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ON THE CLASSIFICATION OF C^*-ALGEBRAS OF REAL RANK ZERO WITH ZERO K_1

HUAXIN LIN

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ABSTRACT. A classification of certain separable C^*-algebras of real rank zero with trivial K_1-group is given. The C^*-algebras considered are those that can be expressed as direct limits of direct sums of matrix algebras, matrix algebras over Cuntz-algebras and matrix algebras over corners of certain extensions of Cuntz-algebras by the compact operators. C^*-algebras in the class are not necessary simple. They are, in general, neither finite nor purely infinite. However, the class includes all AF-algebras and all separable nuclear purely infinite simple C^*-algebras with UCT and trivial K_1. It is closed under stable isomorphism, quotients, hereditary C^*-subalgebras, direct limits and tensor products with AF-algebras.

KEYWORDS: *C^*-algebras, real rank zero, classification.*

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0. INTRODUCTION

Recently there are far reaching advances in the theory of classification of amenable C^*-algebras. In 1976, G.A. Elliott classified AF-algebras by their dimension groups (K_0 with a scale). However, it is around 1990, when G.A. Elliott's work ([12]) on classification of C^*-algebras of real rank zero which are direct limits of circle algebras (by their graded groups $K_* = K_0 \oplus K_1$ together with a distinguished subset) circulated, that the program of classifying amenable C^*-algebras was initiated.

Since then a number of classification results appeared ([2], [4], [7], [8], [9], [10], [13], [14], [15], [16], [17], [18], [19], [20] [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [33], [34], [35], [36], [37], [38], [39], [41], [42], [43], etc. (it is not a complete list)).

In the case of stable rank one, recent works of Dădârlat, Elliott and Gong classify (simple) AH-algebras of real rank zero with slow dimension growth. In the purely infinite simple case, the recent works of Kirchberg ([24]) and Phillips ([34]) now classify all separable nuclear purely infinite simple C^*-algebras with UCT. In this paper, we consider a class of C^*-algebras of real rank zero with zero K_1 which are not necessary simple. C^*-algebras in this class are, in general, neither of finite stable rank, nor purely infinite. However, it does contain all AF-algebras as well as all separable nuclear purely infinite simple C^*-algebras with UCT and zero K_1. C^*-algebras in the class are those that can be expressed as the direct limits of corners and quotients of matrix algebras over E_n with various n, where E_n is the unital essential extension of the Cuntz-algebra $\mathcal{O}_n$ by the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space first introduced by Cuntz in [6]. Every hereditary C^*-subalgebra of a C^*-algebra in the class is still in the class. The class is closed under stable isomorphism, quotients, direct limits and tensor products with AF-algebras.

Since $K_0(E_n) = K_0(E_m) = \mathbb{Z}$ and $K_1(E_n) = K_1(E_m) = 0$, K-theory alone is not a complete invariant. The invariant $(V(A), [1_A])$ that we used for unital case in this paper is the semigroup of Murray-von Neumann equivalence classes of projections in matrices over C^*-algebra A together with the class of the unit.

The key to the proof of our main result is the uniqueness Theorem 3.1. It states in the unital case that two unital homomorphisms φ and ψ from $M_k(E_n)$ into $M_\ell(E_m)$ are approximately unitarily equivalent if and only if they induce the same map $V(M_k(E_n)) \to V(M_\ell(E_m))$. In the case that the compositions of these two maps with the quotient map from $M_\ell(E_m)$ onto $M_\ell(\mathcal{O}_m)$ are injective the statement follows from a very similar argument in [29]. For the case that both homomorphisms map the ideal $M_k(\mathcal{K})$ into $M_\ell(\mathcal{K})$, we need to show that every automorphism α on E_n is approximately inner if $\pi \circ \alpha = \pi$, where $\pi : E_n \to \mathcal{O}_n$ is the quotient map. The proof of this uses Brown's Universal Coefficient Theorem.

1. E-ALGEBRAS AND EXISTENCE OF HOMOMORPHISMS

REMARK 1.1. Let $\mathcal{O}_n$ be the Cuntz algebra, and call its standard generators $s_1, s_2, \ldots, s_n$. Thus $1 = s_i^* s_i$ and $1 = \sum_{i=1}^{n} s_i s_i^*$, $n \geqslant 2$.

Let E_n ($n \geqslant 2$) be the universal unital C^*-algebra on the generators $t_1, t_2, \ldots, t_n$ with the relation that t_i are isometries with orthogonal ranges and $\sum_{i=1}^{n} t_i t_i^* < 1$. Let $I(E_n)$ be the (only closed) ideal generated by the projection $1 - \sum_{i=1}^{n} t_i t_i^*$. It is known that $I(E_n) \cong \mathcal{K}$, the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space (see [6]). Let $\pi : E_n \to \mathcal{O}_n$ be the quotient map. Then $\pi(t_i) = s_i$. We have the short exact sequence

$$0 \to \mathcal{K} \to E_n \to \mathcal{O}_n \to 0.$$

So E_n is a unital essential extension of $\mathcal{O}_n$ by $\mathcal{K}$.

As discussed in the introduction, we will study the direct limits of finite direct sums of matrix algebras over E_n. If A is such a direct limit, we may write $A = \lim_{n\to\infty}(A_n, \varphi_{n,n+1})$, where each A_n is a finite direct sum of matrix algebras over some E_k. If $\varphi_{n,n+1}$ is not injective, then a summand of the image of A_n could be $\mathcal{O}_m$. It is also desirable to include all hereditary C^*-subalgebras. Therefore it is reasonable to allow A_n to be a finite direct sum of corners of matrix algebras over E_k, or $\mathcal{O}_m$.

DEFINITION 1.2. A C^*-algebra A is called an *E-algebra* if it is isomorphic to $pM_k(E_n)p$ for some projection $p \in M_k(E_n)$, or $M_k(\mathcal{O}_n)$, where k is any positive integer and $n \geqslant 2$ is an integer.

A C^*-algebra A is said to be in the class Ω if it is a direct limit of finite direct sums of E-algebras.

It is known that $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_1(\mathcal{O}_n) = 0$. It is also known that $K_0(E_n) = \mathbb{Z}$ and $K_1(E_n) = 0$. So K-theory can not distingush E_2 from E_3. Therefore, more information is needed to distinguish them. A reasonable choice is the semigroup $V(E_n)$.

REMARK 1.3. Let A be a C^*-algebra, we denote by $V(A)$ the Murray-von Neumann equivalence classes of projections in the matrices over A. From [6], it is easy to check that

$$V(E_n) = \mathbb{Z}_+ \sqcup \mathbb{Z},$$

where the addition in $\mathbb{Z}_+$ and $\mathbb{Z}$ are the usual ones, if $x \in \mathbb{Z}_+$ and $\widetilde{y} \in \mathbb{Z}$ then $x + \widetilde{y} = ((1-n)x + y)\widetilde{\ }$. We will denote by $I(V(E_n)) = \mathbb{Z}_+$ the first part of the disjoint union. We will use x for integers in $I(V(E_n))$ and $\widetilde{x}$ for integers in $\mathbf{Z} = V(E_n) \setminus I(V(E_n))$.

We also have

$$V(\mathcal{O}_n) = \{0\} \sqcup \mathbf{Z}/(n-1)\mathbf{Z}.$$

Let V be a finite direct sum of $\mathbf{Z}_+, V(E_n)$, $V(\mathcal{O}_n)$ for various n. Suppose that V_1 is a homomorphic image of V, V_2 is a homomorphic image of $V_1, \dots, V_{k+1}$ is a homomorphic image of $V_k, \dots$. A fact will be used in the proof of Theorem 4.4 is that there is k_0 such that every homomorphism from V_k onto V_{k+1} is an isomorphism for $k > k_0$. One way to see it is to note that both $\mathbf{Z}$ and $\mathbf{Z}/(n-1)\mathbf{Z}$ are noetherian.

REMARK 1.4. Let p be a (nonzero) projection in $M_k(\mathcal{O}_n)$. Suppose that $[p]_0 = \bar{\ell}$ in $K_0(\mathcal{O}_n)$. Then $pM_k(\mathcal{O}_n)p \cong M_\ell(\mathcal{O}_n)$ (or $M_{(n-1)}(\mathcal{O}_n)$, if $\ell = 0$). If p is a (nonzero) projection in $M_k(E_n)$ and $p \notin M_k(I(E_n))$, then, from the above, $[p] = \widetilde{l} \in \mathbf{Z}$. Then p is Murray-von Neumann equivalent to the identity of $M_\ell(E_n)$ (in $M_{l+k}(E_n)$), if $\ell \neq 0$. Therefore $pM_k(E_n)p \cong M_\ell(E_n)$, if $\ell \neq 0$. If $p \in M_k(I(E_n))$, then it is known that $pM_k(E_n)p = pM_k(I(E_n))p \cong M_\ell$ for some ℓ.

LEMMA 1.5. (*cf. Lemma* 2.1 *of* [29]) *Let A be a matrix algebra over E_n or over $\mathcal{O}_n$ and $\mathcal{P}$ be the set of all infinite projections in A. (So $\mathcal{P}$ is the set of all projections in $A\setminus I(A)$ in the case that $A = M_m(E_n)$, or $\mathcal{P}$ is the set of all nonzero projections in A in the case that $A = M_m(\mathcal{O}_n)$.) Then $\mathcal{P}$ satisfies the following conditions:*

(i) *if $p, q \in \mathcal{P}$ and $pq = 0$, then $p + q \in \mathcal{P}$,*

(ii) *if $p \in \mathcal{P}$, $p' \in A$ such that $p' \sim q$, then $p \in \mathcal{P}$,*

(iii) *if $p, q \in \mathcal{P}$, then there exists $p' \in \mathcal{P}$, $p' \sim p$, $p' \leqslant q$ and $q - p' \in \mathcal{P}$,*

(iv) *if $q \in A$ is a projection such that $q \geqslant p$ for some $p \in \mathcal{P}$ then $q \in \mathcal{P}$.*

Furthermore, $K_0(A) = \{[p] \in V(A) : p \in \mathcal{P}\}$. In particular, if $[p]_0 = [q]_0$ in $K_0(A)$ then $[p] = [q]$ in $V(A)$.

REMARK 1.6. From Remark 1.4 and Lemma 1.5, we can list E-algebras as follows:

(1.1) $M_k(E_n)$, $k = 1, 2, \dots$, $n = 2, 3, \dots$,

(1.2) $M_k(\mathcal{O}_n)$, $k = 1, 2, \dots$, $n = 2, 3, \dots$,

(1.3) M_k, $k = 1, 2, \dots$,

(1.4) eE_ne, where $e \in E_n$ and $[e] = \widetilde{0}$, $n = 2, 3, \dots$.

It should be noted that $V(eE_ne) = V(E_n)$ for $[e] = \widetilde{0}$ by Lemma 1.5 (iii). We would also like to point out that the C^*-algebras in cases (1.1) and (1.4) have a unique ideal which we will denote by $I(M_k(E_n))$ ($\cong \mathcal{K}$) or $I(eE_ne)$ ($\cong \mathcal{K}$).

