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A NOTE ON THE REFLEXIVITY OF WEAKLY CLOSED SUBSPACES OF OPERATORS

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ABSTRACT. Many results connect reflexivity and systems of simultaneous equations in the predual (well known by Property $(\mathbb{A}_{m,n})$ and $(\mathbb{B}_{m,n})$) of weakly*-closed subspaces of operators on Hilbert space ([4], [6] and [9]). Here we prove under a suitable hypothesis on the dual space $\mathcal{A}$ (weak*-closed subspace of $\mathcal{L}(\mathcal{H})$) that the dual space generated by $\mathcal{A}$ and a compact operator K is reflexive if the rank of K is greater than 5.

KEYWORDS: *Dual space, predual, property* $(\mathbb{A}_{m,n})$, *reflexivity.*

AMS SUBJECT CLASSIFICATION: Primary 47D15; Secondary 47D25, 47A55.

1. INTRODUCTION

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on the Hilbert space $\mathcal{H}$ and $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$; $Q_{\mathcal{A}}$ denote the predual of $\mathcal{A}$. If $[L] \in Q_{\mathcal{A}}$, $[L]$ has the form $\sum\limits_{n\geqslant 1} [x_n \otimes y_n]$ where $\sum\limits_{n\geqslant 1} \|x_n\|^2 < +\infty$ and $\sum\limits_{n\geqslant 1} \|y_n\|^2 < < +\infty$.

For x and y in $\mathcal{H} \setminus \{0\}$, we write as usual $x \otimes y$ for the rank-one operator on $\mathcal{H}$ defined by $(x \otimes y)(u) = (u, y)x$, for $u \in \mathcal{H}$ and of course $[x \otimes y] \in Q_{\mathcal{A}}$, $\langle T, [x \otimes y]\rangle = (Tx, y)$, $T \in \mathcal{A}$.

DEFINITION 1.1. A weak*-closed subspace of $\mathcal{L}(\mathcal{H})$, $\mathcal{A}$, is said to have the property $(\mathbb{A}_{m,n})$ for m, n cardinal numbers less than or equal to $\aleph_0$, if for every doubly indexed family $([L_{ij}])_{0\leqslant i<m, 0\leqslant j<n}$ in $Q_{\mathcal{A}}$, there exist vectors $(x_i)_{0\leqslant i<m}$ and $(y_j)_{0\leqslant j<n}$ in $\mathcal{H}$ such that $[L_{ij}] = [x_i \otimes y_j]$ for $0 \leqslant i < m$ and $0 \leqslant j < n$.

We usually shorten $(\mathbb{A}_{n,n})$ to $(\mathbb{A}_n)$.

We recall from [12] that a linear subspace $\mathcal{A}$ is reflexive if it contains every operator $T \in \mathcal{L}(\mathcal{H})$ with the property that $Tx \in (\overline{\mathcal{A}x})$ for every $x \in \mathcal{H}$.

This concept of reflexivity was introduced by Loginov and Sulman in [12]. Of course reflexive subspaces are weakly closed and this definition coincides with the usual definition ($\mathcal{A} = \operatorname{AlgLat}\mathcal{A}$) if $\mathcal{A}$ is subalgebra of $\mathcal{L}(\mathcal{H})$.

We say that $T \in \mathcal{L}(\mathcal{H})$ is reflexive if the weak-closed algebra $\mathcal{A}_T$ generated by T is.

Many works show the relationship between the reflexivity of weak*-closed algebras generated by one contraction in the class $\mathbb{A}$ (the Sz.-Nagy–Foiaş functional calculus is an isometry) ([13], Chapter 3)) and the properties $(\mathbb{A}_{m,n})$, using the fact that these algebras are isomorphic to H^∞ ([5] and [9]). The techniques developed in this study yield a main result that every contraction in the class $\mathbb{A}$ is reflexive ([7]).

In [4] and [6], it is established that the notion of reflexivity does not require isomorphism with H^∞. Here we are interested in the reflexivity of perturbation of reflexive linear subspace and then in the extension of properties $(\mathbb{A}_{m,n})$; in this area we obtained two results [2] and [3].

