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QUASIDIAGONALITY AND A GENERALIZED VERSION OF BERG'S TECHNIQUE

LAURENT W. MARCOUX

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ABSTRACT. We illustrate new and elementary proofs of two results concerning quasidiagonality of tensor products with normal operators and quasidiagonal dilations. The proofs here are based upon a generalization due to I.D. Berg and K.R. Davidson of Berg's Technique for weighted shifts.

KEYWORDS: *Berg's technique, block diagonal nilpotents, dilations, quasidiagonality, tensor product.*

AMS SUBJECT CLASSIFICATION: Primary 47A66, 47A80; Secondary 47A58, 47A20.

1. INTRODUCTION

Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded, linear operators acting on a complex, infinite dimensional, separable Hilbert space. By $\mathcal{K}(\mathcal{H})$ we denote the set of compact operators. An element T of $\mathcal{B}(\mathcal{H})$ is said to be *block-diagonal* (resp. *quasidiagonal*) and we write $T \in$ (BD) (resp. $T \in$ (QD)) if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank orthogonal projections tending strongly to the identity operator I such that $P_nT = TP_n$ for all $n \geqslant 1$ (resp. $\lim\limits_{n\to\infty} \|P_nT - TP_n\| = 0$). It is well-known ([8]) that (QD) $= \overline{\text{(BD)}}$, and that $T \in$ (QD) if and only if $T = B + K$ for some $B \in$ (BD) and $K \in \mathcal{K}(\mathcal{H})$. In fact, given $\varepsilon > 0$, B and K may be chosen in such a way that $\|K\| < \varepsilon$.

In [1], W. Arveson demonstrates how the existence of quasicentral approximate units implies that every $T \in \mathcal{B}(\mathcal{H})$ is a direct summand of a quasidiagonal operator. That is, there exists $S \in$ (QD) and $B \in \mathcal{B}(\mathcal{H})$ such that $S = T \oplus B$.

In [9], D.A. Herrero gives a separate proof of this fact, and demonstrates the existence of a universal quasidiagonal operator Q with the property that if $T \in \mathcal{B}(\mathcal{H})$ and $\|T + \mathcal{K}(\mathcal{H})\| \leqslant 1$ in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, there exists a compact operator K_T such that $Q + K_T$ is unitarily equivalent to $Q \oplus T$. Herrero's argument is based upon ([7], Theorem 5.1). It is also deducible from Hadwin's characterization of closures of unitary orbits ([6]), based upon D. Voiculescu's non-commutative Weyl-von Neumann Theorem ([14]).

One of the objects of this paper is to exhibit yet another proof of this result. The present proof is motivated by the fact that it is single-operator theoretic (without recourse to C^*-algebra techniques nor Voiculescu's Theorem) and relatively elementary. In essence, it relies only upon Lemma 3.2 of [4]. That lemma is an extension to general operators of a technique developed in the study of weighted shifts by I.D. Berg ([2]). Berg's original technique was slightly modified by D.A. Herrero [10] (cf. also [5]) and has proven to be of immense value in the study of weighted shifts [10], [12], [13].

Our hope is to show that this technique as generalized in [4] may be used to great advantage in more general approximation problems in Hilbert space. Indeed, before proving the above mentioned result, we shall use this technique to provide an elementary proof of a result easily derived from other works of D. Voiculescu.

In [15], Voiculescu shows that if a quasidiagonal C^*-algebra $\mathfrak{B}$ *homotopically dominates* a C^*-algebra $\mathfrak{A}$ (i.e. there are $*$-homomorphisms $f : \mathfrak{A} \to \mathfrak{B}$ and $g : \mathfrak{B} \to \mathfrak{A}$ such that $g \circ f$ is homotopic to $\mathrm{id}_{\mathfrak{A}}$), then $\mathfrak{A}$ is also quasidiagonal. In [16], he uses this theorem to show that if D is a diagonal operator with eigenvalues $\mathbb{Q} \cap [0, 1]$, and if $T \in \mathcal{B}(\mathcal{H})$ is arbitrary then $D \otimes T$ is quasidiagonal. As he points out, this follows easily from the fact that $C^*(D \otimes T)$ is isomorphic to $\mathcal{C}_0(0, 1] \otimes C^*(T)$, which is contractible. The same argument implies that $M \otimes T \in (\mathrm{QD})$ whenever M is a normal operator with simply connected spectrum containing $\{0\}$. If N is any normal operator such that $0 \in \sigma(N)$ and $\sigma(N)$ is connected, then N can be approximated in norm by operators M as above, since the spectrum of N can be approximated in the Hausdorff metric using simply connected sets containing $\{0\}$. Since $\|N \otimes T - M \otimes T\| = \|N - M\| \, \|T\|$ and (QD) is closed, it follows that $N \otimes T \in (\mathrm{QD})$ as well.

