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# NOTES ON HYPERTRACES AND $C^*$-ALGEBRAS

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ABSTRACT. The concept of a hypertrace on a $C^*$-algebra of bounded linear operators on a Hilbert space has been used, somewhat unexplicitely, in several recent works. In the present paper, we give an introduction to this concept and its relation to amenability and nuclearity, and initiate a study of the naturally emerging classes of so-called weakly hypertracial $C^*$-algebras and hypertracial $C^*$-algebras.

KEYWORDS: *$C^*$-algebra, hypertrace, amenability, nuclearity.*

AMS SUBJECT CLASSIFICATION: 46L05, 46L30, 46L35, 22D25, 43A07.

## INTRODUCTION

Let $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$, the bounded linear operators on some non-zero Hilbert space $\mathcal{H}$. By a *hypertrace* on $\mathcal{A}$ we mean a state $\varphi$ on $\mathcal{B}(\mathcal{H})$ which contains $\mathcal{A}$ in its centralizer, i.e. $\mathcal{A} \subseteq C_\varphi$ where $C_\varphi$ is the $C^*$-algebra given by

$$C_\varphi = \{Y \in \mathcal{B}(\mathcal{H}) \mid \varphi(XY) = \varphi(YX) \quad \text{for all } X \in \mathcal{B}(\mathcal{H})\}.$$

This concept was introduced by Connes in his fundamental paper on the classification of injective factors ([12]) as an important tool in establishing that any injective $\mathrm{II}_1$-factor $\mathcal{N}$ acting on a separable Hilbert space is $*$-isomorphic to the hyperfinite $\mathrm{II}_1$-factor. The key observation was that from the existence of a hypertrace on $\mathcal{N}$ one may deduce that $\mathcal{N}$ satisfies a property analogous to Følner's characterization of the amenability of a group. The reader may also consult [29] or [30] to see how the hyperfiniteness of $\mathcal{N}$ may be obtained from this Følner property. The analogy between the existence of a hypertrace on $\mathcal{N}$ and the existence of an invariant mean on $l^\infty(G)$ (where $G$ denotes a discrete group), i.e. the amenability of $G$, is nicely exposed in [13].

The concept of hypertrace for $C^*$-algebras has also been of some interest, though in a somewhat unexplicit form. For example, Connes uses it to rule out the existence of finitely summable unbounded Fredholm modules on some $C^*$-algebras in [15]. Further, an operator $T \in \mathcal{B}(\mathcal{H})$ is finite in the sense of Williams if there exists a hypertrace on the $C^*$-algebra generated by $T$ (cf. [37], Theorem 4), while a unitary representation $U$ of a locally compact group $G$ is amenable in the sense of Bekka ([6]) if there exists a hypertrace on the $C^*$-algebra generated by $U(G)$. Finally, it has also been useful in the recent work of Kirchberg ([20], [21]).

In the first section of these notes, we point out how [12], [13] and/or [6] may be used to characterize the existence of a hypertrace on $\mathcal{A}$ as being equivalent to the existence of a "Følner net" for $\mathcal{A}$ of non-zero finite dimensional projections in $\mathcal{B}(\mathcal{H})$. Then we explain how the notion of a Følner net for $\mathcal{A}$ is related to [2] and [5] where the problem of approximating the spectrum of some self-adjoint operators in $\mathcal{B}(\mathcal{H})$ is considered. This problem is quite important from the point of view of applications in quantum physics and it is therefore of interest to be able to decide, when given a self-adjoint operator $T$, which of the $C^*$-algebras containing $T$ may be represented on a Hilbert space in such a way that their homomorphic images possess a Følner net, i.e. possess a hypertrace. This leads naturally to the study initiated in the next two sections. The Arveson approach also requires the existence of a unique trace on the actual $C^*$-algebra, but it seems possible to us that this rather strict condition may be somewhat weakened.

In the second section, we explore in some details the following notion, inspired by [6]: if $\mathcal{A}$ is a $C^*$-algebra and $\pi$ is a non-degenerate representation of $\mathcal{A}$, we say that $\pi$ is hypertracial if $\pi(\mathcal{A})$ has a hypertrace, and we say that $\mathcal{A}$ is weakly hypertracial if all faithful non-degenerate representations of $\mathcal{A}$ are hypertracial (it is in fact enough that $\mathcal{A}$ has one hypertracial non-degenerate representation). We show that the class of weakly hypertracial $C^*$-algebras is quite large and includes many familiar examples. Further, it behaves nicely with respect to crossed products by amenable discrete groups and tensor products. Let us also mention here that we recently received a preprint by G. Vaillant ([34]) where he studies $C^*$-growth, Voiculescu's condition, Følner type conditions for $C^*$-algebras and their relation to nuclearity in connection with a program initiated by Voiculescu on the structure of $C^*$-algebras inspired by the work of Gromov and Connes on discrete groups, growth and Fredholm modules. One of his results may be stated as follows: Voiculescu's condition implies a strong Følner type condition, which itself implies weak hypertraciality.

In many aspects, the class of weakly hypertracial $C^*$-algebras is too large and in the third and final section, we study the smaller class of hypertracial $C^*$-algebras: a $C^*$-algebra is hypertracial if all its non-degenerate representations are hypertracial. This class contains all strongly amenable $C^*$-algebras (as defined in [19]) and many nuclear $C^*$-algebras satisfying some finiteness condition. In many cases hypertraciality is quite easy to establish compared to nuclearity. An interesting open question is the following: is a separable unital hypertracial $C^*$-algebra necessarily nuclear? (the answer being no for non-separable $C^*$-algebras).

As general references on operator algebras, we refer to [16], [24], [27], [31] and [32]. Our notation shall mainly be as in [24]. Especially if $\mathcal{A}$ is a $C^*$-algebra, then $\tilde{\mathcal{A}}$ denotes its unitization (defined even if $\mathcal{A}$ is unital). If $\mathcal{A}$ is unital, then $\mathcal{U}(\mathcal{A})$ denotes its unitary group.

Concerning amenability of groups, the reader should consult [26] or [28]. Nuclearity and amenability of $C^*$-algebras are equivalent notions ([14], [17]). For an overview, see [26], 1.31 and also [23]. A review of injectivity of von Neumann algebras and its equivalent formulations is given in [26], 2.35.

Although not essential in this paper which deals mostly with $C^*$-algebras, we conclude this introduction by gathering for completeness some results on hypertraces and finite von Neumann algebras.

Let $\mathcal{N}$ denote a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. If $\mathcal{N}$ is injective, i.e. there exists a conditional expectation $E : \mathcal{B}(\mathcal{H}) \to \mathcal{N}$, then $\tau \circ E$ is a hypertrace on $\mathcal{N}$ for any tracial state $\tau$ on $\mathcal{N}$. Conversely, if $\mathcal{N}$ is countably decomposable and $\mathcal{N}$ has a hypertrace $\varphi$ such that $\varphi|_{\mathcal{N}}$ is faithful, then $\mathcal{N}$ is injective ([13]; [31], 10.27). Further, if $\mathcal{N}$ has a separable predual and $\mathcal{N}$ has a hypertrace, then $\mathcal{N}$ is approximately finite dimensional (this follows from [29] or [30], if one uses the fact that the existence of a hypertrace on $\mathcal{N}$ is independent of the Hilbert space on which $\mathcal{N}$ acts, cf. Theorem 2.1, so that one may assume that $\mathcal{H} = L^2(\mathcal{N}, \tau)$ where $\tau$ is a faithful normal trace on $\mathcal{N}$ as in [29] or [30]). Finally, $\mathcal{N}$ is of course injective whenever $\mathcal{N}$ is approximately finite dimensional.

## 1. HYPERTRACES AND FØLNER NETS

In this section, $\mathcal{A}$ denotes a $C^*$-algebra of bounded operators acting on a Hilbert space $\mathcal{H}$. We assume that $\mathcal{A}$ contains the identity operator $I = I_{\mathcal{H}}$ on $\mathcal{H}$ (if $I \notin \mathcal{A}$, one may just replace $\mathcal{A}$ with $\mathcal{A}+\mathbf{C}\cdot I$). If $T$ is an operator on $\mathcal{H}$, $\|T\|_1$ (resp. $\|T\|_2$) denotes its trace-class norm (resp. its Hilbert-Schmidt norm). The $C^*$-algebra of compact operators on $\mathcal{H}$ is denoted by $\mathcal{K}(\mathcal{H})$.