LEMMA 1.7. *Let $p \in E_n \setminus I(E_n)$ and $k|[p]$ in $\mathbb{Z}$, where $k > 0$. Then there are mutually orthogonal projections $p_1, p_2, \ldots, p_k \in E_n \setminus I(E_n)$ such that $p_1 + p_2 + \cdots + p_k = p$ and $[p_i] = [p]/k$ in $V(E_n)$.*

Proof. Suppose that $km = [p]$. Find k mutually orthogonal projections $q_1, q_2, \ldots, q_k \in M_k(E_n) \setminus M_k(I(E_n))$ such that $[q_i] = \widetilde{m}$. Set $q = \sum_{i=1}^{k} q_i$. By Lemma 1.5, there is a partial isometry $W \in M_k(E_n)$ such that

$$W^*W = q \quad \text{and} \quad WW^* = p.$$

So define $p_i = Wq_iW^*$. ∎

The rest of this section is to prove the following theorem:

THEOREM 1.8. *Let A and D be finite direct sums of E-algebras. Let*

$$\alpha : (V(A), [\mathbf{1}_A]) \to (V(D), [\mathbf{1}_D])$$

be a homomorphism. Then there exists a homomorphism $\varphi : A \to D$ which induces α, i.e. $\varphi_* = \alpha$.

Proof. Write

$$A = \bigoplus_{i=1}^{n} B_i, \quad D = \bigoplus_{j=1}^{m} C_j$$

where each B_i and C_j is an E-algebra. Let $p_1, p_2, \ldots, p_n$ be the corresponding minimal central projections in A and $q_1, q_2, \ldots, q_m$ be the corresponding minimal central projections in D. Let $\pi_j : D \to C_j$ be the projection. It suffices to show that there exists a homomorphism $\varphi_j : A \to C_j$ such that $\varphi_j(\mathbf{1}_A) = q_j$ and $(\varphi_j)_* = (\pi_j)_* \circ \alpha$ on $V(A)$.

So we may assume that $D = C_1$. We claim that there are mutually orthogonal projections $d_1, d_2, \ldots, d_n \in D$ such that $\mathbf{1}_D = \sum_{i=1}^{n} d_i$ and $[d_i] = \alpha([p_i])$. We do have $x_i \in V(D)$ such that $x_i = \alpha([p_i])$ and $\sum_{i=1}^{n} x_i = [\mathbf{1}_D]$. Without loss of generality, we may assume that $x_i \neq 0$. Thus it is obvious that such d_i exists in the case that $D \cong M_\ell$, or $D \cong M_\ell(\mathcal{O}_m)$. For the case that $D \cong M_\ell(E_m)$, or $D \cong pM_\ell(E_m)p$ with $[p] = \widetilde{0}$, we can apply Lemma 1.7.

From the claim, we see that it suffices to show that there are homomorphism $\psi_i : B_i \to d_iDd_i$ such that $\psi_i(p_i) = d_i$ and $(\psi_i)_* = \alpha|V(B_i)$. In other words, we reduce the general case to the case that both A and D have only one summand.

We point out that we do not need to worry about the case that $A \neq M_\ell$ and $D = M_\ell$, since every homomorphism from $V(\mathcal{O}_n) = \{0\} \sqcup \mathbb{Z}/(n-1)\mathbb{Z}$ or from

$V(E_n) = \mathbf{Z}_+ \sqcup \mathbf{Z}$ into $\mathbf{Z}_+$ has to be zero map. The former case is trivial. For the case that $A = M_\ell(E_n)$, note that the only map from $\mathbf{Z}$ into $\mathbf{Z}_+$ is zero. If $x \in V(I(E_n)) = \mathbf{Z}_+$, then

$$\alpha(x) = \alpha(x) + \alpha(\widetilde{0}) = \alpha(((1-n)x)^{\sim}) = 0.$$

Therefore α is zero.

If $A = M_\ell$ and $D = M_m$ or $M_k(\mathcal{O}_m)$, the homomorphism φ is easily established. If $D = M_k(E_m)$, one can apply Lemma 1.7. The other cases will be dealt with below.

(1) Take $A = M_{m_1}(E_n)$ and $D = M_{m_2}(E_m)$ and a homomorphism $\alpha : V(A) \to V(D)$ with $\alpha(\mathbf{1}_{m_1}) = d$ and $[p] = d$, where $\mathbf{1}_{m_1}$ is the identity of $M_{m_1}(E_n)$ and $p \in M_{m_2}(E_m)$ is a projection. There is a homomorphism $\varphi : A \to D$ such that $\varphi_* = \alpha$ and $\varphi(\mathbf{1}_{m_1}) = p$.

We assume $\alpha \neq 0$.

Case (a). $\alpha(I(V(E_n)) = 0$.

For any $\widetilde{x} \in \mathbf{Z} \subset V(E_n)$, if $\widetilde{x}$ is a nonnegative integer,

$$(1-n)\alpha(\widetilde{x}) = \alpha(((1-n)x)^{\sim}) = \alpha(x + \widetilde{0}) = \alpha(x) + \alpha(\widetilde{0}) = \alpha(\widetilde{0}).$$

Since α can not map $\mathbf{Z}$ into $I(V(E_m)) \cong \mathbf{Z}_+$, α maps $\mathbf{Z}$ into $V(E_m) \setminus I(V(E_m))$. Thus $\alpha(\widetilde{0}) = \widetilde{0}$. So the image of α, in this case, is $\{0\} \sqcup \{\widetilde{0}\}$.

So $[p] = \widetilde{0}$. By Lemma 1.7, there are mutually orthogonal and mutually equivalent projections $e_1, e_2, \dots, e_{m_1}$ such that $\sum_{i=1}^{m_1} e_i = p \leqslant \mathbf{1}_{m_2}$ and $[e_i] = \widetilde{0}$. There are $u_{ij} \in M_{m_2}(E_m)$ such that

$$u_{ij}^* u_{ij} = e_i \quad \text{and} \quad u_{ij} u_{ij}^* = e_j, \quad i, j = 1, 2, \dots, m_1.$$

Since $[e_1] = \widetilde{0}$, by Lemma 1.7, there are mutually orthogonal and mutually equivalent projections $q_1, q_2, \dots, q_n \in e_1 M_{m_1}(E_n)$ such that $[q_i] = [e_1] = \widetilde{0}$. Suppose that $v_j \in e_1 M_{m_2}(E_m) e_1$ are partial isometries such that

$$v_j^* v_j = e_1 \quad \text{and} \quad v_j v_j^* = q_j.$$

Then the C^*-subalgebra C generated by v_j is isomorphic to $\mathcal{O}_n$ and the C^*-subalgebra B generated by u_{ij} and C is isomorphic to $M_{m_1}(\mathcal{O}_n)$. Now define φ to be the composition of the following:

$$M_{m_1}(E_n) \to M_{m_1}(\mathcal{O}_n) \to B.$$

One checks that the so defined map meets the requirements.

To save the notation, without loss of generality, from now on, we assume that $m_1 = 1$.

Case (b). $\alpha(I(V(E_n)) \neq 0$ but $\alpha(I(V(E_n)) \subset I(V(E_m))$.

Let $1 \in I(V(E_n)) = \mathbb{Z}_+$ and $\widetilde{1} \in V(E_n) \setminus I(V(E_n)) = \mathbb{Z}$. Suppose that $\alpha(1) = \ell$ and $\alpha(\widetilde{1}) = \widetilde{k}$. Then we must have

$$((1-m)\ell)^{\sim} = ((1-n)k)^{\sim}.$$

Let $e \in \mathcal{K} \subset pDp$ such that $[e] = \ell \in I(V(E_m))$. Take $q \in (p-e)D(p-e)$ such that $[q] = \widetilde{0}$ (see Lemma 1.5). Then $[e+q] = ((1-m)l)^{\sim} = ((1-n)k)^{\sim}$. Note $[p] = k$. We obtain

$$[p-(e+q)] = \widetilde{k} - ((1-n)k)^{\sim} = n\widetilde{k}..$$

By Lemma 1.7, there are mutually orthogonal and mutually equivalent projections $q_1, q_2, \ldots, q_n \leqslant p-(e+q)$ such that $[q_i] = \widetilde{k}$ and $\sum_{i=1}^{n} q_i = p-(e+q)$. Set $p_i = q_i$, $i = 1, 2, \ldots, (n-1)$ and $p_n = q_n + q$. Note that we have $[p_i] = k$, $p_i \perp p_j$ $i \neq j$ and $p - \sum_{i=1}^{n} p_i = e$. There are partial isometries v_j, $j = 1, 2, \ldots, n$ such that

$$v_j^* v_j = p \quad \text{and} \quad v_j v_j^* = p_j.$$

Then the C^*-subalgebra C generated by v_j, $j = 1, 2, \ldots, n$ is isomorphic to E_n. Define $\varphi : E_n \to M_{m_2}(E_n)$ by the isomorphism from E_n to C. One checks that the so defined φ meets the requirements.

Case (c). α maps $V(E_n)$ into $\{0\} \sqcup \mathbb{Z}$ and $\alpha(I(V(E_n)) \neq 0$.

Suppose that $\alpha(\widetilde{1}) = \widetilde{k}$. Then $[p] = \widetilde{k}$. By Lemma 1.5, there are n mutually orthogonal projections $p_1, p_2, \ldots, p_n \in pM_{m_2}(E_m)p$ such that $p - \sum_{i=1}^{n} p_i \in M_{m_2}(E_m) \setminus M_{m_2}(\mathcal{K})$ and $[p_i] = [p]$. Therefore there are partial isometries $v_1, v_2, \ldots, v_n \in pM_{m_2}(E_m)p$ such that

$$v_j^* v_j = p \quad \text{and} \quad v_j v_j^* = p_j, \quad j = 1, 2, \ldots, n.$$

Then the C^*-subalgebra C generated by $\{v_j\}$ is isomorphic to E_n. We let $\varphi : E_n \to C$ be the isomorphism which maps s_j to v_j. Clearly, φ meets the requirements.

(2) Take $A = M_{m_1}(\mathcal{O}_n)$, and $D = M_{m_2}(E_m)$ and a homomorphism $\alpha : V(A) \to V(D)$ with $\alpha([\mathbf{1}_{m_1}]) = d$ and $[p] = d$, where $p \in D$ is a projection. Then there is a homomorphism $\varphi : A \to D$ such that $\varphi_* = \alpha$ and $\varphi(\mathbf{1}_{m_1}) = p$.

It is obvious that the only nonzero homomorphism α is the one which maps 0 to 0 and maps $\widetilde{0}$ to $\widetilde{0}$. Therefore (2) follows from the proof of Case (a) of (1).

(3) Take $A = M_{m_1}(\mathcal{O}_n)$ and $D = M_{m_2}(\mathcal{O}_m)$ and a homomorphism $\alpha : V(A) \to V(D)$ with $\alpha([\mathbf{1}_{m_1}]) = d$ and $[p] = d$, where $p \in D$ is a projection. Then there is a homomorphism $\varphi : A \to D$ such that $\varphi_* = \alpha$ and $\varphi(\mathbf{1}_{m_1}) = p$.

Note that $V(A) = \{0\} \sqcup K_0(A)$ and $V(D) = \{0\} \sqcup K_0(D)$. Thus this case follows from 2.6 of [35].

(4) Take $A = M_{m_1}(E_n)$ and $D = M_{m_2}(\mathcal{O}_m)$ and $\alpha : V(A) \to V(D)$ with $\alpha([\mathbf{1}_{m_1}]) = d$ and $[p] = d$, where $p \in D$ is a projection. Then there is a homomorphism $\varphi : A \to D$ such that $\varphi_* = \alpha$ and $\varphi(\mathbf{1}_{m_1}) = p$.

Case (a). $\alpha(I(V(E_n)) = 0$. In this case α factors through $\{0\} \sqcup \mathbf{Z}/(n-1)\mathbf{Z}$. Therefore this case can be reduced to (3).

