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ON QUOTIENTS OF FUNCTION ALGEBRAS AND OPERATOR ALGEBRA STRUCTURES ON ℓ_p

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ABSTRACT. Recently, the first author gave a characterization of operator algebras up to complete isomorphism. We give here some characterizations of quotients of function algebras (Q-algebras), again up to complete isomorphism. Using these, we examine which operator space structures on ℓ_p (with pointwise product) correspond to operator algebras, and which to Q-algebras. We also give a new approach to the long outstanding similarity problem of Halmos, studying operator space structures on the disc algebra. Finally, we show that the Banach algebra of von Neumann-Schatten p-class operators on a Hilbert space is an operator algebra for all $1 \leqslant p \leqslant \infty$ (with either the usual or the Schur product). That is, these algebras are bicontinuously and algebraically isomorphic to a *norm closed* algebra of operators on some Hilbert space. However, with the usual operator space structures they are not *completely bicontinuously* isomorphic to any closed algebra of operators.

KEYWORDS: *Operator algebras, operator spaces, function algebras, completely bounded maps.*

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1. INTRODUCTION AND PRELIMINARIES

It has been clear for many years that the full understanding of an operator algebra A requires the knowledge of the norms induced by the embedding $A \subset B(H)$ on the spaces $M_n(A)$ of $n \times n$ matrices with entries in A. It should be especially noticed that two such algebras $A_1 \subset B(H_1)$, $A_2 \subset B(H_2)$ can be isomorphic as Banach algebras and have non-comparable matricial norms. The great importance of $M_n(A)$ for nonselfadjoint operator algebras was revealed by the work of Arveson ([1]) and has seen many developments and applications recently (see e.g. [26],

[8], [6], [15], [3]). Paulsen's formulation of Halmos' similarity problem ([25], [23]) is another example of where the importance of matricial norms for an operator algebra may be seen clearly. Hitherto the question as to the variety of possible matricial norms associated with a given operator algebra has been given scant attention. In this paper we give a very geometrical approach to this question. In fact it will be seen here in several examples that whether a given A is an operator algebra or not simply depends on its operator space geometry, or more particularly, on how much the multiplication stretches the unit balls of $M_n(A)$.

The main example considered in this paper is the space ℓ_p, for $1 \leqslant p \leqslant +\infty$. Equipped with the pointwise multiplication this Banach space becomes a Banach algebra. Davie ([12]) and Varopoulos ([33]) proved that ℓ_p is then an operator algebra. In other words, there exists a Hilbert space H and a bicontinuous homomorphism from ℓ_p onto some closed subalgebra of $B(H)$. More precisely, they showed that ℓ_p is a Q-algebra, i.e. we can find a uniform algebra B as well as a closed ideal $J \subset B$ such that ℓ_p and the quotient algebra B/J are isomorphic in the Banach algebra sense. The fact that Q-algebras are actually operator algebras was an earlier theorem of Cole ([37], see also [9], p. 272). The matricial approach to operator algebras, together with the work of Davie and Varopoulos, leads to the following problem:

Given $1 \leqslant p \leqslant +\infty$, what are the matricial norms induced on ℓ_p by its representations as a subalgebra of $B(H)$?

It will appear later that for $1 \leqslant p < +\infty$, the answer is not unique. In order to study this, we need to work in the setting of operator spaces.

We recall that an operator space is a subspace $E \subset B(H)$, where H is a Hilbert space. We will assume throughout that our operator spaces are complete. An operator space is equipped with the matricial norms induced on $M_n(E)$ by regarding this as a subspace of $B(\ell_2^n(H))$. Equivalently, an operator space is a closed subspace E of a C^*-algebra B, with the convention that $M_n(E)$ inherits the norm from the C^*-algebra $M_n(B)$. We also recall that if E and F are operator spaces, then a map $u : E \to F$ is completely bounded (c.b. in short) provided that the maps $u \otimes I_{M_n} : M_n(E) \to M_n(F)$ are uniformly bounded. The c.b. norm of u is then defined by $\|u\|_{\mathrm{cb}} = \sup_{n \geqslant 1} \|u \otimes I_{M_n}\|$. When equipped with $\|\ \|_{\mathrm{cb}}$, the set of all c.b. maps from E into F is a Banach space denoted by $\mathrm{CB}(E, F)$. We say that $u : E \to F$ is a complete isometry if all the maps $u \otimes I_{M_n}$ are isometries and that it is a complete isomorphism if u is an isomorphism such that u and u^{-1} are both c.b. maps. We also call such a map a completely bounded isomorphism or a completely bicontinuous isomorphism. Furthermore two operator spaces E

and F are completely isomorphic (resp. completely isometric) provided that there exists a complete isomorphism (resp. completely isometric isomorphism) between E and F. Of course, operator spaces are always defined and studied up to complete isometry. Given a Banach space E, an *operator space structure* (o.s.s. for short) on E will be a sequence of norms, the n-th norm defined on $M_n(E)$, such that there exists an embedding $J : E \to B(H)$ for which all the $J \otimes I_{M_n}$ are isometries. We refer the reader to [32] for the fundamental characterization of operator space structures.

We will use the quotient and the dual within the category of operator spaces. Given an operator space E and a closed subspace $F \subset E$, the quotient operator space E/F is defined by letting $M_n(E/F)$ be the Banach space $M_n(E)/M_n(F)$ ([32]). The dual Banach space E^* of an operator space E becomes an operator space by letting

$$M_n(E^*) = \mathrm{CB}(E, M_n) \tag{1.1}$$

under natural identification. The reader is referred to [32], [7], [5], [16], [4] for details about quotients and duality and for the basic results of the theory of operator spaces.

Now let A be an operator space. Assume that A is endowed with a Banach algebra multiplication.

DEFINITION 1.1. We will say that A is an *operator algebra* provided that there exist a Hilbert space H, a closed subalgebra $B \subset B(H)$ and a complete isomorphism from A onto B which is also a Banach algebra homomorphism.

If E is a Banach algebra, an operator algebra structure on E will be an o.s.s. on E which makes E an operator algebra in the sense of Definition 1.1. With this terminology, the problem raised above reads:

(1.2) What are the operator algebra structures on ℓ_p?

Likewise we will study Q-algebras in the framework of operator spaces.

DEFINITION 1.2. An operator space A endowed with a Banach algebra multiplication will be called a (*matricial*) *Q-algebra* provided that there exist a uniform algebra B, a closed ideal $J \subset B$ and a complete isomorphism between A and B/J which is a Banach algebra homomorphism.

In Definition 1.2, B has the minimal o.s.s. induced by its representation as a subalgebra of some commutative C^*-algebra (see below for details). We know from [8] that Cole's theorem generalizes as follows: any matricial Q-algebra in the sense of Definition 1.2 is an operator algebra in the sense of Definition 1.1.

If E is a Banach algebra, a matricial Q-algebra structure on E will be an o.s.s. on E which makes E a matricial Q-algebra. When there is no confusion we will drop the word "matricial". Thus a complementary question to (1.2) is now:

(1.3) What are the matricial Q-algebra structures on ℓ_p?

Before going on, we should say that we are very far from being able to describe all the o.s.s. on ℓ_p. One of the main purposes of this work is then to give a partial answer to problems (1.2) and (1.3) by determining which of the most common and tractable o.s.s. on ℓ_p are actually operator algebra structures or even Q-algebra structures. The o.s.s. to be studied later will be mainly defined by using the Min and Max functors, and operator space interpolation. Let us recall the necessary definitions. Let E be a Banach space. The operator spaces $\operatorname{Min} E$ and $\operatorname{Max} E$ ([5]) are respectively the smallest and the greatest o.s.s. on E. They are defined by the formulae (1.4) and (1.5). Let K be the compact set $(B_{E^*}, \sigma(E^*, E))$. Then for any $x = [x_{ij}] \in M_n \otimes E$:

$$\|x\|_{M_n(\operatorname{Min} E)} = \sup\{\|[\zeta(x_{ij})]\|_{M_n} \mid \zeta \in K\} \tag{1.4}$$

$$\|x\|_{M_n(\operatorname{Max} E)} = \sup\{\|[T(x_{ij})]\|_{M_n(B(H))} \mid H \text{ is a Hilbert space, } T \in B(E, B(H)),\ \|T\| \leqslant 1\}. \tag{1.5}$$

It follows quite easily that a bounded map T into $\operatorname{Min} E$, or out of $\operatorname{Max} E$, is completely bounded with $\|T\|_{\mathrm{cb}} = \|T\|$. We shall use this fact several times. The operator space $\operatorname{Min} E$ can equivalently be defined by stating that the canonical embedding $\operatorname{Min} E \subset C(K)$ is a complete isometry. We recall for later use that the minimal and maximal o.s.s. are duals of each other. Namely we have complete isometries ([5]):

$$(\operatorname{Min} E)^* = \operatorname{Max} E^*, \quad (\operatorname{Max} E)^* = \operatorname{Min} E^*. \tag{1.6}$$

We now turn to the (complex) interpolation of operator spaces which was recently introduced by Pisier ([28]). We refer to [2] for background on the interpolation theory of Banach spaces. Let E_0, E_1 be two operator spaces. Assume that (E_0, E_1) is a compatible couple in the sense of the interpolation of Banach spaces. For any $n \geqslant 1$, $(M_n(E_0), M_n(E_1))$ is then a compatible couple. For any $0 \leqslant \theta \leqslant 1$, we can thus equip the interpolated space $E_\theta = [E_0, E_1]_\theta$ with an o.s.s. by letting

$M_n(E_\theta) = [M_n(E_0), M_n(E_1)]_\theta$. We will always assume that E_θ has this o.s.s. It will be worthwhile to have another description of the operator space E_θ. Let $\mathcal{S}$ be the open strip $\{0 < \text{Re} z < 1\}$ and let $\mathcal{F}$ be the space of all the continuous functions f from $\overline{\mathcal{S}}$ into $E_0 + E_1$ which are analytic on $\mathcal{S}$ and such that $t \mapsto f(it)$ (resp. $t \mapsto f(1+it)$) is continuous from $\mathbb{R}$ into E_0 (resp. E_1) and tend to 0 as $|t| \to +\infty$. The space $\mathcal{F}$ is a Banach space with the norm:

$$\|f\| = \max\{\sup_t \|f(it)\|_{E_0}, \sup_t \|f(1+it)\|_{E_1}\}. \tag{1.7}$$

For any $0 \leqslant \theta \leqslant 1$, we then denote by $\mathcal{F}_\theta$ the space $\{f \in \mathcal{F} \mid f(\theta) = 0\}$. Then we have ([2]):

$$E_\theta = \mathcal{F}/\mathcal{F}_\theta. \tag{1.8}$$

Now note that (1.7) means that we have an isometric embedding:

$$\mathcal{F} \subset L^\infty(\mathbb{R}, E_0) \overset{\infty}{\oplus} L^\infty(\mathbb{R}, E_1). \tag{1.9}$$

The right hand side of (1.9) has a natural o.s.s. Namely, if we have $E_0 \subset B(H_0)$, $E_1 \subset B(H_1)$, the space $L^\infty(\mathbb{R}, E_0) \overset{\infty}{\oplus} L^\infty(\mathbb{R}, E_1)$ is clearly a subspace of the C^*-algebra $L^\infty(\mathbb{R}, B(H_0)) \overset{\infty}{\oplus} L^\infty(\mathbb{R}, B(H_1))$, whence the o.s.s. Therefore (1.9) yields an o.s.s. on $\mathcal{F}$. It is fairly clear that the o.s.s. defined on E_θ by (1.8) coincides with the previous one.

Using interpolation, Pisier introduced a distinguished o.s.s. on ℓ_p ([28], [29]) which we will denote by $O\ell_p$, and is defined as:

$$O\ell_p = [\text{Min}\,\ell_\infty, \text{Max}\,\ell_1]_{\frac{1}{p}}. \tag{1.10}$$

One of the main features of these structures is that, letting $\frac{1}{p} + \frac{1}{q} = 1$, we have for any $1 \leqslant p < +\infty$:

$$(O\ell_p)^* = O\ell_q \tag{1.11}$$

completely isometrically. We refer the reader to [29] for a more concrete description of $O\ell_p$. The self-dual Hilbertian operator space $O\ell_2$ is usually denoted by OH. See [28] for the important properties of this operator space.