THEOREM 1.2. ([2]) *If $\mathcal{A}$ is a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$ $(0 < \gamma \leqslant 1)$, and R is a non-trivial finite rank operator, then $\mathcal{A} + \mathbb{C}R$ has the property $(\mathbb{A}_{n,\aleph_0}) \cap (\mathbf{A}_{\aleph_0,n}) \setminus (\mathbb{A}_{n+1})$, where $n = \operatorname{rank}(R)$.*

THEOREM 1.3. ([3]) *Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$, $(0 < \gamma \leqslant 1)$ and a compact operator K of infinite rank, then $\mathcal{A} + \mathbb{C}K$ has the property $(\mathbb{A}_{\aleph_0})$.*

Let $0 \leqslant \theta < 1$; the following subset of the predual of $\mathcal{A}$ were defined in [5] and [10] by $\mathcal{X}_\theta(\mathcal{A})$, the set of all $[L] \in Q_\mathcal{A}$ such that there exist $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $(\mathcal{H})_1$ (the closed unit ball of $\mathcal{H}$) which converge weakly to 0 and satisfy (1.1), (1.2) and (1.3):

$$\limsup_{n\to+\infty} \|[L] - [x_n \otimes y_n]\| \leqslant \theta; \tag{1.1}$$

$$\lim_{n\to+\infty} \|[x_n \otimes w]\| = 0, \quad \forall w \in \mathcal{H}; \tag{1.2}$$

$$\lim_{n\to+\infty} \|[w \otimes y_n]\| = 0, \quad \forall w \in \mathcal{H}. \tag{1.3}$$

Note that $\mathcal{X}_\theta(\mathcal{A})$ is closed and absolutely convex set.

DEFINITION 1.4. ([5]) Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$; $\mathcal{A}$ is said to have the property $X_{\theta,\gamma}$, $(0 \leqslant \theta < \gamma \leqslant 1)$ if $\mathcal{X}_\theta(\mathcal{A}) \supset (Q_{\mathcal{A}})_\gamma$ (the closed ball in $Q_{\mathcal{A}}$ centered at 0 with radius γ).

The following result is established in [8], Chapter 3.

THEOREM 1.5. *If $\mathcal{A}$ is a dual algebra with the property $X_{\theta,\gamma}$ $(0 \leqslant \theta < \gamma \leqslant 1)$, then $\mathcal{A}$ has the property $(\mathbb{A}_{\aleph_0})$.*

This theorem is still true if $\mathcal{A}$ is a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$.

A reflexivity theorem proved in [4] states in particular that a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$, $(0 < \gamma \leqslant 1)$ is reflexive. The purpose of this paper is to show that in this case $\mathcal{A} + \mathbb{C}K$ is reflexive when K is a compact operator such that its rank is greater than 5. It is worthy to note that $\mathcal{A} + \mathbb{C}K$ (with $\operatorname{rank}(K) > 0$) can not have the property $X_{0,\gamma}$ ([3]).

2. PRELIMINARIES

Note that we can also define the space $\mathcal{X}_0(\mathcal{A})$ and the property $X_{0,\gamma}$ for a dual space of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces.

The following result is proved in [1], Proposition 3.1, in the case $n = m$.

PROPOSITION 2.1. *Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$, $(0 < \gamma \leqslant 1)$; $M_{m,n}(\mathcal{A}) = \{(T_{ij})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}; T_{ij} \in \mathcal{A}\}$ has the property $X_{0,\frac{\gamma}{nm}}$, for every $m \geqslant 1$ and $n \geqslant 1$.*

Note that $M_{m,n}(\mathcal{A})$ is naturally identified with a dual space of $\mathcal{L}(\mathcal{H}^{(n)}, \mathcal{H}^{(m)})$ and $Q_{M_{m,n}(\mathcal{A})}$ is identified with $M_{m,n}(Q_{\mathcal{A}})$.

We have the following result, which have been shown in ([8], Chapter 1) when $\mathcal{H}_1 = \mathcal{H}_2$.

PROPOSITION 2.2. *Let $\mathcal{A}$ be a dual space of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ with the property $X_{0,\gamma}$, $(0 < \gamma \leqslant 1)$. Suppose given $[L] \in Q_{\mathcal{A}}$, vectors $a \in \mathcal{H}_1$, $b \in \mathcal{H}_2$, $\mathcal{L}_1$ and $\mathcal{L}_2$ finite codimensional subspace of $\mathcal{H}_1$ and $\mathcal{H}_2$, $\varepsilon > 0$ and δ such that $\|[L] - [a \otimes b]\| < \delta$; then there exist x in $\mathcal{H}_1$ and y in $\mathcal{H}_2$ such that :*

$$[L] = [x \otimes y], \ (x - a) \in \mathcal{L}_1, \ (y - b) \in \mathcal{L}_2,$$

$$\max(\|x - a\|_1, \|y - b\|_2) < \sqrt{\frac{\delta}{\gamma}}.$$

DEFINITION 2.3. Let $\mathcal{A}$ be a dual space. $\mathcal{A}$ has the property (P) if $\mathcal{A}$ has the property $(\mathbb{A}_{1,2})$ and for $x_1, x_2, y_1, y_2 \in \mathcal{H}$, and ε a given positive number, there exist vectors $\xi_1, \xi_2, \eta_1, \eta_2, \eta_3 \in \mathcal{H}$ such that