Based on the work of Connes ([12], [13]) or alternatively on the work of Bekka ([6]) which builds on [12], the existence of a hypertrace on $\mathcal{A}$ may be characterized by properties analogous to Reiter's and Følner's characterizations of the amenability of a group:

THEOREM 1.1. *Consider the following conditions on $\mathcal{A}$, where $p = 1, 2$, and where $\mathcal{U}(\mathcal{A})$ is considered as a discrete group.*

(H) *There exists a hypertrace on $\mathcal{A}$.*

(B) *The unitary representation of $\mathcal{U}(\mathcal{A})$ on $\mathcal{H}$ induced by the identity representation of $\mathcal{A}$ on $\mathcal{H}$ is amenable in the sense of Bekka.*

($\mathrm{P}_p$) *There exists a net $\{S_\alpha\}$ of operators on $\mathcal{H}$ such that $\|S\|_p = 1$, $S \geqslant 0$ and*

$$\lim_\alpha \|S_\alpha A - AS_\alpha\|_p = 0 \quad \textit{for all} \quad A \in \mathcal{A}\,.$$

($\mathrm{F}_p$) *There exists a net $\{Q_\alpha\}$ of non-zero finite dimensional projections in $\mathcal{B}(\mathcal{H})$ such that*

$$\lim_\alpha \frac{\|Q_\alpha A - AQ_\alpha\|_p}{\|Q_\alpha\|_p} = 0 \quad \textit{for all}\ A \in \mathcal{A}\,.$$

*Then conditions* (H), (B), ($\mathrm{P}_1$), ($\mathrm{P}_2$), ($\mathrm{F}_1$), ($\mathrm{F}_2$) *are all equivalent. If $\mathcal{A}$ is separable, then the nets $\{S_\alpha\}$ and $\{Q_\alpha\}$ may be chosen as sequences.*

*Proof.* As every element of $\mathcal{A}$ is a linear combination of elements in $\mathcal{U}(\mathcal{A})$, it is clear that (H) is equivalent to (B). The equivalences ($\mathrm{P}_1$) $\Leftrightarrow$ ($\mathrm{P}_2$) and ($\mathrm{F}_1$) $\Leftrightarrow$ ($\mathrm{F}_2$) are consequences of the Powers-Størmer inequalities (just as in [6], [12] and [13]), while ($\mathrm{F}_p$) $\Rightarrow$ ($\mathrm{P}_p$) and ($\mathrm{P}_1$) $\Rightarrow$ (H) are quite obvious.

Further, the implications (H) $\Rightarrow$ ($\mathrm{P}_1$) and ($\mathrm{P}_2$) $\Rightarrow$ ($\mathrm{F}_2$) are shown by Connes in [13], 2.4–2.6, under the assumption that $\mathcal{A}$ is a $\mathrm{II}_1$-factor acting on $\mathcal{H} = L^2(\mathcal{A}, \tau)$ where $\tau$ denotes the tracial state of $\mathcal{A}$. However, a careful reading of his arguments assures one that the only assumption on $\mathcal{A}$ which is relevant to this part of Connes' article is that $\mathcal{A}$ is a $C^*$-algebra of operators acting on a Hilbert space $\mathcal{H}$ and containing $I_{\mathcal{H}}$. This means precisely that (H) $\Rightarrow$ ($\mathrm{P}_1$) and ($\mathrm{P}_2$) $\Rightarrow$ ($\mathrm{F}_2$) are true in our setting. Alternatively, one may argue as follows to finish the proof: by [6], Theorem 6.2, (B) implies that given $\varepsilon > 0$ and $U_1, U_2, \ldots, U_n \in \mathcal{U}(\mathcal{A})$, there exists a non-zero finite dimensional projection $Q$ in $\mathcal{B}(\mathcal{H})$ such that

$$\|U_j Q U_j^* - Q\|_1 < \varepsilon \|Q\|_1,$$

$j = 1, 2, \ldots, n$, from which ($F_1$) easily follows and also that $\{Q_\alpha\}$ and $\{P_\alpha\}$ may be chosen as sequences if $\mathcal{A}$ is separable. ∎

REMARKS 1.2. (i) By invoking the Jordan decomposition, we have (H) $\Leftrightarrow$ there exists a non-zero bounded linear functional $\psi$ on $\mathcal{B}(\mathcal{H})$ such that $\psi(AX) = \psi(XA)$ for all $A \in \mathcal{A}, X \in \mathcal{B}(\mathcal{H})$ (cf. [9], Proposition 5).

(ii) By taking into account the generalized Powers-Størmer inequalities ([22]), one may show that (H) $\Leftrightarrow$ ($P_p$) $\Leftrightarrow$ ($F_p$) for any $p \geqslant 1$.

(iii) If $\{\mathcal{H}, D\}$ is a finitely summable unbounded Fredholm module over $\mathcal{A}$, Connes shows in [15], Lemma 9, the existence of a net satisfying ($P_1$).

(iv) In the case when $\mathcal{A}$ is separable and $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$, Kirchberg shows in [21], Proposition 3.2, that the existence of a hypertrace on $\mathcal{A}$ is equivalent to the fact that $\mathcal{A}$ has a "liftable" tracial state.

A net $\{Q_\alpha\}$ satisfying ($F_1$) shall be called *a Følner net for* $\mathcal{A}$.

Inspired by Arveson's paper ([2]) on $C^*$-algebras and numerical linear algebra, we introduced in [5] the following notion:

Let $\mathcal{F} = \{\mathcal{H}_n\}_{n \geqslant 1}$ be a filtration of $\mathcal{H}$, i.e. each $\mathcal{H}_n$ is a non-zero finite dimensional subspace of $\mathcal{H}$, $\mathcal{H}_n \subseteq \mathcal{H}_{n+1}$ and $\mathcal{H} = \overline{\bigcup_{n \geqslant 1} \mathcal{H}_n}$. Then $\mathcal{F}$ is called a weak $\mathcal{A}$-filtration if

$$\lim_{n \to +\infty} \frac{\|P_n A - A P_n\|_1}{\|P_n\|_1} = 0 \quad \text{for all } A \in \mathcal{A}, \tag{1.1}$$

where $P_n$ denotes the projection of $\mathcal{H}$ onto $\mathcal{H}_n$. Our motivation was that we could give an extended version of [2], Theorem 4.5, cf. [5], Theorem 1, to the case when $\mathcal{A}$ has a unique tracial state and has a weak $\mathcal{A}$-filtration, and that for some interesting examples of $\mathcal{A}$'s with a unique tracial state there exists a natural weak $\mathcal{A}$-filtration, thus making it possible to put some numerical approximations of the spectrum of self-adjoint operators in $\mathcal{A}$ on a somewhat firm ground.

However, an inspection of the proof of [5], Theorem 1, makes it clear that the projections $P_n$ in fact don't have to be associated to a filtration of $\mathcal{H}$: it suffices that they satisfy condition (1.1) for the proof to go through. Further, the proof is easily adapted to handle nets instead of sequences of projections. This means that the following result holds:

THEOREM 1.3. *Suppose that $\mathcal{A}$ has a hypertrace $\varphi$ such that $\varphi|_{\mathcal{A}}$ is the only tracial state on $\mathcal{A}$. Let $\{Q_\alpha\}$ be a Følner net for $\mathcal{A}$ (such a net exists by Theorem* 1.1). *Then we have*

$$\lim_\alpha \frac{1}{d_\alpha}\Big[f(\lambda_{1,\alpha}) + \cdots + f(\lambda_{d_\alpha,\alpha})\Big] = \varphi(f(A)) = \int_{\mathbf{R}} f(t)\mathrm{d}\mu_A(t)$$

*for all $f \in C_0(\mathbf{R})$, where $d_\alpha = \dim Q_\alpha$, $A$ is a self-adjoint operator in $\mathcal{A}$, $\{\lambda_{1,\alpha}, \ldots, \lambda_{d_\alpha,\alpha}\}$ is the list of eigenvalues (repeated according to multiplicity) of $Q_\alpha A|_{Q_\alpha \mathcal{H}}$ and $\mu_A$ is the spectral measure of $A$ associated with $\varphi$. If $\varphi|_{\mathcal{A}}$ is faithful, then* $\mathrm{sp}(A) = \mathrm{supp}(\mu_A)$.

REMARKS 1.4. (i) In a concrete situation where one wants to approximate numerically the spectrum of a self-adjoint operator $A$ in $\mathcal{B}(\mathcal{H})$, the problem is of course to find if possible an appropriate $\mathcal{A}$ containing $A$ and an explicit sequence $\{Q_n\}$ satisfying the above assumptions.

(ii) If $\mathcal{A}$ has a faithful unique tracial state and $\mathcal{A}$ is infinite dimensional, one gets easily from [3], Proposition 2.2, that the spectrum of a self-adjoint operator in $\mathcal{A}$ coincides with its essential spectrum.

(iii) If $\mathcal{A} = \mathcal{B}(\mathbf{C}^n)$ for some $n \geqslant 1$, then any net $\{Q_\alpha\}$ of non-zero projections in $\mathcal{B}(\mathbf{C}^n)$ is a Følner net for $\mathcal{A}$ if and only if $Q_\alpha = I$ for all $\alpha \geqslant \alpha_0$ (for some $\alpha_0$) and $\mathcal{A}$ has only one hypertrace given by the normalized trace on $\mathcal{B}(\mathbf{C}^n)$. The statement in Theorem 1.3 is then just an obvious corollary of the spectral theorem.

(iv) Another quite trivial example, but more instructive, is the following: Suppose now $\mathcal{H}$ is infinite dimensional and let $\mathcal{A} = \mathcal{K}(\mathcal{H}) + \mathbf{C}\cdot I$. Then $\mathcal{A}$ has a hypertrace. Indeed, a state $\varphi$ on $\mathcal{B}(\mathcal{H})$ is a hypertrace on $\mathcal{A}$ if and only if $\varphi$ is singular (since $\varphi$ is singular $\Leftrightarrow \varphi|_{\mathcal{K}(\mathcal{H})} = 0$, cf. [32]). On the other hand, let $\{Q_\alpha\}$ be any increasing net of non-zero finite dimensional projections in $\mathcal{B}(\mathcal{H})$ converging strongly to $I$. Then $\{Q_\alpha\}$ is a Følner net for $\mathcal{A}$, since

$$\begin{aligned}\frac{\|Q_\alpha A - AQ_\alpha\|_1}{\|Q_\alpha\|_1} &\leqslant \frac{\mathrm{rank}(Q_\alpha A - AQ_\alpha)\|Q_\alpha A - AQ_\alpha\|}{\|Q_\alpha\|_1} \\ &\leqslant 2\|Q_\alpha A - AQ_\alpha\| \to 0\end{aligned}$$

for all $A \in \mathcal{K}(\mathcal{H})$.