Thus our second objective is to provide an elementary proof of this fact, again, based only upon Lemma 3.2 of [4]. We also include a couple of new results regarding approximation by block-diagonal nilpotents.

2. QUASIDIAGONALITY OF THE TENSOR PRODUCT

Our main tool is Lemma 2.3 below, and deals with block-tridiagonal operators.

DEFINITION 2.1. $T \in \mathcal{B}(\mathcal{H})$ is *tridiagonal* with respect to a decomposition $\mathcal{H} = \bigoplus_{n\in I} \mathcal{H}_n$ if its corresponding operator matrix (T_{ij}) where $T_{ij} = P(\mathcal{H}_j)T|\mathcal{H}_i$ satisfies $T_{ij} = 0$ when $|i-j| > 1$.

REMARK 2.2. Suppose $\mathcal{H}$ is separable with orthonormal basis $\{e_n\}_{n\geqslant 1}$. Let $T \in \mathcal{B}(\mathcal{H})$, and let $\mathcal{H}_0$ be an arbitrary finite dimensional subspace of $\mathcal{H}$. Define, for $n \geqslant 0$,

$$\mathcal{H}_{n+1} = \operatorname{span}\Big\{\bigoplus_{i=0}^{n} \mathcal{H}_i,\ T\mathcal{H}_n,\ T^*\mathcal{H}_n,\ e_{n+1}\Big\} \ominus \bigoplus_{i=0}^{n} \mathcal{H}_i.$$

It is easily verified that T is tridiagonal with respect to $\bigoplus_{n=0}^{\infty} \mathcal{H}_n$.

LEMMA 2.3. (Lemma 3.2 of [4]) *Let $A = (A_{ij})$ and $B = (B_{ij})$ be tridiagonal with respect to $\bigoplus_{i=0}^{N} \mathcal{H}_i$ and suppose that $A_{ij} = B_{ij}$ except for $i = j = 0$ and $i = j = N$. Let C and D be tridiagonal with $C_{ij} = D_{ij} = A_{ij}$ for $1 \leqslant i+j \leqslant 2N-1$, $C_{00} = A_{00}$, $D_{00} = B_{00}$, $C_{NN} = B_{NN}$, and $D_{NN} = A_{NN}$. Then there is a unitary operator U such that*

$$\|A \oplus B - U^*(C \oplus D)U\| < \frac{\pi}{N}\|A\|.$$

In matrix notation, we have

$$A = \begin{bmatrix} A_{00} & A_{01} & & & \\ A_{10} & A_{11} & A_{12} & & \\ \ddots & \ddots & \ddots & & \\ & & & & A_{N-1,N} \\ & & & A_{N,N-1} & A_{N,N} \end{bmatrix}$$

$$B = \begin{bmatrix} B_{00} & A_{01} & & & \\ A_{10} & A_{11} & A_{12} & & \\ \ddots & \ddots & \ddots & & \\ & & & & A_{N-1,N} \\ & & & A_{N,N-1} & B_{N,N} \end{bmatrix}$$

$$C = \begin{bmatrix} A_{00} & A_{01} & & & \\ A_{10} & A_{11} & A_{12} & & \\ \ddots & \ddots & \ddots & & \\ & & & & A_{N-1,N} \\ & & & A_{N,N-1} & B_{N,N} \end{bmatrix}$$

$$D = \begin{bmatrix} B_{00} & A_{01} & & & & \\ A_{10} & A_{11} & A_{12} & & & \\ \ddots & \ddots & \ddots & & & \\ & & & & & A_{N-1,N} \\ & & & & A_{N,N-1} & A_{N,N} \end{bmatrix}.$$

The present use of the Berg-Davidson Technique applied to tridiagonal operators mimicks the use of Berg's Technique for weighted shift operators. The reader is referred to [2], [3], [9], [12], [13] for the description of as well as examples of the use of Berg's Technique.

