zeta_1}\,\overline{\psi(q^{-2}\zeta_1)}\right)\Big\}.$$

Then it is seen that the right hand side goes to zero when we take the limit $\zeta_1 \to -\infty$. Therefore we have

$$\int_{-\infty}^{0} \{[P\varphi](\zeta)\overline{\psi(\zeta)} - \varphi(\zeta)\overline{[P\psi](\zeta)}\}\, \mathrm{d}_{q^2}\zeta = 0$$

which means $(P\varphi,\psi) = (\varphi, P\psi)$. This proves the assertion. ∎

Let $(P^*,\mathcal{D}^*)$ be the adjoint of $(P,\mathcal{D})$. Then the above Lemma, in particular, shows $(P,\mathcal{D}) \subset (P^*,\mathcal{D}^*)$.

We recall that the operator P can also be regarded as a difference operator acting on a space of sequences. This means that for any sequence φ, the expression $P\varphi$ makes sense as a sequence over the integers. It is also remarked that the inner product (φ,ψ) makes sense for any sequence ψ if φ is in $C_c(-\infty,0]_{q^2}$. Then, in view of the above proof, the following two statements are easily seen.

COROLLARY 2.5. *For any $\varphi \in C_c(-\infty,0]_{q^2}$ and any sequence ψ, $(P\varphi,\psi) = (\varphi, P\psi)$ holds.*

LEMMA 2.6. *For $\varphi \in \mathcal{D}^*$, the sequence $P\varphi$ is equal to $P^*\varphi$ regarded as a sequence.*

Then we obtain the next statement.

PROPOSITION 2.7. *The operator $(P,\mathcal{D})$ is self-adjoint.*

Proof. The point we have to show is the inclusion $\mathcal{D}^* \subset \mathcal{D}$. Then, by Lemma 2.6, we have $P\psi = P^*\psi \in L^2(-\infty,0]_{q^2}$. This means that if we see

$$\lim_{\zeta\to 0} \sqrt{-\zeta}[D_{q^2}\psi](\zeta) = 0\,, \tag{2.3}$$

we obtain the assertion.

Let $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{D}^*$. Then

$$\int_{-\infty}^{\zeta} \left\{[P\varphi](\eta)\overline{\psi(\eta)} - \varphi(\eta)\overline{[P^*\psi](\eta)}\right\} d_{q^2}\eta = F(\varphi,\psi)(\zeta) . \tag{2.4}$$

Now, we define $\varphi \in \mathcal{D}$ by

$$\varphi(\zeta) = \begin{cases} 1 & \text{for } -1 \leqslant \zeta < 0, \\ 0 & \text{for } \zeta < -1 . \end{cases}$$

Then for $-1 < \zeta < 0$, we have

$$F(\varphi,\psi)(\zeta) = \frac{q^2}{1-q^2}(1-\zeta)\left\{\overline{\psi(\zeta)} - \overline{\psi(q^{-2}\zeta)}\right\}$$

and hence

$$F(\varphi,\psi)(q^2\zeta) = q^2(1-\zeta)(-\zeta)\overline{[D_{q^2}\psi](\zeta)} .$$

Then, due to the fact that the left hand side of (2.4) goes to zero as $\zeta \to 0$, we have

$$\lim_{\zeta\to 0} \zeta[D_{q^2}\psi](\zeta) = 0 .$$

Therefore, Lemma 2.2 applies to ψ and we obtain (2.3). ∎

LEMMA 2.8. *For $\varphi \in \mathcal{D}$, the limit $\varphi(0) := \lim_{\zeta\to 0}\varphi(\zeta)$ exists.*

Proof. We put $\psi(\zeta) = \sqrt{-\zeta}[D_{q^2}\varphi](\zeta)$. Since $\varphi \in \mathcal{D}$, we have $\lim_{\zeta\to 0}\psi(\zeta) = 0$. Therefore, for $-1 < \zeta_1 < \zeta_2 < 0$, we obtain

$$\varphi(\zeta_2) - \varphi(\zeta_1) = \int_{\zeta_1}^{\zeta_2} [D_{q^2}\varphi](\zeta)\, d_{q^2}\zeta = \int_{\zeta_1}^{\zeta_2} \frac{\psi(\zeta)}{\sqrt{-\zeta}}\, d_{q^2}\zeta$$

and hence

$$|\varphi(\zeta_2) - \varphi(\zeta_1)| \leqslant \max_{-1<\zeta<0} |\psi(\zeta)| \int_{\zeta_1}^{\zeta_2} \frac{1}{\sqrt{-\zeta}}\, d_{q^2}\zeta .$$

Then the finiteness of $\max_{-1<\zeta<0}|\psi(\zeta)|$ and the fact that the Jackson integral

$$\int_{\zeta_1}^{\zeta_2} \frac{1}{\sqrt{-\zeta}}\, d_{q^2}\zeta$$

goes to zero as $\zeta_1, \zeta_2 \to 0$, we have the assertion. ∎

LEMMA 2.9. *Let K be the operator on $L^2[-1,0]_{q^2}$ defined by*

$$[K\varphi](\zeta) := \int_{\zeta}^{0} \frac{1}{\xi(1-q^2\xi)} \int_{\xi}^{0} \varphi(\eta)\, d_{q^2}\eta\, d_{q^2}\xi$$

for $\varphi \in L^2[-1,0]_{q^2}$. Then the operator K is bounded.

Proof. For $a, b \in (-\infty, 0]_{q^2} \bigcup \{0\}$, we define the characteristic function $\chi_{[a,b]}$ defined by

$$\chi_{[a,b]}(\zeta) := \begin{cases} 1 & (a \leqslant \zeta \leqslant b), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have

$$\begin{aligned}[K\varphi](\zeta) &:= \int_{\zeta}^{0} \frac{1}{\xi(1-q^2\xi)} \int_{\zeta}^{0} \chi_{[\xi,0]}(\eta)\varphi(\eta)\, d_{q^2}\eta\, d_{q^2}\xi \\ &= \int_{\zeta}^{0} K(\zeta,\eta)\varphi(\eta)\, d_{q^2}\eta\end{aligned}$$

where

$$K(\zeta,\eta) = \int_{\zeta}^{0} \frac{1}{\xi(1-q^2\xi)} \chi_{[\zeta,\eta]}(\xi)\, d_{q^2}\xi\,.$$

Therefore, due to the inequality

$$|K(\zeta,\eta)| = \int_{\zeta}^{q^2\eta} \left|\frac{1}{\xi(1-q^2\xi)}\right| d_{q^2}\xi \leqslant \int_{\zeta}^{q^2\eta} \frac{1}{|\xi|}\, d_{q^2}\xi\,,$$

we see that there exists a constant C independent of ζ, η satisfying

$$|K(\zeta,\eta)| \leqslant C\left|\frac{\zeta}{\eta}\right|^{\frac{1}{4}}$$

for $-1 \leqslant \zeta \leqslant \eta < 0$. Then we have

$$\begin{aligned}|[K\varphi](\zeta)|^2 &\leqslant \int_{\zeta}^{0} |K(\zeta,\eta)|^2\, d_{q^2}\eta \int_{\zeta}^{0} |\varphi(\xi)|^2\, d_{q^2}\xi \\ &\leqslant C\|\varphi\|_{L^2[-1,0]_{q^2}} \sqrt{|\zeta|} \int_{\zeta}^{0} |\eta|^{-\frac{1}{2}}\, d_{q^2}\eta \\ &\leqslant C'\|\varphi\|_{L^2[-1,0]_{q^2}} |\zeta|\end{aligned}$$

with a suitable change of the constant C by C'. Hence we obtain

$$\begin{aligned}\|K\varphi\|_{L^2[-1,0]_{q^2}} &= \int_{-1}^{0} |[K\varphi](\zeta)|^2 \, d_{q^2}\zeta \\ &\leqslant C'\|\varphi\|_{L^2[-1,0]_{q^2}} \int_{-1}^{0} |\zeta| \, d_{q^2}\zeta \\ &\leqslant C''\|\varphi\|_{L^2[-1,0]_{q^2}}\end{aligned}$$

with a suitable change of the constant C' by C'' and the assertion is proved. ∎

PROPOSITION 2.10. *The operator $(P, \mathcal{S}_0)$ is essentially self-adjoint and its self-adjoint extension is given by $(P, \mathcal{D})$.*

Proof. The statement which we have to prove is that for any $\varphi \in \mathcal{D}$, we have a sequence $\{\varphi_n\}_{n=0}^{\infty}$ in $\mathcal{S}_0$ satisfying

$$\varphi_n \to \varphi \quad \text{and} \quad P\varphi_n \to P\varphi \quad \text{in} \quad L^2(-\infty, 0]_{q^2} . \tag{2.5}$$

We prove (2.3) by the following cases according to the condition of the support of φ.

Case 1. First, we consider the case of φ satisfying

$$\varphi(0) := \lim_{\zeta \to 0} \varphi(\zeta) = 0 \quad \text{and} \quad \operatorname{supp}(\varphi) \subset [-q^6, 0]_{q^2} .$$

We put $T_{q^2}P\varphi = \psi$. It follows $\operatorname{supp}(\psi) \subset [-q^2, 0]_{q^2}$ and then define $\psi_n \in \mathcal{S}_0$ by

$$\psi_n(\zeta) := \chi_{[-1,-q^{2n}]}(\zeta)\psi(\zeta), n = 0, 1, \ldots .$$

It is seen that the sequence $\{\psi_n\}_{n=0}^{\infty}$ converges to ψ in $L^2(-\infty, 0]_{q^2}$ as $n \to \infty$. We then define the sequence $\{\varphi_n\}_{n=0}^{\infty}$ by

$$\varphi_n(\zeta) := \chi_{[-1,0]}(\zeta)[K\psi_n](\zeta) \, , n = 0, 1, \ldots .$$

By the definition of the operator K, $[K\psi_n](\zeta) = 0$ for $-q^{2n} < \zeta < 0$ and hence $\varphi_n \in \mathcal{S}_0$ for $n = 0, 1, \ldots$.

By the definition,

$$[T_{q^2}P\varphi](\zeta) = [D_{q^2}\zeta(1 - q^2\zeta)D_{q^2}\varphi](\zeta) = \psi(\zeta) \, ,$$
$$\lim_{\zeta \to 0} [\zeta D_{q^2}\varphi](\zeta) = 0$$

and hence

$$[\zeta(1-q^2\zeta)D_{q^2}\varphi](\zeta) = -\int_\zeta^0 \psi(\eta)\,d_{q^2}\eta\,.$$

On the other hand, φ satisfies $\lim_{\zeta\to 0}\varphi(\zeta) = 0$ by the assumption. Therefore we have

$$\begin{aligned}\varphi(\zeta) &= \varphi(\zeta) - \varphi(0)\\ &= \int_\zeta^0 \frac{1}{\xi(1-q^2\xi)}\int_\zeta^0 \psi(\eta)\,d_{q^2}\eta\,d_{q^2}\xi\\ &= [K\psi](\zeta)\,.\end{aligned}$$

Due to the definition of the operator K, $\varphi_n(\zeta) = \varphi(\zeta) = 0$ for $\zeta < -1$ and $n = 0, 1, \ldots$. Therefore Lemma 2.9 implies

$$\begin{aligned}\|\varphi - \varphi_n\|_{L^2(-\infty,0]_{q^2}} &= \|\varphi - \varphi_n\|_{L^2(-1,0]_{q^2}}\\ &= \|K(\psi - \psi_n)\|_{L^2(-1,0]_{q^2}}\\ &\leqslant \text{const.}\|\psi - \psi_n\|_{L^2(-1,0]_{q^2}}\,.\end{aligned} \tag{2.6}$$

This proves that the sequence $\{\varphi_n\}_{n=0}^\infty$ converges to φ in $L^2(-\infty,0]_{q^2}$.