In Section 3, we will prove that for all $1 \leqslant p \leqslant +\infty$, $O\ell_p$ is an operator algebra. We will also calculate the values of p for which $\text{Min}\,\ell_p$ or $\text{Max}\,\ell_p$ is an operator algebra. We will then start the study of Q-algebra structures. In Section 2, we investigate the particular case ℓ_2, where different natural o.s.s. than

$\mathrm{Min}\,\ell_2$, $\mathrm{Max}\,\ell_2$ or OH arise. We especially show that $R \cap C$ is an operator algebra whereas $R + C$ is not (see [28] and below for the definitions).

Section 4 is devoted to Q-algebras. We give two characterizations of matricial Q-algebras, and a useful sufficient condition. We use these to show that $R \cap C$ is a matricial Q-algebra. As a corollary we obtain that for any $1 < p < +\infty$, there is a continuum of matricial Q-algebra structures on ℓ_p.

In the Section 5, we leave ℓ_p and consider the analogous problem of determining operator algebra structures on the disc algebra $A(\mathbf{D})$. Our motivation is that the existence of an operator algebra structure on $A(\mathbf{D})$ non-completely isomorphic to $\mathrm{Min}\,A(\mathbf{D})$ would provide a counterexample to the famous similarity problem of Halmos. The main theorem in [3] shows that in order to solve Halmos' problem in the negative it suffices to find a new operator *space* structure on $A(\mathbf{D})$ for which multiplication is still a completely bounded bilinear map. It is tempting to guess that for θ sufficiently small and positive, the interpolated o.s.s. $[\mathrm{Min}\,A(\mathbf{D}), \mathrm{Max}\,A(\mathbf{D})]_\theta$ would be such a structure. However, perhaps unfortunately, we are able to show that any o.s.s dominating (or equal to) one of these interpolated o.s.s. cannot be an operator algebra structure.

In Section 6 we show that the trace class S_1 (and indeed all the von Neumann-Schatten p-classes) are bicontinuously isomorphic to (norm closed) operator algebras. This is with respect to either the usual or the Schur product. However, S_1 is not completely boundedly isomorphic to an operator algebra if we equip it with its standard o.s.s.

We end the paper with a list of open problems.

In the rest of this section, we give some background and introduce some notation. Let us first come back to interpolation. With the notation above, assume that E_0 and E_1 are operator algebras such that the multiplications on E_0, E_1 extend to the same Banach algebra multiplication on $E_0 + E_1$. Then $\mathcal{F}$ is a Banach algebra under pointwise multiplication. Clearly, $L^\infty(\mathbf{R}, E_0) \overset{\infty}{\oplus} L^\infty(\mathbf{R}, E_1)$ is an operator algebra and the isometric embedding (1.9) is a homomorphism. Hence $\mathcal{F}$ is an operator algebra. Since any quotient of an operator algebra is still an operator algebra ([8], Corollary 3.2), the alternative definition (1.8) shows that E_θ is also an operator algebra. Therefore:

(1.12) The class of operator algebras is stable under interpolation.

We now turn to a quite convenient characterization of operator algebras which was recently established by the first author ([3]). It will be our main tool in order to deal with problem (1.2). In order to state this result, and for later use, we need the notion of a c.b. bilinear map. Let X, Y, Z be three operator spaces and let $u : X \times Y \to Z$ be a bilinear map. For any $n \geqslant 1$, u induces a map $u^{(n)} : M_n(X) \times M_n(Y) \to M_n(Z)$ defined by:

$$u^{(n)}([x_{ij}],[y_{ij}]) = \left[\sum_{k=1}^{n} u(x_{ik}, y_{kj})\right]_{ij}.$$

We say that u is c.b. provided that $\|u\|_{\mathrm{cb}} = \sup_{n\geqslant 1} \|u^{(n)}\| < +\infty$. The characterization is:

THEOREM 1.3. ([3]) *Let A be an operator space. Assume that A is endowed with a Banach algebra multiplication $m : A \times A \to A$, then*

$$A \text{ is an operator algebra} \Leftrightarrow m \text{ is c.b.}$$

The particular case of "$\Leftarrow$" when A had a normalized unit and $\|m\|_{\mathrm{cb}} \leqslant 1$, had been previously obtained in [8]. Note also that, as observed in [3], the assertion (1.12) can be viewed as a consequence of Theorem 1.3.

Completely bounded bilinear maps are — by now — a well understood subject. We refer the reader to [11], [27], [10], [17], [4] for the main results about them. We merely recall that the complete boundedness of $u : X \times Y \to Z$ corresponds to the complete boundedness of the linearized map (still denoted by) $u : X \overset{h}{\otimes} Y \to Z$, where $X \overset{h}{\otimes} Y$ is the Haagerup tensor product of X and Y. Thus in the setting of Theorem 1.3, A is an operator algebra iff

$$\|m : A \overset{h}{\otimes} A \to A\|_{\mathrm{cb}} < +\infty. \tag{1.13}$$

We will often invoke the simple fact that $\overset{h}{\otimes}$ is a uniform operator space tensor norm in the sense of [7]. Thus if $u_1 : X_1 \to Y_1$ and $u_2 : X_2 \to Y_2$ are c.b. maps between operator spaces, then:

$$u_1 \otimes u_2 : X_1 \overset{h}{\otimes} X_2 \to Y_1 \overset{h}{\otimes} Y_2$$

is c.b. and we have:

$$\|u_1 \otimes u_2\|_{\mathrm{cb}} \leqslant \|u_1\|_{\mathrm{cb}} \|u_2\|_{\mathrm{cb}}. \tag{1.14}$$

In order to study operator algebra structures on ℓ_p, we will often restrict to the finite dimensional spaces ℓ_p^n. We will denote by $m_n : \ell_p^n \otimes \ell_p^n \to \ell_p^n$ the pointwise multiplication (same notation for all $1 \leqslant p \leqslant +\infty$). Assume that we are given an o.s.s. on ℓ_p. This induces an o.s.s. on each ℓ_p^n. Then it is not hard to see using

Theorem 1.3 in the form of (1.13), that the o.s.s. on ℓ_p is an operator algebra structure iff:

$$\exists C \mid \forall n \geqslant 1,\ \|m_n : \ell_p^n \overset{h}{\otimes} \ell_p^n \to \ell_p^n\|_{\mathrm{cb}} \leqslant C. \tag{1.15}$$

We will denote by $(e_i)_{i\geqslant 1}$ the canonical basis of ℓ_p. To avoid confusion with the injective tensor product of Banach spaces we will denote by $X \overset{\smile}{\otimes} Y$ the spatial (or minimal) tensor product of two operator spaces X and Y (see e.g. [7]). Note that $\overset{\smile}{\otimes}$ satisfies the property (1.14) stated for $\overset{h}{\otimes}$.

Lastly, we denote $S_1^n = M_n^*$ the dual operator space of M_n.

2. THE CASE OF ℓ_2

Besides Min ℓ_2, Max ℓ_2 and OH, there are at least four interesting and well-understood o.s.s. on ℓ_2 which are usually denoted by $R, C, R\cap C, R+C$. Let us now recall what these structures are. Given a Hilbert space H, the column operator space H_c is the o.s.s. on H given by the identity $H_c = B(\mathbf{C}, H)$. The notation C stands for $(\ell_2)_c$ and we denote by $C_n = (\ell_2^n)_c$ the n-dimensional version of C. We use similar notations for row structures. Namely, $H_r = B(H^*, \mathbf{C})$, $R = (\ell_2)_r$, $R_n = (\ell_2^n)_r$. The matrix norms on C and R are easy to compute. For any $a_1, \ldots, a_n$ in $B(\ell_2)$:

$$\Big\|\sum_i a_i \otimes e_i\Big\|_{B(\ell_2)\overset{\smile}{\otimes} C} = \Big\|\sum_i a_i^* a_i\Big\|^{\frac{1}{2}}\ , \quad \Big\|\sum_i a_i \otimes e_i\Big\|_{B(\ell_2)\overset{\smile}{\otimes} R} = \Big\|\sum_i a_i a_i^*\Big\|^{\frac{1}{2}}.$$

By definition [28], $R\cap C$ is the o.s.s. on ℓ_2 which satisfies for any $a_1, \ldots, a_n \in B(\ell_2)$:

$$\Big\|\sum_i a_i \otimes e_i\Big\|_{B(\ell_2)\overset{\smile}{\otimes}(R\cap C)} = \max\left\{\Big\|\sum_i a_i^* a_i\Big\|^{\frac{1}{2}}, \Big\|\sum_i a_i a_i^*\Big\|^{\frac{1}{2}}\right\}. \tag{2.1}$$

In other words, $R\cap C$ is the diagonal subspace of the direct sum $R \overset{\infty}{\oplus} C$. Lastly, $R+C$ is the dual operator space of $R\cap C$. We now determine, using condition (1.15), which of the seven o.s.s. on ℓ_2 reviewed above are operator algebra structures.

THEOREM 2.1. (i) *The following are operator algebras: $R, C, OH, R\cap C$,* Max ℓ_2. *Indeed ℓ_2 with any o.s.s. which dominates both R and C is an operator algebra.*

(ii) *The following are not operator algebras: $R+C$,* Min ℓ_2. *Indeed ℓ_2 with any o.s.s. which is dominated by both R and C is not an operator algebra.*

Proof. Let $\mathcal{M}_n$ be the vector space of $n \times n$ matrices, without prescribed norm. We identify $\ell_2^n \otimes \ell_2^n$ with $\mathcal{M}_n$ in the usual way. Given $h = \sum_1^n h_i e_i$ and $k = \sum_1^n k_i e_i$ in ℓ_2^n, we regard $h \otimes k$ as the matrix $[h_i k_j] \in \mathcal{M}_n$. Under this identification, the multiplication on ℓ_2^n becomes the diagonal map $m_n : \mathcal{M}_n \to \mathbb{C}^n$ given by $m_n([t_{ij}]) = (t_{ii})_{1 \leqslant i \leqslant n}$.

(i) Let HS_n be the Hilbert space obtained by equipping $\mathcal{M}_n$ with the Hilbert-Schmidt norm. It is well-known that under the above identification, we have completely isometrically ([7], [17], [4]):

$$C_n \overset{h}{\otimes} C_n = (HS_n)_{\mathrm{c}}, \quad R_n \overset{h}{\otimes} R_n = (HS_n)_{\mathrm{r}}.$$

Hence m_n is a contraction from $C_n \overset{h}{\otimes} C_n$ into C_n and thus (see [17], [4]),

$$\|m_n : C_n \overset{h}{\otimes} C_n \to C_n\|_{\mathrm{cb}} \leqslant 1.$$

This proves that C is an operator algebra. Similarly, R is an operator algebra. From [28], we have $OH = [R, C]_{\frac{1}{2}}$ whence OH is also an operator algebra by (1.12).

Now note that we also have the complete isometry ([7])

$$R_n \overset{h}{\otimes} C_n = S_1^n$$

whence:

$$\|m_n : R_n \overset{h}{\otimes} C_n \to \mathrm{Max}\,\ell_1^n\|_{\mathrm{cb}} \leqslant 1. \tag{2.2}$$

Indeed in (2.2), m_n is the adjoint of the canonical inclusion of $\mathrm{Min}\,\ell_\infty^n$ into M_n as the space of diagonal matrices.

Assume that ℓ_2 is endowed with an o.s.s. for which $\|\mathrm{Id} : \ell_2^n \to R_n\|_{\mathrm{cb}} \leqslant 1$ and $\|\mathrm{Id} : \ell_2^n \to C_n\|_{\mathrm{cb}} \leqslant 1$. Then it follows from (1.14) and (2.2) that $\|m_n : \ell_2^n \overset{h}{\otimes} \ell_2^n \to \ell_2^n\|_{\mathrm{cb}} \leqslant 1$. Since this clearly holds for $R \cap C$ and $\mathrm{Max}\,\ell_2$, these are operator algebras.