Case (b). $\alpha(I(V(E_n)) \neq 0$. From the assumption, $m_1|[p]$, there are mutually orthogonal and mutually equivalent projections $g_1, g_2, \ldots, g_{m_1} \in B$ such that $\sum_{i=1}^{m_1} g_i = p$. As before, in order to save the notation, without loss of generality, we may assume that $m_1 = 1$. Now if $1 \in I(V(E_n)$, since $\alpha(1) \neq 0$ and $\alpha(\widetilde{0}) = \widetilde{0}$,

$$\alpha(1) = \alpha(1) + \alpha(\widetilde{0}) = \alpha(1 + \widetilde{0}) = \alpha(((1-n))^{\sim}) = (1-n)d.$$

Note $\mathcal{O}_m$ is purely infinite and simple. There are mutually orthogonal and mutually equivalent projections $p_1, p_2, \ldots, p_n \in pBp$ such that $[p_i] = [p]$ and $p - \sum_{i=1}^{n} p_i > 0$. We have $\left[p - \sum_{i=1}^{n} p_i\right] = (1-n)d$. There are partial isometries $v_1, v_2, \ldots, v_n \in pBp$ such that

$$v_j^* v_j = p \quad \text{and} \quad v_j v_j * = p_j, \quad j = 1, 2, \ldots, n.$$

So we define $\varphi(s_j) = v_j$, $j = 1, 2, \ldots, n$. One checks easily that so defined φ meets the requirements.

(5) The case that $A = eE_n e$ with $[e] = \widetilde{0}$.

Let 1 be the identity of E_n. Suppose that $\alpha([1]) = d'$. It is easy to see that there is a projection $p' \in D$ such that $[p'] = d'$. By the cases that have been established, there is a homomorphism $\psi : E_n \to D$ such that $\alpha = \psi_*$ and $\psi(1) = d'$. Clearly $[\psi(e)] = \widetilde{0}$ if D is a corner of $M_{m_2}(E_m)$ which is not isomorphic to a matrix algebra, or $[\psi(\mathbf{1}_A)] = \overline{0}$, if $D \cong M_{m_2}(\mathcal{O}_m)$. In all cases, if $p \in D$ with

$[p] = \alpha([e])$, then there exists a partial isometry $w \in D$ such that $w^*w = p$ and $ww^* = p'$. We define $\varphi = w^*\psi w|A$.

(6) The case that $D = eM_{m_2}(E_m)e$ with $[e] = \widetilde{\mathbf{0}}$.
This case follows the cases (1)–(5) by taking $p = e$. ∎

2. AUTOMORPHISMS OF E_n

Recall that an automorphism α on a C^*-algebra A is said to be *derivable* if $\alpha = \exp(\delta)$ for some $*$-derivation δ on A.

The following is an immediate consequence of 8.6.12 of [32].

LEMMA 2.1. (*cf.* 8.6.12 *of* [32]) *Let A be a separable unital C^*-algebra and α a derivable automorphism. Then α is approximately inner.*

Proof. It suffices to show that every derivation on A is approximately inner. But that follows from 8.6.12 of [32]. ∎

REMARK 2.2. Let A be a separable (unital) C^*-algebra and $\tau_1, \tau_2 : A \to M(\mathcal{K})/\mathcal{K}$ be two unital essential extensions. Suppose that $[\tau_1] = [\tau_2]$ in $\mathrm{Ext}(A, \mathcal{K})$. By a result of Voiculescu ([45]) (see [1]), there exist unital essential trivial extensions τ_0', τ_0'' such that $\tau_1 \oplus \tau_0'$ is weakly equivalent to τ_1 and $\tau_2 \oplus \tau_0''$ is equivalent to τ_2. We also have trivial extensions τ_{00}' and τ_{00}'' such that $\tau_1 \oplus \tau_{00}'$ is weakly equivalent to $\tau_2 \oplus \tau_{00}''$. We view τ_{00}' and τ_{00}'' as two homomorphisms from A into $M(\mathcal{K})/\mathcal{K}$. Note that both $\tau_0' \oplus \tau_{00}' \oplus \mathbf{0}$ and $\tau_0'' \oplus \tau_{00}'' \oplus \mathbf{0}$ are essential nonunital trivial extensions, by the same result of Voiculescu again, these two essential extensions are weakly equivalent to $\tau_0' \oplus \mathbf{0}$ and $\tau_0'' \oplus \mathbf{0}$. This implies that $\tau_1 \oplus \mathbf{0}$ is weakly equivalent to $\tau_2 \oplus \mathbf{0}$. It follows that there is a unitary $W \in M(\mathcal{K})/\mathcal{K}$ such that

$$W^*\tau_1(a)W = \tau_2(a) \quad \text{for} \quad a \in A.$$

We will use this result of Voiculescu in the next lemma.

LEMMA 2.3. *Let $\tau : \mathcal{O}_n \to M(\mathcal{K})/\mathcal{K}$ be a unital homomorphism and $u \in M(\mathcal{K})$ be a unitary such that $[\pi(u), a] = 0$ for every $a \in \tau(\mathcal{O}_n)$. Then there is a selfadjoint element $h \in M(\mathcal{K})/\mathcal{K}$ with $\|h\| \leqslant 2\pi$ such that $\pi(u) = \exp(ih)$ and*

$$[h, a] = 0, \quad \text{for all} \quad a \in \tau(\mathcal{O}_n).$$

Proof. If $\mathrm{sp}(\pi(u)) \neq \mathbb{S}^1$, then $\pi(u) = \exp(ih)$ for some selfadjoint h in C^*-subalgebra generated by $\pi(u)$ with $\|h\| \leqslant 2\pi$. So $[h, a] = 0$ for all $a \in \tau(\mathcal{O}_n)$. So now we assume that $\mathrm{sp}(\pi(u)) = \mathbb{S}^1$. Set $h_0 = [(\pi(u) + \pi(u)^*/2)_+]2\pi$ and

$v = \exp(ih_0)$. Note that $[h_0, a] = 0 = [v, a]$ for all $a \in \tau(\mathcal{O}_n)$. Let $\tau_1 : C(\mathbb{S}^1)\otimes\mathcal{O}_n \to M(\mathcal{K})/\mathcal{K}$ defined by C^*-subalgebra $C^*(\pi(u), \tau(\mathcal{O}_n))$ and $\tau_2 : C(\mathbb{S}^1) \otimes \mathcal{O}_n \to M(\mathcal{K})/\mathcal{K}$ defined by the C^*-subalgebra $C^*(\pi(v), \tau(\mathcal{O}_n))$. Note that $\mathcal{O}_n$ is simple and $\mathrm{sp}(\pi(u)) = \mathrm{sp}(\pi(v)) = \mathbb{S}^1$, τ_i is a monomorphism, $i = 1, 2$. Therefore both are unital essential extensions of $C(\mathbb{S}^1) \otimes \mathcal{O}_n$ by $\mathcal{K}$. We compute, by Künneth's formula, that

$$K_0(C(\mathbb{S}^1) \otimes \mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}, \quad K_1(C(\mathbb{S}^1) \otimes \mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$$

and the maps from $K_1(C(\mathbb{S}^1) \otimes \mathcal{O}_n)$ into $K_1(M(\mathcal{K})/\mathcal{K})$ induced by the maps τ_1 and τ_2 are zero. Note that τ_1 and τ_2 are two essential extensions of $C(\mathbb{S}^1) \otimes \mathcal{O}_n$ by $\mathcal{K}$. From Brown's Universal Coeffecient Theorem (see [3]), we compute that

$$[\tau_1], [\tau_2] \in \mathrm{ext}^1_{\mathbb{Z}}(\mathbb{Z}/(n-1)\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/(n-1)\mathbb{Z}.$$

From UCT again, there exists an essential extension σ of $C(\mathbb{S}^1 \otimes \mathcal{O}_n)$ such that

$$[\sigma] \in \mathrm{ext}^1_{\mathbb{Z}}(\mathbb{Z}/(n-1)\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/(n-1)\mathbb{Z}$$

and $[\sigma]$ is a generator for $\mathbb{Z}/(n-1)\mathbb{Z}$. It follows that there are integer k and m such that

$$k[\sigma] = [\tau_1] \quad \text{and} \quad m[\sigma] = [\tau_2].$$

Let σ', τ_1' and τ_2' be the essential extensions of $\mathcal{O}_n$ by restricting σ, τ_1 and τ_2 on the C^*-subalgebra $\mathbb{C} \otimes \mathcal{O}_n$. Since

$$\tau_1' = \tau_2' = \tau,$$

we have

$$k[\sigma'] = m[\sigma'].$$

If τ is not trivial, we obtain

$$k \equiv m(\mathrm{mod}(n-1)).$$

This implies that $[\tau_1] = [\tau_2]$. Since both τ_1 and τ_2 are unital essential extensions, by Remark 2.2, there is a unitary $W \in M(\mathcal{K})/\mathcal{K}$ such that

$$W^*\tau(t_j)W = \tau(t_j) \quad \text{and} \quad W^*vW = u.$$

In particular, $u = \exp(iW^*h_0W)$. Take $h = W^*h_0W$.

Now we consider the case that τ is trivial. We will show that if τ is trivial, τ_i is trivial too, $i = 1, 2$. It then follows from Remark 2.2 that they are weakly equivalent. The proof finishes as in the case that τ is not trivial.

There is a strong unital trivial essential extension σ_0 (meaning that there is a *unital* monomorphism $j : \mathcal{O}_n \to A$ which splits σ_0, where A is the extension determined by σ_0). By Remark 2.2, there is a unitary $W \in M(\mathcal{K})/\mathcal{K}$ such that

$$W^*\tau(a)W = \sigma_0(a)$$

for all $a \in \mathcal{O}_n$. By considering $W^*\tau W$, we may assume, without loss of generality, that τ is strongly unital.

As above,

$$[\tau_i] \in \mathrm{ext}^1_{\mathbb{Z}}(\mathbb{Z}/(n-1)\mathbb{Z}, \mathbb{Z}).$$

Therefore, to show τ_i is trivial, it is enough to show that

$$0 \to \mathbb{Z} \to K_0(E(i)) \to \mathbb{Z}/(n-1)\mathbb{Z} \to 0$$

splits, where the C^*-algebra $E(i)$ is the extension determined by τ_i. To do this, it is enough to show

$$K_0(E(i)) = \mathbb{Z} \oplus \mathbb{Z}/(n-1)\mathbb{Z}.$$

Note that $K_0(E(i))$ has the form $\mathbb{Z} \oplus G$ for some finite torsion group G. We have $[\mathbf{1}] = (m, x)$, where $m \in \mathbb{Z}$ and $x \in G$. Let η be the map from $K_0(E) \to \mathbb{Z}/(n-1)\mathbb{Z}$. Then $\eta((m, x)) = \bar{1}$. If $m = 0$, then G has to map to $\mathbb{Z}/(n-1)\mathbb{Z}$ surjectively. From the short exact sequence, we will conclude that $G \cong \mathbb{Z}/(n-1)\mathbb{Z}$. So it is enough to show that $m = 0$.