(i) $[\xi_i \otimes \eta_j] = [x_i \otimes y_j], \quad 1 \leqslant i,j \leqslant 2, \quad [\xi_1 \otimes \eta_3] = [x_2 \otimes y_2], \quad [\xi_2 \otimes \eta_3] = 0,$
(ii) $\|x_i - \xi_i\| < \varepsilon, \quad \|y_j - \eta_j\| < \varepsilon, \quad 1 \leqslant i,j \leqslant 2.$

Bercovici, Foiaş and Pearcy have shown in [6] the following result (Lemma 14, Theorem 15).

THEOREM 2.4. *Assume that M is a weakly closed subspace of $\mathcal{L}(\mathcal{H})$. If M has property (P), then M is hereditarily reflexive (every weakly closed subspace of M is reflexive).*

3. REFLEXIVITY RESULTS

Let $\mathcal{K}(\mathcal{H})$ denote the set of all compact operators on $\mathcal{H}$. The main result is:

THEOREM 3.1. *Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$ $(0 < \gamma \leqslant 1)$ and $K \in \mathcal{K}(\mathcal{H})$ such that* $\operatorname{rank}(K) \geqslant 5$. *Then $\mathcal{A} + \mathbb{C}K$ is hereditarily reflexive.*

Proof. We will prove that $\mathcal{A} + \mathbb{C}K$ has property (P) and conclude by Theorem 2.4.

We may suppose that $\|K\| = 1$ because $\mathcal{A} + \mathbb{C}K = \mathcal{A} + \mathbb{C}\frac{K}{\|K\|}$. Since the proof is quite technical we will distinguish two cases.

Finite rank case. Let $R = \sum_{i=1}^{n} \lambda_i \varepsilon_i \otimes e_i$ the canonical writing of R. Then $\lambda_1 = 1$ and λ_i is a decreasing sequence of positive numbers.

Note that an element φ of $Q_{\tilde{\mathcal{A}}}$ $(\tilde{\mathcal{A}} = \mathcal{A} + \mathbb{C}R)$ is split into its action $[L]$ $(= \varphi|\mathcal{A})$ on $\mathcal{A}$ and d $(= \varphi(R))$ on $\mathbb{C}R$.

To make the notations easy for the reader, we will note $[L]$ instead of $[L]_{\mathcal{A}}$ for an element of $Q_{\mathcal{A}}$.

Set $\tilde{\psi} \in M_{2,3}(Q_{\tilde{\mathcal{A}}})$, $(\tilde{\mathcal{A}} = \mathcal{A} + \mathbb{C}R)$, $\tilde{\psi} = (\psi_{ij})_{\substack{1\leqslant i\leqslant 2\\ 1\leqslant j\leqslant 3}}$ such that there are vectors $(x_i)_i$ and $(y_j)_j$ in $\mathcal{H}$ with

$$\psi_{ij} = [x_i \otimes y_j]_{\tilde{\mathcal{A}}},\ 1 \leqslant i,j \leqslant 2, \quad \psi_{13} = [x_2 \otimes y_2]_{\tilde{\mathcal{A}}} \quad \text{and} \quad \psi_{23} = 0.$$

We can write $\tilde{\psi} = ([\tilde{L}], (d_{ij})_{1\leqslant i\leqslant 2, 1\leqslant j\leqslant 3})$ where $[\tilde{L}] = ([L_{ij}])_{1\leqslant i\leqslant 2, 1\leqslant j\leqslant 3}$, $[L_{ij}] = \psi_{ij}|\mathcal{A}$ and $d_{ij} = \psi_{ij}(R)$.