In particular, it may be useful to view this technique as a "splicing" of two sequences. Berg ([2]) provides the following diagram to illustrate the effect. The longer the sequence over which we slice, the "less abrupt" the slice is, which corresponds to a perturbation of small norm.

REMARK 2.4. If we think of A as acting upon $\bigoplus_{i=0}^{N} \mathcal{H}_i(A)$ and B as acting upon $\bigoplus_{i=0}^{N} \mathcal{H}_i(B)$, then it is implicit in the construction of U that $U|\mathcal{H}_0(A) = I$. Moreover, when $\mathcal{H}_i$ is finite dimensional for $1 \leqslant i \leqslant N-1$, the perturbation involved in Lemma 2.3 is induced by a finite rank operator.

LEMMA 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose $\varepsilon > 0$ and $\{d_n\}_{n=1}^{r} \subseteq \mathbb{C}$ satisfies:*

(i) $d_r = 0$, $|d_n| \leqslant 1$, $1 \leqslant n \leqslant r$;

(ii) $|d_n - d_{n+1}| < \frac{\varepsilon}{4}$, $1 \leqslant n \leqslant r-1$.

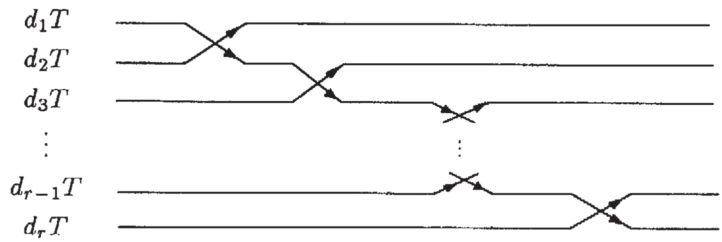
Suppose $T = (T_{ij})$ is tridiagonal with respect to the decomposition $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$. Then for all $M > 0$ there exists a finite rank orthogonal projection P satisfying:

(a) $P \geqslant P_M$, *the orthogonal projection onto* $\bigoplus_{i=1}^{M} \mathcal{H}_i$; *and*

(b) $\left\| \left[P, \left(\bigoplus_{n=1}^{r} d_n T \right) \right] \right\| < \varepsilon \|T\|$.

Proof. Let $T(n) = d_n T$, $1 \leqslant n \leqslant r$. $T(n)$ is assumed to act upon the Hilbert space $\mathcal{H}(n) \cong \mathcal{H}$, and $T(n) = (T_{ij}(n))$ (with $T_{ij}(n) = d_n T_{ij}$ for all i, j, n) is tridiagonal with respect to the decomposition $\mathcal{H}(n) = \bigoplus_{i=1}^{\infty} \mathcal{H}_i(n)$. Of course, $\mathcal{H}_i(n) \cong \mathcal{H}_i$, $1 \leqslant n \leqslant r$, $i \geqslant 1$.

Using Berg's diagram, we illustrate the principle behind our proof.



The idea is to successively perform "splices" that will take us from d_iT to $d_{i+1}T$, $1 \leqslant i \leqslant r-1$. Here we use the fact that $|d_i - d_{i+1}|$ is small, combined with the fact that we perform the "splice" over long sequences to control the norm of the individual perturbations. Then, as is the case with Berg's Technique applied to weighted shifts, the fact that the individual perturbations occur on orthogonal subspaces implies that the norms do not add up.

Choose N large enough so that $N > M$ and $\frac{2\pi}{N} < \frac{\varepsilon}{2}$. Consider the operators $X(n)$, $1 \leqslant n \leqslant r-1$, where

$$X_{ij}(n) = \begin{cases} T_{ij}(n+1) & \text{if } \{i,j\} \cap \{2nN+1, \ldots, 2nN+(N+1)\} \neq \emptyset \\ T_{ij}(n) & \text{otherwise.} \end{cases}$$

Since

$$\|T(n) - T(n+1)\| = \|d_nT - d_{n+1}T\| = |d_n - d_{n+1}|\,\|T\| < \frac{\varepsilon}{4}\|T\|,$$

we conclude that $\|T(n) - X(n)\| < \frac{\varepsilon}{4}\|T\|$. We define $X(r) = T(r) = 0$.