Let $\mathcal{J} : \mathbf{1} \otimes \mathcal{O}_n \to E$ be the unital monomorphism which splits extension τ. From the fact that $\mathbf{1} = \mathcal{J}\big(\mathbf{1} \otimes \sum_{j=1}^{n} s_j s_j^*\big)$, we know that $[\mathbf{1}]$ must be torsion. Therefore, $m = 0$. ∎

LEMMA 2.4. *In Lemma 2.3, for $\varepsilon > 0$, there is a unitary $v \in M(\mathcal{K})$ with $\mathrm{sp}(v) \neq \mathbb{S}^1$ such that*

$$\|u - v\| < \varepsilon$$

and

$$[\pi(v), a] = 0 \quad \text{for all} \quad a \in \tau(\mathcal{O}_n).$$

Proof. For $0 \leqslant \delta \leqslant 1/2$, set $\bar{v} = \exp(ih(1-\delta))$. Then if δ is small enough,

$$\|\pi(u) - \bar{v}\| < \frac{\varepsilon}{2}.$$

Furthermore, $\mathrm{sp}(\overline{v}) \in \{e^{it} : 0 \leqslant t \leqslant (1-\delta)2\pi\}$. Let $w \in M(\mathcal{K})$ such that $\pi(w) = \overline{v}$ and

$$\|u - w\| < \frac{\varepsilon}{2}.$$

We assume that $\varepsilon < 1$. Set $v = w|w|^{-1}$. Then

$$\|u - v\| < \varepsilon,$$

v is a unitary and $\pi(w) = v$. Since the essential spectrum of v is in $\{e^{it} : 0 \leqslant t \leqslant (1-\delta)2\pi\}$, if $\lambda \in \mathrm{sp}(v) \cap (\mathbb{S}^1 \setminus \{e^{it} : 0 \leqslant t \leqslant (1-\delta)2\pi\})$ then λ can only be an isolated point in $\mathrm{sp}(v)$. Therefore $\mathrm{sp}(v) \neq \mathbb{S}^1$. ∎

THEOREM 2.5. *Let*

$$0 \to \mathcal{K} \to E \to \mathcal{O}_n \to 0$$

be a unital essential extension. Let α *be an automorphism on* E. *Suppose that* $\pi \circ \alpha = \pi$, *where* $\pi : E \to \mathcal{O}_n$ *is the quotient map. Then* α *is approximately inner.*

Proof. It is well known that there is a unitary $u \in M(\mathcal{K})$ ($\cong B(\ell^2)$) such that

$$\alpha(x) = u^* x u$$

for all $x \in E$. For any $\varepsilon > 0$, by Lemma 2.4, there is a unitary $v \in M(\mathcal{K})$ such that

$$\|v - u\| < \frac{\varepsilon}{6}, \quad [a, \pi(v)] = 0$$

for all $a \in \pi(E)$ and $\mathrm{sp}(v) \neq \mathbb{S}^1$. Define $\beta(x) = v^* x v$ for $x \in E$. Since $[a, \pi(v)] = 0$, β is an automorphism on E and

$$\|\alpha - \beta\| < \frac{\varepsilon}{2}.$$

There is a continuous function F defined on $\mathrm{sp}(v)$ such that $F(v) \in M(\mathcal{K})/\mathcal{K}$ is selfadjoint and $\exp(F(v)) = v$. Note $[\pi(F(v)), a] = 0$ for all $a \in \pi(E)$. Define $L(x) = F(v)x - xF(v)$ for $x \in E$. Since $F(\pi(v))$ commutes with $\pi(E)$, L is a $*$-derivation of E. So $\beta = \exp(L)$. It follows from Lemma 2.1 that β is approximately inner. Therefore α is approximately inner. ∎

DEFINITION 2.6. Let A and B be C^*-algebras, let G be a finite subset of A, and let φ and ψ be two homomorphisms from A to B. We say that φ and ψ are *approximately unitarily equivalent to within* ε, with respect to G, if there is a unitary $v \in M(B)$, the multiplier algebra of B (if B is unital, $M(B) = B$) such that

$$\|\varphi(g) - v\psi(g)v^*\| < \varepsilon$$

for all $g \in G$. We abbreviate this as

$$\varphi \overset{\varepsilon}{\sim} \psi \quad \text{on } G.$$

We say that φ and ψ are *approximately unitarily equivalent* if $\varphi \overset{\varepsilon}{\sim} \psi$ for all $\varepsilon > 0$. (Of course, this notion does not depend on the choice of G.)

The following is a result of M. Rørdam and N.C. Phillips.

THEOREM 2.7. (*Theorem* 5.1 *of* [35] *and* [33]) *Let D be a unital C^*-algebra of real rank zero containing a proper infinite full projection. Let φ and ψ be two unital homomorphisms from $\mathcal{O}_n \to D$. Then φ and ψ are approximately unitarily equivalent if and only if $[\varphi] = [\psi]$ in $KK(\mathcal{O}_n, D)$. In particular φ and ψ are approximately unitarily equivalent if $K_1(D) = 0$.*

Proof. Note, as remarked in Remark after 6.5 in [36], that D satisfies the condition 1.3 in [36]. So the case that n is even was proved in 5.1 of [35]. By 4.6 of [33], the "decoy Cuntz algebra" Q_n (in 2.1 of [20]) is isomorphic to $\mathcal{O}_n$ for all positive integers $\geqslant 2$. The case that n is odd then follows from 8.1 of [36] (and 4.6 of [33]). ∎

LEMMA 2.8. *Let $\varphi_1, \varphi_2 : A \to B$ be two unital homomorphism, where $A \cong M_{m_1}(E_n)$ or $A \cong eE_ne$ for some $[e] = \widetilde{\mathbf{0}}$ and $B \cong M_{m_2}(E_m)$ or $B \cong dE_md$ for $[d] = \widetilde{\mathbf{0}}$. Suppose that φ_1 and φ_2 induce the same homomorphisms on $V(E_n)$. Suppose also that φ_1 and φ_2 map the ideal $I(A)$ into the ideal $I(B)$. Then φ_1 is approximately unitarily equivalent to φ_2.*

Proof. If $\varphi_i(I(A)) = 0$, then both maps factor through $M_k(\mathcal{O}_n)$ for some k. Suppose that $\varphi_i = \psi_i \circ \pi$, where $\pi : A \to M_k(\mathcal{O}_n)$ is the quotient map and ψ_1 and ψ_2 are two monomorphisms from $M_k(\mathcal{O}_n)$ into $M_{m_2}(E_m)$ or dE_md. It follows from Theorem 2.7, since $K_1(E_m) = \{0\}$, that ψ_1 and ψ_2 are approximately unitarily equivalent. Consequently, φ_1 and φ_2 are approximately unitarily equivalent.

So now we assume that both φ_1 and φ_2 are monomorphisms. We first consider the case that $A = M_{m_1}(E_n)$. The case that $A = eE_ne$ will follow. It is easy to see that we can reduce the general case to the case that that $m_1 = 1$.

Let $\varphi_1(t_j) = y_j$ and $\varphi_2(t_j) = z_j$, $j = 1,2,\dots,n$. By the assumption, $\varphi_i(1-\sum_{j=1}^{n} t_jt_j^*)$ are in $I(B)$ and they are Murray-von Neumann equivalent. Therefore they are unitarily equivalent, by Lemma 1.5. So we may further assume that $1-\sum_{j=1}^{n} y_jy_j^* = 1-\sum_{j=1}^{n} z_jz_j^*$.

Let $\pi : B \to B/I(B) \cong M_k(\mathcal{O}_m)$ be the quotient map ($k = m_2$, if $B \cong M_{m_2}(E_m)$, and $k = (m-1)$ if $B \cong dE_md$ for $[d] = \widetilde{0}$). We now have two homomorphisms from $\mathcal{O}_n$ to $M_{m_2}(\mathcal{O}_m)$, given by $s_j \mapsto \pi(y_j)$ and $s_j \mapsto \pi(z_j)$. Since both maps agree on the identity, they induce the same map on $K_0(\mathcal{O}_n)$. Since $K_1(\pi(p)M_{m_2}(\mathcal{O}_m)\pi(p)) = 0$ for any projection $p \in M_{m_2}(\mathcal{O}_m)$, the Universal Coefficient Theorem ([40], Theorem 1.18) implies that they have the same class in KK-theory. Let $\delta > 0$ and $\eta > 0$ be small numbers (to be chosen below; δ will depend on η). By Theorem 2.7, there is a unitary $v \in M_k(\mathcal{O}_n)$ such that

$$\|v^*\pi(z_j)v - \pi(y_j)\| < \frac{\delta}{2}$$

for $j = 1,2,\dots,n$. Since $U(M_k(\mathcal{O}_m))$ is connected, there is a unitary $u \in B$ such that $\pi(u) = v$. Then there are $a_j \in I(B)$ such that

$$\|u^*z_ju - (y_j + a_j)\| < \delta$$

for $j = 1,2,\dots,n$.

If δ is sufficiently small, then by Lemma 1.3 (2) of [29] there are isometries $y_j' \in E_n$ with orthogonal ranges such that $\pi(y_j') = \pi(y_j)$ and

$$\|y_j + a_j - y_j'\| < \eta$$

for $j = 1,\dots,n$. It follows that

$$\|y_j' - u^*z_ju\| < \eta + \delta.$$

Let H be an infinite dimensional separable Hilbert space. We now identifying B with a C^*-subalgebra of $B(H)$ which identifies $I(B) \cong \mathcal{K}$ with $\mathcal{K}(H)$. Now we have a representation of E_n, $\sigma : E_n \to B \to B(H)$ which is defined by the composition of φ_1 with the above identification. Define the second representation $\sigma' : E_n \to B(H)$ by sending t_j to y_j' (and then using the above identification; but we will identify y_j' with the element in $B(H)$). Suppose that $\sigma\Big(1-\sum_{j=1}^{n} t_jt_j^*\Big)$ has rank k. So does $u^*\Big(1-\sum_{j=1}^{n} z_jz_j^*\Big)u$. Let $\rho : E_n \to B(H)$ be the unique irreducible faithful representation (which is the representation that maps $I(E_n)\,(\cong \mathcal{K})$ onto $\mathcal{K}(H)$). We are going to use Voiculescu's Theorem, as stated in Arveson's paper ([1]), to prove that σ and σ' are approximately unitarily equivalent.

Note that

$$\left\|1-\sum_{j=1}^{n} y_j'(y_j')^* - u^*\Big(1-\sum_{j=1}^{n} z_j z_j^*\Big)u\right\| < n(\eta+\delta),$$

and recall that $1-\sum_{j=1}^{n} z_j z_j^*$ is a rank k projection in $\mathcal{K}(H)$. If $\eta+\delta$ is small enough, it follows that $1-\sum_{j=1}^{n} t_j t_j^*$ is also a rank k projection in $\mathcal{K}(H)$. Since it is not zero, σ is a faithful representation of E_n. Let $H_0=\overline{\sigma(I(E_n))H}$, the essential subspace of $\sigma|J_n$. Note that it is a reducing subspace for σ. Since $\sigma\Big(1-\sum_{j=1}^{n} t_j t_j^*\Big)$ has rank k, we conclude, by standard results in representation theory, that $(\sigma|I(E_n))|H_0$ is a direct sum of k copies of the (unique) faithful irreducible representation of $I(E)\cong\mathcal{K}$. Standard results in representation theory now imply that $\sigma(-)|H_0$, the essential part of σ is unitarily equivalent to the direct sum of k copies of ρ. Similarly, since $1-\sum_{j=1}^{n} t_j'(t_j')^*$ is of rank k, we conclude that the essential part of σ' is unitarily equivalent to the essential part of σ.