$$[\tilde{L}] = \begin{pmatrix} [x_1 \otimes y_1] & [x_1 \otimes y_2] & [x_2 \otimes y_2] \\ [x_2 \otimes y_1] & [x_2 \otimes y_2] & 0 \end{pmatrix} \in M_{2,3}(Q_{\mathcal{A}})$$

and let

$$[\tilde{\tilde{L}}] = \begin{pmatrix} [x_1 \otimes y_1] & [x_1 \otimes y_2] & \delta^2[x_2 \otimes y_2] \\ [x_2 \otimes y_1] & [x_2 \otimes y_2] & 0 \end{pmatrix} \in M_{2,3}(Q_{\mathcal{A}}),$$

where δ is a positive number ($\delta < 1$). Since $\operatorname{rank}(R) \geqslant 5$, take

$$\tilde{u} \in \operatorname{span}\{e_1, \ldots, e_5\} \cap \{R^*Rx_i, R^*y_j; 1 \leqslant i, j \leqslant 2\}^{\perp} \quad \text{and} \quad \|\tilde{u}\| = 1.$$

It easy to see that $\|R\tilde{u}\| \geqslant \lambda_5$. Put $u = \delta\tilde{u}$ and $v = \delta\bar{d}\frac{R\tilde{u}}{\|R\tilde{u}\|^2}$ where $d = d_{13} = (Rx_2, y_2)$; we have $(Ru, v) = \delta^2 d = \delta^2(Rx_2, y_2)$ from which we deduce

$$\begin{aligned} &(R(x_1 + u), y_j) = (Rx_1, y_j), \quad j = 1, 2, \\ &(R(x_1 + u), v) = (Ru, v) = \delta^2 d, \\ &(Rx_2, v) = 0. \end{aligned}$$

Now take the vectors

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}.$$

Using the fact that the norm of an element in $M_{2,3}(\mathcal{A})$ is less or equal than the sum of the norms of each of its entries, we have

$$\begin{aligned} \|[\tilde{\tilde{L}}] - [X \otimes Y]\| &\leqslant \|u\|(\|y_1\| + \|y_2\|) + \|u\|\,\|v\| + (\|x_1\| + \|x_2\|)\|v\| + \delta^2\|[x_2 \otimes y_2]\| \\ &\leqslant \delta\left(\frac{d}{\lambda_5}(1 + \|x_1\| + \|x_2\|) + \|y_1\| + \|y_2\| + \|x_2\|\,\|y_2\|\right) \\ &\leqslant c\delta. \end{aligned}$$

Note that a simple calculation gives

$$\|u\| = \delta, \quad \|v\| \leqslant \frac{d}{\lambda_5}\delta.$$

Suppose now that $0 < \delta < \inf(\frac{\gamma}{24c}\varepsilon^2, 1, \frac{\varepsilon}{4})$, and apply Proposition 2.2 and Proposition 2.1, then there exist $\tilde{X}, \tilde{Y}$ two vectors $\tilde{X} \in \mathcal{H}^{(2)}, \tilde{Y} \in \mathcal{H}^{(3)}$ such that

$$\begin{aligned} &[\tilde{\tilde{L}}] = [\tilde{X} \otimes \tilde{Y}], \quad \max(\|\tilde{X} - X\|, \|\tilde{Y} - Y\|) < \sqrt{\frac{c\delta}{\frac{\gamma}{6}}} < \frac{\varepsilon}{2}, \\ &(\tilde{X} - X) \in ((R\mathcal{H} \cup R^*\mathcal{H})^{\perp})^{(2)}, \\ &(\tilde{Y} - Y) \in ((R\mathcal{H} \cup R^*\mathcal{H})^{\perp})^{(3)}. \end{aligned}$$

This implies that

$$\begin{aligned} &(R\tilde{X}_i, \tilde{Y}_j) = (Rx_i, y_j) = d_{ij}, \quad 1 \leqslant i, j \leqslant 2, \\ &(R\tilde{X}_1, \tilde{Y}_3) = \delta^2 d, \\ &(R\tilde{X}_2, \tilde{Y}_3) = 0. \end{aligned}$$

On the other hand

$$\max\left(\|\tilde{X} - X\|, \|\tilde{Y} - Y\|\right) < \frac{\varepsilon}{2}$$

implies that $\max(\|\tilde{X}_i - x_i\|, \|\tilde{Y}_j - y_j\|) < \varepsilon$ for $1 \leqslant i, j \leqslant 2$; since $\max(\|u\|, \|v\|) < \varepsilon/2$ and $\|\tilde{Y}_3 - v\| < \varepsilon/2$ we have $\|\tilde{Y}_3\| < \varepsilon$. Setting $\xi_i = \tilde{X}_i$, $\eta_j = \tilde{Y}_j$ $1 \leqslant i, j \leqslant 2$, and $\eta_3 = 1/\delta^2 \tilde{Y}_3$. We have $\tilde{\psi} = [\xi \otimes \eta]_{\tilde{\mathcal{A}}}$, where $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ and $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$. Then $\mathcal{A} + \mathbf{C}R$ has the property (P) if $5 \leqslant \operatorname{rank}(R) < +\infty$ and so is reflexive.