We are now in a position to apply Lemma 2.3 simultaneously to the pairs $X(n)$ and $X(n+1)$, $1 \leqslant n \leqslant r-1$. For each such pair we have

$$X_{ij}(n) = X_{ij}(n+1) \quad \text{if } \{i,j\} \cap \{2nN+1, 2nN+2, \ldots, 2nN+(N+1)\} \neq \emptyset.$$

Moreover, since the underlying spaces upon which we shall apply the perturbations arising from Lemma 2.3 are mutually orthogonal, the norms of the perturbations do not add up. As such, we can find a unitary operator U and operators C and D such that

$$\left\| \bigoplus_{n=1}^{r} X(n) - U^*(C \oplus D)U \right\| < \frac{\pi}{N} \left\| \bigoplus_{n=1}^{r} X(n) \right\| < \frac{\pi}{N}(\|T\| + \varepsilon).$$

Now $C = (C_{ij})$ is tridiagonal with respect to $\bigoplus_{i=1}^{\infty} J_i$ where $J_i \cong \mathcal{H}_i$, $i \geqslant 1$. Furthermore,

$$C_{ij} = \begin{cases} X_{ij}(1) & \text{if } \{i,j\} \cap \{1,2,\ldots,2N\} \neq \emptyset \\ X_{ij}(n) & \text{if } \{i,j\} \cap \{2nN+1,\ldots,2(n+1)N\} \neq \emptyset, \\ & \quad 1 \leqslant n \leqslant r-1 \\ 0 & \text{if } i > 2rN \text{ or } j > 2rN. \end{cases}$$

D acts upon the space $\mathcal{M} = \left(\bigoplus_{n=1}^{r} \mathcal{H}(n)\right) \ominus \left(\bigoplus_{i=1}^{\infty} J_i\right)$, but the structure of D will be irrelevant here. Let Q be the orthogonal projection of $\left(\bigoplus_{n=1}^{r} \mathcal{H}(n)\right)$ onto $\bigoplus_{n=1}^{2rN} J_n$. Then Q commutes with $(C \oplus D)$. Let $P = UQU^*$. Since

$$U \mid \bigoplus_{n=1}^{N} \mathcal{H}_n(1) = I,$$

we have $J_n = \mathcal{H}_n(1)$, $1 \leqslant n \leqslant N$. Thus $P \geqslant P_M$. Finally,

$$\begin{aligned} \left\| \left[P, \left(\bigoplus_{n=1}^{r} d_n T\right)\right] \right\| &\leqslant 2\|P\| \left\| \bigoplus_{n=1}^{r} X(n) - T(n) \right\| + \left\| [P, \bigoplus_{n=1}^{r} X(n)] \right\| \\ &\leqslant 2(1) \left(\frac{\varepsilon}{4}\|T\|\right) + 2\|P\| \left\| \bigoplus_{n=1}^{r} X(n) - U^*(C \oplus D)U \right\| \\ &\qquad + \|[P, U^*(C \oplus D)U]\| \\ &< \frac{\varepsilon}{2}\|T\| + \frac{2\pi}{N}\|T\| + \|U^*[Q, (C \oplus D)]U\| \\ &< \frac{\varepsilon}{2}\|T\| + \frac{\varepsilon}{2}\|T\| + 0 \\ &= \varepsilon\|T\|. \end{aligned}$$

This concludes the proof. ∎

THEOREM 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $N \in \mathcal{B}(\mathcal{H})$ is a non-invertible normal operator with connected spectrum. Then $N \otimes T \in$ (QD).*

Proof. The case where $T = 0$ is trivial. Otherwise, by scaling if necessary, $\|N\| \leqslant \|T\| = 1$. By the Weyl–von Neumann–Berg/Sikonia Theorem, given $\varepsilon > 0$ we can find V a unitary operator and D a diagonal operator with $\sigma(D) = \sigma(N)$, and 0 an eigenvalue of infinite multiplicity of D such that $\|N - V^*DV\| < \varepsilon$. Clearly $V^*DV \otimes T = (V^* \otimes I)(D \otimes T)(V \otimes I)$ is quasidiagonal if and only if $D \otimes T \in$ (QD). Since (QD) is closed, it suffices to prove the theorem in the case where $N = D$ is a diagonal as above.