Now, the situation is then devided into the following three cases. In the case that ζ satisfies $-q^4 \leqslant \zeta < 0$, the equality

$$[P\varphi](\zeta) - [P\varphi_n](\zeta) = [T^{-1}\psi](\zeta) - [T^{-1}\psi_n](\zeta)$$

implies

$$\|P\varphi - P\varphi_n\|_{L^2(-q^4,0]_{q^2}} = \|T^{-1}\psi - T^{-1}\psi_n\|_{L^2(-q^4,0]_{q^2}} \tag{2.7}$$

for $n = 0, 1, \ldots$. In the case that ζ is either $-q^{-2}, -1-q^2$, the inequality (2.6) implies that φ_n pointwisely converges to φ and hence, due to the fact that the operator P is a difference operator, we have

$$[P\varphi_n](\zeta) \to [P\varphi](\zeta) \quad \text{for} \quad \zeta = -q^{-2},\ -1\ -q^2. \tag{2.8}$$

Finally, in the case that ζ satisfies $\zeta \leqslant -q^4$, the assumption on φ combined with the definition of φ_n implies

$$P\varphi(\zeta) = P\varphi_n(\zeta) = 0\,. \tag{2.9}$$

In the cases of (2.7), (2.8) and (2.9), we have $P\varphi_n \to P\varphi$ in $L^2(-\infty, 0]_{q^2}$ and this proves the assertion for Case 1.

Case 2. Next, we consider the case of φ satisfying $\operatorname{supp}(\varphi) \subset [-q^6, 0]_{q^2}$. In this case, we put $\tilde{\varphi}(\zeta) = \varphi(\zeta) - \varphi(0)\chi_{[-q^6,0]}(\zeta)$. Then we have $\tilde{\varphi} \in \mathcal{D}$ which satisfies the conditions of Case 1. Hence there exists a sequence $\{\tilde{\varphi}_n\}_{n=0}^{\infty}$ in $\mathcal{S}_0$ satisfying $\tilde{\varphi}_n \to \tilde{\varphi}$ and $P\tilde{\varphi}_n \to P\tilde{\varphi}$ in $L^2(-\infty, 0]_{q^2}$. Therefore the sequence $\{\varphi_n\}_{n=0}^{\infty}$ defined by $\tilde{\varphi}_n = \varphi_n + \varphi(0)\chi_{[-q^6,0]}$ satisfies (2.4) and proves the assertion for Case 2.

Case 3. Finally, we consider the case for general $\varphi \in \mathcal{S}_0$. We put

$$\varphi = \varphi^{(1)} + \varphi^{(2)}$$

where,

$$\varphi^{(1)} = \chi_{[-\infty,-q^4]}\varphi\,, \quad \varphi^{(2)} = \chi_{[-q^6,0]}\varphi\,.$$

Then $\varphi^{(2)}$ satisfies the condition of Case 2. Therefore, we may assume $\varphi = \varphi^{(1)}$ in the following.

We put $\varphi_n = \chi_n\varphi$, $\chi_n = \chi_{[-q^{-2n},0]}$. We see $\operatorname{supp}(\varphi_n) \subset [-q^{-2n}, -q^4]_{q^2}$ and hence $\varphi_n \in \mathcal{D}$. It is also easily seen that the sequence $\{\varphi_n\}_{n=0}^{\infty}$ converges to φ in $L^2(-\infty, 0]_{q^2}$.

By making use of the fact that $|[D_{q^2}\chi_n](\zeta)| = O(|\zeta|^{-1})$, $|[D_{q^2}^2\chi_n](\zeta)| = O(|\zeta|^{-2})$, $|\sqrt{-\zeta}\varphi(\zeta)| = o(1)$ and $|\zeta|^{3/2}|[D_{q^2}\varphi](\zeta)| = o(1)$ as $\zeta \to -\infty$, we see quite easily that the order estimate $|[[P, \chi_n]\varphi](\zeta)| = o(|\zeta|^{-1/2})$ holds as $\zeta \to -\infty$. Moreover, we find that

$$\operatorname{supp}([P, \chi_n]\varphi) \subset [-q^{-2(n+2)}, -q^{-2(n-2)}]\,.$$

Hence we obtain $\|[P, \chi_n]\varphi\| \to 0$ as $n \to \infty$. Therefore we obtain

$$\|P\varphi_n - P\varphi\| \leqslant \|[P, \chi_n]\varphi\| + \|(1 - \chi_n)P\varphi\|$$

with its right hand side converges to zero as $n \to \infty$. This proves the assertion for the Case 3 and then we arrive to the end of the proof. ∎

This proves the statements of Theorem 2.1.

3. THE EIGENFUNCTIONS AND THE GREEN OPERATOR

We start this section by reviewing some of the standard notions of the basic hypergeometric functions which play important roles in this paper. Then we go into the construction of the Green operator of the self-adjoint operator $(P, \mathcal{D})$.

First, we review the notion of the q-shifted factorial. For a suitable complex parameter α and $n = 0, 1, \ldots$, we use the notation

$$(\alpha : q)_n := \prod_{j=0}^{n-1} (1 - \alpha q^j) \tag{3.1}$$

for $n \geqslant 1$ and $(\alpha : q)_n := 1$ for $n = 0$. We call (3.1) the q-shifted factorial of n. Then a q-analogue of the Gauss hypergeometric function which we use in this paper is defined as follows.

DEFINITION 3.1. For suitable parameters α, β and γ, the analytic function ${}_2\varphi_1$ of ζ defined by

$${}_2\varphi_1 \begin{pmatrix} \alpha & & \beta & \\ & \gamma & & : q; \zeta \end{pmatrix} := \sum_{n=0}^{\infty} \frac{(\alpha : q)_n (\beta : q)_n}{(\gamma : q)_n (q : q)_n} \zeta^n$$

is called the *basic hypergeometric function.*

For the purpose of obtaining better descriptions of the eigenfunctions, we define the mapping $\lambda : \mathbb{C} \backslash \{0\} \to \mathbb{C}$ by

$$\lambda(\alpha) := \frac{q^2}{(1 - q^2)^2} (1 + q^2 - (\alpha + \alpha^{-1}) q) . \tag{3.2}$$

The mapping λ satisfies $\lambda(\alpha^{-1}) = \lambda(\alpha)$ for $\alpha \in \mathbb{C} \backslash \{0\}$ and hence λ maps the set $\{0 < |\alpha| \leqslant 1\}$ onto the whole complex plane $\mathbb{C}$. The next proposition is a restatement of the results in [8].

PROPOSITION 3.2. *For* $\alpha \in \mathbb{C} \backslash \{0\}$, *the function of* ζ *defined by*

$$\varphi_\alpha(\zeta) := {}_2\varphi_1 \begin{pmatrix} q\alpha & & q\alpha^{-1} & \\ & q^2 & & : q^2; q^2 \zeta \end{pmatrix}$$

is an analytic function of ζ *for* $|\zeta| \leqslant q^{-2}$ *and satisfies* $(P - \lambda(\alpha))\varphi_\alpha = 0$.

As we have already explained in Section 0, the eigenfunctions of the operator P are described in our previous publication [8]. The statement of Proposition 3.2 is easily seen by the direct calculations. It will be mentioned in the later discussion that the function φ_α of ζ is naturally extended to the whole quantum interval $(-\infty, 0]_{q^2}$ as a sequence satisfying $(P - \lambda(\alpha))\varphi_\alpha = 0$.

REMARK 3.3. If we think of the second order ordinary differential operator of order two, the dimension of the space of the solutions is two. In our situation, also have an eigenfunction of the operator P which belongs to the same eigenvalue and linearly independent from φ_α. This function has an expression of the form:

$$\log_{q^2}(-\zeta) + (\text{an analytic function of } \zeta)\ .$$

However, this type of function is removed by the condition

$$\lim_{\zeta\to 0}\sqrt{-\zeta}[D_{q^2}\varphi](\zeta) = 0$$

which describes the domain of the self-adjoint operator $(P, \mathcal{D})$. This is regarded as a sort of Neumann condition at the boundary $\zeta = 0$. The origin is the only point where we are able to think of the locality in the quantized situation.

The eigenfunctions discussed above are for ζ satisfying $-1 \leqslant \zeta \leqslant 0$. We now pass to the eigenfunctions for $\zeta < -1$. For $\alpha \in \mathbf{C}\setminus\{0\}$, we define the function ψ_α on $(-\infty, 0]_{q^2}$ by

$$\psi_\alpha(\zeta) := \left(\frac{1}{\alpha q}\right)^n \quad \text{for } \zeta = -q^{2n} \in (-\infty, 0]_{q^2}\ .$$

PROPOSITION 3.4. *For $\alpha \notin \{\pm q^m : m \in \mathbf{N}\}$, the functions of ζ defined by*

$$\phi_{\alpha^{\pm1}}(\zeta) := \psi_{\alpha^{\pm1}}(\zeta)\,{}_2\varphi_1\left(\begin{matrix} q\alpha^{\pm1} & & q\alpha^{\pm1} \\ & q^2\alpha^{\pm2} & \end{matrix} ; q^2 ; \frac{1}{\zeta}\right)$$

are eigenfunctions of the operator P which belongs to the eigenvalue $\lambda(\alpha)$ for $-\infty < \zeta < -1$.

This is also a statement which is already implicit in [8] and a proof is obtained by direct calculations.

We remark that the statement of Proposition 3.4 holds for any α even without the assumption $\alpha \notin \{\pm q^m : m = 0, 1, \ldots\}$. For the expansion theorem for the operator P, the important cases are rather for $\alpha = \pm q^m$ for some $m = 0, 1, \ldots$. By the algebraic expression of P, we may think of the operator $(P, \mathcal{D})$ to be non-negative. Then, in view of (3.2), the possible values of α satisfying $\lambda(\alpha) \geqslant 0$ are $-1 \leqslant \alpha < 0$, $0 < \alpha \leqslant 1$ and $\alpha \in \mathbf{C}$, $|\alpha| = 1$.

Now we go into the explicit construction of the Green operator of the self-adjoint operator $(P, \mathcal{D})$. This will be used to prove an expansion theorem in the later section.

We put

$$\begin{aligned}\Phi_\alpha(\zeta) &:= A(\alpha)\phi_\alpha(\zeta)\\ &= A(\alpha)\psi_\alpha(\zeta)\,{}_2\varphi_1\left(\begin{matrix} q\alpha & & q\alpha \\ & q^2\alpha^2 & \end{matrix} : q^2; \frac{1}{\zeta}\right),\end{aligned} \tag{3.3}$$

where

$$A(\alpha) := \frac{(q^2 : q^4)_\infty}{2(q^4 : q^4)_\infty} \frac{(\alpha^{-1}q : q^2)_\infty(-\alpha q : q^2)_\infty}{\alpha q(\alpha^{-2} : q^4)_\infty}.$$

Recall that, for λ in the resolvent set $\rho(P)$ of the operator P, the Green operator $G_\lambda := (P - \lambda I)^{-1}$ of P is a bounded operator on $L^2(-\infty, 0]_{q^2}$ with its range equal to the domain $\mathcal{D}$ of P. (The operator $R(\lambda : P) := -G_\lambda$ is called the resolvent of the operator P.) We also denote by $\sigma(P) := \mathbb{C}\backslash\rho(P)$ the spectrum set. Since our operator P is self-adjoint, the spectrum $\lambda \in \sigma(P)$ is either in the set $\sigma_p(P)$ of point spectrum, or in the set $\sigma_c(P)$ of continuous spectrum.