Now, $\mathcal{A}$ has clearly a unique tracial state $\tau$, so Theorem 1.3 applies. Let $A$ be a self-adjoint compact operator on $\mathcal{H}$. For any $f \in C_0(\mathbf{R})$ we have $\tau(f(A)) = f(0)$, hence $\mu_A = \delta_0$ (the Dirac measure at 0). If $\{\xi_j\}_{j\in J}$ is an orthornormal basis for $\mathcal{H}$ consisting of eigenvectors of $A$ with associated eigenvalues $\{\lambda_j\}$, by choosing $Q_F$

as the projection onto the linear span of $\{\xi_j\}_{j\in F}$ when $F$ is a non-empty finite subset of $J$, we obtain

$$\lim_F \frac{1}{\#(F)} \sum_{j\in F} f(\lambda_j) = f(0) \quad \text{for all } f \in C_0(\mathbb{R}).$$

In the spirit of [2], this might be seen as an indication of the well-known fact that the essential spectrum of $A$ reduces to $\{0\}$.

Let now $G$ denote a locally compact group, $\lambda$ its left regular representation acting on $L^2(G)$, $C^*(\lambda(G))$ the $C^*$-algebra generated by $\lambda(G)$ (so that $C^*(\lambda(G)) = C_r^*(G)$ when $G$ is discrete) and $vN(G) = \lambda(G)''$, the group von Neumann algebra of $G$. The following result is essentially well-known:

PROPOSITION 1.5. *Consider the four conditions:*

(i) *$G$ is amenable;*

(ii) *$C^*(U(G))$ has a hypertrace for all continuous unitary representations $U$ of $G$;*

(iii) *$C^*(\lambda(G))$ has a hypertrace;*

(iv) *$vN(G)$ has a hypertrace.*

*Then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftarrow$ (iv). *If $G$ is discrete, all four conditions are equivalent.*

*Proof.* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is merely a rephrasing of [6], Theorem 2.2, while (iv) $\Rightarrow$ (iii) is trivial. If $G$ is discrete, then $vN(G)$ is a finite von Neumann algebra which is injective whenever $G$ is amenable ([12], Proposition 6.8), hence (i) $\Rightarrow$ (iv) follows. ∎

We also mention a $C^*$-algebraic analog to [12], Proposition 6.8. Let $G$ denote a right amenable discrete semigroup, i.e. there exists a state $M$ on $\ell^\infty(G)$ such that

$$M(f_h) = M(f) \quad \text{for all } f \in \ell^\infty(G),\ h \in G,$$

where $f_h(g) = f(gh)$, $g \in G$.

We shall use the suggestive notation $\int_G f(g)\,\mathrm{d}M(g)$ for $M(f)$.

Suppose there exists a map $U : G \to \mathcal{U}(\mathcal{B}(\mathcal{H}))$ satisfying $U_g \mathcal{A} U_g^* \subseteq \mathcal{A}$ and $U_g U_h U_{gh}^* \in \mathcal{U}(\mathcal{A})$ for all $g, h \in G$. Let $C^*(\mathcal{A}, U(G))$ denote the $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{A}$ and $U(G)$. Then we have:

PROPOSITION 1.6. *If there exists a hypertrace $\varphi$ on $\mathcal{A}$ and $G$ is right-amenable, then there exists a hypertrace on $C^*(\mathcal{A}, U(G))$, also.*

*Proof.* For each $X \in \mathcal{B}(\mathcal{H})$, define

$$\widetilde{\varphi}(X) = \int_G \varphi(U_g X U_g^*)\,\mathrm{d}M(g),$$

which is well-defined since

$$|\varphi(U_g X U_g^*)| \leqslant \|\varphi\|\,\|U_g X U_g^*\| \leqslant \|X\| \quad \text{for all } g \in G.$$

One checks easily that $\widetilde{\varphi}$ is a state on $\mathcal{B}(\mathcal{H})$. Set $V(g,h) = U_g U_h U_{gh}^* \in \mathcal{U}(\mathcal{A})$, $g,h \in G$. For $X \in \mathcal{B}(\mathcal{H})$, $h \in G$, $A \in \mathcal{A}$, we have:

$$\begin{aligned}
\widetilde{\varphi}(U_h X U_h^*) &= \int_G \varphi(U_g U_h X U_h^* U_g^*)\,\mathrm{d}M(g)\\
&= \int_G \varphi(V(g,h) U_{gh} X U_h^* U_g^*)\,\mathrm{d}M(g)\\
&= \int_G \varphi(U_{gh} X U_h^* U_g^* V(g,h))\,\mathrm{d}M(g) \qquad \text{(since } \varphi \text{ is a hypertrace on } \mathcal{A})\\
&= \int_G \varphi(U_{gh} X U_{gh}^*)\,\mathrm{d}M(g)\\
&= \int_G \varphi(U_g X U_g^*)\,\mathrm{d}M(g) \qquad \text{(since } M \text{ is right-invariant)}\\
&= \widetilde{\varphi}(X)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\varphi}(AX) &= \int_G \varphi(U_g A X U_g^*)\,\mathrm{d}M(g)\\
&= \int_G \varphi(U_g A U_g^* U_g X U_g^*)\,\mathrm{d}M(g)\\
&= \int_G \varphi(U_g X U_g^* U_g A U_g^*)\,\mathrm{d}M(g)\\
&= \int_G \varphi(U_g X A U_g^*)\,\mathrm{d}M(g)\\
&= \widetilde{\varphi}(XA).
\end{aligned}$$

Hence it follows that $C^*(\mathcal{A}, U(G))$ is contained in the centralizer of $\widetilde{\varphi}$ and the result follows. ∎

Of course, one may also state a corresponding "left"-version of this proposition. If $G$ is a group, right- and left-amenability are equivalent notions; this is not generally true in the semigroup case (cf. [26]). The semi-group case might be of interest when dealing with twisted crossed products by semigroups of automorphisms, but we shall only consider the group case in the next sections. We just mention here:

COROLLARY 1.7. *Let $G$ denote a discrete group and $Z^2(G, \mathbb{T})$ the set of (normalized) 2-cocycles of $G$ with values in the circle group $\mathbb{T}$. If $u \in Z^2(G, \mathbb{T})$, we denote by $\lambda_u$ the associated projective (left or right) regular representation of $G$ on $\ell^2(G)$ and we set $C^*_r(G,u) = C^*(\lambda_u(G)), vN(G,u) = \lambda_u(G)''$. Then the following four conditions are equivalent:*

(i) *$G$ is amenable;*

(ii) *$C^*(U(G))$ has a hypertrace for all projective unitary representations of $G$;*

(iii) *$C^*_r(G,u)$ has a hypertrace for some $u \in Z^2(G, \mathbb{T})$;*

(iv) *$vN(G,u)$ has a hypertrace for some $u \in Z^2(G, \mathbb{T})$.*

*Proof.* (i) $\Rightarrow$ (ii) follows from Proposition 1.6 by taking $\mathcal{A} = \mathbb{C}$. (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (iii) are trivial. (i) $\Rightarrow$ (iv) follows easily for example from [1], Proposition 3.12. Finally, (iii) $\Rightarrow$ (i) follows in the "classical" way: if $\varphi$ is a hypertrace on $C^*_r(G,u) \subseteq \mathcal{B}(\ell^2(G))$, then a (left or right) invariant mean $M$ on $\ell^\infty(G)$ is given by $M(f) = \varphi(m_f)$, where $m_f$ denotes the multiplication operator by $f \in \ell^\infty(G)$ on $\ell^2(G)$. An easy computation shows indeed that if $\lambda_u$ is for example the right regular representation, then one has $\lambda_u(h) m_f \lambda_u(h)^* = m_{f_h}$ for all $h \in G, f \in \ell^\infty(G)$ as usual. ∎

Finally, there is another situation where the existence of hypertraces is easily shown:

PROPOSITION 1.8. *Suppose $\mathcal{A}$ has a non-zero multiplicative linear functional $\psi$ (e.g. $\mathcal{A}$ is abelian). Then $\mathcal{A}$ has a hypertrace.*

*Proof.* Since $\psi$ preserves adjoints ([24], 2.1.9), it is positive, hence it is a state on $\mathcal{A}$. We may therefore extend it to a state on $\mathcal{B}(\mathcal{H})$ which is easily seen to be a hypertrace on $\mathcal{A}$ by making use of the Cauchy-Schwartz inequality for states. ∎

## 2. WEAKLY HYPERTRACIAL $C^*$-ALGEBRAS

Let $\mathcal{A}$ be a $C^*$-algebra. We shall use the notation $\pi \in \mathrm{Rep}(\mathcal{A})$ to mean that $\pi$ is a non-degenerate $*$-representation of $\mathcal{A}$ on some non-zero Hilbert space $\mathcal{H}_\pi$.

We say that $\pi \in \mathrm{Rep}(\mathcal{A})$ is *hypertracial* if there exists a hypertrace on $\pi(\mathcal{A})$, i.e. there exists a state $\varphi$ on $\mathcal{B}(\mathcal{H}_\pi)$ such that $\pi(\mathcal{A}) \subseteq C_\varphi$.