We have now verified the hypotheses of Theorem 5 (iii) of [1]: σ and σ' have the same kernel (namely $\{0\}$), their compositions with the quotient map from $B(H)$ to the Calkin algebra have the same kernel (namely $I(E_n)$), and the essential parts are unitarily equivalent. That theorem yields a unitary $W\in B(H)$ such that

$$\|W^*y_jW-y_j'\|<\eta \quad \text{and} \quad W^*y_jW-y_j'\in\mathcal{K}(H)$$

for $j=1,2,\ldots,n$. Since $y_j-y_j'\in\mathcal{K}(H)$ and $W^*y_jW-y_j'\in\mathcal{K}(H)$, we obtain $y_j-W^*y_jW\in\mathcal{K}(H)$, whence $W^*y_jW\in B$. Let E be a C^*-subalgebra of pBp generated by $\mathcal{K}(H)$ and y_j. Then E is a unital essential extension of $\mathcal{O}_n$ by $\mathcal{K}$. Furthermore, from the above,

$$\alpha(x)=W^*xW \quad \text{for} \quad x\in E$$

is an automorphism on E. Since $\pi\circ\alpha(y_j)=\pi(y_j)$, $\pi\circ\alpha=\pi$. By Theorem 2.5, there is a unitary $Z\in E\subset B$ such that

$$\|Z^*y_jZ-W^*y_jW\|<\delta$$

for $j = 1, 2, \ldots, n$. We can then combine with earlier estimates to obtain

$$\|Z^* y_j Z - u^* z_j u\| < 2\eta + 2\delta$$

for $j = 1, 2, \ldots, n$. Therefore

$$\|uZ^* \varphi_1(t_j) Z u^* - \varphi_2(t_j)\| < 2\eta + 2\delta$$

for $j = 1, 2, \ldots, n$. We then let $2\eta + 2\delta < \varepsilon$.

For the case that $A = eE_n e$ or $B = dE_m d$, we show that this case follows the case that we have just proved.

If $A = eEe$, define $\widetilde{\varphi}_i = \mathrm{id} \otimes \varphi_i : M_2(A) \to M_2(B)$. Clearly, $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ induce the same homomorphism on $V(A)$. By Lemma 1.5, there is a projection $g \in M_2(A)$ such that $g \geqslant e$ and $[g] = \mathbf{1}$ in $V(A)$. So $gM_2(A)g \cong E_n$ and $A \subset gM_2(A)g$. By what we have established, $\widetilde{\varphi}_i | gM_2(A)g$ are approximately unitarily equivalent. It follows that φ_i are approximately unitarily equivalent. ∎

Note that the part of the proof above using Voiculescu's result is taken from the proof of Lemma 2.2 in [29].

3. UNIQUENESS OF HOMOMORPHISMS OF E-ALGEBRAS

In this section, we will show the following:

THEOREM 3.1. *Let $\varphi_1, \varphi_2 : A \to D$ be two unital homomorphisms, where $A \cong M_{m_1}(E_n)$ or $A \cong eE_n e$ for some projection e with $[e] = \widetilde{0}$ and $D \cong M_{m_2}(E_m)$ or $D \cong dE_m d$ for some projection d with $[d] = \widetilde{0}$. Suppose that φ_1 and φ_2 induce the same homomorphism on $V(E_n)$. Then φ_1 and φ_2 are approximately unitarily equivalent.*

There are two cases. The case that both $\varphi_i\left(1 - \sum_{j=1}^{n} t_j t_j^*\right)$ are in $I(D)$ follows from Lemma 2.8. In this section we will consider the case that both $\varphi_i\left(1 - \sum_{j=1}^{n} t_j t_j^*\right)$ are not in $I(D)$.

This case follows the proof of 3.3 of [29]. The algebra D used in [29] is purely infinite simple. It turns out that proof works for $D = M_{m_2}(E_m)$ or $D \cong dE_m d$ under the assumption that $\varphi_i\left(1 - \sum_{j=1}^{n} t_j t_j^*\right)$ is not in the ideal $M_{m_2}(\mathcal{K})$. We will explain why it works.

The following two definitions are the same as corresponding definitions in [29].

DEFINITION 3.2. Let A be any unital C^*-algebra, and let D be a C^*-algebra. Suppose that the set $\mathcal{P}$ of infinite projections in D is not empty and satisfies condition (i)–(iv) as in Lemma 1.5. Let $\varphi, \psi : A \to D$ be two homomorphisms, and assume that $\varphi(1) \in \mathcal{P}$ and $[\psi(1)]_0 = 0$ in $K_0(D)$. We define a homomorphism $\varphi\widetilde{\oplus}\psi : A \to D$, well defined up to unitary equivalence, by the following construction. Choose a projection $q \in D$ such that $0 < q < \varphi(1)$ and $[q]_0 = 0$. Since $\varphi(1)$ and q are in $\mathcal{P}$, $\varphi(1) - q \neq 0$ and $[\varphi(1) - q]_0 = 0$, there are partial isometries v and w such that $vv^* = \varphi(1) - q$, $v^*v = \varphi(1)$, $ww^* = q$, and $w^*w = \psi(1)$. Now define $(\varphi\widetilde{\oplus}\psi)(a) = v\varphi(a)v^* + w\psi(a)w^*$ for $a \in A$.

DEFINITION 3.3. Let D be a C^*-algebra as in Definition 3.2, let A be an E-algebra, and let $\varphi : A \to D$ be a homomorphism. Then φ is *approximately absorbing* if for every $\psi : A \to D$ such that $[\psi] = 0$ in $KK^0(A, D)$, the homomorphisms φ and $\varphi\widetilde{\oplus}\psi$ are approximately unitarily equivalent.

If $D = M_{m_2}(E_n)$, or $D \cong dE_md$ for some projection d with $[d] = \widetilde{\mathbf{0}}$, then $[\varphi] = 0$ in $KK^0(A, D)$ if and only if the image of φ_* on $V(A)$ is $\{0\} \sqcup \{\widetilde{0}\}$.

PROPOSITION 3.4. (*Compare* 1.7 *of* [29]) *Let $D = M_{m_2}(E_m)$ or $D \cong dE_md$ for some projection d with $[d] = \widetilde{0}$ and let $A = M_{m_1}(E_n)$ or $A \cong eE_ne$ for some projection e with $[e] = \widetilde{0}$. Let φ, $\psi : A \to D$ be two monomorphisms such that $\varphi(\mathbf{1}_A)$ and $\psi(\mathbf{1}_A)$ are unitarily equivalent. If $[\varphi] = [\psi] = 0$ in $KK^0(E_n, D)$, then φ and ψ are approximately unitarily equivalent.*

Proof. First we point out, by conjugating by a unitary, without loss of generality, that we may assume that φ and ψ agree on $\mathbf{1}_A$. By considering $\varphi(\mathbf{1}_A)D\varphi(\mathbf{1}_A)$, we then assume that φ and ψ are unital.

We will first consider the case that $A = M_{m_1}(E_n)$. The other case will follow. By conjugating another unitary, without loss of generality, we may further assume that φ and ψ agree on $M_{m_1}(\mathbf{C})$. By considering $e_{11}\varphi e_{11}$, $e_{11}\psi e_{11}$ and $e_{11}De_{11}$ ($\cong M_{m_2'}(E_m)$), we may also assume that $m_1 = 1$. The proof is almost the same as that of 1.7 of [29]. The condition that $[\varphi] = [\psi] = 0$ in $KK^0(E_n, D)$ implies that φ_* and ψ_* map $V(E_n)$ to $\{\mathbf{0}\} \sqcup \{\widetilde{\mathbf{0}}\}$. In particular, both map $I(E_n)$ into zero or both map $I(E_n) \setminus \{0\}$ into $D \setminus I(D)$. The case that both homomorphisms map $I(E_n)$ into zero follows from Theorem 2.7. So we now assume that both homomorphisms map $I(E_n) \setminus \{0\}$ into $D \setminus I(D)$. We will proceed as in the proof of 1.7 of [29].

Instead of writing $[\varphi] = [\psi] = 0$ in $K_0(D)$ we have $[\varphi(1)] = [\psi(1)] = \widetilde{\mathbf{0}}$ in $V(D)$. The assumption we have made implies that $[q] = \widetilde{\mathbf{0}}$. We also note that $K_1(D) = \{0\}$. The proof of 1.7 of [29] works without any further changes.

Now we will show that the case that $A \cong eE_ne$ follows from the case that has been proved. The argument is the same as that used in the end of the proof of Lemma 2.8. Define $\widetilde{\varphi} = \mathrm{id} \otimes \varphi : M_2(A) \to M_2(D)$ and $\widetilde{\psi} = \mathrm{id} \otimes \psi : M_2(A) \to M_2(B)$. There is a projection $g \in M_2(A)$ with $e \leqslant g$ and $[g] = \widetilde{1}$. So $gM_2(A)g \cong E_n$. We then apply the established case to $\widetilde{\varphi}|gM_2(A)g$ and $\widetilde{\psi}|gM_2(A)g$. ∎

LEMMA 3.5. *(Compare to 2.3 of [29]) Let $D = M_{m_1}(E_m)$, $E \cong dE_md$ for some projection d with $[d] = \widetilde{0}$, or $D = M_{m_2}(\mathcal{O}_m)$, let $A = M_{m_1}(E_n)$ or $A \cong eE_ne$ for some projection e with $[e] = \widetilde{0}$ and let $\varphi : A \to D$ be a monomorphism. Suppose that $\varphi(I(A))) \cap I(D) = \{0\}$. Then φ is approximately absorbing.*

Proof. We will only consider the case that $A = M_{m_1}(E_n)$. The other case will follow exactly the same way as that at the end of the proof of Proposition 3.4.

The proof is almost the same as that of 2.3 in [29]. Replacing D by $\varphi(1)D\varphi(1)$, we may assume that φ is unital. Since it is clear that any nonzero homomorphism of M_{m_1} is approximately absorbing, we may assume, without loss of generality, that $m_1 = 1$. If D is purely infinite, then the lemma follows from 1.7 of [29]. So we now assume that $D = M_{m_2}(E_m)$ or $D \cong dE_md$.

Since $q = \varphi\Big(1 - \sum_{j=1}^{n} t_jt_j^*\Big)$ is in $A \setminus I(M_{m_2}(E_m))$, by Lemma 1.5, there are $n+1$ mutually orthogonal nonzero projections $p_{n+1}, p_{n+2}, \dots, p_{2n}, e \in qDq$ and isometries $\widetilde{t}_{n+1}, \widetilde{t}_{n+2}, \dots, \widetilde{t}_{2n} \in D$ such that $\widetilde{t}_j\widetilde{t}_j^* = p_j$ for $j = n+1, n+2, \dots, 2n$. Now let $A \subset D$ be the C^*-subalgebra generated by $\varphi(t_j)$ for $j = 1, 2, \dots, n$ and $\widetilde{t}_j$ for $j = n+1, n+2, \dots, 2n$. Then A is isomorphic to E_{2n}. By Lemma 2.2 of [29], for any $\varepsilon > 0$ there is a projection $f \in A$ and unital homomorphisms $\psi_1 : \mathcal{O}_{2n} \to fDf$ and $\psi_2 : E_{2n} \to (1-f)D(1-f)$ such that $[f] = 0$ in $K_0(A)$ (and hence in $K_0(D)$), with

$$\|\varphi(t_j) - (\psi_1(s_j) + \psi_2(t_j))\| < \frac{\varepsilon}{3}$$

for $1 \leqslant j \leqslant n$ and

$$\|\widetilde{t}_j - (\psi_1(s_j) + \psi_2(t_j))\| < \frac{\varepsilon}{3}$$

for $n+1 \leqslant j \leqslant 2n$. Define $\varphi_1, \varphi_2 : E_n \to D$ by $\varphi_1(t_j) = \psi_1(s_j)$ and $\varphi_2(t_j) = \psi_2(t_j)$ for $j = 1, \dots, n$. Note that $[\varphi_1] = 0$ in $KK^0(E_n, D)$ by Lemma 1.2 of [28].