As we have already discussed in Propositions 3.2 and 3.4, the functions $\varphi_\alpha(\zeta)$ and $\phi_{\alpha^{\pm 1}}(\zeta)$ (and hence $\Phi_{\alpha^{\pm 1}}(\zeta)$) are eigenfunctions for the operator P belonging to the same eigenvalue $\lambda(\alpha) = \lambda(\alpha^{-1})$ for $\zeta \in [-1, 0]_{q^2}$ and $\zeta \in (-\infty, -q^{-2}]_{q^2}$, respectively. Since P is esssentially a diffrerence operator of order two, we are allowed to think of the extensions of $\varphi_\alpha(\zeta)$ and $\Phi_{\alpha^{\pm 1}}(\zeta)$ to the whole quantum interval $(-\infty, 0]_{q^2}$ as sequences keeping them to be still eigenfunctions of P belonging to the same eigenvalue $\lambda(\alpha) = \lambda(\alpha^{-1})$. We use the same notations for these extended functions. Then we have the following statement due to the connection formula in Mimachi [10].

PROPOSITION 3.5. *For $\alpha \in \mathbb{C}\backslash[\{0\} \bigcup \{q^n : n \in \mathbb{Z}\}]$, the equality*

$$\varphi_\alpha(\zeta) = \Phi_\alpha(\zeta) + \Phi_{\alpha^{-1}}(\zeta), \quad \zeta \in (-\infty, 0]_{q^2}$$

holds.

We first consider the case that the complex parameter $\alpha \in \mathbb{C}$ satisfies $0 < |\alpha| < 1$ and $\alpha \notin \{\pm q^m : m = 0, 1, \dots\}$. We then consider the function $\widetilde{G}(\zeta, \eta; \alpha)$ of $(\zeta, \eta) \in (-\infty, 0]_{q^2} \times (-\infty, 0]_{q^2}$ defined by

$$\widetilde{G}(\zeta, \eta; \alpha) := \begin{cases} \varphi_\alpha(\zeta)\Phi_\alpha(\eta) & \zeta \geqslant \eta, \\ \Phi_\alpha(\zeta)\varphi_\alpha(\eta) & \zeta < \eta. \end{cases}$$

LEMMA 3.6. *For $\zeta \neq \eta$, $((P_\zeta - \lambda(\alpha))\widetilde{G})(\zeta, \eta; \alpha) = 0$.*

Proof. Due to the Propositions 3.2 and 3.4, the functions $\varphi_\alpha(\zeta)$ and $\Phi_\alpha(\zeta)$ are eigenfunctions of the operator $P = P_\zeta$ corresponding to the eigenvalue $\lambda(\alpha)$. Then the assertion follows from the definition of $\widetilde{G}(\zeta, \eta; \alpha)$. ∎

LEMMA 3.7. *The expression* $((P_\zeta - \lambda(\alpha))\widetilde{G})(\eta, \eta; \alpha) \times \eta$ *is independent of* η, *where* P_ζ *is considered to be an operator acting on the first variable* ζ.

Proof. In terms of the q-shift operator, we have the expression:

$$P - \lambda(\alpha) = \frac{q^2}{(1-q^2)^2}\left\{\left(\frac{1}{\zeta} - q^2\right) T_{q^2} + \left(\frac{1}{\zeta} - 1\right) T_{q^2}^{-1} + \left(q(\alpha + \alpha^{-1}) - \frac{2}{\zeta}\right)\right\}.$$

Therefore we obtain

$$\begin{aligned}
&\frac{(1-q^2)^2}{q^2}((P_\zeta - \lambda(\alpha))\widetilde{G})(\eta, \eta; \alpha) \\
&\quad = \left(\frac{1}{\eta} - q^2\right)\widetilde{G}(q^2\eta, \eta; \alpha) + \left(\left(q(\alpha+\alpha^{-1}) - \frac{2}{\eta}\right)\widetilde{G}\right)(\eta, \eta; \alpha) \\
&\qquad + \left(\frac{1}{\eta} - 1\right)\widetilde{G}(q^{-2}\eta, \eta; \alpha) \\
&\quad = \left(\frac{1}{\eta} - q^2\right)\varphi_\alpha(q^2\eta)\Phi_\alpha(\eta) + \left(q(\alpha+\alpha^{-1}) - \frac{2}{\eta}\right)\varphi_\alpha(\eta)\Phi_\alpha(\eta) \\
&\qquad + \left(\frac{1}{\eta} - 1\right)\varphi_\alpha(q^{-2}\eta)\Phi_\alpha(\eta).
\end{aligned} \tag{3.4}$$

On the other hand, we have

$$\left(\frac{1}{\eta} - q^2\right)\Phi_\alpha(q^2\eta) + \left(q(\alpha+\alpha^{-1}) - \frac{2}{\eta}\right)\Phi_\alpha(\eta) + \left(\frac{1}{\eta} - 1\right)\Phi_\alpha(q^{-2}\eta) = 0 \tag{3.5}$$

due to $(P - \lambda(\alpha))\Phi_\alpha(\eta) = 0$. Therefore the combination (3.4) $-$ (3.5)$\times\varphi_\alpha(\eta)$ gives us

$$\frac{(1-q^2)^2}{q^2}((P_\zeta - \lambda(\alpha))\widetilde{G})(\eta, \eta; \alpha) = \left(\frac{1}{\eta} - q^2\right)\left\{\varphi_\alpha(q^2\eta)\Phi_\alpha(\eta) - \Phi_\alpha(q^2\eta)\varphi_\alpha(\eta)\right\}.$$

Therefore we have

$$\begin{aligned}
&((P_\zeta - \lambda(\alpha))\widetilde{G})(\eta, \eta; \alpha) \times \eta \\
&\quad = \frac{q^2}{(1-q^2)^2}(1 - q^2\eta)\left\{\varphi_\alpha(q^2\eta)\Phi_\alpha(\eta) - \Phi_\alpha(q^2\eta)\varphi_\alpha(\eta)\right\} \\
&\quad = \frac{q^2}{(1-q^2)^2}F[\Phi_\alpha, \overline{\varphi_\alpha}](q^2\eta)
\end{aligned} \tag{3.6}$$

where F is the bilinear form defined by (2.2). Then, by the definition of F, we see

$$\begin{aligned}
[T_{q^2}^{-1} D_{q^2} F[\Phi_\alpha, \overline{\varphi_\alpha}]](\zeta) &= [P\Phi_\alpha](\zeta)\varphi_\alpha(\zeta) - \Phi_\alpha(\zeta)[P\varphi_\alpha](\zeta) \\
&= \lambda(\alpha)\Phi_\alpha(\zeta)\varphi_\alpha(\zeta) - \Phi_\alpha(\zeta)\lambda(\alpha)\varphi_\alpha(\zeta) = 0.
\end{aligned}$$

This proves $D_{q^2}\left\{((P_\zeta - \lambda(\alpha))\widetilde{G})(\eta, \eta; \alpha) \times \eta\right\} = 0$ and obtain the assertion. ∎

This lemma allows us to define a function of α given by

$$\widetilde{M}(\alpha) := -\frac{1}{((P_\zeta - \lambda(\alpha))\widetilde{G})(\eta, \eta; \alpha) \times \eta}.$$

We next give an expression for $\widetilde{M}(\alpha)$.

LEMMA 3.8.

$$\widetilde{M}(\alpha) = \frac{(1-q^2)^2}{q^3} \times \frac{1}{A(\alpha)A(\alpha^{-1})(\alpha - \alpha^{-1})}.$$

Proof. By Lemma 3.7, we are allowed to look for the value of $\widetilde{M}(\alpha)$ by taking a limit with respect to the variable η. Therefore, by (3.3) we have

$$\begin{aligned}
-\frac{1}{\widetilde{M}(\alpha)} &= \lim_{\eta\to-\infty}((P_\zeta - \lambda(\alpha))\widetilde{G})(\eta, \eta; \alpha) \times \eta \\
&= \frac{q^2}{(1-q^2)^2} \lim_{\eta\to-\infty} \{\varphi_\alpha(q^2\eta)\Phi_\alpha(\eta) - \Phi_\alpha(q^2\eta)\varphi_\alpha(\eta)\} \\
&= -\frac{q^2}{(1-q^2)^2} \lim_{\eta\to-\infty} \eta \{\varphi_\alpha(\eta)\Phi_\alpha(q^{-2}\eta) - \Phi_\alpha(\eta)\varphi_\alpha(q^{-2}\eta)\} \\
&= -\frac{q^2}{(1-q^2)^2} \lim_{\eta\to-\infty} \eta\{(\Phi_\alpha(\eta) + \Phi_{\alpha^{-1}}(\eta))\Phi_\alpha(q^{-2}\eta) \\
&\quad - \Phi_\alpha(\eta)(\Phi_\alpha(q^{-2}\eta) + \Phi_{\alpha^{-1}}(q^{-2}\eta))\}
\end{aligned} \tag{3.7}$$

where we used (3.6) for the second equality and Proposition 3.5 for the final equality. Then by making use of the fact:

$$\lim_{\eta\to-\infty} {}_2\varphi_1 \left(\begin{matrix} q\alpha^{\pm1} & & q\alpha^{\pm1} \\ & q^2\alpha^{\pm2} & \end{matrix} : q^2; \frac{1}{\eta} \right) = 1,$$

we see that the right hand side of (3.7) is equal to

$$\begin{aligned}
&-\frac{q^2}{(1-q^2)^2} \lim_{\eta\to-\infty} \eta\Big\{(A(\alpha)\psi_\alpha(\eta) + A(\alpha^{-1})\psi_{\alpha^{-1}}(\eta))A(\alpha)\psi_\alpha(q^{-2}\eta) \\
&\qquad - A(\alpha)\psi_\alpha(\eta)(A(\alpha)\psi_\alpha(q^{-2}\eta) + A(\alpha^{-1})\psi_{\alpha^{-1}}(q^{-2}\eta))\Big\} \\
&= -\frac{q^2}{(1-q^2)^2} A(\alpha)A(\alpha^{-1}) \lim_{\eta\to-\infty} \eta\{\psi_{\alpha^{-1}}(\eta)\psi_\alpha(q^{-2}\eta) - \psi_\alpha(\eta)\psi_{\alpha^{-1}}(q^{-2}\eta)\} \\
&= -\frac{q^2}{(1-q^2)^2} A(\alpha)A(\alpha^{-1}) \lim_{N\to+\infty} (-q^{-2N})\Big\{\alpha^{-N}q^N\alpha^{N+1}q^{N+1} \\
&\qquad - \alpha^N q^N \alpha^{-(N+1)} q^{N+1}\Big\} \\
&= \frac{q^3}{(1-q^2)^2} A(\alpha)A(\alpha^{-1})(\alpha - \alpha^{-1})
\end{aligned}$$

where we used the fact that for $|\alpha| < 1$, $\psi_\alpha(\eta)$ goes to zero as η goes to $-\infty$ for the first equality, and we also put $\eta = -q^{-2N}$ for the second equality. This proves the assertion. ∎