(ii) This time we use the following identity ([7]):

$$C_n \overset{h}{\otimes} R_n = M_n. \tag{2.3}$$

Since $\|m_n : M_n \to \ell_2^n\| = \sqrt{n}$, an o.s.s. on ℓ_2 for which $\|\mathrm{Id} : R_n \to \ell_2^n\|_{\mathrm{cb}} \leqslant 1$ and $\|\mathrm{Id} : C_n \to \ell_2^n\|_{\mathrm{cb}} \leqslant 1$ is not an operator algebra structure. Consequently, $R + C$ and $\mathrm{Min}\,\ell_2$ are not operator algebras. ∎

Among the five o.s.s. listed in (i), $R \cap C$ is the only one which is completely isomorphic with a quotient of a minimal operator space (see [22] and below). It turns out that $R \cap C$ is actually a matricial Q-algebra but that is not clear at this level. This will be proved in the fourth section.

3. REGARDING $\operatorname{Min}\ell_p, \operatorname{Max}\ell_p, O\ell_p$

The main purpose of this section is to determine the values of $p \in [1, +\infty]$ for which $\operatorname{Min}\ell_p$ or $\operatorname{Max}\ell_p$ or $O\ell_p$ is an operator algebra. We begin with the two extreme cases $p = 1$ and $p = +\infty$. These two situations turn out to be quite dichotomous.

THEOREM 3.1. (i) *$\operatorname{Min}\ell_\infty$ is, up to complete isomorphism, the only operator algebra structure on ℓ_∞.*

(ii) *Any* o.s.s. *on ℓ_1 is an operator algebra structure.*

Thus, in some sense, ℓ_1 is a "better" operator algebra than ℓ_∞! It is convenient to separate the following estimate.

LEMMA 3.2. *We have* $\|\mathrm{Id} : \operatorname{Min}\ell_1 \to R \cap C\|_{\mathrm{cb}} = 1$.

Proof. Let $a_1, \ldots, a_n$ in $B(\ell_2)$. Then by (1.4),

$$\Big\|\sum_j a_j \otimes e_j\Big\|_{B(\ell_2)\check{\otimes}\operatorname{Min}\ell_1} = \sup\Big\{\Big\|\sum_j \beta_j a_j\Big\|_{B(\ell_2)} \;\Big|\; \sup_{1\leqslant j\leqslant n} |\beta_j| \leqslant 1\Big\}.$$

Moreover, we can write:

$$\sum_j a_j^* a_j = \frac{1}{2\pi}\int_0^{2\pi} \Big(\sum_j a_j e^{ijt}\Big)^* \Big(\sum_j a_j e^{ijt}\Big)\, dt$$

hence

$$\Big\|\sum_j a_j^* a_j\Big\| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \Big\|\sum_j a_j e^{ijt}\Big\|^2 dt.$$

Therefore $\Big\|\sum_j a_j^* a_j\Big\|^{\frac12} \leqslant \Big\|\sum_j a_j \otimes e_j\Big\|_{B(\ell_2)\check{\otimes}\operatorname{Min}\ell_1}$. Similarly, $\Big\|\sum_j a_j a_j^*\Big\|^{\frac12} \leqslant \Big\|\sum_j a_j \otimes e_j\Big\|_{B(\ell_2)\check{\otimes}\operatorname{Min}\ell_1}$. By (2.1), this proves the lemma. ∎

Proof of Theorem 3.1. (i) This is a straightforward consequence of Kadison's theorem which asserts that any bounded homomorphism from the C^*-algebra $\operatorname{Min}\ell_\infty$ into some $B(H)$ is always completely bounded (see e.g. [23], Theorem 8.7).

(ii) Applying Lemma 3.2, (1.14) and (2.2) we obtain that for all $n \geqslant 1$:

$$\|m_n : \operatorname{Min}\ell_1^n \overset{h}{\otimes} \operatorname{Min}\ell_1^n \to \operatorname{Max}\ell_1^n\|_{\mathrm{cb}} \leqslant 1.$$

This proves the assertion. ∎

In view of definition (1.10), Theorem 3.1 and (1.12) we immediately have:

COROLLARY 3.3. *For any* $1 \leqslant p \leqslant +\infty$, $O\ell_p$ *is an operator algebra.*

We now look at minimal and maximal structures ((1.4), (1.5)).

THEOREM 3.4. *Let* $1 \leqslant p \leqslant +\infty$. *Then we have*
(i) $\mathrm{Min}\,\ell_p$ *is an operator algebra* $\Leftrightarrow p = 1$ *or* $p = +\infty$.
(ii) $\mathrm{Max}\,\ell_p$ *is an operator algebra* $\Leftrightarrow 1 \leqslant p \leqslant 2$.

Proof. (i) We fix $1 < p < +\infty$ and set

$$\sigma_n = \|m_n : \mathrm{Min}\,\ell_p^n \overset{h}{\otimes} \mathrm{Min}\,\ell_p^n \to \ell_p^n\|.$$

Let $u = \sum_1^n e_i \otimes e_i \in \mathrm{Min}\,\ell_p^n \overset{h}{\otimes} \mathrm{Min}\,\ell_p^n$. Then $\|m_n(u)\| = n^{1/p}$. Recall that for any Banach space E, $\mathrm{Min}\,E \overset{h}{\otimes} \mathrm{Min}\,E$ is isometrically isomorphic to $E \overset{\gamma_2}{\otimes} E$ where γ_2 is the Grothendieck norm of factorization through Hilbert space (see e.g. [30], Chapter 2). Therefore letting $\frac{1}{p} + \frac{1}{q} = 1$, $\|u\|$ is the γ_2-norm of the identity map $\mathrm{Id} : \ell_q^n \to \ell_p^n$. By trivial factorization $\ell_q^n \xrightarrow{\mathrm{Id}} \ell_2^n \xrightarrow{\mathrm{Id}} \ell_p^n$, we obtain that when $p \geqslant 2$, $\|u\| = 1$ and thus $\sigma_n \geqslant n^{1/p}$, whereas when $p \leqslant 2$, $\|u\| \leqslant n^{\frac{2}{p}-1}$ and thus $\sigma_n \geqslant n^{1/q}$. In any case, $\lim\limits_{+\infty} \sigma_n = +\infty$ hence $\mathrm{Min}\,\ell_p$ is not an operator algebra.

(ii) Let $1 \leqslant p \leqslant 2$. It follows from (2.2) that

$$\|m : \mathrm{Max}\,\ell_2 \overset{h}{\otimes} \mathrm{Max}\,\ell_2 \to \mathrm{Max}\,\ell_1\|_{\mathrm{cb}} \leqslant 1.$$

The contractive inclusions $\ell_1 \to \ell_p \to \ell_2$ together with (1.14) then yields:

$$\|m : \mathrm{Max}\,\ell_p \overset{h}{\otimes} \mathrm{Max}\,\ell_p \to \mathrm{Max}\,\ell_p\|_{\mathrm{cb}} \leqslant 1$$

hence $\mathrm{Max}\,\ell_p$ is an operator algebra.

Now let $2 < p \leqslant \infty$. In order to prove that $\mathrm{Max}\,\ell_p$ is not an operator algebra, we will use Clifford matrices. The idea of using Clifford matrices for the study of maximal operator spaces goes back to Paulsen ([24], see also [18]). Given $n \geqslant 1$, Clifford matrices are unitaries $u_1, \ldots, u_n \in M_{2^n}$ (which do exist) satisfying:

$$\text{(3.1)} \quad \begin{aligned} &\text{(i) } \forall \beta_i \in \mathbb{C}, \ \Big\|\sum_1^n \beta_i u_i\Big\| \leqslant \sqrt{2}\Big(\sum_1^n |\beta_i|^2\Big)^{\frac{1}{2}} \\ &\text{(ii) } \Big\|\sum_1^n u_i \otimes u_i\Big\|_{M_{2^n}(M_{2^n})} = n. \end{aligned}$$

We let $\frac{1}{p}+\frac{1}{q}=1$ $(1 \leqslant q \leqslant 2)$ and set:

$$\tau_n = \|m_n : \operatorname{Max} \ell_p^n \overset{h}{\otimes} \operatorname{Max} \ell_p^n \to \operatorname{Max} \ell_p^n\|_{\mathrm{cb}}.$$

Since the identity map $\mathrm{Id} : \ell_q^n \to \ell_1^n$ has norm $n^{1/p}$, it follows from Lemma 3.2 that $\|\mathrm{Id} : \operatorname{Min} \ell_q^n \to R_n \cap C_n\|_{\mathrm{cb}} \leqslant n^{1/p}$. Therefore, duality (1.6) implies the two estimates:

$$\|\mathrm{Id} : C_n \to \operatorname{Max} \ell_p^n\|_{\mathrm{cb}} \leqslant n^{\frac{1}{p}}$$

$$\|\mathrm{Id} : R_n \to \operatorname{Max} \ell_p^n\|_{\mathrm{cb}} \leqslant n^{\frac{1}{p}}.$$

Applying (1.14), we then deduce:

$$\|m_n : C_n \overset{h}{\otimes} R_n \to \operatorname{Max} \ell_p^n\|_{\mathrm{cb}} \leqslant n^{\frac{2}{p}} \tau_n. \tag{3.2}$$

By (2.3), the map $\theta_n : \operatorname{Min} \ell_\infty^n \to C_n \overset{h}{\otimes} R_n$ defined by $\theta(e_j) = e_j \otimes e_j$ is a complete isometry. Since $m_n \theta_n$ is just the identity map, it now follows from (3.2) that:

$$\|\mathrm{Id} : \operatorname{Min} \ell_\infty^n \to \operatorname{Max} \ell_p^n\|_{\mathrm{cb}} \leqslant n^{\frac{2}{p}} \tau_n. \tag{3.3}$$

We wish to find a lower bound for the left hand side of (3.3). Let $u = \sum_1^n u_i \otimes e_i$. Since each u_i has norm 1, formula (1.4) yields $\|u\|_{M_{2^n}(\operatorname{Min} \ell_\infty^n)} = 1$. The identity $M_{2^n}(\operatorname{Max} \ell_p^n) = \mathrm{CB}(\operatorname{Min} \ell_q^n, M_{2^n})$ resulting from (1.1) and (1.6) means that $\|u\|_{M_{2^n}(\operatorname{Max} \ell_p^n)}$ is the c.b. norm of the map $T : \operatorname{Min} \ell_q^n \to M_{2^n}$ defined by $T(e_j) = u_j$. By (1.4) and (3.1) (i), $\|u\|_{M_{2^n}(\operatorname{Min} \ell_q^n)} \leqslant \sqrt{2}\, n^{\frac{1}{2}-\frac{1}{p}}$ and by (3.1) (ii), $\|(T \otimes I_{M_{2^n}})(u)\|_{M_{2^n}(M_{2^n})} = n$. Therefore we have:

$$\|u\|_{M_{2^n}(\operatorname{Max} \ell_p^n)} = \|T\|_{\mathrm{cb}} \geqslant n \left(\sqrt{2}\, n^{\frac{1}{2}-\frac{1}{p}}\right)^{-1} = \frac{n^{\frac{1}{2}+\frac{1}{p}}}{\sqrt{2}}.$$

This shows that $\|\mathrm{Id} : \operatorname{Min} \ell_\infty^n \to \operatorname{Max} \ell_p^n\|_{\mathrm{cb}} \geqslant \frac{n^{\frac{1}{2}+\frac{1}{p}}}{\sqrt{2}}$. Going back to (3.3), we obtain $\tau_n \geqslant \frac{n^{\frac{1}{2}-\frac{1}{p}}}{\sqrt{2}}$ whence $\lim\limits_{n \to \infty} \tau_n = +\infty$. This completes the proof that $\operatorname{Max} \ell_p$ is not an operator algebra. ∎

We now turn to Q-algebras. We know from [12] and [33] that for all $1 \leqslant p \leqslant +\infty$, there exists at least one matricial Q-algebra structure on ℓ_p. Our aim is then to exhibit such structures and to determine which of the operator algebras provided by Theorems 2.1, 3.1, 3.4 and Corollary 3.3 are actually Q-algebras. The situation is trivial for $p = +\infty$. We then look at the case $p = 1$ where once again, the picture is clear.