If $D = \operatorname{diag}\{d_n\}_{n=1}^{\infty}$, then $D \otimes T = \bigoplus_{n=1}^{\infty} d_n T$. To see that $D \otimes T \in$ (QD), we need only find a sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank orthogonal projections increasing strongly to the identity such that $\lim_{n\to\infty} \|[P_n, D \otimes T]\| = 0$.

As in the previous lemma, we let $T(n) = d_n T$, and suppose that $T(n)$ acts on $\mathcal{H}(n)$ and is tridiagonal with respect to $\bigoplus_{i=1}^{\infty} \mathcal{H}_i(n)$. Either $\sigma(D) = \{0\}$, in which case $D \otimes T = 0 \in$ (QD), or $\sigma(D)$ is infinite (being connected and containing $\{0\}$). In the latter case we use the following argument.

Let $\varepsilon > 0$ and $M > 0$ be an integer. Set R_M to be the orthogonal projection of $\bigoplus_{n=1}^{\infty} \mathcal{H}(n)$ onto $\bigoplus_{n=1}^{M} \bigoplus_{i=1}^{M} \mathcal{H}_i(n)$. We shall produce a finite rank projection P such that $P \geqslant R_M$ and $\|[P, D \otimes T]\| < \varepsilon$. Once this is done, set $\varepsilon_M = \dfrac{1}{M}$ and let M tend to infinity to produce the desired sequence.

To this end, we observe that with $r_0 = 0$, for each $1 \leqslant k \leqslant M$ we can choose $r_k > 0$ and a finite subsequence $\{d_{n_j}\}_{j=r_{k-1}+1}^{r_k}$ satisfying

(i) $n_{r_{k-1}+1} = k$, $d_{n_{r_k}} = 0$;

(ii) $n_j > \max\{M, \{n_i\}_{i=1}^{r_{k-1}}\}$, $r_{k-1} + 2 \leqslant j \leqslant r_k$, $1 < k \leqslant M$, $n_j > M$ if $k = 1$, $2 \leqslant j \leqslant r_1$;

(iii) $|d_{n_j} - d_{n_{j+1}}| < \dfrac{\varepsilon}{4}$, $r_{k-1} + 1 \leqslant j \leqslant r_k$.

Then for each $1 \leqslant k \leqslant M$, we can now apply Lemma 2.3 with the sequence $\{d_{n_j}\}_{j=r_{k-1}+1}^{r_k}$ to obtain a finite rank projection P_k with support contained in $\bigoplus_{j=r_{k-1}+1}^{r_k} \mathcal{H}(n_j)$ satisfying

(a) $P_k \geqslant Q_M(k)$, the orthogonal projection onto $\bigoplus_{i=1}^{M} \mathcal{H}_i(k)$;

(b) $\left\| \left[P_k, \bigoplus_{j=r_{k-1}+1}^{r_k} d_{n_j} T\right] \right\| < \varepsilon \|T\| < \varepsilon$.

Since the n_j's were chosen such that $\{n_j\}_{j=r_{k-1}+1}^{r_k}$ is disjoint from $\{n_j\}_{j=r_{\ell-1}+1}^{r_\ell}$ if $k \neq \ell$, the underlying spaces indexed by the n_j's and the k's are orthogonal. Letting $P = \bigoplus_{k=1}^{M} P_k$, we find $P \geqslant R_M = \bigoplus_{k=1}^{M} Q_M(k)$. Furthermore

$$\|[P, D \otimes T]\| = \left\| \left[P, \bigoplus_{n=1}^{\infty} d_n T\right] \right\| = \max_{1 \leqslant k \leqslant M} \left\| \left[P_k, \bigoplus_{j=r_{k-1}+1}^{r_k} d_{n_j} T\right] \right\| < \varepsilon. \quad \blacksquare$$

REMARK 2.7. The above proof is based upon Lemma 2.5, which in turn relies only upon Lemma 2.3. As such, we need not consider only multiples of a fixed operator T, so long as the operators $T(n)$ are in some generalized sense "block-balanced" — i.e. a finite subsequence of the tridiagonal operator weights "almost pairs up" as required for Lemma 2.3, and one of the $T(n)$'s is 0.

Using this, one can easily construct a wide variety of sets of operators, none of whose elements are quasidiagonal, although their direct sum is. At this point, however, it seems rather doubtful that one could hope to describe when two operators are "block-balanced" in this generalized sense.