The combination of the statements of Lemmas 3.6, 3.7 and 3.8, we obtain:

$$[(P_\zeta - \lambda(\alpha))(\widetilde{M}(\alpha)\widetilde{G})](\zeta,\eta;\alpha) = \frac{1}{-\eta}\delta_\eta(\zeta),$$

where the delta function δ_η of ζ is defined by

$$\delta_\eta(\zeta) := \begin{cases} 1 & \text{for } \zeta = \eta, \\ 0 & \text{for } \zeta \neq \eta. \end{cases}$$

Therefore, we have the following theorem:

THEOREM 3.9. *For a complex parameter $\alpha \in \mathbf{C}$ satisfying $0 < |\alpha| < 1$ and $\Im\lambda(\alpha) \neq 0$, we put*

$$G(\zeta,\eta;\alpha) := \begin{cases} M(\alpha)\varphi_\alpha(\zeta)\Phi_\alpha(\eta) & \text{for } \zeta \geq \eta, \\ M(\alpha)\Phi_\alpha(\zeta)\varphi_\alpha(\eta) & \text{for } \zeta < \eta, \end{cases}$$

where

$$M(\alpha) := \frac{1}{1-q^2}\widetilde{M}(\alpha) = \frac{1-q^2}{q^3} \times \frac{1}{A(\alpha)A(\alpha^{-1})(\alpha^{-1}-\alpha)}.$$

Then the followings hold:

(i) *For a fixed $\eta \in (-\infty, 0]_{q^2}$, $G(\zeta,\eta;\alpha)$ belongs to the domain $\mathcal{D}$ of the operator P as a function of ζ. Conversely, for a fixed $\zeta \in (-\infty, 0]_{q^2}$, $G(\zeta,\eta;\alpha)$ also belongs to the domain $\mathcal{D}$ as a function of η.*

(ii) *For any $\varphi \in L^2(-\infty, 0]_{q^2}$, we have*

$$(P - \lambda(\alpha)) \int_{-\infty}^{0} G(\zeta,\eta;\alpha)\varphi(\eta)\, d_{q^2}\eta = \varphi(\zeta).$$

Proof. (i) In the case that the parameter α satisfies $0 < |\alpha| < 1$, the functions φ_α and Φ_α of ζ are locally square summable (in the sense of Jackson integral) at the neighbourhoods of zero and $-\infty$. Then it was also seen that these are eigenfunctions of the operator P belonging to the eigenvalue $\lambda(\alpha)$. Finally, It is observed that the function φ_α satisfies the boundary condition (2.4). This proves the assertion.

(ii) The operator $P - \lambda(\alpha)$ is a difference operator acting on the space of sequences and hence commutes with the operation of taking the Jackson integral. Therefore, Lemma 3.6 enables us to have:

$$\begin{aligned}(P - \lambda(\alpha)) \int_{-\infty}^{0} G(\zeta,\eta;\alpha)\varphi(\eta)\, d_{q^2}\eta &= \int_{-\infty}^{0} [(P_\zeta - \lambda(\alpha))G](\zeta,\eta;\alpha)\varphi(\eta)\, d_{q^2}\eta \\ &= \int_{-\infty}^{0} (1-q^2)\frac{1}{-\eta}\delta_\eta(\zeta)\varphi(\eta)\, d_{q^2}\eta = \varphi(\zeta).\end{aligned}$$

This proves the equality. ∎

Let $G_{\lambda(\alpha)}$ be the integral operator given by $G(\zeta,\eta;\alpha)$ as the integral kernel i.e.

$$[G_{\lambda(\alpha)}\varphi](\zeta) := \int_{-\infty}^{0} G(\zeta,\eta;\alpha)\varphi(\eta)\,d_{q^2}\eta \quad \text{for} \quad \varphi \in L^2(-\infty,0]_{q^2}.$$

Then we have the final conclusion of this section that the operator $G_{\lambda(\alpha)}$ is the Green operator of $(P-\lambda(\alpha))$ i.e. we have the following theorem.

THEOREM 3.10. (i) *The operator $G_{\lambda(\alpha)}$ is defined on the whole Hilbert space $L^2(-\infty,0]_{q^2}$ and maps onto the domain $\mathcal{D}$ of the operator P. Furthermore, the operator $G_{\lambda(\alpha)}$ is continuous with respect to the norm topology in $L^2(-\infty,0]_{q^2}$ and the graph norm topology in $\mathcal{D}$.*

(ii) *The operator $(P-\lambda(\alpha))G_{\lambda(\alpha)}$ is the identity operator on $L^2(-\infty,0]_{q^2}$.*

4. THE SPECTRUM OF THE OPERATOR

This is the section in which we give a detailed description of the self-adjoint operator P using the Green function which we described in the previous section. We start by listing up some elementary properties of the mapping $\mathbf{C}\backslash\{0\} \ni \alpha \mapsto \lambda(\alpha) \in \mathbf{C}$ defined by (3.2). The following properties are quite easily seen.

LEMMA 4.1. *The mapping λ satisfies the following properties:*

(i) *The mapping λ maps the punctured disk $\{\alpha \in \mathbf{C}; 0 < |\alpha| \leqslant 1\}$ onto the whole complex plane $\mathbf{C}$.*

(ii) *The mapping λ maps the set $\{\alpha \in \mathbf{C}; |\alpha| = 1\,,\, 0 < \arg(\alpha) < \pi\}$ to the bounded open interval $\left(\frac{q^2}{(1+q)^2}, \frac{q^2}{(1-q)^2}\right)$ homeomorphically.*

(iii) *The mapping λ maps the punctured open disk $\{\alpha \in \mathbf{C}; 0 < |\alpha| < 1\}$ to the domain $\mathbf{C}\backslash\left[\frac{q^2}{(1+q)^2}, \frac{q^2}{(1-q)^2}\right]$ biholomorphically.*

We next study the property of the Green kernel $G(\zeta,\eta;\alpha)$ as an analytic function of α. The following lemma is easily seen by Theorem 3.9.

LEMMA 4.2. *For fixed ζ and η in $(-\infty,0]_{q^2}$, the Green kernel $G(\zeta,\eta;\alpha)$ regarded as a function of α extends to give a meromorphic function defined on the open punctured disk $\{\alpha \in \mathbf{C} : 0 < |\alpha| < 1\}$, in which the poles of $G(\zeta,\eta;\alpha)$ locate at the points $\alpha = -q^{2m+1}$, $m = 0,1,\ldots$. The degrees of these poles are all equal to one.*

REMARK 4.3. It is seen that for fixed ζ and η in $(-\infty,0]_{q^2}$, the Green kernel $G(\zeta,\eta;\alpha)$ regarded as a function of α extends to give a meromorphic function defined on $\mathbf{C}\backslash\{0\}$ and the poles of $G(\zeta,\eta;\alpha)$ locate at the points $\alpha=-q^{2m+1}$ and $\alpha=q^{-(2m+1)}$, $m=0,1,\ldots$. The degrees of these poles are all equal to one.

LEMMA 4.4. *The set $\sigma_c(P)$ of continuous spectrum includes the closed interval $\left[\frac{q^2}{(1+q)^2},\frac{q^2}{(1-q)^2}\right]$.*

Proof. Any non trivial solution φ of the difference equation $(P-\lambda(e^{i\theta}))\varphi=0$ $(0\leqslant\theta\leqslant\pi)$ does not converge to zero as $\zeta\to-\infty$ and hence does not belong to the Hilbert space $L^2(-\infty,0]_{q^2}$. As a result, there exists no point spectrum in the interval $\left[\frac{q^2}{(1+q)^2},\frac{q^2}{(1-q)^2}\right]=\{\lambda(e^{i\theta});0\leqslant\theta\leqslant\pi\}$. Then by setting $\lambda_0=\lambda(e^{i\theta})$, we put

$$\varphi^N_{\lambda_0}(\zeta):=\begin{cases}\psi_{e^{i\theta}}(\zeta) & \text{for } \zeta\in[-q^{-2N},-1]_{q^2},\\ 0 & \text{otherwise}.\end{cases}$$

After an easy computation, we have

$$\|(P-\lambda_0)\varphi^N_{\lambda_0}\|/\|\varphi^N_{\lambda_0}\|\to 0 \quad \text{as} \quad N\to\infty,$$

which proves the assertion. ∎

It was stated in Lemma 4.1 (iii) that the mapping $\alpha\mapsto\lambda(\alpha)$ maps the set $\{\alpha\in\mathbf{C};0<|\alpha|<1\}$ to the set $\mathbf{C}\backslash\left[\frac{q^2}{(1+q)^2},\frac{q^2}{(1-q)^2}\right]$ biholomorphically. Let $\lambda\mapsto\alpha(\lambda)$ be the mapping inverse to the mapping $\alpha\mapsto\lambda(\alpha)$. The next statement is important to determine the spectrum of the operator P.

LEMMA 4.5. *The values $\lambda(1)=\frac{q^2}{(1+q)^2}$ and $\lambda(-1)=\frac{q^2}{(1-q)^2}$ do not belong to the set $\sigma_p(P)$ of the point spectrum of the operator P.*

Proof. We prove the assertion for the case $\lambda(1)$. We put

$$\phi_\alpha(\zeta):=\psi_\alpha(\zeta)\,{}_2\varphi_1\left(\begin{matrix}\alpha q & & \alpha q\\ & \alpha^2q^2 & \end{matrix}:q^2;\frac{1}{\zeta}\right)$$

for a complex parameter $\alpha\in\mathbf{C}\backslash\{0\}$.

For $\alpha\in\mathbf{C}\backslash[\{0\}\bigcup\{\pm q^m:m=0,1,\ldots\}]$, the functions ϕ_α and $\phi_{\alpha^{-1}}$ are the linear independent solutions of the equation $(P-\lambda(1))\varphi=0$. Therefore, by taking the limit $\alpha\to1$, we see that the linearly independent solutions of the difference equation $(P-\lambda(1))\varphi=0$ are given by ϕ_1 and

$$\widetilde{\phi}:=\lim_{\alpha\to1}\frac{1}{\alpha-\alpha^{-1}}\{\phi_\alpha-\phi_{\alpha^{-1}}\}.$$

By easy computations, we have

$$q^n|\phi_1(-q^{2n})-1|=O(q^{-n}),\quad \text{and}\quad q^n|\widetilde{\phi}(-q^{2n})+n|=O(q^{-n}),$$

as $n\to-\infty$. On the other hand, for a function $\phi\in L^2(-\infty,0]_{q^2}$, we have $q^n\phi(-q^{2n})\to0$, as $n\to-\infty$. Therefore, any linear combination of ϕ_1 and $\widetilde{\phi}$ does not belong to $L^2(-\infty,0]_{q^2}$, which proves the assertion. ∎

PROPOSITION 4.6. (i) *The set* $\sigma_c(P)$ *of the continuous spectrum of* P *coinsides with* $\left[\frac{q^2}{(1+q)^2}, \frac{q^2}{(1-q)^2}\right]$.

(ii) *The set* $\sigma_p(P)$ *of the point spectrum of* P *coinsides with* $\{\lambda(-q^{2m+1}) : m = 0, 1, \ldots\}$.