PROPOSITION 3.5. *$\mathrm{Min}\,\ell_1$ is, up to complete isomorphism, the only Q-algebra structure on ℓ_1.*

Proof. Assume that ℓ_1 is endowed with a matricial Q-algebra structure. By Definition 1.2, we have an onto homomorphism $\rho : \ell_1 \to B/J$ for some uniform algebra B and some closed ideal $J \subset B$, such that $\|\rho\|_{\mathrm{cb}}, \|\rho^{-1}\|_{\mathrm{cb}} < +\infty$. Let $q : B \to B/J$ be the quotient map. From the lifting property of ℓ_1, we know that there exists a bounded map $\tilde{\rho} : \ell_1 \to B$ with $q\tilde{\rho} = \rho$. Since $B = \mathrm{Min}\,B$, $\tilde{\rho}$ is c.b. from $\mathrm{Min}\,\ell_1$ into B. Writing the identity map $\mathrm{Id} : \ell_1 \to \ell_1$ as $\mathrm{Id} = (\rho^{-1}q)\tilde{\rho}$, we thus obtain that $\|\mathrm{Id} : \mathrm{Min}\,\ell_1 \to \ell_1\|_{\mathrm{cb}} < +\infty$. Hence ℓ_1 has, up to constants, the minimal operator space structure. By [12] and [33] (the assertion that ℓ_1 is a Q-algebra), the converse is obvious. ∎

Before going further, it will be worthwhile to introduce the following definition, which generalizes that of matricial Q-algebra.

DEFINITION 3.6. Let E be an operator space. We say that E is a *Q-space* provided that E is completely isomorphic to a quotient of a minimal operator space.

Let E, F be two operator spaces. We define :

$$d_{\mathrm{cb}}(E,F) = \inf\{\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}}\}$$

where the infimum runs over all possible isomorphisms between E and F. If E and F are not completely isomorphic, we write $d_{\mathrm{cb}}(E,F) = \infty$. In order to measure the "Q-space structure" of an operator space we then define :

$$d_Q(E) = \inf\left\{ d_{\mathrm{cb}}\left(E, \frac{\mathrm{Min}\,F_1}{\mathrm{Min}\,F_2}\right)\right\}$$

where the infimum runs over all included Banach spaces $F_2 \subset F_1$. Clearly, $d_Q(E) < +\infty$ iff E is a Q-space. These Q-spaces were recently characterized by Junge ([19], see also [31]) following a conjecture of Pisier. In Section 4 we shall provide another proof.

THEOREM 3.7. ([19], [31]) *Let E be an operator space. Then E is a Q-space if and only if there exists $C > 0$ such that for any $n, m \geqslant 1$ and any $T : M_n \to M_m$,*

$$\|T \otimes I_E : M_n(E) \to M_m(E)\| \leqslant C\|T\|. \tag{3.4}$$

Moreover, $d_Q(E) = \inf C$ such that (3.4) holds.

The class of Q-spaces is stable under interpolation. Indeed keeping the notation in Section 1, let (E_0, E_1) be a compatible couple of operator spaces.

Assume that E_0, E_1 are both Q-spaces. Since given Banach spaces $Y \subset X$, $L^\infty(\mathbf{R}, X)/L^\infty(\mathbf{R}, Y) = L^\infty(\mathbf{R}, X/Y)$, the spaces $L^\infty(\mathbf{R}, E_0)$ and $L^\infty(\mathbf{R}, E_1)$ are both Q-spaces. Therefore, $L^\infty(\mathbf{R}, E_0) \overset{\infty}{\oplus} L^\infty(\mathbf{R}, E_1)$ is also a Q-space. Hence by (1.9), $\mathcal{F}$ is a Q-space. By (1.8), this clearly implies that each E_θ is a Q-space. This stability property can also be deduced from Theorem 3.7 and this in fact yields the precise estimate:

$$d_Q(E_\theta) \leqslant d_Q(E_0)^{1-\theta} d_Q(E_1)^\theta. \tag{3.5}$$

Note for further use that similarly, when E_0, E_1 are matricial Q-algebras, the construction leading to (1.8) shows that for all $0 \leqslant \theta \leqslant 1$, E_θ is a matricial Q-algebra.

Let us compute $d_Q(O\ell_p^n)$. The following is already known in the case $p = 2$ ([28], [19]).

PROPOSITION 3.8. *For any* $1 \leqslant p \leqslant +\infty$,

$$\frac{n^{\frac{1}{2p}}}{\sqrt{2}} \leqslant d_Q(O\ell_p^n) \leqslant n^{\frac{1}{2p}}.$$

Proof. Obviously, $d_Q(\operatorname{Min} \ell_\infty^n) = 1$. Furthermore, Paulsen ([24]) proved that $d_{\mathrm{cb}}(\operatorname{Max} \ell_1^n, \operatorname{Min} \ell_1^n) \leqslant \sqrt{n}$ hence $d_Q(\operatorname{Max} \ell_1^n) \leqslant \sqrt{n}$. Therefore the interpolation formula (3.5) gives the upper estimate.

Let us now prove the more interesting lower estimate. We shall apply (the easy part of) Theorem 3.7 to the map $T : C_n \to M_{2^n}$ defined by $T(e_j) = u_j$, where $u_1, \ldots, u_n$ are the Clifford matrices introduced in the proof of Theorem 3.4. From (3.1)(i), we know that $\|T\| \leqslant \sqrt{2}$. Let $\frac{1}{p} + \frac{1}{q} = 1$. It is easy to check that $\|\mathrm{Id} : \operatorname{Max} \ell_1^n \to C_n\|_{\mathrm{cb}} = 1$ and $\|\mathrm{Id} : \operatorname{Min} \ell_\infty^n \to C_n\|_{\mathrm{cb}} = \sqrt{n}$ whence

$$\|\mathrm{Id} : O\ell_q^n \to C_n\|_{\mathrm{cb}} \leqslant n^{\frac{1}{2p}}$$

by interpolation, and (1.10). By (1.11), this means that $\|\sum_1^n e_i \otimes e_i\|_{C_n(O\ell_p^n)} \leqslant n^{1/2p}$. Now $(T \otimes I_{O\ell_p^n})(\sum_1^n e_i \otimes e_i) = \sum_1^n u_i \otimes e_i$. Therefore invoking Theorem 3.7, it remains to prove:

$$\Big\| \sum_1^n u_i \otimes e_i \Big\|_{M_{2^n}(O\ell_p^n)} \geqslant n^{\frac{1}{p}}. \tag{3.6}$$

To see this, define $u : \mathbf{C}^n \to M_{2^n}$ by letting $u(e_j) = u_j$. Obviously

$$\|u : \operatorname{Min} \ell_\infty^n \to M_{2^n}\|_{\mathrm{cb}} \leqslant n, \quad \|u : \operatorname{Max} \ell_1^n \to M_{2^n}\|_{\mathrm{cb}} \leqslant 1$$

hence by interpolation

$$\|u : O\ell_p^n \to M_{2^n}\|_{\mathrm{cb}} \leqslant n^{1-\frac{1}{p}}. \tag{3.7}$$

Since $(I_{M_n} \otimes u)(\sum_1^n u_i \otimes e_i) = \sum_1^n u_i \otimes u_i$, the estimate (3.6) follows from (3.1) (ii) and (3.7). ∎

Following similar arguments as above, it is not hard to prove that for any $1 \leqslant p \leqslant +\infty$, $d_Q(\mathrm{Max}\,\ell_p^n) \geqslant c\sqrt{n}$ for some absolute constant $c > 0$. In any case this can be viewed as a corollary of [20], Theorem 3.2. This and Proposition 3.8 lead to:

COROLLARY 3.9. (i) *Let $1 \leqslant p < +\infty$. The operator algebra $O\ell_p$ is not a matricial Q-algebra.*

(ii) *Let $1 \leqslant p \leqslant 2$. The operator algebra $\mathrm{Max}\,\ell_p$ is not a matricial Q-algebra.*

The only matricial Q-algebra structures on ℓ_p ($p \neq 1, \infty$) that we can exhibit at this time come from interpolation (see the discussion before Proposition 3.8). Let $\theta = \frac{1}{p}$; then $[\mathrm{Min}\,\ell_\infty, \mathrm{Min}\,\ell_1]_\theta$ is a Q-algebra representation of ℓ_p. We will see below (Corollary 4.8) that for any $1 < p < +\infty$, there is actually a continuum of matricial Q-algebra structures on ℓ_p.

4. CHARACTERIZATIONS OF Q-ALGEBRAS AND APPLICATION TO $R \cap C$

We wanted to know which of the operator algebras listed in Theorems 2.1, 3.1, 3.4 and Corollary 3.3 are matricial Q-algebras. In view of the previous results, it remains to look at R, C and $R \cap C$. It is clear and well-known that R and C are not Q-spaces. For instance, (3.4) is wrong with $E = C$ or R for the transposition map T. On the contrary, $R \cap C$ is a Q-space. This is a consequence of the main result of [22]. Note that this can also be viewed as a formal combination of Junge's Theorem 3.7 with the Grothendieck-Pisier inequality ([30], Corollary 9.5). As a corollary of the methods of this section we shall prove:

THEOREM 4.1. *$R \cap C$ is a matricial Q-algebra.*

We first give an abstract criterion which can be considered as an operator space version of Davie's characterization of Q-algebras ([12], Theorem 3.3). We will need a little more notation. Suppose that A is an operator algebra. Let n, N, r be three positive integers and let $u : M_n \times \cdots \times M_n \to M_N$ be an r-linear map. We denote by $u_A : M_n(A) \times \cdots \times M_n(A) \to M_N(A)$ the r-linear map defined by tensoring u with the r-fold multiplication on A. Namely for any $x_1, \ldots, x_r$ in M_n and $a_1, \ldots, a_r$ in A, we have $u_A(x_1 \otimes a_1, \cdots, x_r \otimes a_r) = u(x_1, \ldots, x_r) \otimes (a_1 \ldots a_r)$.

THEOREM 4.2. *Let A be a commutative operator algebra. The following conditions are equivalent:*

(i) *A is a matricial Q-algebra.*

(ii) *There exists a constant $K > 0$ such that for any $n, N, r \geqslant 1$ and any r-linear $u : M_n \times \cdots \times M_n \to M_N$, we have $\|u_A\| \leqslant K^r \|u\|$.*

The proof of Theorem 4.2 will combine ideas from [12] and [3]. We will especially make use of a symmetrization principle observed by Davie.

LEMMA 4.3. ([12]) *Let X, Y be Banach spaces and let $v : X \times \cdots \times X \to Y$ be an r-linear map which is symmetric with respect to permutations of the variables. Then $\|v\| \leqslant (2e)^r \sup\{\|v(x, \ldots, x)\| \mid x \in X,\ \|x\| \leqslant 1\}$.*

Proof. We refer the reader to [12], Lemma 2.1. Davie deals with the case $X = \ell_\infty, Y = \mathbb{C}$ but his argument works as well in the general case. ∎

We now come to quantum variables. We will use commutative ones which will be sufficient for our purpose. Given $n \geqslant 1$, we let $\mathcal{PP}_n$ be the algebra of all polynomials in n^2 commuting variables X_{ij} $(1 \leqslant i,\ j \leqslant n)$ without constant term. We equip $\mathcal{PP}_n$ with the following norm:

$$\|P\| = \sup\{|P((x_{ij}))| \mid \|[x_{ij}]\|_{M_n} \leqslant 1\}.$$

Furthermore we endow $\mathcal{PP}_n$ with the minimal structure which makes it (its completion) an operator space.

Let A be a commutative operator algebra and let $\delta > 0$. For any $n, N \geqslant 1$, given $P \in M_N \otimes \mathcal{PP}_n$, we set:

$$\|P\|_{A,\delta} = \sup\{\|P((a_{ij}))\|_{M_N(A)},\ a_{ij} \in A,\ \|[a_{ij}]\|_{M_n(A)} \leqslant \delta\}. \tag{4.1}$$

With this notation we have $\|P\| = \|P\|_{\mathbb{C},1}$. In the following, by a homogeneous $P \in M_N \otimes \mathcal{PP}_n$, we mean a matrix of polynomials whose entries are homogeneous polynomials of same degree.