Recall that when $\rho, \pi \in \mathrm{Rep}(\mathcal{A})$, one says that $\rho$ is weakly contained in $\pi$ (resp. is weakly equivalent to $\pi$) whenever $\ker\pi \subseteq \ker\rho$ (resp. $\ker\pi = \ker\rho$). The following result is inspired by [9], Proposition 7:

THEOREM 2.1. *Suppose $\rho, \pi \in \mathrm{Rep}(\mathcal{A})$, $\rho$ is weakly contained in $\pi$ and $\rho$ is hypertracial. Then $\pi$ is hypertracial.*

*Proof.* Set $\mathcal{B} = \pi(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{H}_\pi)$. Since $\ker\pi \subseteq \ker\rho$ we may define $\psi : \mathcal{B} \to \mathcal{B}(\mathcal{H}_\rho)$ by

$$\psi(\pi(A)) = \rho(A), \quad A \in \mathcal{A}.$$

It is clear that $\psi \in \mathrm{Rep}(\mathcal{B})$ with $\mathcal{H}_\psi = \mathcal{H}_\rho$. By [24], Theorem 5.5.1, we may extend $\psi$ to a non-degenerate $*$-representation $\widetilde{\psi} : \mathcal{B}(\mathcal{H}_\pi) \to \mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space such that there exists a closed subspace $\mathcal{H}'_\rho$ of $\mathcal{K}$ which is invariant for $\widetilde{\psi}(\mathcal{B})$ and a unitary operator $U$ from $\mathcal{H}_\rho$ onto $\mathcal{H}'_\rho$ satisfying

$$\widetilde{\psi}(B)|_{\mathcal{H}'_\rho} = U\psi(B)U^* \quad \text{for all } B \in \mathcal{B},$$

i.e. $\widetilde{\psi}(\pi(A))|_{\mathcal{H}'_\rho} = U\rho(A)U^*$ for all $A \in \mathcal{A}$.

We let $P$ denote the projection of $\mathcal{K}$ onto $\mathcal{H}'_\rho$. Since $\rho$ is hypertracial, there exists a state $\varphi$ on $\mathcal{B}(\mathcal{H}_\rho)$ such that

$$\varphi(\rho(A)X) = \varphi(X\rho(A)) \quad \text{for all } A \in \mathcal{A},\ X \in \mathcal{B}(\mathcal{H}_\rho).$$

Define $\omega$ on $\mathcal{B}(\mathcal{H}_\pi)$ by

$$\omega(Y) = \varphi(U^*P\widetilde{\psi}(Y)|_{\mathcal{H}'_\rho}U), \quad Y \in \mathcal{B}(\mathcal{H}_\pi).$$

Then $\omega$ is a state on $\mathcal{B}(\mathcal{H}_\pi)$ (since $\|\omega\| = \omega(I) = 1$ is easily verified).

Further, for $A \in \mathcal{A}$, $Y \in \mathcal{B}(\mathcal{H}_\pi)$, we have

$$\begin{aligned}
\omega(Y\pi(A)) &= \varphi[U^*P\widetilde{\psi}(Y\pi(A))|_{\mathcal{H}'_\rho}U] \\
&= \varphi[U^*P\widetilde{\psi}(Y)|_{\mathcal{H}'_\rho}UU^*\widetilde{\psi}(\pi(A))|_{\mathcal{H}'_\rho}U] \\
&= \varphi[U^*P\widetilde{\psi}(Y)|_{\mathcal{H}'_\rho}U\rho(A)] \\
&= \varphi[\rho(A)U^*P\widetilde{\psi}(Y)|_{\mathcal{H}'_\rho}U] \\
&= \varphi[U^*\widetilde{\psi}(\pi(A))|_{\mathcal{H}'_\rho}P\widetilde{\psi}(Y)|_{\mathcal{H}'_\rho}U] \\
&= \varphi[U^*P\widetilde{\psi}(\pi(A))\widetilde{\psi}(Y)|_{\mathcal{H}'_\rho}U] \\
&= \omega(\pi(A)Y).
\end{aligned}$$

Hence $\omega$ is a hypertrace on $\pi(\mathcal{A})$, i.e. $\pi$ is hypertracial. ∎

COROLLARY 2.2. *Hypertraciality is invariant under weak equivalence.*

COROLLARY 2.3. *The following conditions are equivalent:*
(i) *There exists* $\rho \in \mathrm{Rep}(\mathcal{A})$ *which is hypertracial;*
(ii) *There exists a faithful* $\pi \in \mathrm{Rep}(\mathcal{A})$ *which is hypertracial;*
(iii) *All faithful* $\pi \in \mathrm{Rep}(\mathcal{A})$ *are hypertracial.*

We say that $\mathcal{A}$ is *weakly hypertracial* if $\mathcal{A}$ satisfies one of the conditions in Corollary 2.3. This property is clearly invariant under $*$-isomorphism.

REMARK 2.4. The following facts are easily deduced:
(i) Any non-zero finite dimensional $C^*$-algebra is weakly hypertracial.
(ii) If $J$ is a proper ideal in $\mathcal{A}$ and $\mathcal{A}/J$ is weakly hypertracial, then $\mathcal{A}$ is weakly hypertracial.
(iii) Any $C^*$-algebra possessing a non-zero finite dimensional representation is weakly hypertracial. Especially, any non-zero abelian $C^*$-algebra is weakly hypertracial (cf. also Proposition 1.8).
(iv) Any quasidiagonal $C^*$-algebra (cf. [36]) is weakly hypertracial.
(v) Any injective finite von Neumann algebra is weakly hypertracial.

Before giving more examples of weakly hypertracial $C^*$-algebras, we point out that this notion is of interest only for unital $C^*$-algebras.

PROPOSITION 2.5. *Any non-zero non-unital $C^*$-algebra $\mathcal{B}$ is weakly hypertracial.*

*Proof.* Let $\pi \in \mathrm{Rep}(\mathcal{B})$ be faithful. Then $I = I_{\mathcal{H}_\pi} \notin \pi(\mathcal{B})$, hence $\pi(\mathcal{B}) + \mathbb{C}\cdot I \simeq \widetilde{\pi(\mathcal{B})}$ is a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$ which has a hypertrace $\varphi$ induced by the obvious non-zero multiplicative linear functional on $\pi(\mathcal{B}) + \mathbb{C}\cdot I$ (cf. Proposition 1.8). This implies that $\pi$ is hypertracial, so $\mathcal{B}$ is weakly hypertracial. ∎

In the notation of the above proof, we have $\varphi|_{\pi(\mathcal{B})} = 0$. This cannot happen with unital $C^*$-algebras: if $\mathcal{A}$ is unital, $\pi \in \mathrm{Rep}(\mathcal{A})$ and $\varphi$ is a hypertrace on $\pi(\mathcal{A})$, then $\tau = \varphi|\pi(\mathcal{A})$ is a tracial state on $\pi(\mathcal{A})$ (since $I_{\mathcal{H}_\pi} = \pi(1) \in \pi(\mathcal{A})$ and $\tau(I_{\mathcal{H}_\pi}) = 1$). The next proposition follows readily.

PROPOSITION 2.6. *If $\mathcal{A}$ is a unital weakly hypertracial $C^*$-algebra, then $\mathcal{A}$ has at least one tracial state.*

Thus any unital $C^*$-algebra without any tracial state, such as the Cuntz algebras $\mathcal{O}_n$ $(n \geqslant 2)$ or the type III factors of countable type or $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ infinitely dimensional, are not weakly hypertracial. We shall soon see that the converse of Proposition 2.6 is not true.

PROPOSITION 2.7. *The following $C^*$-algebras are weakly hypertracial:*

(i) *All* AF-*algebras;*

(ii) *All inductive limits of postliminal (= type* I*) $C^*$-algebras;*

(iii) *All strongly amenable (in the sense of* [19]*) $C^*$-algebras;*

(iv) *All nuclear (= amenable) $C^*$-algebras with at least one tracial state.*

*Proof.* We may assume that the $C^*$-algebras in consideration are unital. (i) is a special case of (ii) which itself is a special case of (iii) (cf. [19]), which again is a special case of (iv). However, (iii) follows immediately from [8], Proposition 1. If one uses the amenability definition of a nuclear $C^*$-algebra, (iv) follows from [8], Proposition 2. It may also be deduced from Remark 2.4.(v): If $\tau$ is a tracial state on a unital nuclear $C^*$-algebra $\mathcal{A}$, then $M = \pi_\tau(\mathcal{A})'' \subseteq \mathcal{B}(\mathcal{H}_\tau)$ is an injective finite von Neumann algebra (cf. [23] or [26]), hence $M$ (and therefore $\pi_\tau(\mathcal{A})$) has a hypertrace, so $\pi_\tau$ is hypertracial. ∎

Note that by [18] all stably finite nuclear unital $C^*$-algebras have at least one tracial state. Combined with Remark 2.4.(i), (i) may also be obtained from:

PROPOSITION 2.8. *Let $\mathcal{S}$ be a non-empty set of weakly hypertracial $C^*$-subalgebras of a $C^*$-algebra $\mathcal{A}$ which is upwards directed by inclusion and such that $\bigcup_{S\in\mathcal{S}} S$ is dense in $\mathcal{A}$. Then $\mathcal{A}$ is weakly hypertracial also.*

*Proof.* We may assume that $\mathcal{A}$ and all elements in $\mathcal{S}$ are unital with the same unit. Let $\pi \in \mathrm{Rep}(\mathcal{A})$ be faithful. Then for each $S \in \mathcal{S}$ we have $\pi|_S : S \to \mathcal{B}(\mathcal{H}_\pi)$ is a faithful element in $\mathrm{Rep}(S)$, so that there exists a hypertrace $\varphi_S$ on $\pi(S)$. But clearly, any weak*-limit point of the net $\{\varphi_S\}_{S\in\mathcal{S}}$ is a hypertrace on $\pi(\mathcal{A})$, i.e. $\pi$ is hypertracial as desired. ∎