Proof. By Theorem 3.9, we find that the Green kernel $G(\zeta, \eta; \alpha(\lambda))$ of P is holomorphic with recpect to λ except on the set $\left[\frac{q^2}{(1+q)^2}, \frac{q^2}{(1-q)^2}\right] \bigcup \{\lambda(-q^{2m+1}) : m = 0, 1, \ldots\}$. Moreover, $G(\zeta, \eta; \alpha(\lambda))$ as a function of λ has poles at $\lambda = \lambda(-q^{2m+1}) : m = 0, 1, 2 \ldots$. By the combination of these facts and Lemmas 4.4, and 4.5, we have the assertion. ∎

We next show that the values $\lambda = \lambda(-q^{2m+1})$, $m = 0, 1, \ldots$ are the eigenvalues of the operator P. This is equivalent to show that the function φ_α of ζ for $\alpha = -q^{2m+1}$ for $m = 0, 1, \ldots$ is in the Hilbert space $L^2(-\infty, 0]_{q^2}$. For that purpose, the connection formula (Proposition 3.5) combined with the process of limit plays an important role.

PROPOSITION 4.7. $\varphi_{-q^{2m+1}}(\zeta) = 2\Phi_{-q^{2m+1}}(\zeta)$.

Proof. The connection formula $\varphi_\alpha(\zeta) = \Phi_\alpha(\zeta) + \Phi_{\alpha^{-1}}(\zeta)$ holds for $\alpha \in \mathbb{C}$, $0 < |\alpha| \leqslant 1$, $\alpha \neq \pm q^m$ for $m = 0, 1, \ldots$. Therefore, by fixing $m = 0, 1, \ldots$, we take the limit $\alpha \to -q^{2m+1}$ inside the set that the connection formula makes sense. Then due to the regularity of φ_α and Φ_α on the sets $\mathbb{C}\backslash\{0\}$ and $\{\alpha \in \mathbb{C} : 0 < |\alpha| < \frac{1}{q^2}\}$, respectively, we have

$$\lim_{\alpha \to -q^{2m+1}} \varphi_\alpha(\zeta) = \varphi_{-q^{2m+1}}(\zeta) \text{ and } \lim_{\alpha \to -q^{2m+1}} \Phi_\alpha(\zeta) = \Phi_{-q^{2m+1}}(\zeta)\,.$$

Therefore it is sufficient to obtain the value of the limit $\lim_{\alpha \to -q^{2m+1}} \Phi_{\alpha^{-1}}(\zeta)$. Now we remember

$$\Phi_{\alpha^{-1}}(\zeta) = \frac{(q^2 : q^4)_\infty}{2(q^4 : q^4)_\infty} \frac{(\alpha q : q^2)_\infty (-\alpha^{-1} q : q^2)_\infty}{\alpha^{-1} q (q^2 : q^4)_\infty} \psi_\alpha(\zeta) \times \sum_{n=0}^{\infty} \frac{(\alpha^{-1} q : q^2)_n^2}{(\alpha^{-2} q^2 : q^2)_n (q^2 : q^2)_n} \zeta^{-n}$$

and hence $(-\alpha^{-1} q : q^2)_\infty \to 0$ and $(-\alpha^{-2} q^2 : q^2)_n \to 0$ as $\alpha \to -q^{2m+1}$ for $n \geqslant 2m - 1$. In the following discussions we put $k = 2m - 1$. We have

$$\lim_{\alpha \to -q^k} \Phi_{\alpha^{-1}}(\zeta) = \frac{(q^2 : q^4)_\infty}{2(q^4 : q^4)_\infty} \frac{(-q^{k+1} : q^2)_\infty}{-q^{-k+1} (q^{2k} : q^4)_\infty} \psi_{q^{-k}}(\zeta) \times \lim_{\alpha \to -q^k} (-\alpha^{-1} q : q^2)_\infty \sum_{n=k}^{\infty} \frac{(-q^{-k+1} : q^2)_n^2}{(\alpha^{-2} q^2 : q^2)_n (q^2 : q^2)_n} \zeta^{-n}\,.$$

Then by making use of the equalities $(q^2 : q^2)_n = (q^2 : q^2)_k (q^{2k+2}; q^2)_{n-k}$ for $n \geqslant k$ and $(q^{-k+1} : q^2)_n = (-q^{-k+1} : q^2)_k^2 (-q^{k+1} : q^2)_{n-k}$, we have

$$\lim_{\alpha \to -q^k} \frac{(-\alpha^{-1} q : q^2)_\infty}{(\alpha^{-2} q^2 : q^2)_n} = \frac{1}{(q^2 : q^2)_{n-k}} \lim_{\alpha \to -q^k} \frac{(-\alpha^{-1} q : q^2)_\infty}{(\alpha^{-2} q^2 : q^2)_k}$$
$$= \frac{1}{(q^2 : q^2)_{n-k}} \frac{(q^{-k+1} : q^2)_{m-1} (q^2 : q^2)_\infty}{2(q^{-2k+2} : q^2)_{k-1}} .$$

Therefore we obtain

$$\lim_{\alpha \to -q^k} \Phi_{\alpha^{-1}}(\zeta) = A \psi_{-q^k}(\zeta) \sum_{n=k}^{\infty} \frac{(-q^{k+1} : q^2)_{n-k}}{(q^{2k+2} : q^2)_{n-k} (q^2 : q^2)_{n-k}} \zeta^{-n}$$
$$= A \psi_{-q^k}(\zeta) \zeta^{-k} {}_2\varphi_1 \begin{pmatrix} -q^{k+1} & & -q^{k+1} & \\ & q^{2k+2} & & : q^2; \frac{1}{\zeta} \end{pmatrix}$$

where A is the constant given by

$$A = \frac{(q^2 : q^4)_\infty}{2(q^4 : q^4)_\infty} \frac{(-q^{k+1} : q^2)_\infty (-q^{-k+1} : q^2)_k}{(-q^{-k+1})(q^2 : q^2)_k (q^{2k} : q^2)_\infty} \frac{(-q^{-k+1} : q^2)_{m-1} (q^2 : q^2)_\infty}{2(q^{-2k+2} : q^2)_{k-1}} .$$

Since $k = 2m - 1$ is an odd number, we have $\psi_{-q^{-k}} = -\psi_{-q^k}$. Furthermore, after a straightforward computation, we have $A = -A(-q^k)$. Therefore, we obtain $\lim_{\alpha \to -q^k} \Phi_{\alpha^{-1}}(\zeta) = \Phi_{-q^k}(\zeta)$,which proves the assertion. ∎

COROLLARY 4.8. *The functions $\varphi_{-q^{2m+1}}$, $m = 0, 1, \ldots$ of ζ are the eigenfunctions of the operator P and the corresponding eigenvalues are given by*

$$\lambda(-q^{2m+1}) = \frac{q^2}{(1 - q^2)^2} (1 + q^{2m})(1 + q^{2m-2}), m = 0, 1, \ldots .$$

Furthermore, these are point spectra of multiplicity free.

Proof. Since the function φ_α of ζ satisfies the equation $P\varphi_\alpha = \lambda(\alpha)\varphi_\alpha$ and hence, we see that the function $\varphi_{-q^{2m+1}}$ belongs to the local Hilbert space $L^2[-1, 0]_{q^2}$. Then due to the equality shown in Lemma 3.7, we have

$$\varphi_{-q^{2m+1}}(\zeta) = 2\Phi_{-q^{2m+1}}(\zeta) = 2A(-q^{2m+1}) \psi_{-q^{2m+1}}(\zeta)$$
$$\times {}_2\varphi_1 \begin{pmatrix} -q^{2m+2} & & -q^{2m+2} & \\ & q^{4m+4} & & : q^2; \frac{1}{\zeta} \end{pmatrix}$$

for $\zeta \in (-\infty, -q^{-2}]_{q^2}$ and hence $\varphi_{-q^{2m+1}}$ again belongs to the local Hilbert space $L^2(-\infty, -q^{-2}]_{q^2}$. We also remark here that the values $A(-q^{2m+1})$ are non-zero for $m = 0, 1, \ldots$. This proves that the function $\varphi_{-q^{2m+1}}$ of ζ belongs to the Hilbert space $L^2(-\infty, 0]_{q^2}$. ∎

5. THE EIGENFUNCTION EXPANSION

This section is devoted to the discussion on the spectral expansion theorem corresponding to the continuous and the point spectrum, respectively. Let $\{E(\lambda)\}_{\lambda\in\mathbf{R}}$ be the decomposition of the unity associated with the self-adjoint operator $(P,\mathcal{D})$. (For the detailed property of the decomposition of the unity, see [15]). By making use of $\{E(\lambda)\}_{\lambda\in\mathbf{R}}$, we have the following two types of projections:

$$E([\lambda(1),\lambda(-1)]) = E\left(\left[\frac{q^2}{(1+q)^2},\frac{q^2}{(1-q)^2}\right]\right), \tag{5.1}$$

$$E(\{\lambda(-q^{2n+1})\}) = E\left(\left\{\frac{q^2}{(1-q)^2}(1+q^{2n-2})(1+q^{-2n})\right\}\right) \tag{5.2}$$

Here, the projections (5.1) and (5.2) correspond to the continuous and the point spectrum, respectively. We first give explicit expressions for these spectral projections.

It is seen by Lemma 4.5 that the values $\lambda(1)$ and $\lambda(-1)$ are not in the point spectrum. We define $E^{(\varepsilon)}$ by making use of the Green operator as:

$$E^{(\varepsilon)} := \frac{1}{2\pi \mathrm{i}} \int_{\lambda(1)}^{\lambda(-1)} \{G_{\mu+\mathrm{i}\varepsilon} - G_{\mu-\mathrm{i}\varepsilon}\}\,\mathrm{d}\mu\,,$$

then, by the spectral decomposition theorem for a self adjoint operator, we have

$$\begin{aligned} E([\lambda(1),\lambda(-1)]) &= E((\lambda(1),\lambda(-1))) \\ &= \frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\downarrow 0} \int_{\lambda(1)}^{\lambda(-1)} \{G_{\mu+\mathrm{i}\varepsilon} - G_{\mu-\mathrm{i}\varepsilon}\}\,\mathrm{d}\mu \\ &= \lim_{\varepsilon\downarrow 0} E^{(\varepsilon)}\,. \end{aligned}$$

Let $E^{(\varepsilon)}(\zeta,\eta)$ be the integral kernel of the operator $E^{(\varepsilon)}$. Then Theorem 3.9 asserts us that the following equality holds:

$$E^{(\varepsilon)}(\zeta,\eta) = \frac{1}{2\pi\mathrm{i}} \int_{\lambda(1)}^{\lambda(-1)} \{G(\zeta,\eta;\lambda(\mu+\mathrm{i}\varepsilon)) - G(\zeta,\eta;\lambda(\mu-\mathrm{i}\varepsilon))\}\,\mathrm{d}\mu\,. \tag{5.3}$$

We now apply the change of the variable μ to θ in the integral (5.3) by

$$\mu = \mu(e^{i\theta}) := \frac{q^2}{(1-q^2)^2}(1+q^2-(e^{i\theta}+e^{-i\theta})q).$$

Then this transformation maps $\theta \in [0,\pi]$ to $\mu \in [\lambda(1),\lambda(-1)]$ and the formula (5.3) reduces to

$$E^{(\epsilon)}(\zeta,\eta) = \frac{1}{2\pi i}\int_0^\pi \left\{G(\zeta,\eta;\alpha_\epsilon(\theta)) - G(\zeta,\eta;\overline{\alpha_\epsilon(\theta)})\right\}\frac{d\mu}{d\theta}$$

$$= \frac{1}{2\pi i}\frac{q^3}{(1-q^2)^2}\int_0^\pi \left\{G(\zeta,\eta;\alpha_\epsilon(\theta)) - G(\zeta,\eta;\overline{\alpha_\epsilon(\theta)})\right\}(e^{-i\theta}-e^{i\theta})\,d\theta$$

where we put $\alpha_\epsilon(\theta) := \alpha(\mu(\theta)+i\epsilon)$ for $\epsilon > 0$ and $\theta \in (0,\pi)$. To see the behaviour of $\epsilon \downarrow 0$, we need the next straightforward lemma.