Infinite rank case. We keep the same notations as before. For $\varepsilon > 0$ given, if we take a suitable sequence of finite rank operators which approches K we will show by induction that there exists vectors $(X^k)_{k=1}^n \subset \mathcal{H}^{(2)}$ and $(Y^k)_{k=1}^n \subset \mathcal{H}^{(3)}$ such that

(3.1) $[\tilde{L}] = [X^k \otimes Y^k]$, for $k \geqslant 1$;

(3.2) $(R_k X_i^k, Y_j^k) = d_{ij}$, for $k \geqslant 1, i = 1, 2$ and $1 \leqslant j \leqslant 3$;

(3.3) $\max(\|X_i^k - X_i^{k-1}\|, \|Y_i^k - Y_i^{k-1}\|) < \frac{\varepsilon}{2^k}$, for $i = 1, 2$ and $k \geqslant 1$.

Put

$$X^0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad Y^0 = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}.$$

Suppose that the canonical writing of K is $K = \sum_{i \geqslant 1} \lambda_i \varepsilon_i \otimes e_i$, where $(\lambda_i)_i$ is a decreasing sequence to 0, $\lambda_1 = 1$ and denote $\lambda_7 = \lambda$.

Put $R_n = \sum_{i=1}^{p_n - 1} \lambda_i \varepsilon_i \otimes e_i$, $\tilde{R}_n = R_n - R_{n-1}$ and $r_n = \|\tilde{R}_n\| = \lambda_{p_{n-1}}$ for $n \geqslant 2$; we may choose R_1 such that $\operatorname{rank}(R_1) \geqslant 7$ $(p_1 \geqslant 8)$ and

$$r_n < \frac{\gamma^2 \lambda}{(3M)^3(1+M)} \frac{1}{2^{4(n+2)+1}} \varepsilon^4, \tag{3.4}$$

where $M = \max(1 + \|X^0\|, 1 + \|Y^0\|)$, which is possible since $(\lambda_i)_{i \geqslant 1}$ is decreasing sequence of positive numbers.

Let $(\beta_n)_{n \geqslant 1}$ the following sequence

$$\beta_n = \frac{\gamma}{3M} \frac{1}{2^{2(n+3)}} \varepsilon^2. \tag{3.5}$$

Step 1. $(R_1 x_i, y_j) = d_{11} - ((K - R_1) x_i, y_j)$.

Put $\alpha_{ij}^1 = (\overline{(K - R_1)x_i, y_j})$ and let $\mathcal{L} = \text{span}\{e_1, e_2, \ldots, e_7\} \cap \{R_1^* R_1 x_i, R_1^* y_i, i = 1,2\}^\perp$. $R_1|\mathcal{L} : \mathcal{L} \to \mathcal{H}$ has a canonical writing $R_1|\mathcal{L} = \sum_{i=1}^{m_1} \varepsilon_i^1 \otimes e_i^1$, where $(e_i^1)_{1 \leqslant i \leqslant m}$ is an orthonormal system and $(\varepsilon_i^1)_{1 \leqslant i \leqslant m}$ an orthogonal system. It is clear that $2 \leqslant m_1 \leqslant 7$ and $\|\varepsilon_i^1\| \geqslant \lambda_7 =: \lambda$ for $1 \leqslant i \leqslant m_1$. Set $U^1 = \begin{pmatrix} U_1^1 \\ U_2^1 \end{pmatrix} = \begin{pmatrix} \frac{\delta}{\sqrt{2}} e_1^1 \\ \frac{\delta}{\sqrt{2}} e_2^1 \end{pmatrix}$ and $V^1 = \begin{pmatrix} V_1^1 \\ V_2^1 \\ V_3^1 \end{pmatrix}$, where

$$V_j^1 = \frac{\sqrt{2}}{\delta}\left(\alpha_{1j}^1 \frac{\varepsilon_1^1}{\|\varepsilon_1^1\|^2} + \alpha_{2j}^1 \frac{\varepsilon_2^1}{\|\varepsilon_2^1\|^2}\right), \quad j = 1, 2$$

and

$$V_3^1 = \sqrt{2}\delta \bar{d} \frac{\varepsilon_1^1}{\|\varepsilon_1^1\|^2}, \quad \text{where} \quad d = d_{13}.$$