The class of weakly hypertracial $C^*$-algebras has the following interesting property:

PROPOSITION 2.9. *Let $\mathcal{A}$ be weakly hypertracial $C^*$-algebra and $\mathcal{B}$ be a $C^*$-subalgebra of $\mathcal{A}$. Assume that $\mathcal{A}$ is unital and $\mathcal{B}$ contains the unit of $\mathcal{A}$. Then $\mathcal{B}$ is weakly hypertracial.*

*Proof.* Choose $\pi \in \mathrm{Rep}(\mathcal{A})$, $\pi$ hypertracial. Under the above assumption, we have $\pi|_{\mathcal{B}} \in \mathrm{Rep}(\mathcal{B})$. As $\pi|_{\mathcal{B}}$ is obviously hypertracial, the result follows. ∎

REMARKS 2.10. (i) The assertion in Proposition 2.9 is not necessarily true if the assumption that $\mathcal{B}$ contains the unit of $\mathcal{A}$ is not satisfied: consider for example $\mathcal{A} = \mathcal{O}_2 \oplus \mathbb{C}$, $\mathcal{B} = \mathcal{O}_2 \oplus \mathbf{0}$.

(ii) Let $\mathcal{A}$ be a unital AF-algebra and $\mathcal{B}$ a $C^*$-subalgebra of $\mathcal{A}$ containing the unit of $\mathcal{A}$. Blackadar has shown in [7], Theorem 1, that any non type I $C^*$-algebra $\mathcal{S}$ contains a non-nuclear $C^*$-subalgebra which may be chosen to contain the unit of $\mathcal{S}$ if $\mathcal{S}$ is unital. Especially, this means that $\mathcal{B}$ is not necessarily nuclear if $\mathcal{A}$ is infinite dimensional. However, $\mathcal{B}$ is weakly hypertracial by Proposition 2.10.

(iii) The fact that the class of unital weakly hypertracial $C^*$-algebras is larger than the class of unital nuclear $C^*$-algebras possessing at least one tracial state may also be seen by considering the hyperfinite $\mathrm{II}_1$-factor on a separable Hilbert space, which is known to be non-nuclear ([35], Corollary 1.9), or by considering the group $C^*$-algebra $C^*(G)$ of a non-amenable discrete group $G$ possessing a non-zero finite dimensional representation (e.g. $G = \mathbb{F}_2 =$ the free group on two generators) since $C^*(G)$ is then non-nuclear ([23]).

When $G$ is a discrete group and $u \in Z^2(G, \mathbb{T})$, we obtain from Corollary 1.7 that $G$ is amenable $\Leftrightarrow$ $C^*_r(G,u)$ is weakly hypertracial $\Leftrightarrow$ $vN(G,u)$ is weakly hypertracial, and these conditions imply that $C^*(G,u)$ is weakly hypertracial. When combined with Proposition 2.9 and by taking into account Remark 1.2.(iii), the following result which is a slight generalization of [15], Theorem 12 (see also [5], Theorem 7) is obtained:

COROLLARY 2.11. *Let $\mathcal{A}$ be a unital $C^*$-algebra containing a unitally embedded copy of $C^*_r(G,u)$ for some non-amenable discrete group $G$ and some $u \in Z^2(G, \mathbb{T})$. Then $\mathcal{A}$ is not weakly hypertracial. Especially, there exists no finitely summable unbounded Fredholm module over $\mathcal{A}$.*

Let now $(\mathcal{A}, G, \alpha, u)$ denote a twisted $C^*$-dynamical system as considered in [25] and in [4], where $\mathcal{A}$ is a unital $C^*$-algebra and $G$ is a discrete group. Note that since $G$ is discrete, no separability assumptions are required (cf. [38] in the case when the two-cocycle $u$ takes values in the unitary group of the center at $\mathcal{A}$). We denote by $C^*(\mathcal{A}, G, \alpha, u)$ (resp. $C^*_r(\mathcal{A}, G, \alpha, u)$) the associated (resp. reduced) twisted $C^*$-crossed product. Recall that these two $C^*$-algebras coincide when $G$ is amenable (cf. [25], Theorem 3.11).

PROPOSITION 2.12. *Let $(\mathcal{A}, G, \alpha, u)$ be as above.*

(i) *Suppose $\mathcal{A}$ is weakly hypertracial and $G$ is amenable.*

*Then $C^*(\mathcal{A}, G, \alpha, u) \simeq C^*_r(\mathcal{A}, G, \alpha, u)$ is weakly hypertracial.*

(ii) *If $C^*_r(\mathcal{A}, G, \alpha, u)$ or $C^*(\mathcal{A}, G, \alpha, u)$ is weakly hypertracial, then $\mathcal{A}$ is weakly hypertracial.*

(iii) *If* $C_r^*(\mathcal{A}, G, \alpha, u)$ *is weakly hypertracial and* $u \in Z^2(G, \mathbf{T})$, *then* $G$ *is amenable.*

*Proof.* (i) Suppose $\mathcal{A}$ is weakly hypertracial and $G$ is amenable. Pick some faithful $\pi \in \mathrm{Rep}(\mathcal{A})$. Then the associated regular representation $\tilde{\pi} \times R$ of $C^*(\mathcal{A}, G, \alpha, u)$ is a non-degenerate $*$-representation of $C^*(\mathcal{A}, G, \alpha, u)$ on $\ell^2(G, \mathcal{H}_\pi)$ such that $C_r^*(\mathcal{A}, G, \alpha, u) \simeq \tilde{\pi} \times R(C^*(\mathcal{A}, G, \alpha, u)) = C^*(\tilde{\pi}(\mathcal{A}), R(G))$ (cf. [25], 3.10, 3.12). Further $\tilde{\pi} \in \mathrm{Rep}(\mathcal{A})$ is faithful, so $\tilde{\pi}$ is hypertracial. By Proposition 1.6 we get that there exists a hypertrace on $C^*(\tilde{\pi}(\mathcal{A}), R(G))$, hence $C_r^*(\mathcal{A}, G, \alpha, u)$ is weakly hypertracial.

(ii) Follows from Proposition 2.9 since $\mathcal{A}$ is unitally embedded in $C_r^*(\mathcal{A}, G, \alpha, u)$ and in $C^*(\mathcal{A}, G, \alpha, u)$.

(iii) When $u \in Z^2(G, \mathbf{T})$, then $C_r^*(G, u)$ is unitally embedded in $C_r^*(\mathcal{A}, G, \alpha, u)$, so the result follows from Corollary 2.11. ∎

Remark that a crossed product $C^*(\mathcal{A}, G, \alpha)$ may be weakly hypertracial for non-amenable $G$ (take for example $\mathcal{A} = \mathbf{C}$, $G = \mathbf{F}_2$, $\alpha = \mathrm{id}$). It is a simple consequence of Proposition 2.12.(i) that the rotation algebras $\mathcal{A}_\theta$ are all weakly hypertracial.

Tensor products behave nicely with respect to weak hypertraciality:

PROPOSITION 2.13. *Let* $\mathcal{A}, \mathcal{B}$ *denote two unital* $C^*$*-algebras and let* $\gamma$ *denote a* $C^*$*-norm on the algebraic tensor product* $\mathcal{A} \odot \mathcal{B}$. *Then* $\mathcal{A} \underset{\gamma}{\otimes} \mathcal{B}$ *is weakly hypertracial* $\Leftrightarrow$ $\mathcal{A}$ *and* $\mathcal{B}$ *are weakly hypertracial.*

*Proof.* ($\Rightarrow$) Both $\mathcal{A}$ and $\mathcal{B}$ embed unitally in $\mathcal{A} \underset{\gamma}{\otimes} \mathcal{B}$, so that this implication is a consequence of Proposition 2.9.

($\Leftarrow$) Suppose $\mathcal{A}$ and $\mathcal{B}$ are weakly hypertracial. Let $\sigma$ denote the spatial (= minimal) $C^*$-norm on $\mathcal{A} \odot \mathcal{B}$. Since $\mathcal{A} \underset{\sigma}{\otimes} \mathcal{B}$ is a quotient of $\mathcal{A} \underset{\gamma}{\otimes} \mathcal{B}$, it suffices to show that $\mathcal{A} \underset{\sigma}{\otimes} \mathcal{B}$ is weakly hypertracial. Let $\pi_1 \in \mathrm{Rep}(\mathcal{A}), \pi_2 \in \mathrm{Rep}(\mathcal{B})$, $\pi_1$ and $\pi_2$ faithful. Then we have

$$\begin{aligned}\pi_1 \underset{\sigma}{\otimes} \pi_2(\mathcal{A} \underset{\sigma}{\otimes} \mathcal{B}) &= C^*(\{\pi_1(A) \otimes \pi_2(B) \mid A \in \mathcal{A}, B \in \mathcal{B}\}) \\ &= C^*(\{U \otimes V \mid U \in \mathcal{U}(\pi_1(\mathcal{A})), V \in \mathcal{U}(\pi_2(\mathcal{B}))\}) \\ &\subseteq \mathcal{B}(\mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2}).\end{aligned}$$