LEMMA 5.1. *For a fixed $\theta \in (0,\pi)$, we have $\lim_{\epsilon\downarrow 0}\alpha_\epsilon(\theta) = e^{i\theta}$.*

Now we fix the parameters ζ, η and $\theta \in (0,\pi)$. We first assume that the condition $\eta \leqslant \zeta$ is satisfied. (The case $\zeta < \eta$ is similar.) Then we have

$$G(\zeta,\eta;\alpha_\epsilon(\theta))(e^{-i\theta}-e^{i\theta})$$
$$= \frac{1-q^2}{q^3}\frac{1}{A(\alpha_\epsilon(\theta))A(\alpha_\epsilon(\theta)^{-1})}\frac{e^{-i\theta}-e^{i\theta}}{\alpha_\epsilon(\theta)^{-1}-\alpha_\epsilon(\theta)}\varphi_{\alpha_\epsilon}(\theta)(\zeta)\Phi_{\alpha_\epsilon(\theta)}(\zeta)$$

which converges to

$$\frac{1-q^2}{q^3}\frac{1}{A(e^{i\theta})A(e^{-i\theta})}\varphi_{e^{i\theta}}(\zeta)\Phi_{e^{i\theta}}(\eta)$$

as $\epsilon \downarrow 0$ due to Lemma 5.1. We also have the similar argument for the factor $G(\zeta,\eta;\overline{\alpha_\epsilon(\theta)})$ to make sure that under the same condition of taking limit, we obtain

$$G(\zeta,\eta;\overline{\alpha_\epsilon(\theta)}) \to -\frac{1-q^2}{q^3}\frac{1}{A(e^{i\theta})A(e^{-i\theta})}\varphi_{e^{i\theta}}(\zeta)\Phi_{e^{-i\theta}}(\eta)$$

where we used the fact that $\varphi_\alpha = \varphi_{\alpha^{-1}}$ for $\alpha = e^{i\theta}$. Therefore we see that the integrand

$$\left\{G(\zeta,\eta;\alpha_\epsilon(\theta)) - G(\zeta,\eta;\overline{\alpha_\epsilon(\theta)})\right\}$$

converges to

$$\frac{1-q^2}{q^3}\frac{1}{A(e^{i\theta})A(e^{-i\theta})}\varphi_{e^{i\theta}}(\zeta)\{\Phi_{e^{i\theta}}(\eta)+\Phi_{e^{-i\theta}}(\eta)\}$$
$$= \frac{1-q^2}{q^3}\frac{1}{A(e^{i\theta})A(e^{-i\theta})}\varphi_{e^{i\theta}}(\zeta)\varphi_{e^{i\theta}}(\eta)$$

as $\varepsilon \downarrow 0$, where we used the connection formula (Proposition 3.5) for the last equality.

For the purpose of computing the strong limit of $E^{(\varepsilon)}$ as $\varepsilon \downarrow 0$, we need to have some properties of the integral kernel. Lemma 4.2 tells us that the Green kernel $G(\zeta, \eta; \alpha)$ is a meromorphic function for $\alpha \in \mathbb{C}\backslash\{0\}$ with its poles given by $\alpha = -q^{2m+1}, m = 1, 2, \ldots$ and $\alpha = \pm q^{-m}, m = 1, 2, \ldots$. In particular, on a suitable compact subset like $\{\alpha \in \mathbb{C} : q^{1/2} \leqslant |\alpha| \leqslant q^{-1/2}\}$ containing the set $\{\alpha \in \mathbb{C} : |\alpha| = 1\}$, the function of α is uniformly bounded. This fact combined with the Lebesgue's convergence theorem lead us to the following Proposition.

PROPOSITION 5.2. *For any fixed ζ and η, we have*

$$\lim_{\varepsilon \downarrow 0} E^{(\varepsilon)}(\zeta, \eta) = \frac{1}{2\pi} \frac{1}{1 - q^2} \int_0^\pi \frac{1}{A(e^{i\theta})A(e^{-i\theta})} \varphi_{e^{i\theta}}(\zeta) \varphi_{e^{i\theta}}(\eta) \, d\theta .$$

Now we put

$$c(\theta) := \frac{1}{2\pi} \frac{1}{1 - q^2} \frac{1}{A(e^{i\theta})A(e^{-i\theta})} .$$

Then we have the following straightforward Lemma.

LEMMA 5.3.

$$c(\theta) = \frac{2}{\pi} \frac{q^2}{1 - q^2} \left\{ \frac{(q^4 : q^4)_\infty}{(q^2 : q^4)_\infty} \right\}^2 \frac{(e^{2i\theta} : q^4)_\infty (e^{-2i\theta} : q^4)_\infty}{(e^{2i\theta} q^2 : q^4)_\infty (e^{-2i\theta} q^2 : q^4)_\infty} .$$

The combination of the discussions which we gave, we get to the following statement which is one of the main theorem in this section.

THEOREM 5.4. *Let $c(\theta)$ be the function discussed above. Then for any element $f \in L^2(-\infty, 0]_{q^2}$, we have*

$$[E((\lambda(1), \lambda(-1))f](\zeta) = \int_0^\pi \left\{ \int_{-\infty}^0 c(\theta) \varphi_{e^{i\theta}}(\zeta) \varphi_{e^{i\theta}}(\eta) f(\eta) \, d_{q^2}\eta \right\} d\theta$$

Proof. Let $f \in C_c(-\infty, 0]_{q^2}$ be an arbitrary fixed element. Then due to the Proposition 5.2, we have

$$\lim_{\varepsilon \downarrow 0} [E^{(\varepsilon)} f](\zeta) = \int_0^\pi \left\{ \int_{-\infty}^0 c(\theta) \varphi_{e^{i\theta}}(\zeta) \varphi_{e^{i\theta}}(\eta) f(\eta) \, d_{q^2}\eta \right\} d\theta .$$

Since the strong limit of $E^{(\epsilon)}$ as $\epsilon \downarrow 0$ goes to $E((\lambda(1),\lambda(-1))$, the left hand side of the above formula coincides with $[E((\lambda(1),\lambda(-1))f](\zeta)$. This means that for $f \in C_c(-\infty,0]_{q^2}$, we have

$$[E((\lambda(1),\lambda(-1)f](\zeta) = \int_0^{\pi} \left\{ \int_{-\infty}^{0} c(\theta)\varphi_{e^{i\theta}}(\zeta)\varphi_{e^{i\theta}}(\eta)f(\eta)\,\mathrm{d}_{q^2}\eta \right\} \mathrm{d}\theta \,.$$

Now, the operator $E((\lambda(1),\lambda(-1))$ is bounded on the Hilbert space $L^2(-\infty,0]_{q^2}$ so that by the density of $C_c(-\infty,0]_{q^2}$ in $L^2(-\infty,0]_{q^2}$, we see that the mapping

$$f \mapsto \int_0^{\pi} \left\{ \int_{-\infty}^{0} c(\theta)\varphi_{e^{i\theta}}(\zeta)\varphi_{e^{i\theta}}(\eta)f(\eta)\,\mathrm{d}_{q^2}\eta \right\} \mathrm{d}\theta$$

uniquely extends to give a bounded linear operator on the whole Hilbert space $L^2(-\infty,0]_{q^2}$ and then this proves the assertion. ∎

Next we pass to the discussions involving the point spectrum of the operator P. For the purpose of simplifying the notations, we put $E_n := E(\{\lambda(-q^{2n+1})\})$ for $n = 1,2,\ldots$. Proposition 4.6 implies that the points $\lambda(-q^{2n+1})$, $n = 1,2,\ldots$ are isolate points of the spectrum set $\sigma(P)$ of the operator P. Therefore, due to the spectral decomposition theorem, we have

$$E_n = \frac{1}{2\pi \mathrm{i}} \oint_{C_n} G_\lambda \,\mathrm{d}\lambda \,,$$

where C_n is a contour separating the point $\lambda(-q^{2n+1})$ from the spectrum set $\sigma(P)$. Then by the change of the variable, we have

$$E_n = \frac{1}{2\pi \mathrm{i}} \oint_{\widetilde{C}_n} G_{\lambda(\alpha)} \frac{\mathrm{d}\lambda}{\mathrm{d}\alpha} \mathrm{d}\alpha \,,$$

where $\widetilde{C}_n$ is a contour separating the point $-q^{2n+1}$ from the set $\alpha(\sigma(P))$. Therefore by putting the integral kernel of the operator E_n by $E_n(\zeta,\eta)$, we have

$$E_n(\zeta,\eta) = \frac{1}{2\pi \mathrm{i}} \oint_{\widetilde{C}_n} G(\zeta,\eta;\lambda(\alpha)) \frac{\mathrm{d}\lambda}{\mathrm{d}\alpha} \mathrm{d}\alpha \,.$$

The function of α given by $G(\zeta,\eta;\alpha) = G(\zeta,\eta;\lambda(\alpha))$ has simple poles at the points $\alpha = \alpha_n = -q^{2n+1}$ for $n = 0, 1, \ldots$. Therefore the residue calculus gives us

$$\begin{aligned} E_n(\zeta,\eta) &= \frac{q^3}{(1-q^2)^2} \operatorname{Res}_{\alpha=\alpha_n} \ \{G(\zeta,\eta;\alpha)(1-\alpha^{-2})\} \\ &= \frac{q^3}{(1-q^2)^2} \lim_{\alpha\to\alpha_n} \{(\alpha-\alpha_n)G(\zeta,\eta;\alpha)(1-\alpha^{-2})\} . \end{aligned}$$

In the following discussions, we assume $\eta \leqslant \zeta$. Then Theorem 3.9 applies to obtain

$$E_n(\zeta,\eta) = -\frac{1}{1-q^2} \lim_{\alpha\to\alpha_n} \left\{ \frac{\alpha^{-1}(\alpha-\alpha_n)\varphi_\alpha(\zeta)\Phi_\alpha(\eta)}{A(\alpha)A(\alpha^{-1})} \right\} .$$

Since $A(\alpha^{-1})$ has poles of order one at $\alpha_n := -q^{2n+1}$ for $n = 0, 1, \ldots$. Hence we obtain

$$\begin{aligned} E_n(\zeta,\eta) &= -\frac{1}{1-q^2} \frac{\alpha_n^{-1}}{A(\alpha_n)} \varphi_{\alpha_n}(\zeta)\Phi_{\alpha_n}(\eta) \lim_{\alpha\to\alpha_n} \frac{\alpha-\alpha_n}{A(\alpha^{-1})} \\ &= -\frac{4}{1-q^2} \left\{ \frac{(q^4:q^4)_\infty}{(q^2:q^4)_\infty} \right\}^2 \frac{q^2}{(q^2:q^2)_\infty} \frac{1}{(q^{-2}:q^{-2})_n} \\ &\quad \times \frac{(\alpha_n^2:q^4)_\infty(\alpha_n^{-2}:q^4)_\infty}{(\alpha_n q:q^2)_\infty(\alpha_n^{-1}q:q^2)_\infty(-\alpha_n q:q^2)_\infty} \varphi_{\alpha_n}(\zeta)\Phi_{\alpha_n}(\eta) \\ &= -\frac{2}{1-q^2} q^{-2n}(-1) \frac{(q^2:q^4)_{n+1}(q^4:q^4)_n}{(q^2:q^2)_n(-q^2:q^2)_n(q^2:q^4)_n} \varphi_{\alpha_n}(\zeta)\Phi_{\alpha_n}(\eta) \\ &= 2\frac{q^{2n+1}-q^{-(2n+1)}}{q-q^{-1}} \varphi_{\alpha_n}(\zeta)\Phi_{\alpha_n}(\eta) \\ &= 2[2n+1]_q \varphi_{\alpha_n}(\zeta)\Phi_{\alpha_n}(\eta) . \end{aligned}$$

Now, Proposition 4.7 applies for the case $\alpha_n = -q^{2n+1}$ applies to obtain the equality $\varphi_{-q^{2n+1}} = 2\Phi_{-q^{2n+1}}$ and hence we obtain

$$E_n(\zeta,\eta) = [2n+1]_q \varphi_{-q^{2n+1}}(\zeta)\varphi_{-q^{2n+1}}(\eta) .$$

In the above discussions, we have assumed the condition $\eta \leqslant \zeta$. But the discussion is similar for the case $\eta > \zeta$ to obtain the same equality.