The statement below can be regarded as the "quantized version" of Craw's Lemma.

PROPOSITION 4.4. *Let A be a commutative operator algebra. The following conditions are equivalent:*

(i) *A is a matricial Q-algebra.*

(ii) *There are two constants $M, \delta > 0$ such that for any $N, n \geqslant 1$ and any $P \in M_N \otimes \mathcal{PP}_n$*

$$\|P\|_{A,\delta} \leqslant M\,\|P\|.$$

(iii) *Same as* (ii) *for homogeneous $P \in M_N \otimes \mathcal{PP}_n$.*

We remark that our proof will show that A is completely isometric to a quotient of a function algebra iff (ii) holds with $M = \delta = 1$.

Proof. Clearly for a uniform algebra B, $\|P\|_{B,1} \leqslant \|P\|$ for any P. It is then easy to check that (i) $\Rightarrow$(ii). The implication (iii) $\Rightarrow$ (ii) is an old trick. Assume (iii) with $M, \delta > 0$. Any $P \in M_N \otimes \mathcal{PP}_n$ can be written as a finite sum $P = \sum\limits_{r\geqslant 1} P_r$ where P_r is zero or a homogeneous polynomial of degree r. Moreover, $\|P_r\| \leqslant \|P\|$ for all $r \geqslant 1$ as in [3]. Then, $\|P\|_{A,\frac{\delta}{2}} \leqslant \sum\limits_{r\geqslant 1} \|P_r\|_{A,\frac{\delta}{2}} \leqslant M \sum\limits_{r\geqslant 1} 2^{-r}\|P_r\| \leqslant M\|P\|$. Hence (ii) is fulfilled with M and $\frac{\delta}{2}$.

Now assume that (ii) holds. Let T be the closed unit ball of $B(\ell_2)$ endowed with the usual w*-topology. Any $t \in T$ will be considered as a infinite matrix $(t_{ij})_{i,j\geqslant 1}$. Denoting by K the space of all compact operators on ℓ_2, we let Λ be the closed ball of $K \overset{\smile}{\otimes} A$ of radius δ. Then we let $\Gamma = T^\Lambda$ be the set of all functions from Λ into T, with product topology. By Tychonov's theorem the set Γ is compact. For any $a \in \Lambda$ and any integers $i, j \geqslant 1$, we define $f_{ij}^a : \Gamma \to \mathbb{C}$ by letting $f_{ij}^a(\varphi) = \varphi(a)_{ij}$. Obviously, $f_{ij}^a \in C(\Gamma)$. Now let V be the algebra generated by all the f_{ij}^a ($a \in \Lambda$; $i, j \geqslant 1$) and let $E = \overline{V} \subset C(\Gamma)$. By construction, E is a uniform algebra.

We wish to define a c.b. homomorphism $q : E \to A$ by letting $q(f_{ij}^a) = a_{ij}$ for all a, i, j. In order to prove that this is indeed possible we consider $v = [v_{k\ell}] \in M_N \otimes V$. There exist distinct $a(1), \cdots, a(r)$ in Λ and $m \geqslant 1$ such that each $v_{k\ell}$ lies in the algebra generated by $\{f_{ij}^{a(p)} \mid 1 \leqslant p \leqslant r,\ 1 \leqslant i,\ j \leqslant m\}$. Let $n = mr$. Each $v_{k\ell}$ is a polynomial in the variables $f_{ij}^{a(p)}$ ($1 \leqslant p \leqslant r$, $1 \leqslant i$, $j \leqslant m$). Therefore making the substitution

$$[X_{ij}] \leftrightarrow \begin{pmatrix} [f_{ij}^{a(1)}] & & \\ & \ddots & \\ & & [f_{ij}^{a(r)}] \end{pmatrix}$$

we obtain $P_{k\ell} \in \mathcal{PP}_n$ such that

$$v_{k\ell} = P_{k\ell}\left([f_{ij}^{a(1)}] \oplus \cdots \oplus [f_{ij}^{a(r)}]\right).$$

We may suppose in addition that $P_{k\ell}$ only depends on the variables which $v_{k\ell}$ depends on. Let $P = [P_{k\ell}] \in M_N \otimes \mathcal{PP}_n$, we claim that

$$\|P\| = \|v\|. \tag{4.2}$$

Indeed $\|P\| = \sup\{\|P(\alpha_1 \oplus \cdots \oplus \alpha_r)\|_{M_N} \mid \alpha_p \in M_m,\ \sup\limits_p \|\alpha_p\| \leqslant 1\}$ whereas

$$\begin{aligned} \|v\| &= \sup\{\|[v_{k\ell}(\varphi)]\|_{M_N} \mid \varphi \in \Gamma\} \\ &= \sup\{\|P([f_{ij}^{a(1)}(\varphi)] \oplus \cdots \oplus [f_{ij}^{a(r)}(\varphi)])\|_{M_N} \mid \varphi \in \Gamma\} \\ &= \sup\{\|P([\varphi(a(1))_{ij}] \oplus \cdots \oplus [\varphi(a(r))_{ij}])\|_{M_N} \mid \varphi \in \Gamma\} \end{aligned}$$

whence the result.

Given $1 \leqslant p \leqslant r$, let us denote by $b(p) \in M_m(A)$ the $m \times m$ upper left corner of $a(p)$. We set $c = \begin{pmatrix} b(1) & & \\ & \ddots & \\ & & b(r) \end{pmatrix} \in M_n(A)$. Then $\|c\| = \sup_p \|b(p)\| \leqslant \sup_p \|a(p)\| \leqslant \delta$. Thus our assumption gives $\|P(c)\|_{M_N(A)} \leqslant M\|P\|$. Now $P(c) = [q(v_{k\ell})]$, hence we deduce from (4.2) that

$$\|[q(v_{k\ell})]\| \leqslant M\|v\|.$$

This shows that q is a well-defined and c.b. homomorphism from E into A.

Let $b \in M_n(A)$ with $\|b\| = \delta$. We can consider b as an element of Λ by the usual embedding. Then $v = [f_{ij}^b] \in M_n(V)$ has norm 1 and $q(v) = b$. Therefore q is onto and $q^{-1} : A \to E/\ker q$ is c.b. We thus obtain (i). ∎

Proof of Theorem 4.2. Assume (ii). We will prove that the condition (iii) in Proposition 4.4 is fulfilled. Suppose that we are given a homogeneous $P \in M_N \otimes \mathcal{PP}_n$ of degree $r \geqslant 1$. Let $I = (\{1,\ldots,n\} \times \{1,\ldots,n\})^r$. For any $\alpha = ((i_1,j_1),\ldots,(i_r,j_r)) \in I$, we set $\alpha_p = (i_p,j_p)$. We now write: $P = \sum_{\alpha \in I} \lambda_\alpha \otimes X_{\alpha_1} \cdots X_{\alpha_r}$ with $\lambda_\alpha \in M_N$. We can assume that for any α, β in I, whenever $X_{\alpha_1} \cdots X_{\alpha_r} = X_{\beta_1} \cdots X_{\beta_r}$, we have $\lambda_\alpha = \lambda_\beta$. Let $(E_{ij})_{1 \leqslant i,j \leqslant n}$ be the canonical basis of M_n. We define an r-linear map $u : M_n \times \cdots \times M_n \to M_N$ by letting $\lambda_\alpha = u(E_{\alpha_1},\ldots,E_{\alpha_r})$. By construction, u is a symmetric map. Since $\|P\| = \sup\{\|u(x,\ldots,x)\| \mid x \in M_n, \|x\| \leqslant 1\}$, Lemma 4.3 implies that $\|u\| \leqslant (2\mathrm{e})^r\|P\|$. Therefore $\|u_A\| \leqslant (2\mathrm{e}K)^r\|P\|$. Since A is commutative, we have $\|P\|_{A,1} \leqslant \|u_A\|$. Hence $\|P\|_{A,\delta} \leqslant \|P\|$ for $\delta = (2\mathrm{e}K)^{-1}$, whence the result. The converse implication (i) $\Rightarrow$ (ii) is easy and left to the reader. ∎

REMARK 4.5. Following Varopoulos's approach to Craw's Lemma ([34], see also Dixon [14]), we can establish a non-commutative and more general version of Proposition 4.4 with essentially the same proof. For this purpose we need a non-commutative analogue of $\mathcal{PP}_n$. For any $n \geqslant 1$, we let $\widetilde{\mathcal{PP}}_n$ be the algebra of all polynomials in n^2 non-commuting variables X_{ij} $(1 \leqslant i,\ j \leqslant n)$ without constant term. Let A be an operator space which is also a Banach algebra. As above, for any $P \in M_N \otimes \widetilde{\mathcal{PP}}_N$ and any $\delta > 0$, we define $\|P\|_{A,\delta}$ by (4.1). Now suppose B is an operator algebra which is also a dual operator space. The following are equivalent:

(i) There exist $M,\ \delta > 0$ such that for all $N, n \geqslant 1$ and any $P \in M_N \otimes \widetilde{\mathcal{PP}}_n$, $\|P\|_{A,\delta} \leqslant M\|P\|_{B,1}$;

(ii) There exist a compact set Γ and a subalgebra E of $C(\Gamma, B)$ such that A is c.b. homomorphic to a quotient algebra of E.

Note that in this statement, we do not assume that A is an operator algebra. This was already the case in Proposition 4.4 although not mentioned there. Since the quotient of an operator algebra is completely isometrically isomorphic to an operator algebra ([8]) we thus obtain a characterization which slightly improves ([3], Corollary 3.2):

> An operator space A which is a Banach algebra is an operator algebra iff it satisfies (i) for $B = B(\ell_2)$.

Clearly there is also a completely isometric version of these results. We refer the reader to [21] for possible generalizations.

REMARK 4.6. Junge's Theorem 3.7 can be viewed as a particular case of Theorem 4.2. Indeed let E be an operator space. Let us equip E with the trivial multiplication, i.e. $xy = 0$ for all $x, y \in E$. Clearly E can be regarded as an operator algebra. Indeed, if $E \subset B(H)$, E is completely isometric to $A = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in E \right\} \subset B(H \oplus H)$, which is a suitable operator algebra. Now assume that E satisfies (3.4). Then A satisfies the condition (ii) of Theorem 4.2. Therefore, Theorem 4.2 implies that E is a Q-space. Conversely, a Q-space equipped with the trivial multiplication is a matricial Q-algebra.

It should be noticed that if C is the constant appearing in (3.4), the construction in the proof of Proposition 4.4 yields a c.b. isomorphism of exactly constant C between E and a quotient of a minimal operator space. Thus (with minor changes), the proof of our Proposition 4.4 provides a new proof of Theorem 3.7.

The following consequence of Theorem 4.2 should be compared with [33], Theorem 1.

COROLLARY 4.7. *Let A be a commutative operator algebra. Assume that A is a Q-space and that the multiplication m on A satisfies: $\|m : A \overset{\vee}{\otimes} A \to A\|_{cb} < +\infty$. Then A is a matricial Q-algebra.*

Proof. Since A is a Q-space, there exist a Banach space E and a quotient map $q : E \to A$ such that :

$$C_1 = \|q : \mathrm{Min}\, E \to A\|_{cb} < +\infty \,, \quad C_2 = \left\| q^{-1} : A \to \frac{\mathrm{Min}\, E}{\mathrm{Min}\,(\mathrm{Ker}\, q)} \right\|_{cb} < +\infty.$$

Furthermore we set $C_3 = \|m : A \overset{\vee}{\otimes} A \to A\|_{cb}$.