Let $G_1 = \mathcal{U}(\pi_1(\mathcal{A}))$, $G_2 = \mathcal{U}(\pi_2(\mathcal{A}))$ considered as discrete groups, and denote by $i_1$ (resp. $i_2$) the identity unitary representation of $G_1$ on $\mathcal{H}_{\pi_1}$ (resp. $G_2$ on $\mathcal{H}_{\pi_2}$). Since $\pi_1$ and $\pi_2$ are hypertracial this means that $i_1$ and $i_2$ are amenable in the sense of Bekka ([6]). By [6], Corollary 5.4, we get that $i_1 \otimes i_2$ is amenable, hence that there exists a state $\varphi$ on $\mathcal{B}(\mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2})$ such that

$$\{U \otimes V \mid U \in G_1,\ V \in G_2\} \subseteq C_\varphi.$$

This implies that $\pi_1 \underset{\sigma}{\otimes} \pi_2(\mathcal{A} \underset{\sigma}{\otimes} \mathcal{B}) \subseteq C_\varphi$, i.e. that $\pi_1 \underset{\sigma}{\otimes} \pi_2$ is hypertracial, which shows that $\mathcal{A} \underset{\sigma}{\otimes} \mathcal{B}$ is weakly hypertracial as desired. ∎

Instead of invoking [6], Corollary 5.4, at the end of the above proof, one may use Theorem 1.1: When $\pi_1$ (resp. $\pi_2$) is hypertracial, there exists a Følner net $\{Q^1_\alpha\}$ for $\pi_1(\mathcal{A})$ in $\mathcal{B}(\mathcal{H}_{\pi_1})$ (resp. $\{Q^2_\alpha\}$ for $\pi_2(\mathcal{B})$ in $\mathcal{B}(\mathcal{H}_{\pi_2})$), and one may then check without too much difficulty that $\{Q_{(\alpha,\beta)}\}$, where $Q_{(\alpha,\beta)} = Q^1_\alpha \otimes Q^2_\beta$, is a Følner net (under the product order) for $\pi_1 \underset{\sigma}{\otimes} \pi_2(\mathcal{A} \underset{\sigma}{\otimes} \mathcal{B})$ in $\mathcal{B}(\mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2})$, from which the hypertraciality of $\pi_1 \underset{\sigma}{\otimes} \pi_2$ follows.

## 3. HYPERTRACIAL $C^*$-ALGEBRAS

We say that a $C^*$-algebra $\mathcal{A}$ is *hypertracial* if $\pi$ is hypertracial for all $\pi \in \mathrm{Rep}(\mathcal{A})$. Hypertraciality is clearly preserved under $*$-isomorphism. Further, a simple $C^*$-algebra is hypertracial if and only if it is weakly hypertracial. Especially, a simple non-unital $C^*$-algebra is hypertracial. So it seems that the concept of hypertraciality will be mostly of interest for unital $C^*$-algebras.

The class of hypertracial $C^*$-algebras is obviously smaller than the class of weakly hypertracial $C^*$-algebras. For example, we have:

PROPOSITION 3.1. *Suppose $\mathcal{A}$ is hypertracial and $\mathcal{J}$ is a non-zero proper ideal in $\mathcal{A}$. Then $\mathcal{J}$ and $\mathcal{A}/\mathcal{J}$ are hypertracial.*

*Proof.* As we may extend any $\pi \in \mathrm{Rep}(\mathcal{J})$ to a $\tilde{\pi} \in \mathrm{Rep}(\mathcal{A})$ satisfying $\mathcal{H}_{\tilde{\pi}} = \mathcal{H}_\pi$, it follows readily from the hypertraciality of $\mathcal{A}$ that $\mathcal{J}$ is hypertracial. Further, let $\pi \in \mathrm{Rep}(\mathcal{A}/\mathcal{J})$ and set $\dot{\pi} = \pi \circ \psi$ where $\psi$ denotes the canonical $*$-homomorphism from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{J}$. Then $\dot{\pi} \in \mathrm{Rep}(\mathcal{A})$ with $\mathcal{H}_{\dot{\pi}} = H_\pi$ and $\dot{\pi}$ is hypertracial since $\mathcal{A}$ is hypertracial. As $\dot{\pi}(\mathcal{A}) = \pi(\mathcal{A}/\mathcal{J})$, we get that $\pi$ is hypertracial. Hence, $\mathcal{A}/\mathcal{J}$ is hypertracial. ∎

We don't know whether the converse of Proposition 3.1 is true.

COROLLARY 3.2. *$\mathcal{A}$ is hypertracial*
*$\Leftrightarrow$ $\mathcal{A}/\mathcal{J}$ is weakly hypertracial for all proper ideals $\mathcal{J}$ in $\mathcal{A}$*
*$\Leftrightarrow$ $\mathcal{A}/\mathcal{J}$ is hypertracial for all proper ideals $\mathcal{J}$ in $\mathcal{A}$.*

In light of Proposition 2.5, it is enough to consider modular ideals $\mathcal{J}$ in $\mathcal{A}$ (i.e. such that $\mathcal{A}/\mathcal{J}$ is unital) in the above corollary.

COROLLARY 3.3. *$\mathcal{A}$ is hypertracial $\Leftrightarrow$ $\widetilde{\mathcal{A}}$ is hypertracial.*

*Proof.* ($\Rightarrow$) Let $\rho \in \mathrm{Rep}(\widetilde{\mathcal{A}})$, so that $\rho(I) = I_{\mathcal{H}_\rho}$ where $I$ denotes the unit of $\widetilde{\mathcal{A}}$. Set $\pi = \rho|_{\mathcal{A}}$. If $I_{\mathcal{H}_\rho} \in \pi(\mathcal{A})$, then $\pi(\mathcal{A}) = \rho(\widetilde{\mathcal{A}})$ and $\pi$ is non-degenerate, hence $\rho$ is hypertracial if $\mathcal{A}$ is hypertracial. Suppose now that $I_{\mathcal{H}_\rho} \notin \pi(\mathcal{A})$. Then we have $\rho(\widetilde{\mathcal{A}}) \simeq \pi(\mathcal{A})^{\sim}$. But $\pi(\mathcal{A})^{\sim}$ has a non-zero 1-dimensional representation, thus it follows from Remark 2.4.(iii) that $\rho(\widetilde{\mathcal{A}})$ is weakly hypertracial, i.e. $\rho$ is hypertracial. So we have shown that $\rho$ is hypertracial for all $\rho \in \mathrm{Rep}(\widetilde{\mathcal{A}})$ as desired.

($\Leftarrow$) This implication follows from Proposition 3.1. ∎

To check the hypertraciality of $\mathcal{A}$, it is enough to consider irreducible representations of $\mathcal{A}$:

PROPOSITION 3.4. *$\mathcal{A}$ is hypertracial $\Leftrightarrow$ $\rho$ is hypertracial for all irreducible $\rho \in \mathrm{Rep}(\mathcal{A})$.*

*Proof.* The forward implication is trivial. So assume that $\rho$ is hypertracial for all irreducible $\rho \in \mathrm{Rep}(\mathcal{A})$. Let $\pi \in \mathrm{Rep}(\mathcal{A})$. Choose some irreducible $\rho' \in \mathrm{Rep}(\pi(\mathcal{A}))$ and set $\rho = \rho' \circ \pi \in \mathrm{Rep}(\mathcal{A})$. Then $\rho$ is irreducible and $\rho$ is weakly contained in $\pi$. Hence we get from Theorem 2.1 that $\pi$ is hypertracial since $\rho$ is hypertracial. This shows that $\mathcal{A}$ is hypertracial. ∎

An immediate consequence of Proposition 3.4 is that all liminal $C^*$-algebras are hypertracial. But in fact, since any quotient of a finite dimensional (resp. abelian) (resp. postliminal) (resp. $AF$-algebra) (resp. strongly amenable) $C^*$-algebra inherits the respective property (cf. [24] and [19], 7.3), we get from Corollary 3.2 and the results in Section 2 that the class of hypertracial $C^*$-algebras includes all finite dimensional $C^*$-algebras, all abelian $C^*$-algebras, all postliminal $C^*$-algebras and more generally all strongly amenable $C^*$-algebras. It also includes the irrational rotation algebras and the hyperfinite $\mathrm{II}_1$-factor (since these are simple weakly hypertracial $C^*$-algebras). Further, we have:

PROPOSITION 3.5. *Suppose $\mathcal{A}$ is a nuclear $C^*$-algebra satisfying the following finiteness condition: for all proper (modular) ideals $\mathcal{J}$ in $\mathcal{A}$, $\mathcal{A}/\mathcal{J}$ has a tracial state. Then $\mathcal{A}$ is hypertracial.*

*Proof.* Let $\mathcal{J}$ be a proper (modular) ideal in $\mathcal{A}$. By [11], Corollary 4, $\mathcal{A}/\mathcal{J}$ is nuclear, and it has a tracial state by assumption. Hence $\mathcal{A}/\mathcal{J}$ is weakly hypertracial by Proposition 2.7. By Corollary 3.2 (and its accompanying remark), this shows that $\mathcal{A}$ is hypertracial. ∎

It follows easily from [19], 7.3 and [8], Proposition 1, that any strongly amenable $C^*$-algebra satisfies the assumptions in Proposition 3.5. We don't know whether the converse is true. On the other hand, the converse of Proposition 3.5 is not true, at least for non-separable $C^*$-algebras: indeed, the hyperfinite $\mathrm{II}_1$-factor is hypertracial, but as pointed out before, it is not nuclear. However, it would be interesting to know whether any (unital) separable hypertracial $C^*$-algebra is nuclear.