DEFINITION 2.8. We define the set of *block-diagonal nilpotent operators* as

$$(\mathrm{BDN}) = \{R \in (\mathrm{BD}) : R^k = 0 \text{ for some } k \geqslant 1\}.$$

REMARK 2.9. It is an interesting open problem to characterize $\overline{(\mathrm{BDN})}$. What we shall use below is a result of D.A. Herrero ([9], Theorem 5.4) that says that a normal operator N is in $\overline{(\mathrm{BDN})}$ if and only if $\sigma(N)$ is connected and contains the origin.

PROPOSITION 2.10. *Let $T \in (\mathrm{QD})$ and $R \in \overline{(\mathrm{BDN})}$. Then $R \otimes T \in \overline{(\mathrm{BDN})}$.*

Proof. First note that is suffices to prove the proposition in the case $T \in (\mathrm{BD})$. Indeed, suppose $R \otimes X \in \overline{(\mathrm{BDN})}$ for all $X \in (\mathrm{BD})$. Then we may choose a sequence $\{T_n\}_{n=1}^{\infty} \subseteq (\mathrm{BD})$ with $T = \lim_{n\to\infty} T_n$. Then $R \otimes T_n \in \overline{(\mathrm{BDN})}$ by hypothesis, and $R \otimes T = \lim_{n\to\infty} R \otimes T_n$, implying $R \otimes T \in \overline{(\mathrm{BDN})}$.

Thus we assume $T \in (\mathrm{BD})$, say $T = \bigoplus_{n=1}^{\infty} T(n)$, where each $T(n)$ is a finite dimensional matrix. Since $R \in \overline{(\mathrm{BDN})}$, for all $\varepsilon > 0$ there exists $R_\varepsilon = \bigoplus_{n=1}^{\infty} R_\varepsilon(n) \in (\mathrm{BDN})$ (again, we assume each $R_\varepsilon(n)$ is a matrix) such that $\|R - R_\varepsilon\| < \varepsilon$. Let $p(\varepsilon)$ be the order of nilpotence of R_ε. Now

$$\begin{aligned} R_\varepsilon \otimes T &= \Big(\bigoplus_{n=1}^{\infty} R_\varepsilon(n)\Big) \otimes \Big(\bigoplus_{k=1}^{\infty} T(k)\Big) \\ &\cong \bigoplus_{n=1}^{\infty} \bigoplus_{k=1}^{\infty} \Big(R_\varepsilon(n) \otimes T(k)\Big). \end{aligned}$$

But $R_\varepsilon(n)$, $T(k)$ are matrices, implying $R_\varepsilon(n) \otimes T(k)$ is also a matrix, and so $R_\varepsilon \otimes T \in (\mathrm{BD})$. Furthermore, $(R_\varepsilon \otimes T)^{p(\varepsilon)} = R_\varepsilon^{p(\varepsilon)} \otimes T^{p(\varepsilon)} = 0 \otimes T^{p(\varepsilon)} = 0$. Thus $R_\varepsilon \otimes T \in (\mathrm{BDN})$. Since $\|R_\varepsilon \otimes T - R \otimes T\| = \|R_\varepsilon - R\|\,\|T\| < \varepsilon\|T\|$, letting ε tend to zero yields $R \otimes T \in \overline{(\mathrm{BDN})}$. ∎

COROLLARY 2.11. *Let $T \in \mathcal{B}(\mathcal{H})$ and N be a normal operator such that $\sigma(N) = \overline{\mathbf{D}} = \{z \in \mathbf{C} : |z| \leqslant 1\}$. Then $N \otimes T \in \overline{(\mathrm{BDN})}$.*

Proof. First observe that N is approximately unitarily equivalent to $N \otimes N$, since both are normal operators with spectrum equal to $\overline{\mathbf{D}}$ (Weyl–von Neumann–Berg/Sikonia Theorem). Thus $N \otimes T$ is approximately unitarily equivalent to $(N \otimes N) \otimes T \cong N \otimes (N \otimes T)$. By Theorem 2.6, $N \otimes T \in (\mathrm{QD})$, and as pointed out in Remark 2.9, $N \in \overline{(\mathrm{BDN})}$. It follows from Proposition 2.10 that $N \otimes (N \otimes T) \in \overline{(\mathrm{BDN})}$, and hence $N \otimes T \in \overline{(\mathrm{BDN})}$. ∎