We have $\|U^1\| = \delta$; like in finite rank case take $\delta = \inf(1, \frac{\gamma\lambda}{960M^3}\varepsilon^2, \frac{\varepsilon}{4})$. Thus

$$|\alpha_{ij}^1| \leqslant \|K - R_1\| \, \|x_i\| \, \|y_j\| \leqslant r_2 M^2 \leqslant \delta^2 M^2$$

and

$$|d| = |(Kx_2, y_2)| \leqslant \|K\| \, \|x_2\| \, \|y_2\| \leqslant M^2.$$

Then $\|V_j^1\| < \frac{2}{\lambda}\delta M^2$, $j = 1, 2$ and $\|V_3^1\| \leqslant \frac{\sqrt{2}\delta}{\lambda}|d| \leqslant \frac{\sqrt{2}M^2}{\lambda}\delta$. We deduce $\|V^1\| < \frac{4\delta}{\lambda}M^2$.

One may easily verify that $(R_1(X_i^0 + U_i^1), Y_j^0 + V_j^1) = d_{ij}$ if $(i,j) \neq (1,3)$ and $(R_1(x_1 + U_1^1), V_3^1) = \delta^2 d$.

Let $\tilde{X} = X^0 + U^1$ and $\tilde{Y} = Y^0 + V^1$. We have

$$\begin{aligned} \|[\tilde{\tilde{L}}] - [\tilde{X} \otimes \tilde{Y}]\| &\leqslant \|U^1\| \, \|Y^0\| + \|X^0\| \, \|V^1\| + \|U^1\| \, \|V^1\| + \delta^2 \|x_2\| \, \|y_2\| \\ &\leqslant \delta M^2 \left(1 + \frac{4}{\lambda}M + \frac{4}{\lambda} + 1\right) < \frac{10}{\lambda} M^3 \delta. \end{aligned}$$

Since $M_{2,3}(\mathcal{A})$ has the property $X_{0,\frac{\gamma}{6}}$ by Proposition 2.1, then Proposition 2.2 provides vectors $X^1 \in \mathcal{H}^{(2)}$ and $Z^1 \in \mathcal{H}^{(3)}$ such that $[\tilde{\tilde{L}}] = [X^1 \otimes Z^1]$,

$$\max(\|X^1 - \tilde{X}\|, \|Z^1 - \tilde{Y}\|) < \sqrt{\frac{\frac{10M^3}{\lambda}\delta}{\frac{\gamma}{6}}} < \frac{\varepsilon}{4},$$

$$(X^1 - \tilde{X}) \in ((R_1\mathcal{H} \cup R_1^*\mathcal{H})^\perp)^{(2)}$$

and

$$(Z^1 - \tilde{Y}) \in ((R_1\mathcal{H} \cup R_1^*\mathcal{H})^\perp)^{(3)}.$$

This implies that

$$(R_1X_i^1, Z_j^1) = d_{ij}, \quad i = 1,2, \quad 1 \leqslant j \leqslant 3, \quad (i,j) \neq (1,3)$$

and

$$(R_1X_1^1, Z_3^1) = \delta^2 d_{13}.$$

If we take $Y^1 = \begin{pmatrix} Z_1^1 \\ Z_2^1 \\ \frac{1}{\delta^2} Z_3^1 \end{pmatrix}$, one can easily check that

$$[X^1 \otimes Y^1] = [\tilde{L}],$$

$$(R_1X_i^1, Y_j^1) = d_{ij}, \quad i = 1,2, \quad 1 \leqslant j \leqslant 3,$$

and

$$\|X_i^1 - x_i\| < \frac{\varepsilon}{2},$$

$$\|Y_i^1 - y_i\| < \frac{\varepsilon}{2}, \quad i = 1,2.$$

Step (n+1).

Suppose now that we can find vectors $(X^k)_{1\leqslant k\leqslant n} \subset \mathcal{H}^{(2)}$ and $(Y^k)_{1\leqslant k\leqslant n} \subset \mathcal{H}^{(3)}$ satisfying (3.1), (3.2) and (3.3). It is clear that $\|X^k\| < M$, $\|Y^k\| < M$, and the action of R_{n+1} is

$$(R_{n+1}X_i^n, Y_j^n) = d_{ij} + (\tilde{R}_{n+1}X_i^n, Y_j^n).$$

Put $\alpha_{ij}^{n+1} = -\overline{(\tilde{R}_{n+1}X_i^n, Y_j^n)}$.