Let us also remark that a $C^*$-subalgebra of a hypertracial $C^*$-algebra is not necessarily hypertracial: Blackadar shows in [7], Theorem 2, that any non type I $C^*$-algebra $\mathcal{A}$ has a $C^*$-subalgebra $\mathcal{B}$ (containing the unit of $\mathcal{A}$ if $\mathcal{A}$ is unital) which has $\mathcal{O}_2$ as a quotient, so that $\mathcal{B}$ is not hypertracial by Corollary 3.2.

The class of hypertracial $C^*$-algebras behaves nicely with respect to inductive limits and crossed products by amenable discrete groups.

PROPOSITION 3.6. (i) *Let $\mathcal{S}$ be a non-empty set of hypertracial $C^*$-subalgebras of a $C^*$-algebra $\mathcal{A}$. Suppose that $\mathcal{S}$ is upwards directed by inclusion and $\bigcup_{S\in\mathcal{S}} S$ is dense in $\mathcal{A}$. Then $\mathcal{A}$ is hypertracial also.*

(ii) *Suppose that $(\mathcal{A}_n, \varphi_n)_{n=1}^{\infty}$ is a direct sequence of hypertracial $C^*$-algebras. Then the direct limit $\varinjlim \mathcal{A}_n$ is hypertracial also.*

*Proof.* (i) Let $\pi \in \mathrm{Rep}(\mathcal{A})$. Then $\mathcal{S}' = \{\pi(S) \mid S \in \mathcal{S}\}$ is upwards directed by inclusion and $\overline{\bigcup_{S'\in\mathcal{S}'} S'} = \pi(\mathcal{A})$.

Since every element in $\mathcal{S}'$ is weakly hypertracial by assumption, it follows from Proposition 2.8 that $\pi(\mathcal{A})$ is weakly hypertracial, hence that $\pi$ is hypertracial. So $\mathcal{A}$ is hypertracial as asserted.

(ii) Let $\varphi^n : \mathcal{A}_n \to \mathcal{A}$ be the natural map, where $\mathcal{A} = \varinjlim \mathcal{A}_n$ (cf. [24]). Then $\mathcal{S} = \{\varphi^n(\mathcal{A}_n) \mid n \geqslant 1\}$ is an upwards directed family of hypertracial $C^*$-subalgebras of $\mathcal{A}$ whose union is dense in $\mathcal{A}$, so that $\mathcal{A}$ is hypertracial by (i). ∎

Let $G$ denote a discrete group. It is not difficult to deduce from [6], Theorem 2.2, that $G$ is amenable $\Leftrightarrow$ $C^*(G)$ is hypertracial $\Leftrightarrow$ $C^*_r(G)$ is hypertracial. More generally, we have:

PROPOSITION 3.7. *Let $(\mathcal{A}, G, \alpha, u)$ denote a twisted $C^*$-dynamical system as in Proposition 2.12, but where we do not assume that $\mathcal{A}$ is unital if $u$ is trivial. Consider the following conditions:*

(i) *$G$ is amenable and $\mathcal{A}$ is hypertracial;*

(ii) *$C^*(\mathcal{A}, G, \alpha, u)$ is hypertracial;*

(iii) *$C^*_r(\mathcal{A}, G, \alpha, u)$ is hypertracial;*

(iv) *$G$ is amenable.*

*Then we have* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *If* $u \in Z^2(G, \mathbf{T})$, *then we have* (iii) $\Rightarrow$ (iv).

*Proof.* (i) $\Rightarrow$ (ii) Suppose $G$ is amenable and $\mathcal{A}$ is hypertracial. Let $\mathcal{B} = C^*(\mathcal{A}, G, \alpha, u)$ and assume first that $\mathcal{A}$ is unital. Let $\rho \in \mathrm{Rep}(\mathcal{B})$. Recall from [25] that we may write $\rho = \pi \times U$, where $\pi \times U$ is the non-degenerate representation of $\mathcal{B}$ associated to a covariant representation $(\pi, U)$ of $(\mathcal{A}, G, \alpha, u)$ on a Hilbert space $\mathcal{H}$, i.e. $\pi \in \mathrm{Rep}(\mathcal{A})$, $\mathcal{H}_\pi = \mathcal{H}$ and $U : G \to \mathcal{U}(\mathcal{B}(\mathcal{H}))$ is a map such that

$$U_g U_h = \pi(u(g,h))U_{gh} \quad \text{and} \quad \pi(\alpha_g(A)) = U_g \pi(A) U_g^*$$

for all $g, h \in G$, $A \in \mathcal{A}$, and that we have $\rho(\mathcal{B}) = \pi \times U(\mathcal{B}) = C^*(\pi(\mathcal{A}), U(G))$. We may now apply Proposition 1.6 and deduce that $\rho$ is hypertracial. This shows that $\mathcal{B}$ is hypertracial. Now, if $u$ is trivial and $\mathcal{A}$ is non-unital, then we may extend $\alpha$ to an action of $G$ on $\widetilde{\mathcal{A}}$ in the obvious way. As $\widetilde{\mathcal{A}}$ is hypertracial by Corollary 3.3, we get from the first part that $C^*(\widetilde{\mathcal{A}}, G, \alpha)$ is hypertracial. But $C^*(\mathcal{A}, G, \alpha)$ is an ideal of $C^*(\widetilde{\mathcal{A}}, G, \alpha)$, so that $C^*(\mathcal{A}, G, \alpha)$ is hypertracial by Corollary 3.2.

(ii) $\Rightarrow$ (iii) follows from Corollary 3.2 since $C_r^*(\mathcal{A}, G, \alpha, u)$ is a quotient of $C^*(\mathcal{A}, G, \alpha, u)$.

Finally, suppose that $u \in Z^2(G, \mathbf{T})$ and that $C_r^*(\mathcal{A}, G, \alpha, u)$ is hypertracial. Then $C_r^*(\mathcal{A}, G, \alpha, u)$ is especially weakly hypertracial, so that $G$ is amenable by Proposition 2.12.(iii). ∎

COROLLARY 3.8. *Let* $(\mathcal{A}, G, \alpha, u)$ *denote a* $C^*$*-dynamical system where* $G$ *is a discrete group,* $\mathcal{A}$ *is a unital* $C^*$*-algebra, and* $u \in Z^2(G, \mathbf{T})$.

(i) *Suppose* $\mathcal{A}$ *is hypertracial. Then* $G$ *is amenable*

$\Leftrightarrow$ $C^*(\mathcal{A}, G, \alpha, u)$ *is hypertracial*

$\Leftrightarrow$ $C_r^*(\mathcal{A}, G, \alpha, u)$ *is hypertracial.*

(ii) *Suppose* $\mathcal{A}$ *is simple. Then* $G$ *is amenable and* $\mathcal{A}$ *is weakly hypertracial*

$\Leftrightarrow$ $C^*(\mathcal{A}, G, \alpha, u)$ *is hypertracial*

$\Leftrightarrow$ $C_r^*(\mathcal{A}, G, \alpha, u)$ *is hypertracial.*

*Proof.* (i) follows from Proposition 3.7. (ii) follows from Proposition 3.7 combined with Proposition 2.9. ∎

The behaviour of hypertraciality with respect to tensor products seems more difficult to handle. We can only show the following.

PROPOSITION 3.9. *Let $\mathcal{A}$ be a hypertracial $C^*$-algebra and $\mathcal{B}$ a unital $C^*$-algebra such that there exists a subgroup $G$ of $\mathcal{U}(\mathcal{B})$ which generates $\mathcal{B}$ as a $C^*$-algebra and which is amenable as a discrete group. Let $\gamma$ denote a $C^*$-norm on $\mathcal{A}\odot\mathcal{B}$. Then $\mathcal{A}\underset{\gamma}{\otimes}\mathcal{B}$ is hypertracial (especially, $\mathcal{B}$ is hypertracial under the above condition).*

*Proof.* As the $C^*$-norm $\gamma$ on $\mathcal{A}\odot\mathcal{B}$ may be extended to a $C^*$-norm on $\tilde{\mathcal{A}}\odot\mathcal{B}$ and $\mathcal{A}\underset{\gamma}{\otimes}\mathcal{B}$ is then an ideal in $\tilde{\mathcal{A}}\underset{\gamma}{\otimes}\mathcal{B}$, we may assume that $\mathcal{A}$ is unital (cf. Proposition 3.1 and Corollary 3.3). Now, let $\pi\in\mathrm{Rep}(\mathcal{A}\underset{\gamma}{\otimes}\mathcal{B})$. Then there exist $\pi_1\in\mathrm{Rep}(\mathcal{A}),\pi_2\in\mathrm{Rep}(\mathcal{B})$ with $\mathcal{H}_{\pi_1}=\mathcal{H}_{\pi_2}=\mathcal{H}_\pi$ such that

$$\pi(A\otimes B)=\pi_1(A)\pi_2(B)=\pi_2(B)\pi_1(A),\quad A\in\mathcal{A},\ B\in\mathcal{B},$$

and

$$\pi(\mathcal{A}\underset{\gamma}{\otimes}\mathcal{B})=C^*(\pi_1(\mathcal{A}),\pi_2(\mathcal{B})).$$

The restriction of $\pi_2$ to $G$ is a unitary representation of $G$ on $\mathcal{H}_\pi$. Further, as $G$ generates $\mathcal{B}$ as a $C^*$-algebra, it is clear that $\pi_2(G)$ generates $\pi_2(\mathcal{B})$ as a $C^*$-algebra, so that we have

$$\pi(\mathcal{A}\underset{\gamma}{\otimes}\mathcal{B})=C^*(\pi_1(\mathcal{A}),\pi_2(G)).$$

As $G$ is amenable and $\pi_1$ is hypertracial since $\mathcal{A}$ is hypertracial, we may invoke Proposition 1.6 and obtain that $\pi$ is hypertracial. This shows that $\mathcal{A}\underset{\gamma}{\otimes}\mathcal{B}$ is hypertracial. ∎

As pointed out to us by the referee, a $C^*$-algebra $\mathcal{B}$ as in Proposition 3.9 is nuclear, since it is a quotient of $C^*(G)$ (which is nuclear since $G$ is amenable). This means that $\gamma$ is in fact the spatial $C^*$-norm on $\mathcal{A}\odot\mathcal{B}$. An analogous assumption on $\mathcal{B}$ has been considered in [10] in the case when $\mathcal{B}$ is a von Neumann algebra.