REMARK 2.12. In [9], Herrero defines the following "universal quasidiagonal" operator Q of norm 1 and proves:

PROPOSITION. ([9], Corollary 4.2) *Let* $Q = \bigoplus_{n,k=1}^{\infty} D_{n,k}$, *where* $\{D_{n,k}\}_{k=1}^{\infty}$ *is a dense subset of the unit ball of* $\mathcal{B}(\mathbb{C}^n)$, $n \geqslant 1$. *If* $T \in \mathcal{B}(\mathcal{H})$ *and* $\|T\| \leqslant 1$, *then* Q *is approximately unitarily equivalent to* $Q \oplus T$.

(NOTE. while this is not the precise statement of that result, it follows easily from the result and the techniques of that paper. The next section contains a different proof of this fact.)

COROLLARY 2.13. *Let* Q *be the operator defined above. Then* $Q \in \overline{(\mathrm{BDN})}$.

Proof. It is relatively easy to see that Q is approximately unitarily equivalent to $N \otimes Q$, where N is a normal operator with $\sigma(N) = \overline{\mathbb{D}}$. (The skeptical reader may apply the result of Hadwin ([6])). The corollary now follows immediately from Corollary 2.11. ∎

3. QUASIDIAGONAL DILATIONS

We now prove the result promised in Section 1, namely,

THEOREM 3.1. *Let* $\mathcal{H}$ *be a separable Hilbert space and* $T \in \mathcal{B}(\mathcal{H})$. *Then there exists* $S \in (\mathrm{QD})$ *and* $D \in \mathcal{B}(\mathcal{H})$ *such that* $S \cong T \oplus D$.

Proof. The proof is very similar to that of Lemma 2.5. Indeed, suppose $T = (T_{ij})$ is tridiagonal as before.

Consider the operators $X(n)$, $n \geqslant 1$, where

$$X_{ij}(n) = \begin{cases} T_{ij} & \text{if } i + j \leqslant 2^{n+1} + 2 \\ 0 & \text{otherwise.} \end{cases}$$

In matrix form, we have

$$X_{ij}(n) = \begin{bmatrix} T_{00} & T_{01} & & & & \\ T_{10} & T_{11} & T_{12} & & & \\ & T_{21} & T_{22} & T_{23} & & \\ & \ddots & \ddots & \ddots & & \\ & & & & T_{2^n,2^n} & T_{2^n,2^n+1} \\ & & & & T_{2^n+1,2^n} & T_{2^n+1,2^n+1} \end{bmatrix} \oplus 0,$$

which we rewrite as

(3.1)

$$X_{ij}(n) = \begin{bmatrix} A_{00}(n) & T_{2^{n-1},\,2^{n-1}+1} & & & \\ T_{2^{n-1}+1,\,2^{n-1}} & T_{2^{n-1}+1,\,2^{n-1}+1} & & & \\ \ddots & \ddots & \ddots & & \\ & & & T_{2^n-1,\,2^n} & \\ & & T_{2^n,\,2^n-1} & T_{2^n,\,2^n} & T_{2^n,\,2^n+1} \\ & & & T_{2^n+1,\,2^n} & A_{2^n+1,\,2^n+1}(n) \end{bmatrix}$$

where

$$A_{00}(n) = \begin{bmatrix} T_{00} & T_{01} & & \\ T_{10} & T_{11} & T_{12} & \\ \ddots & \ddots & \ddots & \\ & & & T_{2^{n-1}-1,\,2^{n-1}} \\ & & T_{2^{n-1},\,2^{n-1}-1} & T_{2^{n-1},\,2^{n-1}} \end{bmatrix}$$

and

$$A_{2^n+1,\,2^n+1}(n) = \begin{bmatrix} T_{2^n+1,\,2^n+1} & 0 \\ 0 & 0 \end{bmatrix}.$$