The above arguments give us the following Theorem.

THEOREM 5.5. *We have* $E(\{\lambda(-q^{2n+1})\})f = [2n+1]_q (f, \varphi_{-q^{2n+1}})\varphi_{-q^{2n+1}}$ *i.e.*

$$[E(\{\lambda(-q^{2n+1})\})f](\zeta) = [2n+1]_q \left\{ \int_{-\infty}^{0} f(\eta)\varphi_{-q^{2n+1}}(\eta)\, d_{q^2}\eta \right\} \varphi_{-q^{2n+1}}(\zeta)$$

for any element $f \in L^2(-\infty, 0]_{q^2}$.

Combined with Proposition 4.6, the Theorems 5.4, and 5.5, we have the following spectral decomposition theorem.

THEOREM 5.6. *Any element f in the Hilbert space $L^2(-\infty,0]_{q^2}$ is expressed in terms of the system of the eigenfunctions $\{\varphi_{-q^{2n+1}}\}_{n=0}^{\infty}$ and the generalized eigenfunctions $\{\varphi_{e^{i\theta}} : 0 < \theta < \pi\}$ of the self-adjoint operator P as follows:*

$$f(\zeta) = \int_0^{\pi} c(\theta)\varphi_{e^{i\theta}}(\zeta)\left\{\int_{-\infty}^{0} f(\eta)\varphi_{e^{i\theta}}(\eta)\,\mathrm{d}_{q^2}\eta\right\}\mathrm{d}\theta$$
$$+\sum_{n=0}^{\infty}[2n+1]_q\varphi_{-q^{2n+1}}(\zeta)\left\{\int_{-\infty}^{0} f(\eta)\varphi_{-q^{2n+1}}(\eta)\,\mathrm{d}_{q^2}\eta\right\}$$

where $[n]_q$ is the (homogeneous) q-integer of $n \in \mathbb{Z}$ and $c(\theta)$ is the c-function described in Lemma 5.3.

COROLLARY 5.7. *The closed interval $[\frac{q^2}{(1+q)^2}, \frac{q^2}{(1-q)^2}]$ included in the spectrum set $\sigma(P)$ consists of absolutely continuous spectrum of the operator P.*

REMARK 5.8. As we emphasized in the introduction, the theorem above claimes the appearance of the point spectrum. This fact strongly suggests the existence of the strange series representations in the quantized case.

6. PLANCHEREL FORMULA

In this section, we discuss about the Plancherel type of formula associated with the spectral decomposition of the self-adjoint operator P. We denote by $\mathcal{H}_c$ the closed subspace of the whole Hilbert space $\mathcal{H} := L^2(-\infty,0]_{q^2}$ corresponding to the continuous spectrum of the operator P. Then we have $\mathcal{H}_c = E((\lambda(1),\lambda(-1)))\mathcal{H}$. We denote by $\mathcal{F} : \mathcal{H} \longrightarrow L^2(0,\pi)$ and $\widetilde{\mathcal{F}} : L^2(0,\pi) \longrightarrow \mathcal{H}$ the continuous linear mappings which we discussed in the previous section i.e.

$$[\mathcal{F}f](\theta) = \lim_{N\to\infty}\int_{-q^{-2N}}^{0} f(\eta)\varphi_{e^{i\theta}}(\eta)D(\theta)\,\mathrm{d}_{q^2}\eta$$

for $f \in \mathcal{H}$ and

$$[\widetilde{\mathcal{F}}g](\zeta) = \int_0^{\pi} g(\theta)\varphi_{e^{i\theta}}(\eta)D(\theta)\,\mathrm{d}\theta$$

for $g \in L^2(0,\pi)$, where $D(\theta)$ is the density defined by

$$D(\theta) := \sqrt{|c(\theta)|} = \sqrt{\frac{2}{\pi}\frac{q^2}{1-q^2}\frac{(q^4:q^4)_\infty}{(q^2:q^4)_\infty}\frac{|(e^{2i\theta}:q^4)_\infty|}{|(e^{2i\theta}q^2:q^4)_\infty|}}\,.$$

In the previous section we proved that the equality $\widetilde{\mathcal{F}}\mathcal{F} = E((\lambda(1), \lambda(-1)))$ holds. Then the aim of this section is to prove that the two mappings $\mathcal{F} : \mathcal{H}_c \longrightarrow L^2(0,\pi)$ and $\widetilde{\mathcal{F}} : L^2(0,\pi) \longrightarrow \mathcal{H}_c$ are isomorphic. This is the same as to say that the Plancherel theorem holds.

Our first step is to prove that the equality $\mathcal{F}\widetilde{\mathcal{F}} = \mathrm{Id}$ on the Hilbert space $L^2(0,\pi)$. Due to the density argument, it is sufficient to prove that the equality $\mathcal{F}\widetilde{\mathcal{F}}g = g$ for $g \in C_0^\infty(0,\pi)$. Let g be any fixed element in $C_0^\infty(0,\pi)$. Then we have

$$[\mathcal{F}\widetilde{\mathcal{F}}g](\theta) = \lim_{N\to\infty} \int_{-q^{-2N}}^{0} \varphi_{e^{i\theta}}(\eta) D(\theta) \left\{ \int_0^\pi g(\vartheta)\varphi_{e^{i\vartheta}}(\eta) D(\vartheta)\, d\vartheta \right\} d_{q^2}\eta$$

$$= \lim_{N\to\infty} \int_0^\pi \left\{ \int_{-q^{-2N}}^{0} \varphi_{e^{i\theta}}(\eta)\varphi_{e^{i\vartheta}}(\eta)\, d_{q^2}\eta \right\} g(\vartheta)\, d\vartheta$$

which we put

$$\lim_{N\to\infty} \int_0^\pi \Phi(\theta,\vartheta;N) D(\theta) D(\vartheta)\, d\vartheta\,.$$

This is to say that the function $\Phi(\theta,\vartheta;N)$ is defined by

$$\Phi(\theta,\vartheta;N) := \int_{-q^{-2N}}^{0} \varphi_{e^{i\theta}}(\eta)\varphi_{e^{i\vartheta}}(\eta)\, d_{q^2}\eta\,.$$

If we see that the function $\Phi(\theta,\vartheta;N)D(\theta)D(\vartheta)$ converges to $\delta(\theta-\vartheta)$ as $N \to \infty$ in the sense of distribution, we have $\mathcal{F}\widetilde{\mathcal{F}}g = g$ for $f \in C_0^\infty(0,\pi)$. This fact, combined with the continuity of the linear mappings $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, the equality $\mathcal{F}\widetilde{\mathcal{F}} = \mathrm{Id}$ on $L^2(0,\pi)$ follows from the density of $C_0^\infty(0,\pi)$ in $L^2(0,\pi)$. Now the proof reduces to the properties of the function $\Phi(\theta,\vartheta;N)$ which will be discussed in the next proposition.

PROPOSITION 6.1. *The function $\Phi(\theta,\vartheta;N)$ is of the following form:*

$$\begin{aligned}\Phi(\theta,\vartheta;N) &= (1+q^{2N})(1-q^2)A(e^{i\theta})A(e^{-i\vartheta})\frac{\sin(N+\frac{1}{2})(\theta-\vartheta)}{\sin\frac{1}{2}(\theta-\vartheta)} \\ &+ (1+q^{2N})B_1(\theta,\vartheta)e^{iN(\theta+\vartheta)} + (1+q^{2N})B_2(\theta,\vartheta)e^{-iN(\theta+\vartheta)} \\ &+ (1+q^{2N})B_3(\theta,\vartheta)\left\{e^{-i(N+1)\theta+iN\vartheta} - e^{-iN\theta+i(N+1)\vartheta}\right\} \\ &+ c(\theta,\vartheta;N)\end{aligned}$$

where $B_j(\theta,\vartheta)$, $j=1,2,3$ are the functions satisfying that $B_j(\theta,\vartheta)D(\theta)D(\vartheta)$ are of class C^∞ and 2π-periodic with respect to the variables θ and ϑ. Then $c(\theta,\vartheta;N)$ is again a function of class C^∞ and 2π-periodic with respect to the variables θ and ϑ with an extra condition that the equality

$$\lim_{N\to\infty}\sup_{0\leqslant\theta,\,\vartheta\leqslant\pi}|c(\theta,\vartheta;N)D(\theta)D(\vartheta)|=0$$

holds.

Proof. By making use of the equality $P\varphi_{e^{i\theta}}=\lambda(e^{i\theta})\varphi_{e^{i\theta}}$ and the formulas of the boundary form which we discussed in Section 2, we have

$$\begin{aligned}\Phi(\theta,\vartheta;N)&\{\lambda(e^{i\theta})-\lambda(e^{i\vartheta})\}\\&=\int_{-q^{-2N}}^{0}\{[P\varphi_{e^{i\theta}}](\eta)\varphi_{e^{i\vartheta}}(\eta)-\varphi_{e^{i\theta}}(\eta)[P\varphi_{e^{i\vartheta}}](\eta)\}\,d_{q^2}\eta\\&=F[\varphi_{e^{i\theta}},\varphi_{e^{i\vartheta}}](0)-F[\varphi_{e^{i\theta}},\varphi_{e^{i\vartheta}}](-q^{-2N})\\&=-F[\varphi_{e^{i\theta}},\varphi_{e^{i\vartheta}}](-q^{-2N})\,.\end{aligned}$$

Then by making use of the connection formula, this is equal to

$$\begin{aligned}&-F[\Phi_{e^{i\theta}},\Phi_{e^{i\vartheta}}](-q^{-2N})-F[\Phi_{e^{-i\theta}},\Phi_{e^{i\vartheta}}](-q^{-2N})\\&\quad-F[\Phi_{e^{i\theta}},\Phi_{e^{-i\vartheta}}](-q^{-2N})-F[\Phi_{e^{-i\theta}},\Phi_{e^{-i\vartheta}}](-q^{-2N})\,.\end{aligned}$$