COROLLARY 3.10. *Suppose $\mathcal{A}$ is a hypertracial $C^*$-algebra. Then*
(i) *$\mathcal{A}\otimes M_n(\mathbb{C})$ is hypertracial for all $n\geqslant 1$;*
(ii) *$\mathcal{A}\otimes\mathcal{B}$ is hypertracial whenever $\mathcal{B}$ is an* AF*-algebra;*
(iii) *$\mathcal{A}\otimes\mathcal{B}$ is hypertracial whenever $\mathcal{B}$ is an abelian $C^*$-algebra.*

*Proof.* (i) It is not difficult to show that there exists a discrete amenable subgroup of $\mathcal{U}(M_n(\mathbb{C}))$ which generates $M_n(\mathbb{C})$ as a $C^*$-algebra for any $n\geqslant 1$, so that this assertion follows from Proposition 3.9. However, this may also be shown directly; if $\pi\in\mathrm{Rep}(\mathcal{A}\otimes M_n(\mathbb{C}))$, then $\pi(\mathcal{A}\otimes M_n(\mathbb{C}))\simeq\pi(\mathcal{A}\otimes\mathbf{1})\otimes M_n(\mathbb{C})$, so that $\pi$ is hypertracial (by Proposition 2.13 if $\pi(\mathcal{A}\otimes\mathbf{1})$ is unital, trivially otherwise).

(ii) This follows easily from Proposition 3.6 combined with (i).

(iii) Assume $\mathcal{B}$ is abelian and unital. Then $\mathcal{U}(\mathcal{B})$ is abelian, hence amenable as a discrete group, so that the result follows from Proposition 3.9. The non-unital case follows by unitization. ∎

There has been recently considerable interest in the study of nuclear $C^*$-algebras which may be written as an inductive limit $\mathcal{A} = \varinjlim \mathcal{A}_n$ where each $\mathcal{A}_n$ is a finite direct sum of $C^*$-algebras of the form $C_0(X) \otimes M_k(\mathbf{C})$, $X$ being a locally compact Hausdorff space and $k \in \mathbf{N}$.

COROLLARY 3.11. *Let $\mathcal{A}$ be as above. Then $\mathcal{A}$ is hypertracial.*

*Proof.* Each direct summand of $\mathcal{A}_n$ being hypertracial by Corollary 3.10, it is clear that each $\mathcal{A}_n$ is hypertracial. Hence $\mathcal{A}$ is hypertracial by Proposition 3.6. ∎

One could also argue that $\mathcal{A}$ is (well known to be) strongly amenable and therefore hypertracial, but the above proof is more elementary. In some cases, such $\mathcal{A}$'s may also be written as a twisted crossed product of an abelian $C^*$-algebra by an abelian discrete group (cf. [33]) and the hypertraciality of these $\mathcal{A}$'s may then also be deduced from Proposition 3.7.

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REFERENCES

1. C. ANANTHARAMAN-DELAROCHE, Action moyennable d'un groupe localement compact sur une algèbre de von Neumann, *Math. Scand.* **45**(1979), 289–304.
2. W. ARVESON, $C^*$-algebras and numerical linear algebra, *J. Funct. Anal.* **122**(1994), 333–360.
3. W. ARVESON, Improper filtrations for $C^*$-algebras: Spectra of unilateral tridiagonal operators, *Acta Sci. Math. (Szeged)* **57**(1993), 11-24.
4. E. BÉDOS, Discrete groups and simple $C^*$-algebras, *Math. Proc. Cambridge Philos. Soc.* **109**(1991), 521–537.
5. E. BÉDOS, On filtrations for $C^*$-algebras, *Houston J. Math.* **20**(1994), 63–74.
6. M. BEKKA, Amenable unitary representations of locally compact groups, *Invent. Math.* **100**(1990), 383–401.
7. B. BLACKADAR, Nonnuclear subalgebras of $C^*$-algebras, *J. Operator Theory* **14** (1985), 347–350.
8. J. BUNCE, Characterizations of amenable and strongly amenable $C^*$-algebras, *Pacific J. Math.* **43**(1972), 563–572.
9. J. BUNCE, Finite operators and amenable $C^*$-algebras, *Proc. Amer. Math. Soc.* **56**(1976), 145–151.

10. H. CHODA, M. ECHIGO, A new algebraical property of certain von Neumann algebras, *Proc. Japan Acad. Ser. A Math. Sci.* **39**(1963), 651–655.
11. M.D. CHOI, E.G. EFFROS, Nuclear $C^*$-algebras and injectivity; the general case, *Indiana Univ. Math. J.* **26**(1977), 443–446.
12. A. CONNES, Classification of injective factors, *Ann. Math.* **104**(1976), 73–115.
13. A. CONNES, On the classification of von Neumann algebras and their automorphisms, *Sympos. Math.*, vol. 20, pp. 435–478, 1976.
14. A. CONNES, On the cohomology of operator algebras, *J. Funct. Anal.* **28**(1978), 248–253.
15. A. CONNES, Compact metric spaces, Fredholm modules and hyperfiniteness, *Ergodic Theory Dynamical Systems* **9**(1989), 207–220.
16. J. DIXMIER, *Les $C^*$-algèbres et leurs représentations*, 2nd ed. Gauthier-Villars, 1969.
17. U. HAAGERUP, All nuclear $C^*$-algebras are amenable, *Invent. Math.* **74**(1983), 305–319.
18. U. HAAGERUP, Quasitraces on exact $C^*$-algebras are traces, Talk given at the NATO conference "Operator algebras, mathematical physics and low dimensional topology", Istanbul, July 1991 (Hand-written notes).
19. B.E. JOHNSON, Cohomology in Banach algebras, *Mem. Amer. Math. Soc.* **127**(1972), 1–96.
20. E. KIRCHBERG, On non semisplit extensions, tensor products and exactness of group $C^*$-algebras, *Invent. Math.* **112**(1993), 449-489.
21. E. KIRCHBERG, Discrete groups with Kazhdan's property $T$ and factorization property are residually finite, *Math. Ann.* **299**(1994), 551-563.
22. H. KOSAKI, Generalized Powers-Størmer inequalities, Appendix to "Distances between unitary orbits in von Neumann algebras" by F. Hiai and Y. Nakamura, *Pacific J. Math.* **138**(1989), 288–292.
23. E.C. LANCE, Tensor products and $C^*$-algebras, *Proc. Sympos. Pure Math.*, vol. 38, pp. 379–399, 1982.
24. G.J. MURPHY, *$C^*$-Algebras and Operator Theory*, Academic Press, 1990.
25. J. PACKER, I. RAEBURN, Twisted crossed products of $C^*$-algebras, *Math. Proc. Cambridge Philos. Soc.* **106**(1989), 293–311.
26. A.L. PATERSON, *Amenability*, Math. Surveys Monographs, vol. 29, Amer. Math. Soc., 1988.
27. G.K. PEDERSEN, *$C^*$-Algebras and their Automorphism Groups*, LMS Monographs 14, Academic Press, 1979.
28. J.P. PIER, *Amenable Locally Compact Groups*, J. Wiley & Sons, 1984.
29. S. POPA, A short proof of injectivity implies hyperfiniteness for finite von Neumann algebras, *J. Operator Theory* **16**(1985), 261–272.
30. S. POPA, On amenability in type $\mathrm{II}_1$-factors, in *Operator Algebras and Applications*, vol. 2, pp. 173–183, London Math. Soc. Lecture Note Ser. **136**, Cambridge Univ. Press, 1988.
31. Ş. STRĂTILĂ, *Modular theory in operator algebras*, Editura Academiei — Abacus Press, 1981.
32. M. TAKESAKI, *Theory of operator algebras I*, Springer Verlag, 1979.
33. K. THOMSEN, Diagonalization in inductive limits of circle algebras, *J. Operator Theory* **27**(1992), 325–340.
34. G. VAILLANT, Følner conditions, nuclearity and subexponential growth in $C^*$-algebras, preprint, 1994.

35. S. WASSERMANN, On tensor products of certain group $C^*$-algebras, *J. Funct. Anal.* **23**(1976), 239–254.
36. S. WASSERMANN, *Exact $C^*$-algebras and related topics*, Lecture Notes Ser., vol. 19, GARC, Seoul National University, 1994.
37. J.P. WILLIAMS, Finite operators, *Proc. Amer. Math. Soc.* **26**(1970), 129–136.
38. G. ZELLER-MEIER, Produits croisés d'une $C^*$-algèbre par un groupe d'automorphismes, *J. Math. Pures Appl. (9)* **47**(1968), 101–239.

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