We may also write

(3.2)

$$X_{ij}(n+1) = \begin{bmatrix} B_{00}(n) & T_{2^{n-1},\,2^{n-1}+1} & & & \\ T_{2^{n-1}+1,\,2^{n-1}} & T_{2^{n-1}+1,\,2^{n-1}+1} & & & \\ \ddots & \ddots & \ddots & & \\ & & & T_{2^n-1,\,2^n} & \\ & & T_{2^n,\,2^n-1} & T_{2^n,\,2^n} & T_{2^n,\,2^n+1} \\ & & & T_{2^n+1,\,2^n} & B_{2^n+1,\,2^n+1}(n) \end{bmatrix}$$

where

$$B_{00}(n) = A_{00}(n)$$

and

$$B_{2^n+1,\,2^n+1}(n) = \begin{bmatrix} T_{2^n+1,\,2^n+1} & T_{2^n+1,\,2^n+2} & & \\ T_{2^n+2,\,2^n+1} & T_{2^n+2,\,2^n+2} & T_{2^n+2,\,2^n+3} & \\ \ddots & \ddots & \ddots & \\ & & & T_{2^{n+1},\,2^{n+1}+1} \\ & & T_{2^{n+1}+1,\,2^{n+1}} & T_{2^{n+1}+1,\,2^{n+1}+1} \end{bmatrix} \oplus 0.$$

In particular, $X_{ij}(n) = X_{ij}(n+1) = T_{ij}$ if $2^n + 1 \leqslant i + j \leqslant 2^{n+1} + 1$. We can therefore apply Lemma 2.3 simultaneously to each pair $X(n)$ and $X(n+1)$, using the matrices (3.1) and (3.2). For each application of Lemma 2.3, the underlying spaces are orthogonal. As such, the norms of the perturbations required by Lemma 2.3 do not add up. Rather, since the individual perturbations are induced by finite rank operators, and since the norms of the individual perturbations are on the order of $\frac{\pi}{2^{n-1}}\|T\|$ which tends to 0 as n tends to ∞, the total perturbation due to the simultaneous application of Lemma 2.3 to each pair must be induced by a compact operator.

In this case, the Berg-style diagram looks essentially the same as that for Lemma 2.5, with the exception that there is no longer a "bottom row". This in turn is due to the fact that we are performing the splices on infinitely many pairs.

The end result is that we can find a unitary operator U and operators C and D such that

$$\Big\|\bigoplus_{n=1}^{\infty} X(n) - U^*(C \oplus D)U\Big\| < \pi \Big\|\bigoplus_{n=1}^{\infty} X(n)\Big\| \leqslant \pi\|T\|.$$

Here,

$$C_{ij} = \begin{cases} X_{ij}(n) & \text{if } 2^n + 1 \leqslant i + j \leqslant 2^{n+1} \\ X_{ij}(1) & \text{if } i + j \leqslant 2. \end{cases}$$

Since $X_{ij}(n) = T_{ij}$ when $i + j \leqslant 2^{n+1} + 2$, we get $C_{ij} = T_{ij}$ for all i and j. Thus $C \cong T$. Let $S_0 = U\Big(\bigoplus_{n=1}^{\infty} X(n)\Big)U^*$ to obtain $\|S_0 - (C \oplus D)\| < \pi\|T\|$, and $S_0 - (C \oplus D) \in \mathcal{K}(\mathcal{H})$. Since S_0 is clearly quasidiagonal (as each $X(n)$ is finite rank), so is $S := S_0 - K = C \oplus D \cong T \oplus D$, as claimed. ∎

It is worth pointing out that one can even control the norm of the perturbation K above, simply by applying Lemma 2.3 to the pairs $X(n)$ and $X(n+1)$, for $n \geqslant N$.

Then the perturbation is on the order of $\frac{\pi}{2^{N-1}}\|T\|$, so that by choosing N large, the perturbation can be made to have small norm.

If one attempts to apply Lemma 2.3 recursively to the pairs $X(n)$ and $X(n+1)$ rather than simultaneously to all such pairs, then at the N^{th} stage, the perturbed system can be seen to be unitarily equivalent to $\bigoplus_{n=1}^{\infty} X(n)$, the original system!

One could appeal to the results of Hadwin on closures of unitary orbits ([6]) to obtain the conclusion, however this defeats the purpose of the present article, which is to obtain a direct and elementary proof of this result.

REMARK 3.2. If Q is Herrero's universal quasidiagonal operator (cf. Corollary 2.11), then it follows essentially by inspection that S_0 is an approximate direct summand of Q, and hence from above, so is T.

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LAURENT W. MARCOUX
Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1
CANADA

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