Our aim is to show that A verifies the condition (ii) of Theorem 4.2. Suppose that we are given an r-linear map $u : M_n \times \cdots \times M_n \to M_N$. We can assume that $r \geqslant 2$. Let $a_1, \ldots, a_r$ in $M_n(A)$ such that $\|a_i\| < 1$ for all $1 \leqslant i \leqslant r$. We will prove:

$$\|u_A(a_1, \ldots, a_r)\| \leqslant (C_1C_2C_4)^r \|u\| \tag{4.3}$$

where $C_4 = \max\{C_3, 1\}$. Let us denote by $\widehat{\otimes}$ and $\check{\otimes}$ the projective and injective tensor products of Banach spaces (see e.g. [13]). Recall that we denote by S_1^n the dual (operator) space of M_n. Then the multilinear map u canonically induces a map $U : M_n \widehat{\otimes} \cdots \widehat{\otimes} M_n \to M_N$ whose adjoint $U^* : S_1^N \to S_1^n \check{\otimes} \cdots \check{\otimes} S_1^n$ has norm equal to $\|u\|$. Note also that the r-fold tensor product $Q = q \otimes \cdots \otimes q$ satisfies

$$\|Q : \operatorname{Min} E \overset{\smile}{\otimes} \cdots \overset{\smile}{\otimes} \operatorname{Min} E \to A \overset{\smile}{\otimes} \cdots \overset{\smile}{\otimes} A\|_{\mathrm{cb}} \leqslant C_1^r. \tag{4.4}$$

Now let $1 \leqslant i \leqslant r$. Since $M_n(\operatorname{Min} E) = M_n \check{\otimes} E$, there exist $f_i \in M_n \check{\otimes} E$ such that $(q \otimes I_{M_n})(f_i) = a_i$ and $\|f_i\| \leqslant C_2$. Under the classical identification $M_n \check{\otimes} E = B(S_1^n, E)$, f_i corresponds to a linear map $T_i : S_1^n \to E$ of norm less than C_2. Then $T = T_1 \otimes \cdots \otimes T_r$ satisfies:

$$\|T : S_1^n \check{\otimes} \cdots \check{\otimes} S_1^n \to E \check{\otimes} \cdots \check{\otimes} E\| \leqslant C_2^r. \tag{4.5}$$

Under the identity $M_N(A) = \mathrm{CB}(S_1^N, A)$ (see (1.1)) the A-valued matrix $u_A(a_1, \ldots a_r)$ corresponds to some $\theta : S_1^N \to A$. Let $m_r : A \overset{\smile}{\otimes} \cdots \overset{\smile}{\otimes} A \to A$ be the r-fold multiplication on A. Then it is an easy algebraic exercise to check that

$$\theta = m_r \cdot Q \cdot T \cdot U^*. \tag{4.6}$$

Recall that $\operatorname{Min}(E \check{\otimes} \cdots \check{\otimes} E) = \operatorname{Min} E \overset{\smile}{\otimes} \cdots \overset{\smile}{\otimes} \operatorname{Min} E$, so that $Q \cdot T$ makes sense. From (4.5), we have $\|TU^* : S_1^N \to \operatorname{Min}(E \check{\otimes} \cdots \check{\otimes} E)\|_{\mathrm{cb}} \leqslant C_2^r \|U\|$. Since $\|m_r\|_{\mathrm{cb}} \leqslant C_3^{r-1}$, (4.3) then follows from (4.4) and (4.6). This completes the proof that the condition (ii) of Theorem 4.2 is fulfilled. ∎

We remark that the condition of Corollary 4.7 is not a necessary condition for A to be a matricial Q-algebra. That is we have examples (for instance $A = [\operatorname{Min} \ell_1, \operatorname{Min} \ell_\infty]_{\frac{1}{2}}$) to show that not every matricial Q-algebra A satisfies $\|m : A \overset{\smile}{\otimes} A \to A\|_{\mathrm{cb}} < +\infty$.

Proof of Theorem 4.1. Recall that $R \cap C$ is a Q-space ([22], see also [29] for explanation of this). Thus applying Corollary 4.7, it remains to show that:

$$\|m : (R \cap C) \overset{\smile}{\otimes} (R \cap C) \to R \cap C\|_{\mathrm{cb}} \leqslant 1. \tag{4.7}$$

We saw (proof of Theorem 2.1) that $\|m : C \overset{h}{\otimes} C \to C\|_{\mathrm{cb}} \leqslant 1$. However, $C \overset{h}{\otimes} C = C \overset{\smile}{\otimes} C$ completely isometrically ([17], [4]), hence $\|m : C \overset{\smile}{\otimes} C \to C\|_{\mathrm{cb}} \leqslant 1$. Therefore, $\|m : (R \cap C) \overset{\smile}{\otimes} (R \cap C) \to C\|_{\mathrm{cb}} \leqslant 1$. Similarly, $\|m : (R \cap C) \overset{\smile}{\otimes} (R \cap C) \to R\|_{\mathrm{cb}} \leqslant 1$, whence (4.7). ∎

COROLLARY 4.8. *Let $1 < p < +\infty$. There is a continuum of Q-algebra structures on ℓ_p.*

Proof. We merely sketch the proof. We let $E_0(p) = [\mathrm{Min}\,\ell_\infty, \mathrm{Min}\,\ell_1]_{\frac{1}{p}}$. For any $2 \leqslant p < +\infty$, we set $E_1(p) = [\mathrm{Min}\,\ell_\infty, R \cap C]_{\frac{2}{p}}$ and for any $1 < p \leqslant 2$, we set $E_1(p) = [\mathrm{Min}\,\ell_1, R \cap C]_{\frac{2}{q}}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Lastly we define for any $0 \leqslant \theta \leqslant 1$:

$$E_\theta(p) = [E_0(p), E_1(p)]_\theta.$$

By Proposition 3.5, Theorem 4.1 and interpolation, all the $E_\theta(p)$ are matricial Q-algebras. Moreover for any $1 < p < +\infty$, it is not hard to show that the spaces $E_\theta(p)$ $(0 \leqslant \theta \leqslant 1)$ are mutually distinct. ∎

REMARK 4.9. Varopoulos ([33], [35]) defined a DQ-algebra to be a Banach algebra A for which there exists a uniform algebra B, a continuous surjective homomorphism $f : B \to A$, and a continuous linear one sided inverse map $g : A \to B$ (so fg =Id). He showed that A is a DQ-algebra if and only if A is an injective algebra, that is if and only if the product map induces a map $A \otimes A \to A$ which is continuous with respect to the Banach space injective tensor norm. We observe that if A is a DQ-algebra, then it is fairly clear that Min A is a matricial Q-algebra in the sense of our paper. This is the case for $A = \ell_1$ for example.

REMARK 4.10. It is interesting to look at another example of a Q-algebra which appeared in Davie's paper ([12]). Given $r \in \mathbb{N}$, we let $C^r(I)$ be the space of all r-continuously differentiable complex functions on a compact interval I. This is a Banach space with the norm

$$\|f\| = \sup\{\|f^{(k)}\|_\infty \mid 0 \leqslant k \leqslant r\}$$

where $\|g\|_\infty = \sup_{t \in I} |g(t)|$. Davie showed that $C^r(I)$ can be represented as a Q-algebra, whence the question of exhibiting a matricial Q-algebra structure on $C^r(I)$ follows. It turns out that:

$$\mathrm{Min}\, C^r(I) \text{ is a matricial } Q\text{-algebra.} \tag{4.8}$$

To check (4.8), note that the map $J : C^r(I) \to \ell_\infty^{r+1}(C(I))$ defined by $J(f) = (f, f', \ldots, f^{(r)})$ is an isometry. Using this it is not hard to verify that

$$m : C^r(I) \overset{\vee}{\otimes} C^r(I) \to C^r(I) \tag{4.9}$$

is bounded (with norm $\leqslant 2^r$). Hence $C^r(I)$ is an injective algebra in the sense of Remark 4.9. Therefore, it follows from that remark that $\mathrm{Min}\, C^r(I)$ is a matricial

Q-algebra. Note that the matrix norms on $\operatorname{Min} C^r(I)$ may be written in a more appealing form. Since $\ell_\infty^{r+1}(C(I))$ is a commutative C^*-algebra, the map J defined above induces the minimal operator space structure on $C^r(I)$. Consequently for any $f_{ij} \in C^r(I)$, $1 \leqslant i, j \leqslant n$, we have:

$$\|[f_{ij}]\|_{M_n(\operatorname{Min} C^r(I))} = \sup_{0 \leqslant k \leqslant r} \{\|[f_{ij}^{(k)}]\|_{M_N(C(I))}\}.$$

Finally note that (4.8) can also be viewed as a consequence of our work. Indeed (4.9) means that $\operatorname{Min} C^r(I)$ satisfies Corollary 4.7.

5. OPERATOR ALGEBRA STRUCTURES ON THE DISC ALGEBRA

By definition, the disc algebra $A(\mathbf{D})$ is the closure of the vector space $\mathcal{P}$ of all complex polynomials in $C(\mathbf{T})$ with $\mathbf{T} = \{t \in \mathbf{C} \mid |t| = 1\}$. Although $A(\mathbf{D})$ is naturally a uniform algebra, it makes sense to look for operator algebra structures on $A(\mathbf{D})$ different from the minimal one. This problem is particularly interesting in connection with the similarity problem of Halmos.

Let $T \in B(H)$. We recall that by definition, T is polynomially bounded provided that there is a constant $C > 0$ such that $\|P(T)\| \leqslant C\|P\|$ for any $P \in \mathcal{P}$. The Halmos problem is the question of whether every polynomially bounded T is similar to a contraction, i.e. there exists an invertible map $S \in B(H)$ such that $\|STS^{-1}\| \leqslant 1$. A striking Theorem of Paulsen ([25], [23]) asserts that this similarity problem is equivalent to the following:

(5.1) Is every bounded homomorphism from $A(\mathbf{D})$ into $B(H)$ necessarily completely bounded?

It has probably been observed by several people that (5.1) has the following equivalent formulation.

(5.2) Let B be an operator algebra which is bicontinuously homomorphic to $A(\mathbf{D})$. Is B necessarily completely isomorphic to $\operatorname{Min} B$?

The restatement (5.2) of the similarity problem leads to the study of operator algebra structures on $A(\mathbf{D})$. Note that thanks to Theorem 1.3, a counterexample to Halmos's problem would be obtained if we could:

(5.3) Find an o.s.s. on $A(\mathbf{D})$ non-completely isomorphic to the minimal one, for which $m : A(\mathbf{D}) \overset{h}{\otimes} A(\mathbf{D}) \to A(\mathbf{D})$ is c.b.

What can be showed at this time is that except $\operatorname{Min} A(\mathbb{D})$, all the interpolated o.s.s. between $\operatorname{Min} A(\mathbb{D})$ and $\operatorname{Max} A(\mathbb{D})$ are not operator algebra structures. These are obvious candidates to try, in connection with (5.3). For any $0 \leqslant \theta \leqslant 1$, we set:

$$A_\theta = [\operatorname{Min} A(\mathbb{D}), \operatorname{Max} A(\mathbb{D})]_\theta .$$

THEOREM 5.1. (i) *For any* $\theta \in]0,1]$, *the multiplication*

$$m : \operatorname{Max} A(\mathbb{D}) \overset{h}{\otimes} \operatorname{Max} A(\mathbb{D}) \to A_\theta$$

is not c.b.

(ii) *Let A be the space $A(\mathbb{D})$ endowed with an* o.s.s. *Assume that there exists* $\theta \in]0,1]$ *for which* $\operatorname{Id} : A \to A_\theta$ *is* c.b. *Then A is not an operator algebra.*

(iii) $\forall\, \theta \in]0,1]$, *$A_\theta$ is not an operator algebra.*

In order to establish Theorem 5.1, we will appeal to the following well-known lemma, which can be proved by a classical use of outer functions. We omit the proof.