Now let $\varphi_0 : E_n \to D$ be any homomorphism with $[\varphi_0] = 0$ in $KK^0(E_n, D)$. Without loss of generality, we may assume $\varphi_0(1) \leqslant \varphi_1(1)$. Then $\varphi_1 \overset{\varepsilon/3}{\sim} \varphi_1 \widetilde{\oplus} \varphi_0$ by Proposition 3.4. Therefore

$$\varphi \overset{\varepsilon/3}{\sim} \varphi_1 + \varphi_2 \overset{\varepsilon/3}{\sim} (\varphi_1 \widetilde{\oplus} \varphi_0) + \varphi_2' \overset{\varepsilon/3}{\sim} \varphi \widetilde{\oplus} \varphi_0 \quad \text{on } G,$$

so $\varphi \overset{\varepsilon}{\sim} \varphi \widetilde{\oplus} \varphi_0$ on G as desired, where G is the standard generators of E_n. ∎

DEFINITION 3.6. Let $\varphi : E_n \to D$ be a homomorphism, where $D = M_{m_2}(E_m)$ or $D \cong dE_m d$ for some projection d with $[d] = \widetilde{0}$. Let $p_j = \varphi(t_j t_j^*)$ for $j = 1, 2, \ldots$. Suppose that $\varphi(I(E_n)) \cap I(D) = \{0\}$. So $\varphi\Big(1 - \sum_{j=1}^{n} t_j t_j^*\Big) \in D \setminus I(D)$. By Lemma 1.5, there are $g_{n-1}, g_n \in D$ such that

$$g_{n-i}^* g_{n-i} = \varphi(1), \quad g_{n-i} g_{n-i}^* \leqslant \varphi\Big(1 - \sum_{j=1}^{n} t_j t_j^*\Big), \quad i = 0, 1$$

and

$$\varphi\Big(1 - \sum_{j=1}^{n} t_j t_j^*\Big) - g_{n-1} g_{n-1}^* - g_n g_n^* \notin I(D).$$

Define $\overline{\varphi} : E_n \to (1 - p_1 - p_2) D (1 - p_1 - p_2)$ by

$$\overline{\varphi}(t_j) = \varphi(t_{j+1})(1 - p_1 - p_2)$$

for $j = 1, 2, \ldots, n-1$ and

$$\overline{\varphi}(t_{n-i}) = g_{n-i}, \quad i = 0, 1.$$

If $\varphi : M_{m_1}(E_n) \to D$, we may write $\varphi = \phi_1 \otimes \phi_2$, where ϕ_1 is a homomorphism from $M_{m_1} \otimes \mathbb{C} \to D$ and ϕ_2 is a homomorphism from E_n into $\varphi(e_{11}) D \varphi(e_{11})$, $\{e_{ij}\}$ being a matrix unit for M_{m_1}. We define $\overline{\varphi} = \phi_1 \otimes \overline{\phi}_2$.

If $\varphi : eE_n e \to D$, we first extend it to $\widetilde{\varphi} = \mathrm{id} \otimes \varphi : M_2(A) \to M_2(D)$. Then take a projection $g \in M_2(A)$ with $e \leqslant g$ and $[g] = \widetilde{1}$. Let $\varphi_1 = \widetilde{\varphi} | g M_2(A) g$. We define $\overline{\varphi} = \overline{\varphi}_1 | A$.

LEMMA 3.7. *Let φ and $\overline{\varphi}$ be as in Definition* 3.6, *and set*

$$D_0 = [\mathbf{1} \oplus (\mathbf{1} - p_1 - p_2)] M_2(D) [\mathbf{1} \oplus (1 - p_1 - p_2)].$$

Then the direct sum

$$\varphi \oplus \overline{\varphi} : A \to D_0$$

satisfies $[\varphi \oplus \overline{\varphi}] = 0$ *in* $KK^0(A, D_0)$.

Proof. Clearly $[(\varphi \oplus \overline{\varphi})(1)] = 0$ in $K_0(D_0)$. The result is now immediate from Lemma 1.2 of [29]. ∎

Proof of Theorem 3.1. By Lemma 2.8, we may assume that $\varphi_i\Big(1 - \sum_{j=1}^{n} t_j t_j^*\Big) \notin I(D)$.

We follow the notation of Definition 3.6. By Lemma 1.5, there is a partial isometry $W \in M_3(D)$ such that

$$W^* W = \mathbf{1} \oplus (\mathbf{1} - p_1 - p_2) \oplus \mathbf{1} \quad \text{and} \quad WW^* = \mathbf{1} \oplus \mathbf{0} \oplus \mathbf{0}.$$

Since both $[\varphi \oplus \overline{\varphi}]$ and $[\overline{\varphi} \oplus \psi]$ are zero in $KK^0(E_n, D)$, Lemma 3.5 implies that

$$\varphi \overset{\varepsilon/2}{\sim} W(\varphi \oplus \overline{\varphi} \oplus \psi) W^* \overset{\varepsilon/2}{\sim} \psi. \quad ∎$$

4. UNIQUENESS AND EXISTENCE THEOREMS

The following is a well known result.

LEMMA 4.1. *Let φ, $\psi : M_\ell \to D$ be two nonzero homomorphisms such that $\varphi_* = \psi_*$ on $V(M_\ell)$. Then they are unitarily equivalent.*

LEMMA 4.2. *Let p, $q \in A$ be two (nonzero) projections with $[p] = [q]$ in $V(A)$ where $A = M_\ell(E_n)$ or $A \cong eE_ne$ for some projection e with $[e] = \widetilde{0}$. Suppose that both $1-p$ and $1-q$ are not zero and both $1-p$ and $1-q$ are or both are not in $I(A)$. Then p and q are unitarily equivalent.*

Proof. Suppose that $1-q$ and $1-p$ are in $I(E_n)$. Then

$$\widetilde{0} + [1-p] = \widetilde{0} + [1-q] = ((1-n)k)^{\sim},$$

since $[p] = [q]$. This implies that $[1-p] = [1-q]$ in $V(M_\ell(E_n))$. Therefore p and q are unitarily equivalent.

Suppose that $1-p$ and $1-q$ are both in $A \setminus I(A)$. Then $1-p$, $1-q \in \mathcal{P}$. It follows from Lemma 1.5 that $[1-p] = [1-q]$. ∎

LEMMA 4.3. *Let A and B be finite direct sums of E-algebras. Let φ and ψ be two homomorphisms from A into B. Then φ and ψ are approximately unitarily equivalent if and only if they induce the same map $V(A) \to V(B)$ and $\varphi(\mathbf{1}_A)$ is unitarily equivalent to $\psi(\mathbf{1}_A)$.*

Proof. If φ and ψ are approximately unitarily equivalent, then they induce the same maps $V(A) \to V(B)$ and $\varphi(1)$ is unitarily equivalent to $\psi(1)$. Suppose that φ and ψ induce the same maps $V(A) \to V(B)$ and $\varphi(1)$ and $\psi(1)$ are unitarily equivalent. By conjugating with a unitary, if necessary, we may assume that $\varphi(1) = \psi(1)$. Since a corner of an E-algebra is still an E-algebra, replacing B by $\varphi(1)B\varphi(1)$, we may further assume that both φ and ψ are unital.

Write

$$A = \bigoplus_{\ell=1}^{r} C_\ell,$$

where each C_ℓ is an E-algebra. Let $\mathcal{J}_\ell : D_\ell \to A$ be the natural inclusion and $p_1, p_1, \ldots, p_r$ be the minimal central projections in A, so that $p_\ell = \mathcal{J}_\ell(1)$.

We claim that $\varphi(p_\ell)$ and $\psi(p_\ell)$ are unitarily equivalent. We write

$$B = \bigoplus_{m=1}^{L} D_m,$$

where each D_m is an E-algebra.

Let $\pi_m : B \to D_m$ be the projection. To prove the claim, it is enough to show that $\pi_m \circ \varphi(p_\ell)$ and $\pi_m \circ \psi(p_\ell)$ are unitarily equivalent. By the assumption, $[\pi_m \circ \varphi(p_\ell)] = [\pi_m \circ \psi(p_\ell)]$ in $V(D_m)$. So it is enough to show that both are unital or both are not unital in D_m and satisfy the condition that if $D_m = M_k(E_n)$, $\mathbf{1}_{D_m} - \pi_m \circ \varphi(p_\ell)$ and $\mathbf{1}_{D_m} - \pi_m \circ \psi(p_\ell)$ are both or are both not in $I(D_m)$, by Lemma 4.2. If one of them is unital, say, $\pi_m \circ \varphi(p_\ell)$ is unital, then $\pi_m \circ \psi(p_\ell)$ has to be unital too. If it were not, then, since ψ is unital,

$$\pi_m \circ \psi\Big(1 - \sum_{i \neq \ell} p_i\Big) \neq \mathbf{1}_{D_m} - \pi_m \circ \varphi(p_\ell) \neq 0.$$

However,

$$\pi_m \circ \psi\Big(1 - \sum_{i \neq \ell} p_i\Big) = 0$$

since $\pi_m \circ \psi(p_\ell) = \mathbf{1}_{D_m}$. Therefore we would have

$$\Big[\varphi\Big(1 - \sum_{i \neq \ell} p_i\Big)\Big] \neq \Big[\psi\Big(1 - \sum_{i \neq \ell} p_i\Big)\Big]$$

in $V(B)$. So $\pi_m \circ \psi(p_\ell)$ has to be unital.

We now assume that both $\pi_m \circ \varphi(p_\ell)$ and $\pi_m \circ \psi(p_\ell)$ are not unital. Since

$$\pi_m \Big[\varphi\Big(1 - \sum_{i \neq \ell} p_i\Big)\Big] = \Big[\pi_m \circ \psi\Big(1 - \sum_{i \neq \ell} p_i\Big)\Big]$$

and both φ and ψ are unital, $\mathbf{1}_{D_m} - \pi_m \circ \varphi(p_\ell)$ and $\mathbf{1}_{D_m} - \pi_m \circ \psi(p_\ell)$ both are in $I(D_m)$ or both are not. This proves the claim.

We may therefore assume that $\varphi(p_\ell) = \psi(p_\ell)$ for each ℓ. It is now clear that it suffices to show that, for each ℓ and m, $\pi_m \circ \varphi \circ j_\ell$ and $\pi_m \circ \psi \circ j_\ell$ are approximately unitarily equivalent. The assumption implies that

$$(\pi_m \circ \varphi \circ j_\ell)_* = (\pi_m \circ \psi \circ j_\ell)_*$$

on $V(C_\ell)$. Therefore the result follows from Theorems 3.1, 2.7 and Lemma 4.1. ∎

THEOREM 4.4. *Let A and B be two C^*-algebras in Ω and let φ and ψ be two homomorphisms from A into B satisfying the following:*

(i) *if A is unital, $\varphi(1)$ is unitarily equivalent to $\psi(1)$, or*

(ii) *if A is not unital then B is also not unital and*

$$[\varphi(e_n)] = [\psi(e_n)],$$

where $\left\{\sum_{i=1}^{n} e_i\right\}$, $\left\{\sum_{i=1}^{n} \varphi(e_i)\right\}$ *and* $\left\{\sum_{i=1}^{n} \psi(e_i)\right\}$ *are approximate identities for* A *and* B *consisting projections.*

Then φ *and* ψ *are approximately unitarily equivalent if and only if they induce the same map* $V(A) \to V(B)$.

