J. OPERATOR THEORY
34(1995), 213–216

THE NUMERICAL RANGE OF A TOEPLITZ OPERATOR WITH HARMONIC SYMBOL

J.K. THUKRAL

Communicated by Norberto Salinas

ABSTRACT. A complete description of the numerical range of a Toeplitz operator acting on the Bergman space of the unit disk with harmonic symbol is given.

KEYWORDS: *Toeplitz operator, numerical range, spectrum, self-adjoint, normal, subnormal, convexoid, harmonic.*

AMS SUBJECT CLASSIFICATION: Primary 47B35; Secondary 47A15.

Let T be a bounded linear operator on a Hilbert space H. T is called *self-adjoint* if $T = T^*$; *normal* if $TT^* = T^*T$; *subnormal* if there exists a normal operator N on a Hilbert space K containing H such that H is invariant under N and $N|H = T$. The *spectrum* of T is denoted by $\sigma(T)$. We write $\operatorname{conv} M$ to denote the *convex hull* of a set M. The *numerical range* of T, denoted $W(T)$, is the set $W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$. The following facts about numerical range are frequently used in the paper:

(i) $W(T)$ is always convex and its closure, $\overline{W(T)}$, contains the spectrum of T;

(ii) if T is subnormal, then T is convexoid, that is, $\overline{W(T)} = \operatorname{conv}\sigma(T)$;

(iii) each extreme point of $W(T)$ is an eigenvalue of T and

(iv) W is a linear function of T, that is, $W(\alpha T+\beta) = \alpha W(T)+\beta$ for complex numbers α and β.

For proofs, see [3], Chapter 17 and [6].

Let $\mathrm{d}A$ denote the usual area measure on the open unit disk $\mathbf{D}$ of the complex plane $\mathbf{C}$. The *Bergman space* $L^2_a(\mathbf{D})$ is the subspace of $L^2(\mathbf{D}, \mathrm{d}A)$ consisting of those functions in $L^2(\mathbf{D}, \mathrm{d}A)$ which are analytic on $\mathbf{D}$. Let P denote the orthogonal projection of $L^2(\mathbf{D}, \mathrm{d}A)$ onto $L^2_a(\mathbf{D})$. For $\varphi \in L^\infty(\mathbf{D}, \mathrm{d}A)$, the *Toeplitz operator with symbol* φ, denoted T_φ, is the operator from $L^2_a(\mathbf{D})$ to $L^2_a(\mathbf{D})$ defined by $T_\varphi f = P(\varphi f)$ for f in $L^2_a(\mathbf{D})$. We denote the space of bounded analytic functions on $\mathbf{D}$ by H^∞. By a *harmonic function* we mean a complex-valued function on $\mathbf{D}$ whose Laplacian is identically 0.

In general, the behavior of Toeplitz operators on the Bergman space may be quite different from that of the Toeplitz operators on the Hardy space H^2 of the unit circle. However, it is shown in [1] and [5] that Toeplitz operators on $L^2_a(\mathbf{D})$ with harmonic symbols behave quite similarly to those on H^2, and one can prove analogues for this class of many results about Toeplitz operators on H^2.

The purpose of this paper is to completely describe the numerical range of a Toeplitz operator on $L^2_a(\mathbf{D})$ with harmonic symbol. The description of the numerical range of an arbitrary Toeplitz operator acting on the Hardy space H^2 was given by Klein ([4]). See also Brown and Halmos ([2], p. 100).

We begin with the following:

LEMMA 1. *Suppose φ is a nonconstant real-valued bounded harmonic function on $\mathbf{D}$, then $W(T_\varphi) = (\inf\varphi, \sup\varphi)$.*

Proof. By a result of McDonald and Sundberg ([5], Proposition 12), $\sigma(T_\varphi) = [\inf\varphi, \sup\varphi]$. As φ is real-valued, T_φ is self-adjoint and hence convexoid. Thus $\overline{W(T_\varphi)} = [\inf\varphi, \sup\varphi]$. Since $W(T_\varphi)$ is convex, it follows that $(\inf\varphi, \sup\varphi) \subset W(T_\varphi)$. If either $\inf\varphi$ or $\sup\varphi$ belongs to $W(T_\varphi)$, it is an extreme point of $W(T_\varphi)$ and thus an eigenvalue of T_φ. But a Bergman Toeplitz operator with real-valued harmonic symbol can not have an eigenvalue unless it is a scalar ([5], Proposition 13). Thus φ is constant, contradicting the hypothesis. Hence $W(T_\varphi) = (\inf\varphi, \sup\varphi)$. ∎

THEOREM 2. *Let φ be a nonconstant bounded harmonic function on $\mathbf{D}$ and let T_φ be normal. Then there exist constants a and b such that $\sigma(T_\varphi)$ is the closed line segment $[a, b]$ and $W(T_\varphi)$ is the corresponding open segment (a, b).*

Proof. Because T_φ is normal, by a recent result of Axler and Čučkovič ([1], Corollary 17), $\varphi(\mathbf{D})$ lies on some line in $\mathbf{C}$. Then there exist constants α, β and a real-valued function ψ on $\mathbf{D}$ such that $\varphi = \alpha\psi + \beta$. Thus $T_\varphi = \alpha T_\psi + \beta$. By Lemma 1, $\sigma(T_\psi) = [m, M]$ and $W(T_\psi) = (m, M)$, where m is the infimum and M is the supremum of ψ. Hence $\sigma(T_\varphi) = [\alpha m + \beta, \alpha M + \beta]$ and $W(T_\varphi) = (\alpha m + \beta, \alpha M + \beta)$. This completes the proof. ∎