Let $\mathcal{L}_n = \operatorname{span}\{e_1, \ldots, e_7\} \cap \{R_{n+1}^* Y_j^n, R_{n+1}^* R_{n+1} X_i^n,\ 1 \leqslant i \leqslant 2 \text{ and } 1 \leqslant j \leqslant 3\}^\perp$ and $R_1|\mathcal{L}_n = \sum_{i=1}^{m_{n+1}} \varepsilon_i^{n+1} \otimes e_i^{n+1}$ be the canonical writting of $R_1|\mathcal{L}_n : \mathcal{L}_n \to \mathcal{H}$ where $(e_i^{n+1})_{1\leqslant i\leqslant m_{n+1}}$ is an orthonormal system and $(\varepsilon_i^{n+1})_{1\leqslant i\leqslant m_{n+1}}$ is an orthogonal system. Then it is not hard to see that $2 \leqslant m_{n+1} \leqslant 7$ and $\|\varepsilon_i^{n+1}\| \geqslant \lambda_7 \stackrel{\text{def}}{=} \lambda$ for $1 \leqslant i \leqslant m_{n+1}$. Setting

$$U^n = \begin{pmatrix} U_1^n \\ U_2^n \end{pmatrix} = \begin{pmatrix} \frac{\beta_n}{\sqrt{2}} & e_1^{n+1} \\ \frac{\beta_n}{\sqrt{2}} & e_2^{n+1} \end{pmatrix}$$

and

$$V^n = \begin{pmatrix} V_1^n \\ V_2^n \\ V_3^n \end{pmatrix}, \text{ where } V_j^n = \frac{\sqrt{2}}{\beta_n}\Big(\alpha_{1j}^{n+1} \frac{\varepsilon_1^{n+1}}{\|\varepsilon_1^{n+1}\|^2} + \alpha_{2j}^{n+1} \frac{\varepsilon_2^{n+1}}{\|\varepsilon_2^{n+1}\|^2}\Big), \ 1 \leqslant j \leqslant 3$$

we have by simple calculation:

$$\begin{aligned}(R_{n+1}U_i^n, V_j^n) &= ((R_1/\mathcal{L}_n)(U_i^n), V_j^n)\\ &= \left(\varepsilon_i^{n+1}, \alpha_{1j}^{n+1}\frac{\varepsilon_1^{n+1}}{\|\varepsilon_1^{n+1}\|^2} + \alpha_{2j}^{n+1}\frac{\varepsilon_2^{n+1}}{\|\varepsilon_2^{n+1}\|^2}\right) = \overline{\alpha_{ij}^{n+1}}.\end{aligned}$$

Thus $(R_{n+1}(X_i^n + U_i^n), (Y_j^n + V_j^n)) = d_{ij}$ (since $U_i^n \in \mathcal{L}_n$).

Put $\tilde{X}^n = X^n + U^n$ and $\tilde{Y}^n = Y^n + V^n$. We have

$$\begin{aligned}\|[\tilde{L}] - [\tilde{X}^n \otimes \tilde{Y}^n]\| &\leqslant \|[U^n \otimes Y^n]\| + \|[U^n \otimes V^n]\| + \|[X^n \otimes V^n]\|\\ &\leqslant \|U^n\|\,\|Y^n\| + \|U^n\|\,\|V^n\| + \|X^n\|\,\|V^n\|.\end{aligned}$$

Let us seek uper bounds for $\|U^n\|$ and $\|V^n\|$. It is easy to see that $\|U^n\| = \beta_n$ and from the definition of α_{ij}^{n+1}, we clearly have

$$|\alpha_{ij}^{n+1}| \leqslant r_{n+1}\|X^n\|\,\|Y^n\| \leqslant M^2 r_{n+1}.$$

Then $\|V_j^n\| \leqslant \frac{2}{\beta_n\lambda}M^2 r_{n+1}$. We deduce $\|V^n\| \leqslant \frac{6}{\beta_n\lambda}M^2 r_{n+1}$ and it follows then that

$$\begin{aligned}\|[\tilde{L}] - [\tilde{X}^n \otimes \tilde{Y}^n\| &\leqslant M\beta_n + \frac{6}{\lambda}M^2 r_{n+1} + \frac{6M^3}{\lambda\beta_n}r_{n+1}\\ &\leqslant M\beta_n + \frac{6}{\lambda\beta_n}M^2(1+M)r_{n+1}.\end{aligned}$$