Proof. If φ and ψ are approximately unitarily equivalent, then they induce the same maps $V(A) \to V(B)$. Suppose that φ and ψ induce the same maps $V(A) \to V(B)$. Write

$$A = \overline{\bigcup_{i=1}^{\infty} A_i}, \quad B = \overline{\bigcup_{n=1}^{\infty} B_n}$$

where $A_1 \subset A_2 \subset \cdots \subset A$, $B_1 \subset B_2 \subset \cdots \subset B$ and each A_i or B_n is isomorphic to a finite direct sum of E-algebras. It suffices to show that the restrictions of φ and ψ on A_i are approximately unitarily equivalent for every i.

Let $\mathcal{G}$ be a finite subset of A_i which contains its generators. It is enough to show that, for every $\varepsilon > 0$, there exists a unitary $u \in B$ such that

$$\|u^*\varphi(f)u - \psi(f)\| < \varepsilon$$

for all $f \in \mathcal{G}$. Let $\mathcal{G} = \{f_1, f_1, \ldots, f_m\}$. For any $\delta > 0$ there are $g_1, g_2, \ldots, g_m$, $g_1', g_2', \ldots, g_m' \in B_\ell$ for some $\ell > 0$ such that

$$\|g_i - \varphi(f_i)\| < \delta \quad \text{and} \quad \|g_i' - \psi(f_i)\| < \delta$$

$i = 1, 2, \ldots, m$. It follows from 1.3 of [28] and 5.7 of [31] that, if δ is small enough, there are unital monomorphisms φ' and ψ' from A_i into B_ℓ such that

$$\|\varphi(f_i) - \varphi'(f_i)\| < \frac{\varepsilon}{3}$$

and

$$\|\psi(f_i) - \psi'(f_i)\| < \frac{\varepsilon}{3}.$$

We have,

$$(\varphi|A_i)_* = \varphi_*' = (\psi|A_i)_* = \psi_*'$$

as maps from $V(A_i)$ into $V(B)$. Let $\alpha_{n,k} : V(A_n) \to V(A_k)$ and $\beta_{n,k} : V(B_n) \to V(B_k)$, where $k > n$, be the homomorphisms induced by the inductive systems. By choosing a larger l, since $V(A_i)$ is finitely generated, we may assume that $\varphi_*' \circ \alpha_{i,\infty}(V(A_i)) \subset V(B_\ell)$. By Remark 1.3 there is an integer k such that $\beta_{\ell+k,\infty}$ is injective on $\beta_{\ell,\ell+k}(V(B_\ell))$. Denote by σ the inverse of $\beta_{\ell+k,\infty}$ restricted on

$\beta_{\ell,\ell+k}(V(B_\ell))$. Then $\sigma \circ \varphi'_* = \sigma \circ \psi'_*$. By choosing an even larger ℓ and changing notation, without loss of generality, we may assume that

$$\varphi'_* = \psi'_*$$

as maps from $V(A_i)$ into $V(B_\ell)$. If both A and B are unital, we may assume that $\mathbf{1}_{A_i} = \mathbf{1}_A$ and $\mathbf{1}_{B_n} = \mathbf{1}_B$ for all i and n. Since, in this case, $\varphi(1)$ and $\psi(1)$ are unitarily equivalent, we may assume, by conjugating with a unitary in B, that $\varphi(1) = \psi(1)$. Since $\varphi(1)B\varphi(1)$ is still in Ω , we may further assume, from the beginning, that both φ and ψ are unital. Therefore, in this case, we can apply Lemma 4.3. We obtain a unitary $u \in B_\ell \subset B$ such that

$$\|u^*\varphi(f_i)u - \psi'(f_i)\| < \frac{\varepsilon}{3}.$$

Therefore $\varphi \overset{\varepsilon}{\sim} \psi$ on $\mathcal{G}$.

Now we consider the case that both A and B are not unital. It is enough to show that the restriction of φ and ψ on p_nAp_n are approximately unitarily equivalent for each n (and i), where $p_n = \sum_{i=1}^{n} e_n$. There is a partial isometry $w_1 \in B$ such that

$$w_1^*w_1 = \varphi(p_n) \quad \text{and} \quad w_1w_1^* = \psi(e_n),$$

by the assumption (ii). By considering $\varphi|e_nAe_n$, $w_1^*\psi w_1|e_nAe_n$, e_nAe_n and $\varphi(e_n)B\varphi(e_n)$, and applying the unital case that we have just established, we conclude that $\varphi|e_nAe_n$ and $w_1^*\psi w_1|e_nAe_n$ are approximately unitarily equivalent. There are partial isometries $w_i \in B$ such that

$$w_i^*w_i = \varphi(e_{n+i}) \quad \text{and} \quad w_iw_i^* = \psi(e_{n+i}), \quad i = 1, 2, \dots .$$

Set $w = \sum_{i=1}^{\infty} w_i$. Since both $\left\{\sum_{i=1}^{k} \varphi(e_i)\right\}$ and $\left\{\sum_{i=1}^{k} \psi(e_i)\right\}$ are approximate identities for B consisting of projections, it is standard to check that the sum converges in the strict topology of B and $w \in M(B)$. So the above implies that $\varphi|e_nAe_n$ and $\mathrm{Ad}w \circ \psi|e_nAe_n$ are approximately unitarily equivalent. Thus $\varphi|e_nAe_n$ and $\psi|e_nAe_n$ are approximately unitarily equivalent. ∎

THEOREM 4.5. *Let A and D be two C^*-algebras in Ω. Suppose that*

$$\alpha : V(A) \to V(D)$$

is a homomorphism. If A is unital and $\alpha([\mathbf{1}_A]) = [p]$ for some projection $p \in D$, then there exists a unital homomorphism $\varphi : A \to D$ which induces α, i.e. $\varphi_* = \alpha$ such that $\varphi(\mathbf{1}_A) = p$.

If both A and B are nonunital and $\alpha([e_n]) = [d_n] \neq 0$, where $\left\{\sum_{i=1}^{n} e_i\right\}$ and $\left\{\sum_{i=1}^{n} d_i\right\}$ are approximate identities for A and D respectively, consisting of projections, then there exists a homomorphism $\varphi : A \to B$ which induces α such that $\varphi(e_n) = d_n$.

Proof. In the case that A is unital, by considering pDp, we may assume that $[\alpha(\mathbf{1}_A)] = [\mathbf{1}_D]$. We consider the unital case first. Let

$$A_1 \xrightarrow{f_{1,2}} A_1 \xrightarrow{f_{2,3}} A_3 \longrightarrow \cdots$$

be a sequence of finite direct sums of E-algebras with limit A. Let $f_i : A_i \to A$ be the canonical homomorphism given by the direct limit. Since $f_i(A_i)$ is also a finite direct sum of E-algebra, by replacing A_i by $f_i(A_i)$, we may assume that

$$A_1 \subset A_2 \subset \cdots A$$

and $A = \overline{\bigcup_{i=1}^{\infty} A_i}$. Similarly, we also assume that

$$D = \overline{\bigcup_{i=1}^{\infty} D_i}, \quad D_1 \subset D_2 \subset \cdots D$$

and each D_i is a finite direct sum of E-algebras. Since each $V(A_i)$ is finitely generated, by passing to a subsequence and changing the notation we may assume that $\alpha|V(A_i) \subset V(D_i)$. Furthermore, in the unital case, we assume that $\mathbf{1}_{A_i} = \mathbf{1}_A$ and $\mathbf{1}_{D_i} = \mathbf{1}_D$.

The homomorphisms

$$\alpha \circ (f_i)_* : V(A_i) \to V(D_i)$$

are induced by a unital homomorphisms $\psi_i : A_i \to D_i$ by Theorem 1.8. The two homomorphisms $\psi_{i+1}|A_i$ and ψ_i from A_i into D_{i+1} induce the same maps $V(A_i) \to V(D_{i+1})$. From our construction, they are both unital. Hence they are approximately unitarily equivalent by Lemma 4.3. It follows that there are unitaries $u_i \in D$ so that the system

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots \\ \psi_1 \downarrow & & \psi_2 \downarrow & & \psi_3 \downarrow & & \\ D & \xrightarrow{\mathrm{Ad}\,u_1} & D & \xrightarrow{\mathrm{Ad}\,u_2} & D & \longrightarrow & \cdots \end{array}$$

is a one-sided approximate intertwining in the sense of Elliott (Remark 2.3 of [11]), see also Lemma 2 of [44].

Put

$$v_1 = \mathrm{id}_D, \quad v_{i+1} = v_i \circ \mathrm{Ad}(u_i^*).$$

Then the sequence $\{v_k \circ \psi_k(a)\}_{k=i}^{\infty}$ is Cauchy in D for every $i \in \mathbb{N}$ and every $a \in A_i$ (see [44]). Let $\varphi_i(a)$ be the limit of this sequence. Then $\varphi_i : A_i \to D$ is a homomorphism and

$$\begin{array}{ccc} A_i & \longrightarrow & A_{i+1} \\ \varphi_i \searrow & & \swarrow \varphi_{i+1} \\ & D & \end{array}$$

commutative. Thus we get a homomorphism $\varphi : A \to D$ making the diagram

$$\begin{array}{ccc} & A_i & \\ f_i \swarrow & & \searrow \varphi_i \\ A & \xrightarrow{\varphi} & D \end{array}$$

commutative. If $e \in A_i$ is a projection, then since v_k is inner and by the choice of ψ_k, we have

$$[v_k \circ \psi_k(e)] = \alpha \circ (f_i)_*([e])$$

for all k. Since near by projections are unitarily equivalent, this implies that

$$[\varphi_i(e)] = \alpha \circ (f_i)_*([e]).$$

From the commuting diagram above, it is seen that $[\varphi(e)] = \alpha([e])$ for every projection e in

$$\bigcup_{i=1}^{\infty} f_i(A_i) \subset A.$$

Since every projection in A is unitarily equivalent to some projection in this dense subset of A, it follows that $\varphi_* = \alpha$.

For nonunital case, we let $A_i = p_i A p_i$, where $p_i = \sum_{k=1}^{i} e_k$ and $D_i = q_i D q_i$, where $q_i = \sum_{k=1}^{i} d_k$. We will repeat the proof of the unital case with some changes. We use the conclusion of the unital case to obtain $\psi_i : A_i \to D_i$ which also maps p_i to q_i, $i = 1, 2, \ldots$. To obtain a unitary $u_i \in M(D)$, we notice that $\alpha \circ (f_{i+1})_*(e_{i+1}) = [d_{i+1}]$. So $[\psi_{i+1}(p_{i+1} - p_i)] = [d_{i+1}]$. This implies that $\psi_i(p_i)$ is unitarily equivalent to $\psi_{i+1}(p_i)$ in D_{i+1}. By Theorem 4.4, ψ_i is approximately unitarily equivalent to $\psi_{i+1}|A_i$. Thus $u_i \,(\in M(D))$ exists. To obtain $\varphi : A \to D$ which induces α, we follow the rest of the proof in the unital case.