LEMMA 5.2. *Let $n \geqslant 1$ and let $\varepsilon > 0$. Then there exist $f_1, \ldots, f_n$ in $A(\mathbb{D})$ such that:*

$$\forall\, 1 \leqslant j \leqslant n, \quad \|f_j\| = 1; \tag{5.4}$$

$$\forall\, t \in \mathbb{T}, \quad \sum_{j=1}^{n} |f_j(t)| \leqslant 1 + \varepsilon. \tag{5.5}$$

Proof of Theorem 5.1. Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hence we only have to prove (i). We set $A = A(\mathbb{D})$ and fix $0 < \varepsilon < \frac{1}{2}$. We assume that $m : \operatorname{Max} A \overset{h}{\otimes} \operatorname{Max} A \to A_\theta$ is c.b. and wish to show that $\theta = 0$, i.e. $A_\theta = \operatorname{Min} A$. Let

$$C = \|m : \operatorname{Max} A \overset{h}{\otimes} \operatorname{Max} A \to A_\theta\|_{\mathrm{cb}} .$$

The factorization theorem for c.b. multilinear maps ([11], [27]) yields:

(5.6) For any $N \geqslant 1$ and any bounded $u : A \to M_N$, there exists a Hilbert space H as well as bounded maps $\alpha : A \to B(\ell_2^N, H), \beta : A \to B(H, \ell_2^N)$ such that $\|\alpha\|\,\|\beta\| \leqslant C \|u : A_\theta \to M_N\|_{\mathrm{cb}}$ and such that $u(ba) = \beta(b)\alpha(a)$ for all $a, b \in A$.

We now give ourselves a linear map $T : \ell_\infty^n \to M_N$. Appealing to Lemma 5.2 we then define a linear map $V : \ell_\infty^n \to A$ by letting $V(e_j) = f_j^2$. We also let

$E = V(\ell_\infty^n) \subset A$. Let $\gamma = \sum_i \gamma_i e_i \in \ell_\infty^n$. Given $1 \leqslant j \leqslant n$, we know from (5.4) that there exists $t_0 \in \mathbf{T}$ for which $|f_j(t_0)| = 1$. Then we have:

$$|\gamma_j| = |\gamma_j f_j^2(t_0)| \leqslant \Big\|\sum_{i=1}^{n} \gamma_i f_i^2\Big\| + \Big|\sum_{\substack{1\leqslant i\leqslant n\\ i\neq j}} \gamma_i f_i^2(t_0)\Big|.$$

Since each f_i has norm one, we have $|f_i(t)|^2 \leqslant |f_i(t)|$ for all $t \in \mathbf{T}$. Therefore the inequality (5.5) implies:

$$\sum_{\substack{1\leqslant i\leqslant n\\ i\neq j}} |f_i^2(t_0)| \leqslant \varepsilon.$$

Hence $|\gamma_j| \leqslant \|\sum_i \gamma_i f_i^2\| + \varepsilon \sup|\gamma_i|$. Consequently, we have $(1-\varepsilon)\|\gamma\| \leqslant \|V(\gamma)\|$, which means that V is one-one, with $V^{-1} : E \to \ell_\infty^n$ satisfying $\|V^{-1}\| \leqslant 1 + 2\varepsilon$. By Hahn-Banach, V^{-1} has an extension $W : A \to \ell_\infty^n$ with same norm. Let us apply (5.6) to $u = TW$. Let $\ell_{\infty,\theta}^n = [\mathrm{Min}\,\ell_\infty^n, \mathrm{Max}\,\ell_\infty^n]_\theta$. Then by interpolation:

$$\|W : A_\theta \to \ell_{\infty,\theta}^n\|_{\mathrm{cb}} \leqslant \|W\| = \|V^{-1}\| \leqslant 1 + 2\varepsilon.$$

Hence we obtain α, β factorizing u as in (5.6) with:

$$\|\alpha\|\,\|\beta\| \leqslant C(1+2\varepsilon)\|T : \ell_{\infty,\theta}^n \to M_N\|_{\mathrm{cb}}. \tag{5.7}$$

In what follows, the notation $\|T\|_{\mathrm{cb}}$ stands for the c.b. norm of T acting on the classical $\mathrm{Min}\,\ell_\infty^n$. For any $1 \leqslant j \leqslant n$, $T(e_j) = u(f_j^2)$ whence:

$$T(e_j) = \beta(f_j)\alpha(f_j). \tag{5.8}$$

We claim that:

$$\|T\|_{\mathrm{cb}} \leqslant (1+\varepsilon)^2 \|\alpha\|\,\|\beta\|. \tag{5.9}$$

Combining (5.7) and (5.9), we thus obtain that

$$\|T\|_{\mathrm{cb}} \leqslant K\|T : \ell_{\infty,\theta}^n \to M_N\|_{\mathrm{cb}}$$

for some absolute constant K not depending on n, N. This readily implies that $\theta = 0$.

It thus remains to check (5.9). Let $\pi : \ell_\infty^n \to M_n$ be the canonical representation of the C^*-algebra ℓ_∞^n as the diagonal matrices. It follows from (5.8) that for all $\gamma \in \ell_\infty^n$:

$$T(\gamma) = (\beta(f_1), \ldots, \beta(f_n))(\pi(\gamma) \otimes \mathrm{Id}_H)\begin{pmatrix}\alpha(f_1)\\ \vdots \\ \alpha(f_n)\end{pmatrix}.$$

Therefore

$$\|T\|_{\mathrm{cb}} \leqslant \Big\|\sum_j \beta(f_j)\beta(f_j)^*\Big\|^{\frac{1}{2}} \Big\|\sum_j \alpha(f_j)^*\alpha(f_j)\Big\|^{\frac{1}{2}}.$$

Let $J : \ell_\infty^n \to A$ be defined by $J(e_j) = f_j$. For any $\gamma \in \ell_\infty^n$, $\|\sum_j \gamma_j f_j\| = \sup_{t\in\mathbf{T}} |\sum_j \gamma_j f_j(t)| \leqslant (1+\varepsilon)\|\gamma\|$ by (5.5). Hence $\|J\| \leqslant 1+\varepsilon$. Now by Lemma 3.2,

$$\Big\|\sum_j \alpha(f_j)^*\alpha(f_j)\Big\|^{\frac{1}{2}} \leqslant \|\alpha J\|$$

hence $\|\sum_j \alpha(f_j)^*\alpha(f_j)\|^{1/2} \leqslant (1+\varepsilon)\|\alpha\|$. Similarly, $\|\sum_j \beta(f_j)\beta(f_j)^*\|^{1/2} \leqslant (1+\varepsilon)\|\beta\|$, whence (5.9). ∎

Another curious statement, whose validity is equivalent to Halmos' problem by the new characterization of operator algebras, is the following:

(5.10) Suppose $T : A(\mathbf{D}) \to B(H)$ is a bounded *linear* map for which there is a constant K such that for all n and all matrices $[f_{ij}], [g_{ij}]$ in $M_n(A(\mathbf{D}))$ we have:

$$\Big\|\Big[T\Big(\sum_{k=1}^n f_{ik}g_{kj}\Big)\Big]\Big\| \leqslant K\|[T(f_{ij})]\|\,\|[T(g_{ij})]\|.$$

Then T is completely bounded.

This last condition is also equivalent to the same condition but now with $K = 1$. This seems to be an apparent weakening of Paulsen's condition.

6. OPERATOR ALGEBRA STRUCTURES ON THE TRACE CLASS OPERATORS

We end by examining the von Neumann-Schatten p-classes $S_p = S_p(H)$ defined on an infinite dimensional Hilbert space H for $1 \leqslant p \leqslant +\infty$. We shall mainly focus on the trace class Banach space S_1, the dual Banach space to $K(H)$. Each $S_p(H)$ has two natural Banach algebra multiplications, the usual product of operators, and the Schur product. We assume that $H = \ell_2$, the general case for all our theorems is identical. We believe that it was shown first in [36] that S_1 with Schur product is a Banach algebra (although this also follows from our considerations below). As was pointed out to us by P.G. Dixon, one reason why it is of interest to know whether S_1 with usual product is bicontinuously an operator algebra, is because it is easily shown to be a Banach algebra satisfying the non-unital von Neumann inequality. It was an open question for a long time whether the non-unital von Neumann inequality was sufficient to characterize operator algebras amongst all non-unital Banach algebras; a very nice counterexample was given in 1993 by Dixon ([38]). The unital version of this question is still open.

THEOREM 6.1. *Let S_1 be equipped with either the usual or the Schur product. Then S_1 is bicontinuously isomorphic to an operator algebra. In fact for all p such that $1 \leqslant p \leqslant 2$, $\operatorname{Max} S_p$ with Schur product is an operator algebra in the sense of Definition* 1.1.

Proof. To prove the assertion for the usual product we endow S_1 with an o.s.s. for which (1.13) holds. Namely, consider the operator space $\operatorname{Max}(\ell_2) \overset{h}{\otimes} \operatorname{Max}(\ell_2)$. This is isometrically isomorphic to S_1 as a Banach algebra, if we equip $\ell_2 \otimes \ell_2$ with the product which concatenates the middle terms:

$$(x \otimes y)(x' \otimes y') = (y \cdot x')x \otimes y'.$$

Here "$\cdot$" is the bilinear "dot-product" on ℓ_2. The fact that the isomorphism above is isometric follows quickly from the general fact that for Banach spaces E, F, the Haagerup norm on $\operatorname{Max}(E) \otimes \operatorname{Max}(F)$ equals the γ_2^* norm as Banach space norms ([4]), although a direct proof is easily given.

We now appeal to (1.13). We need to check that multiplication

$$(\operatorname{Max}(\ell_2) \overset{h}{\otimes} \operatorname{Max}(\ell_2)) \overset{h}{\otimes} (\operatorname{Max}(\ell_2) \overset{h}{\otimes} \operatorname{Max}(\ell_2)) \to \operatorname{Max}(\ell_2) \overset{h}{\otimes} \operatorname{Max}(\ell_2)$$

is completely contractive. However the domain of this map reordered by associativity is $\operatorname{Max}(\ell_2) \overset{h}{\otimes} (\operatorname{Max}(\ell_2) \overset{h}{\otimes} \operatorname{Max}(\ell_2)) \overset{h}{\otimes} \operatorname{Max}(\ell_2)$. The scalar valued concatenation of the two middle factors is just the trace on S_1, which is contractive, and hence automatically completely contractive since it is scalar valued.

For the assertion about the Schur product notice that as Banach algebras S_2 with Schur product is the same as $\ell_2(\mathbb{N} \times \mathbb{N})$ with pointwise product. Then (2.2) gives a complete contraction $(S_2)_r \overset{h}{\otimes} (S_2)_c \to \operatorname{Max}(\ell_1(\mathbb{N}^2))$. However the "identity map" from $\operatorname{Max}(\ell_1(\mathbb{N}^2)) \to S_1$ is contractive and therefore automatically completely contractive whatever o.s.s. we give to this last copy of S_1. Thus, the Schur product s satisfies

$$\|s : (S_2)_r \overset{h}{\otimes} (S_2)_c \to \operatorname{Max} S_1\|_{cb} \leqslant 1. \tag{6.1}$$

Hence it is completely contractive from $\operatorname{Max} S_2 \overset{h}{\otimes} \operatorname{Max} S_2$ into $\operatorname{Max} S_1$. For any p between 1 and 2, if we compose this last map with the identity maps $\operatorname{Max} S_p \to \operatorname{Max} S_2$ and $\operatorname{Max} S_1 \to \operatorname{Max} S_p$, we see from (1.13) that $\operatorname{Max} S_p$ is an operator algebra for the Schur product. ∎

By interpolation we obtain:

COROLLARY 6.2. *For $1 \leqslant p \leqslant \infty$, the von Neumann-Schatten p-class S_p is bicontinuously isomorphic to an operator algebra (with either the usual or the Schur product).*

As a corollary we deduce that all of these algebras are Arens regular. As far as we aware this and Corollary 6.2 were unknown for $1 \leqslant p < 2$.

For the remainder of this section we will write OS_p for the "usual" o.s.s. on S_p, namely the interpolated o.s.s. (introduced by Pisier in [29]) between $S_\infty = K(\ell_2)$ and S_1 (the latter viewed as the operator space dual of $K(\ell_2)$, and also as a subspace of $K(\ell_2) = C \overset{h}{\otimes} R$, but with the $R \overset{h}{\otimes} C$ matrix norms). We will write OS_1^t for the o.s.s. transpose of OS_1 (that is, the operator space with matrix norms $\|[x_{ij}]\|_{M_n(OS_1^t)} = \|[x_{ji}]\|_{M_n(OS_1)}$).

Since ℓ_1, with its o.s.s. as the operator dual of c_0, is completely boundedly isomorphic to an operator algebra, one might conjecture that OS_1 is also completely boundedly isomorphic to an operator algebra. Note that in this analogy it is not clear whether to take the usual or the Schur product on S_1.