To deal with the case when T_φ is nonnormal, we first prove the following:

LEMMA 3. *Suppose φ is a bounded harmonic function on $\mathbb{D}$. Then $\varphi \geqslant 0$ if and only if $T_\varphi \geqslant 0$.*

Proof. The inequality $\varphi \geqslant 0$ clearly implies $T_\varphi \geqslant 0$. To prove the other direction, suppose now that $T_\varphi \geqslant 0$. In particular T_φ is self-adjoint, that is, $T_\varphi = T_{\bar\varphi}$ which shows that φ is real-valued. Hence by [5], Proposition 12, $\sigma(T_\varphi) = [\inf\varphi, \sup\varphi]$ which is contained in $[0,\infty)$. Thus $\varphi \geqslant 0$. ∎

LEMMA 4. *Let φ be a bounded harmonic function on $\mathbb{D}$. Suppose the numerical range of T_φ lies in the upper half-plane and contains the origin. Then the operator T_φ is self-adjoint.*

Proof. We are to prove that if $\operatorname{Im} T_\varphi \geqslant 0$ or equivalently $\operatorname{Im}\varphi \geqslant 0$ (Lemma 3) and $0 = \langle T_\varphi f, f\rangle$ for some unit vector f in $L^2_{\mathrm{a}}(\mathbb{D})$, then $\operatorname{Im}\varphi = 0$. Since $0 = \langle T_\varphi f, f\rangle = \langle \varphi f, f\rangle = \int_{\mathbb{D}} \varphi |f|^2 \,\mathrm{d}A$, it follows that $\int_{\mathbb{D}} (\operatorname{Im}\varphi)|f|^2\,\mathrm{d}A = 0$, and hence $(\operatorname{Im}\varphi)|f|^2 = 0$ on $\mathbb{D}$. Now since $f \in L^2_{\mathrm{a}}(\mathbb{D})$ and $\|f\| = 1$, we therefore have $\operatorname{Im}\varphi = 0$ on $\mathbb{D}$. ∎

By applying an appropriate linear function, we immediately have the following:

COROLLARY 5. *If a line of support of the numerical range of a Toeplitz operator with bounded harmonic symbol contains a point of the numerical range, then it contains the entire spectrum and hence the entire numerical range.*

THEOREM 6. *Let φ be a bounded harmonic function on $\mathbb{D}$ and let T_φ be nonnormal. Then $W(T_\varphi)$ is an open convex set which is the interior of its closure.*

Proof. We need only show that $W(T_\varphi)$ is open. The required result then follows from the fact that an open convex set is the interior of its closure. Suppose that $W(T_\varphi)$ is not open. Then there exists a point λ in $W(T_\varphi)$ that belongs to $\partial W(T_\varphi)$, the boundary of $W(T_\varphi)$. That means $0 \in W(T_{\varphi-\lambda})$ and is in $\partial W(T_{\varphi-\lambda})$. Since $W(T_{\varphi-\lambda})$ is convex, we may rotate $W(T_{\varphi-\lambda})$ so that it lies in the upper half-plane. In other words, we can find a complex number α of unit modulus such that $W(T_{\alpha\varphi-\alpha\lambda})$ lies in the upper half-plane. By Lemma 4, there exists a real-valued function ψ on $\mathbb{D}$ such that $\alpha\varphi - \alpha\lambda = \psi$. This implies that $\varphi = c\psi + d$ for some constants c and d. Thus $T_\varphi = cT_\psi + d$ is normal, which contradicts the hypothesis. ∎

It is known that if $\varphi \in H^\infty$, then $\sigma(T_\varphi) = \overline{\varphi(\mathbb{D})}$.

COROLLARY 7. *If $\varphi \in H^\infty$, then $W(T_\varphi)$ is the convex hull of $\varphi(\mathbf{D})$.*

Proof. We may assume that φ is nonconstant. Then T_φ is not normal by [1], Theorem 1. By Theorem 6, $W(T_\varphi)$ is an open convex set which is the interior of its closure. As $\varphi \in H^\infty$, T_φ is subnormal and hence convexoid. Thus $\overline{W(T_\varphi)} = \operatorname{conv} \overline{\varphi(\mathbf{D})}$. Thus $W(T_\varphi)$ is the interior of $\operatorname{conv} \overline{\varphi(\mathbf{D})}$. The convex hull of a compact set is compact, the convex hull of an open set is open. From these facts, it follows that $W(T_\varphi)$ is the convex hull of $\varphi(\mathbf{D})$. ∎

Acknowledgements. I would like to thank Professor Sheldon Axler for helpful discussions on this material.

REFERENCES

1. S. AXLER, Z. ČUČKOVIĆ, Commuting Toeplitz operators with harmonic symbol, *Integral Equations Operator Theory* **14**(1991), 1–12.
2. A. BROWN, P.R. HALMOS, Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.* **213**(1963/64), 89–102.
3. P.R. HALMOS, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, N.J. 1967.
4. E.M. KLEIN, The numerical range of a Toeplitz operator, *Proc. Amer. Math. Soc.* **35**(1972), 101–103.
5. G. MCDONALD, C. SUNDBERG, Toeplitz operators on the disc, *Indiana Univ. Math. J.* **28**(1979), 595–611.
6. J.G. STAMPFLI, Extreme points of the numerical range of a hyponormal operator, *Michigan Math. J.* **13**(1966), 87–89.

J.K. THUKRAL
Department of Mathematics
Shri Ram College of Commerce
University of Delhi
Delhi – 110007
INDIA

Received June 17, 1991.