Since we have the relations (3.4) and (3.5),

$$\begin{aligned}\|[\tilde{L}] - [\tilde{X}^n \otimes \tilde{Y}^n]\| &< \frac{\gamma}{3}\frac{1}{2^{2(n+3)}}\varepsilon^2 + \frac{\gamma}{6}\frac{\varepsilon^2}{2^{2(n+2)+1}}\\ &< \frac{1}{2^{2(n+2)}}\frac{\gamma\varepsilon^2}{6} \stackrel{\text{def}}{=} \rho_n.\end{aligned}$$

By applying Proposition 2.1 and Proposition 2.2 we can find vectors $X^{n+1} \in \mathcal{H}^{(2)}$ and $Y^{n+1} \in \mathcal{H}^{(3)}$ such that

$$[\tilde{L}] = [X^{n+1} \otimes Y^{n+1}],$$

$$\max(\|X^{n+1} - \tilde{X}^n\|,\ \|Y^{n+1} - \tilde{Y}^n\|) < \sqrt{\frac{\rho_n}{\frac{\gamma}{6}}} = \frac{1}{2^{n+2}}\varepsilon,$$

$$(X^{n+1} - \tilde{X}^n) \in ((R_{n+1}\mathcal{H} \cup R_{n+1}^*\mathcal{H})^\perp)^{(2)}, \tag{3.6}$$

$$(Y^{n+1} - \tilde{Y}^n) \in ((R_{n+1}\mathcal{H} \cup R_{n+1}^*\mathcal{H})^\perp)^{(3)}. \tag{3.7}$$

From (3.6) and (3.7), it follows that $(R_{n+1}X_i^{n+1}, Y_j^{n+1}) = d_{ij}$ for $i = 1,2$ and $1 \leqslant j \leqslant 3$. Furthermore, since $\max(\|U^n\|, \|V^n\|) < \frac{\varepsilon}{2^{n+2}}$, we obtain

$$\begin{aligned}\|X^{n+1} - X^n\| &\leqslant \|X^{n+1} - \tilde{X}^n\| + \|U^n\| \\ &< \frac{1}{2^{n+1}}\varepsilon,\end{aligned}$$

and

$$\begin{aligned}\|Y^{n+1} - Y^n\| &\leqslant \|Y^{n+1} - \tilde{Y}^n\| + \|V^n\| \\ &< \frac{1}{2^{n+1}}\varepsilon.\end{aligned}$$

If we consider $(X_i^n)_n$ and $(Y_j^n)_n$, there are two Cauchy sequences and thus converge. Let ξ_i and η_j their respective limits; we have

$$[\tilde{L}] = \lim_{n\to+\infty}[X^n \otimes Y^n] = [\xi \otimes \eta] \quad \text{where,} \quad \xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix},$$

$$d_{ij} = \lim_{n\to+\infty}(R_n X_i^n, Y_j^n) = (K\xi_i, \eta_j), \quad \text{for} \quad i = 1,2 \quad \text{and} \quad 1 \leqslant j \leqslant 3.$$

This means that

$$\tilde{\psi} = [\xi \otimes \eta]_{\tilde{\mathcal{A}}}. \tag{3.8}$$

On the other hand

$$\begin{aligned}\|\xi_i - x_i\| &= \lim_{n\to+\infty}\|X_i^n - x_i\| \\ &= \lim_{n\to+\infty}\Big\|\sum_{k=2}^{n}(X_i^k - X_i^{k-1}) + X_i^1 - x_i\Big\| \\ &\leqslant \lim_{n\to+\infty}\sum_{k=2}^{n}\|X^k - X^{k-1}\| + \|X_i^1 - x_i\| \\ &< \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{2^k}\varepsilon = \varepsilon \quad \text{for,} \quad i = 1,2.\end{aligned} \tag{3.9}$$

The same argument shows that

$$\|\eta_i - y_i\| < \varepsilon \quad \text{for,} \quad i = 1,2. \tag{3.10}$$

The relations (3.8), (3.9) and (3.10) mean that $\tilde{\mathcal{A}} = \mathcal{A} + \mathbb{C}K$ has the property (P) and then is hereditarily reflexive by the Theorem 2.4.

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