Now, we see that the function $\widetilde{\varphi}_\theta$ of ζ defined by $\widetilde{\varphi}_\theta(\zeta)=\zeta\{\Phi_{e^{i\theta}}-A(e^{i\theta})\psi_{e^{i\theta}}(\zeta)\}$ is a two variable analytic function of $(\zeta,\theta)\in\{\zeta\in\mathbf{C}:|\zeta|<1\}\times\mathbf{R}$ and in particular, the function obtained by the partial derivative of the function $(\zeta,\theta)\to\widetilde{\varphi}_\theta(\zeta)$ with respect to the parameter θ is bounded. It is also seen that the function $\partial_\theta\psi_{e^{i\theta}}(-q^{-2N})$ obtained by the partial differentiation of the function $(\zeta,\theta)\to\psi_{e^{i\theta}}(\zeta)$ with respect to the parameter θ with ζ replaced by $-q^{-2N}$ is of order $O(N)$ as $N\to\infty$. In view of these points, we have

$$\begin{aligned}\Phi(\theta,\vartheta;N)=&-A(e^{i\theta})A(e^{i\vartheta})F[\Phi_{e^{i\theta}},\Phi_{e^{i\vartheta}}](-q^{-2N})\\&-A(e^{-i\theta})A(e^{i\vartheta})F[\Phi_{e^{-i\theta}},\Phi_{e^{i\vartheta}}](-q^{-2N})\\&-A(e^{i\theta})A(e^{-i\vartheta})F[\Phi_{e^{i\theta}},\Phi_{e^{-i\vartheta}}](-q^{-2N})\\&-A(e^{-i\theta})A(e^{-i\vartheta})F[\Phi_{e^{-i\theta}},\Phi_{e^{-i\vartheta}}](-q^{-2N})\\&+\Phi_1(\theta,\vartheta;N)\,,\end{aligned}\tag{6.1}$$

where the remainder term $\Phi_1(\theta,\vartheta;N)$ is a C^∞-function of (θ,ϑ) satisfying

$$\lim_{N\to\infty}\sup_{0\leqslant\theta,\,\vartheta\leqslant\pi}\left|\frac{\Phi_1(\theta,\vartheta;N)D(\theta)D(\vartheta)}{\theta-\vartheta}\right|=0\,.\tag{6.2}$$

Then by the definition of ψ and the boundary form F, we have

$$F[\psi_{e^{\pm i\theta}}, \psi_{e^{\pm i\vartheta}}](-q^{-2N}) = \left\{e^{\pm i(N+1)\theta \pm iN\vartheta} - e^{\pm iN\theta \pm i(N+1)\vartheta}\right\} \frac{q^3}{1-q^2}(1+q^{2N})$$

for + and − signs, respectively. We apply these equalities to the second and the third term of the right hand side of (6.1) and we have

$$\begin{aligned} &A(e^{-i\theta})A(e^{i\vartheta})F[\Phi_{e^{-i\theta}}, \Phi_{e^{i\vartheta}}](-q^{-2N}) + A(e^{i\theta})A(e^{-i\vartheta})F[\Phi_{e^{i\theta}}, \Phi_{e^{-i\vartheta}}](-q^{-2N}) \\ &\quad = \frac{4q^3}{1-q^2}(1+q^{2N})A(e^{i\theta})A(e^{-i\vartheta}) \sin\frac{\theta+\vartheta}{2} \sin\left\{\left(N+\frac{1}{2}\right)(\theta-\vartheta)\right\} \\ &\qquad + \frac{q^3}{1-q^2}(1+q^{2N})\{A(e^{i\theta})A(e^{-i\vartheta}) - A(e^{-i\theta})A(e^{i\vartheta})\} \\ &\qquad \times \left\{e^{-i(N+1)\theta+iN\vartheta} - e^{-iN\theta+i(N+1)\vartheta}\right\} . \end{aligned}$$

Therefore we have

$$\begin{aligned} &\Phi(\theta,\vartheta;N)\{\lambda(e^{i\theta}) - \lambda(e^{i\vartheta})\} \\ &\quad = \frac{4q^3}{1-q^2}(1+q^{2N})A(e^{i\theta})A(e^{-i\vartheta}) \sin\frac{\theta+\vartheta}{2} \sin\left\{\left(N+\frac{1}{2}\right)(\theta-\vartheta)\right\} \\ &\qquad - \frac{q^3}{1-q^2}(1+q^{2N})A(e^{i\theta})A(e^{i\vartheta})(e^{i\theta} - e^{i\vartheta})e^{iN(\theta+\vartheta)} \\ &\qquad - \frac{q^3}{1-q^2}(1+q^{2N})A(e^{-i\theta})A(e^{-i\vartheta})(e^{-i\theta} - e^{-i\vartheta})e^{-iN(\theta+\vartheta)} \\ &\qquad + \frac{q^3}{1-q^2}(1+q^{2N})\{A(e^{i\theta})A(e^{-i\vartheta}) - A(e^{-i\theta})A(e^{i\vartheta})\} \\ &\qquad \times \left\{e^{-i(N+1)\theta+iN\vartheta} - e^{-iN\theta+i(N+1)\vartheta}\right\} \\ &\qquad + \Phi_1(\theta,\vartheta;N) . \end{aligned}$$

Since

$$\lambda(e^{i\theta}) - \lambda(e^{i\vartheta}) = \frac{4q^3}{(1-q^2)^2} \sin\frac{\theta+\vartheta}{2} \sin\frac{\theta-\vartheta}{2} ,$$

we define

$$\begin{aligned} B_1(\theta,\vartheta) &= -(1-q^2)A(e^{i\theta})A(e^{i\vartheta}) \frac{e^{i\theta} - e^{i\vartheta}}{\cos\vartheta - \cos\theta} , \\ B_2(\theta,\vartheta) &= -\frac{(1-q^2)}{2} A(e^{-i\theta})A(e^{-i\vartheta}) \frac{e^{-i\theta} - e^{-i\vartheta}}{\cos\vartheta - \cos\theta} , \\ B_3(\theta,\vartheta) &= \frac{(1-q^2)}{2} \frac{A(e^{i\theta})A(e^{-i\vartheta}) - A(e^{-i\theta})A(e^{i\vartheta})}{\cos\vartheta - \cos\theta} \quad \text{and} \\ c(\theta,\vartheta;N) &= \frac{(1-q^2)^2}{2q^3} \frac{\Phi_1(\theta,\vartheta;N)}{\cos\vartheta - \cos\theta} . \end{aligned}$$

Then the functions B_1, B_2, B_3 and c satisfy the condition of the statement. The fact that the function c satisfies the condition follows from (6.2). This proves the assertion. ∎

PROPOSITION 6.2. *For any element f in $C_0^\infty(0,\pi)$, we have*

(i)

$$\lim_{N\to\infty}\int_0^\pi \Phi(\theta,\vartheta;N)D(\theta)D(\vartheta)f(\vartheta)\,d\vartheta = f(\theta)\,.$$

The convergence of the limit is uniform for $\theta \in [0,\pi]$

(ii)

$$\lim_{N\to\infty}\int_0^\pi\int_0^\pi \Phi(\theta,\vartheta;N)D(\theta)D(\vartheta)f(\theta)\overline{f(\vartheta)}\,d\theta\,d\vartheta$$

$$= \int_0^\pi |f(\theta)|^2\,d\theta \equiv \|f\|^2_{L^2(0,\pi)}\,.$$

(3) *$\Phi(\theta,\vartheta;N)D(\theta)D(\vartheta)$ as a function of θ and ϑ converges to the Dirac's delta function $\delta(\theta-\vartheta)$ as $N\to\infty$ in the sense of distribution.*

Proof. We prove (i). Then the statements (ii) and (iii) follow directly from the statement (i). The proof of (i) goes as follows. The function

$$\rho_N(\theta) := \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$$

is the Dirichlet kernel. Therefore the Riemann-Lebesgue theorem combined with Proposition 6.1 implies

$$\lim_{N\to\infty}\int_0^\pi \Phi(\theta,\vartheta;N)D(\theta)D(\vartheta)f(\vartheta)\,d\vartheta$$

$$= \lim_{N\to\infty}\int_0^\pi (1-q^2)(1+q^{2N})A(e^{i\theta})A(e^{-i\vartheta})D(\theta)D(\vartheta)f(\vartheta)\rho_N(\theta-\vartheta)\,d\vartheta$$

$$= 2\pi(1-q^2)A(e^{i\theta})A(e^{-i\theta})|D(\theta)|^2 f(\theta) = f(\theta)\,.$$

The uniformity of the convergence for $\theta\in[0,\pi]$ follows from the properties of the Dirichlet kernel and the Riemann-Lebesgue theorem. ∎

THEOREM 6.3. (i) *The mappings $\mathcal{F}:\mathcal{H}_c \longrightarrow L^2(0,\pi)$ and $\tilde{\mathcal{F}}: L^2(0,\pi)\longrightarrow \mathcal{H}_c$ are isomorphisms satisfying $\tilde{\mathcal{F}}\mathcal{F} = \mathrm{Id}$ on $\mathcal{H}_c$ and $\mathcal{F}\tilde{\mathcal{F}} = \mathrm{Id}$ on $L^2(0,\pi)$.*

(ii) *For φ, $\psi\in\mathcal{H}_c$, the equality*

$$(\varphi,\psi)_{L^2(-\infty,0]_{q^2}} = (\mathcal{F}\varphi,\mathcal{F}\psi)_{L^2(0,\pi)}$$

holds. This is to say that the mapping $\mathcal{F}$ is a unitary operator from the Hilbert space $\mathcal{H}_c$ onto the Hilbert space $L^2(0,\pi)$.

Proof. The equality $\widetilde{\mathcal{F}}\mathcal{F} = \mathrm{Id}$ on $\mathcal{H}_c$ follows from Theorem 5.6. Therefore, we prove the equality $\mathcal{F}\widetilde{\mathcal{F}} = \mathrm{Id}$ on $L^2(0,\pi)$. By Proposition 6.2, we have $\mathcal{F}\widetilde{\mathcal{F}}f = f$ for any $f \in C_0^\infty(0,\pi)$. Hence, the density of $C_0^\infty(0,\pi)$ in $L^2(0,\pi)$ proves the assertion (i). This shows that for the proof of (ii), it is sufficient to see that the mapping $\widetilde{\mathcal{F}}$ is isometry i.e. the equality $\|\widetilde{\mathcal{F}}f\|_{L^2(-\infty,0]_{q^2}} = \|f\|_{L^2(0,\pi)}$ for $f \in C_0^\infty(0,\pi)$. However, this was already proved in the second statement of Proposition 6.2. ∎

7. SOME REMARKS

If we think of taking the limit $q \to 1$ formally to the formula which we gave in the statement of Proposition 4.6, the end point of the continuous spectrum $\frac{q^2}{(1-q)^2}$ goes to the infinity and the point spectrum set $\sigma_p(P)$ is necessarily put away to the infinity. Then, only the continuous spectrum set $\sigma_c(P)$ remains as the half open interval $[\frac{1}{4},\infty)$. This means that the spectrum of the radial part of the Casimir operator C on the quantum group $SU_q(1,1)$ *formally converges* to that of the corresponding object on the classical Lie group $SU(1,1)$. Moreover, in the process of taking the formal limit $q \to 1$ in the expression of the c-function $c(\theta)$ (see Lemma 5.3 and Theorem 5.6), we observe that the spectral representation of the operator C on the zonal Hilbert space *approximates* to that on the corresponding classical object. These arguments lead us to think about the rigorous meaning of *classical limit.*

The further discussions concerning these points as well as the detailed part of the unitary representations of the quantum group $SU_q(1,1)$ will be discussed in our forthcoming publications.

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