THEOREM 6.3. *With notation as above, OS_1 and OS_1^t are not operator algebras (in the sense of Definition 1.1 with either the usual or the Schur product. Indeed with the usual product, OS_p is not an operator algebra for any $1 \leqslant p < \infty$.*

Proof. Again we appeal to (1.13). For the case of the usual product, and usual o.s.s, the linearization of multiplication is not even bounded on the Haagerup tensor product, as is easily seen by taking $a_i = E_{1i}, b_i = E_{i1}$ for $1 \leqslant i \leqslant n$. Then $\|\sum_i a_i b_i\|_{S_1} = \|\sum_i E_{11}\|_{S_1} = n$. On the other hand, as an element of $R_n(OS_1)$, we have $\|[a_1 \cdots a_n]\| = 1$, and as an element of $C_n(OS_1)$, we have $\|[b_1 \cdots b_n]^t\| = 1$. Thus $\|\sum_i a_i \otimes b_i\|_h \leqslant 1$.

The same argument shows that OS_p is not an operator algebra for any $1 \leqslant p < \infty$. To show the assertion for OS_1^t with usual product, we simply switch a_i and b_i in the above.

Now we consider a multiplication $m : OS_1 \otimes OS_1 \to OS_1$. Suppose this was completely bounded w.r.t the Haagerup tensor product (see (1.13)). By the elementary operator space identifications ([7], [17], [4]), we see that m lies in

$$\mathrm{CB}((R \overset{h}{\otimes} C) \overset{h}{\otimes} (R \overset{h}{\otimes} C), OS_1) = \mathrm{CB}(R \overset{h}{\otimes} (C \overset{h}{\otimes} R) \overset{h}{\otimes} C, OS_1).$$

However, denoting by $\widehat{\otimes}$ the projective operator space tensor product ([7], [16]),

$$R \overset{h}{\otimes} (C \overset{h}{\otimes} R) \overset{h}{\otimes} C = R \widehat{\otimes} K(\ell_2) \widehat{\otimes} C = K(\ell_2) \widehat{\otimes} OS_1.$$

Therefore m lies in

$$\begin{aligned}\mathrm{CB}(K(\ell_2)\widehat{\otimes} OS_1, OS_1) &= \mathrm{CB}(K(\ell_2), \mathrm{CB}(OS_1, OS_1))\\ &= \mathrm{CB}(K(\ell_2), \mathrm{CB}(K(\ell_2), B(\ell_2))).\end{aligned}$$

This last equality is because for any operator spaces E, F we have $\mathrm{CB}(E\widehat{\otimes} F, \mathbf{C}) = \mathrm{CB}(E, F^*) = \mathrm{CB}(F, E^*)$. If we take m to be the Schur product, and unravel these correspondences, we find that m corresponds to the map $S \mapsto (T \mapsto S^{\mathrm{t}} \circ T))$ which is supposed to be in $\mathrm{CB}(K(\ell_2), \mathrm{CB}(K(\ell_2), B(\ell_2)))$. Here "$\circ$" is the Schur product on $B(\ell_2)$ and "$^{\mathrm{t}}$" is transposition, with respect to a fixed orthonormal basis of ℓ_2. We are grateful to Roger Smith for an example which shows that for fixed n there is a unitary $n\times n$ matrix U_n, such that the map $S \mapsto S^{\mathrm{t}} \circ U_n$ has c.b. norm $\geqslant \sqrt{n}$. (To be more specific, if ω is a primitive n-th root of 1, let $U_n = \frac{1}{\sqrt{n}}[\omega^{ij}]$. Let $A = [\omega^{-ij} e_{ji}] \in M_n(M_n)$. Then A is a unitary, but the action described above acting on A yields norm $\sqrt{n}$). This yields the desired contradiction.

The same argument gives the result for OS_1^{t} with Schur product. ∎

The last few results bring up the question of determining, analoguously to our work in Sections 2 and 3 for ℓ_p, the possible operator algebra structures on S_p.

We saw that $\mathrm{Max}(\ell_2) \overset{h}{\otimes} \mathrm{Max}(\ell_2)$ is an operator algebra with the usual product. Another example of an o.s.s on S_1 making it an operator algebra is $(R\cap C) \overset{h}{\otimes} (R\cap C)$. The proof of this is just like the proof of the first part of Theorem 6.1. We observe that $(R\cap C) \overset{h}{\otimes} (R\cap C)$ is isometric to S_1, this is because of the following *contractive* factorization of the identity map:

$$S_1 \to \mathrm{Max}(\ell_2) \overset{h}{\otimes} \mathrm{Max}(\ell_2) \to (R\cap C) \overset{h}{\otimes} (R\cap C) \to R \overset{h}{\otimes} C \cong S_1.$$

The same argument shows that for *any* two o.s.s. E_1 and E_2 on ℓ_2 larger than $R\cap C$, we have that $S_1 \cong E_1 \overset{h}{\otimes} E_2$ isometrically, and that with this o.s.s. S_1 is an operator algebra for the usual product. In particular, various interpolation structures between $R\cap C$ and $\mathrm{Max}(\ell_2)$ yield operator algebra structures on S_1. It is also possible to show that all the o.s.s.'s on S_1 discussed in this paragraph are operator algebras for the Schur product. This is left to the reader.

On S_2 equipped with the usual product, the situation remains unclear. It is easy to see that $(S_2)_{\mathrm{r}} = R \overset{h}{\otimes} R$ and $(S_2)_{\mathrm{c}} = C \overset{h}{\otimes} C$ are not operator algebras, since the 'two middle term concatenation' is clearly not bounded. We saw earlier that OS_2 is not an operator algebra. Let $(S_2)_{\mathrm{r}\cap\mathrm{c}} = (S_2)_{\mathrm{r}} \cap (S_2)_{\mathrm{c}}$. We are also able to show that this space is not an operator algebra for the usual product. In fact the simplest operator algebra structure on S_2 we know is $(S_2)_{\mathrm{r}\cap\mathrm{c}} \cap K(\ell_2)$, as communicated to us by Haskell Rosenthal. To define this space, we consider S_2 as a subspace of $K(\ell_2)$ and endow S_2 with the largest matrix norm coming from $(S_2)_{\mathrm{r}\cap\mathrm{c}}$ and $K(\ell_2)$. We claim that $A = (S_2)_{\mathrm{r}\cap\mathrm{c}} \cap K(\ell_2)$ is an operator algebra structure on S_2. Indeed it satisfies (1.13). This is because $K(\ell_2)$ is an operator algebra, and because of the following completely contractive factorization (and its companion column version):

$$A \overset{h}{\otimes} A \to (S_2)_{\mathrm{r}} \overset{h}{\otimes} K(H) \cong (R \overset{h}{\otimes} R) \overset{h}{\otimes} (C \overset{h}{\otimes} R)$$
$$\cong R \overset{h}{\otimes} (R \overset{h}{\otimes} C) \overset{h}{\otimes} R \to R \overset{h}{\otimes} R \cong (S_2)_{\mathrm{r}}.$$

After hearing our announcement of Theorem 6.1, Haskell Rosenthal has found and shown us another o.s.s. on S_1 which makes it an operator algebra for the usual product. This structure is $\mathrm{Min}\,(S_1) \cap (S_2)_{\mathrm{r}\cap\mathrm{c}} \cap K(\ell_2)$. It should be emphasized that as a Banach algebra this is just usual S_1, it is the higher level matrix norms which are different. We now give a proof using standard operator space identifications ([7], [17], [4]) of Rosenthal's result. The usual multiplication gives a completely contractive map

$$\|m : (S_2)_{\mathrm{r}} \overset{h}{\otimes} (S_2)_{\mathrm{c}} \to OS_1\|_{\mathrm{cb}} \leqslant 1. \tag{6.2}$$

This is because we can factor the above map

$$(S_2)_{\mathrm{r}} \overset{h}{\otimes} (S_2)_{\mathrm{c}} \cong (R \overset{h}{\otimes} R) \overset{h}{\otimes} (C \overset{h}{\otimes} C) \cong R \overset{h}{\otimes} (R \overset{h}{\otimes} C) \overset{h}{\otimes} C \to R \overset{h}{\otimes} C \cong OS_1$$

where the "$\to$" is given by the scalar valued (completely contractive) concatenation of the two middle terms (see the proof of Theorem 6.1). We saw that $A = (S_2)_{\mathrm{r}\cap\mathrm{c}} \cap K(\ell_2)$ satisfies (1.13). Combining with (6.2) we see that $OS_1 \cap (S_2)_{\mathrm{r}\cap\mathrm{c}} \cap K(\ell_2)$ satisfies (1.13) and so is an operator algebra structure on S_1.

It is also clear from the above that for any operator space structure on S_1 smaller than OS_1, $S_1 \cap (S_2)_{\mathrm{r}\cap\mathrm{c}} \cap K(\ell_2)$ is an operator algebra structure on S_1 for the usual product.

We now turn to the Schur product. By our analysis in Section 2, $(S_2)_{\mathrm{r}}$, $(S_2)_{\mathrm{c}}$, $(S_2)_{\mathrm{r}\cap\mathrm{c}}$ and OS_2 are all operator algebras with the Schur product, since as Banach algebras S_2 with Schur product is just $\ell_2(\mathbf{N} \times \mathbf{N})$ with pointwise product. Moreover:

COROLLARY 6.4. *For all $2 \leqslant p \leqslant \infty$, and with the Schur product, OS_p is an operator algebra. For $1 \leqslant p \leqslant 2$, and for any operator space structure on S_p, $S_p \cap (S_2)_{\mathrm{r}\cap\mathrm{c}}$ is an operator algebra structure on S_p, for the Schur product.*

Proof. In [3] we showed that $K(\ell_2)$ is an operator algebra with Schur product. As remarked above we have that OS_2 with Schur product is an operator algebra. The first result follows by interpolation. The proof of the second assertion follows from statement (6.1) using the trick we used above to show Rosenthal's result. ∎

There is another device whereby to manufacture operator algebra structures on various operator ideals with the Schur product, namely by taking the spatial tensor product of the operator algebras found in Sections 2 and 3. For instance, $\mathrm{Max}(\ell_2)$ was found to be an operator algebra, therefore $\mathrm{Max}(\ell_2) \check{\otimes} \mathrm{Max}(\ell_2)$ is also one. However, results of Paulsen ([39]), Junge and Pisier ([20]) show that $\mathrm{Max}(\ell_2) \check{\otimes} \mathrm{Max}(\ell_2)$ is bicontinuously isomorphic to S_1, which gives another operator algebra structure on S_1 with Schur product.

QUESTIONS 6.5. Some open questions that arose in our study of S_p:

(1) Is $\mathrm{Max}(S_2)$ an operator algebra for the usual product?

(2) Let $1 < p < 2$. Is OS_p an operator algebra for the Schur product?

(3) Could S_p with Schur product be a Q-algebra for any p not equal to 2? Indeed notice that by Corollary 6.4, $\mathrm{Min}(S_1) \cap (S_2)_{\mathrm{r}\cap\mathrm{c}}$ is a Q-space structure on S_1 which is an operator algebra for the Schur product. Is it a matricial Q-algebra?

QUESTIONS 6.6. General open questions:

(1) Is $B(H)$ or ℓ_p $(2 < p < \infty)$, with an arbitrary Banach algebra multiplication, bicontinuously isomorphic to an operator algebra? (This is an old question of Varopoulos.)

(2) If A is a commutative Banach algebra such that $\mathrm{Min}\,A$ is an operator algebra, then is A a Q-algebra? Is $\mathrm{Min}\,A$ a matricial Q-algebra? If $\mathrm{Min}\,A$ is an operator algebra and a Q-algebra then is $\mathrm{Min}\,A$ a matricial Q-algebra? If $\mathrm{Min}\,A$ is a matricial Q-algebra, then is it an injective algebra in the sense of Varopoulos (see Remark 4.9)?

(3) If A is a unital Banach algebra satisfying von Neumann's inequality, then is A bicontinuously isomorphic to an operator algebra? (See remarks at beginning of Section 6.)

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