To obtain a homomorphism φ' which not only induces α but also maps e_n to d_n, we let $d'_n = \varphi(e_n)$. Then $[d'_n] = [d_n]$. So there are partial isometries $v_n \in D$ such that

$$v_n^* v_n = d_n \quad \text{and} \quad v_i v_i^* = d'_n, \quad n = 1, 2, \dots .$$

Set $v = \sum_{n=1}^{\infty} v_n$. Let B be the hereditary C^*-subalgebra generated by $\{d'_n\}$. It is standard to check that $v^*bv \in D$ for all $b \in B$ and $vdv^* \in B$ for all $d \in D$. Then the homomorphism φ' defined by $\varphi(a) = v^*\varphi(a)v$ for $a \in A$ meets the requirements. ∎

5. CONCLUSIONS

PROPOSITION 5.1. (*Proposition* A *in* [36] *see also* [11]) *If A and B are two separable C^*-algebras and there are homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \varphi$ is approximately unitarily equivalent to* id_A *and $\varphi \circ \psi$ is approximately unitarily equivalent to* id_B, *then A and B are isomorphic. Furthermore, there is an isomorphism $\phi : A \to B$ such that $\phi_* = \varphi_*$ as maps from $V(A)$ onto $V(B)$.*

Proof. This follows from Proposition A in [36]. Note that ϕ could be chosen as the pointwise limit of $\{\mathrm{Ad}v_k \circ \varphi\}$ (see the proof of Proposition A in [36]). Therefore, if $p \in A$ is a projection, $\varphi(p)$ is equivalent to $\phi(p)$. ∎

THEOREM 5.2. *Let A and B be two unital C^*-algebras in Ω. Suppose that there is an isomorphism*

$$\alpha : (V(A), [\mathbf{1}_A]) \to (V(B), [\mathbf{1}_B]).$$

Then there is a isomorphism $\varphi : A \to B$ which induces α, i.e. $\varphi_* = \alpha$.

Proof. Let $\beta : (V(B), [\mathbf{1}_B]) \to (V(A), [\mathbf{1}_A])$ be the inverse of α. By Theorem 2.4 there are unital homomorphisms $\varphi' : A \to B$ and $\psi : B \to A$ with $\varphi'_* = \alpha$ and $\psi_* = \beta$. We note that $\psi \circ \varphi'$ and $\varphi' \circ \psi$ induce the identity maps on $V(A)$ and $V(B)$, respectively. It follows from Theorem 4.4 that $\psi \circ \varphi'$ is approximately unitarily equivalent to id_A and $\varphi' \circ \psi$ is approximately unitarily equivalent to id_B. It follows from Proposition 5.1 that there exists an isomorphism $\varphi : A \to B$ which induces α. ∎

THEOREM 5.3. *Let A and B be two nonunital C^*-algebras in Ω. Let $\{e_n\}$ and $\{d_n\}$ be approximate identities for A and B, respectively, consisting of projections. Suppose that there is an isomorphism*

$$\alpha : V(A) \to V(B)$$

such that $\alpha([e_n - e_{n-1}]) = [d_n - d_{n-1}]$ ($e_0 = d_0 = 0$). Then there is an isomorphism $\varphi : A \to B$ which induces α, i.e. $\varphi_* = \alpha$ such that $\varphi(e_n) = d_n$.

Proof. Let $\beta : V(B) \to V(A)$ be the inverse of α. It follows from Theorem 4.5 that there are homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi_* = \alpha$, $\varphi(e_n) = d_n$, $\psi_* = \beta$ and $\psi(d_n) = e_n$. It follows from Theorem 4.4 that $\varphi \circ \psi$ and id_A are approximately unitarily equivalent, $\psi \circ \varphi$ and id_B are approximately unitarily equivalent. Then, by Proposition 5.1, A is isomorphic to B.

Suppose that $\varphi' : A \to B$ is an isomorphism. Let $d'_n = \varphi'(e_n)$. By Proposition 5.1, we may assume that $[d'_n - d'_{n-1}] = [d_n - d_{n-1}]$ ($d_0 = d'_0 = 0$). Therefore there exists a sequence of partial isometries $w_n \in B$ such that

$$w_n^*(d_n - d_{n-1})w_n = d'_n - d'_{n-1} \quad \text{and} \quad w_n(d'_n - d'_{n-1})w_n^* = d_n - d_{n-1},$$

$n = 1, 2, \ldots$. Set $w = \sum_{n=1}^{\infty} w_n$. Then it is standard that $\sum_{n=1}^{\infty} w_n$ convergences in the strict topology and $w \in M(B)$, the multipier algebra of B. We define $\varphi = \mathrm{Ad}w^* \varphi'$. ∎

PROPOSITION 5.4. *Every quotient of a C^*-algebra A in Ω is in Ω.*

Proof. We may assume that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$, where each A_n is a finite direct sum of E-algebras and $A_n \subset A_{n+1}$. Let $\phi : A \to A/I$ be a quotient map. Then $A/I = \overline{\bigcup_{n=1}^{\infty} \phi(A_n)}$. Since each $\phi(A_n)$ is a finite direct sum of E-algebras, $A/I \in \Omega$. ∎

THEOREM 5.5. *Ω is closed under direct limits.*

Proof. Suppose that $A = \lim_{n\to\infty}(A_n, f_{n,n+1})$, where each $A_n \in \Omega$. Let $f_n : A_n \to A$ be the canonical homomorphism induced by the direct limit. By Proposition 5.4, $f_n(A_n) \in \Omega$. We may then assume, by replacing A_n by $f_n(A_n)$, that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ and $A_n \subset A_{n+1}$. Suppose that $A_n = \overline{\bigcup_{n=1}^{\infty} B_k^{(n)}}$, $B_k^{(n)} \subset B_{k+1}^{(n)}$ and every $B_k^{(n)}$ is a finite direct sum of E-algebras. Let $\{a_n\}$ be a dense subset of the unit ball of A. Let $G_1 = \{g_1, g_2, \ldots, g_{m_1}\}$ be a (finite) set of generators of B_1. There are k_2, n_2 and $G'_1 \subset B_{k_2}^{(n_2)} \subset A_{n_2}$ such that $\|g_i - g'_i\| < \varepsilon$ for $i = 1, 2, \ldots, m_1$ and $\mathrm{dist}(a_1, B_{k_1}^{(n_1)}) < 1/2$. If follows from 1.3 in [29] and 5.1 of [31], if ε is small enough, that there are a C^*-subalgebra $C_{k_1} \subset B_{k_1}^{(n_1)}$ and an isomorphism $\psi_1 : C_{k_1} \to B_1$ with

$$\|g_i - \psi_1(g_i)\| < \frac{1}{2}, \quad i = 1, 2, \ldots, m_1.$$

Let $G_2 = \psi_1(G_1) \cup F_2$, where F_2 is a (finite) set of generators of $B_{k_2}^{(n_2)}$. By repeating the above argument, we obtain a monomorphism $\psi_2 : B_{k_2}^{(n_2)} \to B_{k_3}^{(n_3)}$ such that

$$\|g - \psi_2(g)\| < \frac{1}{4}$$

for all $g \in G_2$ and

$$\operatorname{dist}(\{a_1, a_2\}, B_{k_2}^{(n_2)}) < \frac{1}{4}.$$

After defining $\psi_\ell : B_{k_{\ell-1}}^{(n_{\ell-1})} \to B_{k_\ell}^{(n_\ell)}$, we let $G_\ell = \psi_\ell(G_{\ell-1}) \cup F_\ell$, where F_ℓ is a (finite) set of generators of $B_{k_\ell}^{(n_\ell)}$. We then obtain a monomorphism $\psi_{\ell+1}$: $B_{k_\ell}^{(n_\ell)} \to B_{k_{\ell+1}}^{(n_{\ell+1})}$ for some $k_{\ell+1}$ and $n_{\ell+1}$ such that

$$\|g - \psi_{\ell+1}(g)\| < \left(\frac{1}{2}\right)^\ell$$

for all $g \in G_\ell$ and

$$\operatorname{dist}(\{a_1, a_2, \ldots, a_\ell\}, B_{k_{\ell+1}}^{(n_{\ell+1})}) < \left(\frac{1}{2}\right)^\ell.$$

Thus we have the following one-sided approximate intertwining system:

$$\begin{array}{ccccccc} B_1 & \xrightarrow{\psi_1} & B_{k_2}^{(n_2)} & \xrightarrow{\psi_2} & B_{k_3}^{(n_3)} & \longrightarrow & \cdots \\ j_1 \downarrow & & j_2 \downarrow & & j_3 \downarrow & & \\ A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\mathrm{id}_A} & A & \longrightarrow & \cdots \end{array}$$

where j_i is the inclusion. Set $B = \lim_{m\to\infty}(B_{k_m}^{(n_m)}, \psi_m)$. Then $B \in \Omega$. As the argument used in the proof of Theorem 4.5, the sequence

$$\{j_m \circ \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_i(b)\}$$

define a homomorphism φ_i from $B_{k_i}^{(n_i)}$ into A. Furthermore, we get a homomorphism $\varphi : B \to A$ from φ_i. We claim that φ is surjective. For any $\varepsilon > 0$ and a_m, there is $b_\ell \in B_{k_\ell}^{(n_\ell)}$ such that

$$\|a_m - b_\ell\| < \left(\frac{1}{2}\right)^\ell < \frac{\varepsilon}{4}.$$

We also have

$$\|j_{m+k} \circ \psi_{m+k} \circ \psi_{m+k-1} \circ \cdots \circ \psi_\ell(b_\ell) - b_\ell\| < \sum_{i=1}^{k} \left(\frac{1}{2}\right)^{\ell+i} < \left(\frac{1}{2}\right)^{\ell-1} < \frac{\varepsilon}{2}.$$

This implies, by identifying b_ℓ with the image of it in B, that

$$\|\varphi(b_\ell) - b_\ell\| \leqslant \frac{\varepsilon}{2}.$$

Therefore

$$\|a_m - \varphi(b_\ell)\| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon.$$

Since $\{a_n\}$ is dense in the unit ball of A, the above inequality implies that φ is surjective. Since $B \in \Omega$, by Proposition 5.4, $A \in \Omega$. ∎

COROLLARY 5.6. *Every hereditary C^*-subalgebra of a C^*-algebra A in Ω is in Ω.*

Proof. Suppose that B is a hereditary C^*-subalgebra of A. Then, since A has real rank zero and is separable, B has an approximate identity $\{e_n\}$ consisting of projections. So $B = \overline{\bigcup_{n=1}^{\infty} e_n A e_n}$. Clearly, every $e_n A e_n \in \Omega$. Therefore the corollary follows from Theorem 5.5. ∎

COROLLARY 5.7. *The class Ω is closed under tensor products with any* AF-*algebras.*

Proof. Let $A \subset \Omega$ and B be an AF-algebra. Write $B = \lim_{n\to\infty}(B_n, \varphi_{n,n+1})$, where each B_n is a finite direct sum of matrices. Clearly, $A \otimes B = \lim_{n\to\infty}(A \otimes B_n, \mathrm{id}_A \otimes \varphi_{n,n+1})$. Since every $A \otimes B_n$ is clearly in Ω we conclude that $A \otimes B \in \Omega$. ∎

As we pointed earlier, we know that C^*-algebras in Ω are typically neither stably finite nor purely infinite. E_n is a typical example. However, in extreme cases, it contains all AF-algebras and all separable nuclear purely infinite simple C^*-algebras with UCT and trivial K_1.

PROPOSITION 5.8. *Every* AF-*algebra is in Ω.*

Proof. This is because every matrix algebra is an E-algebra. ∎

REMARK 5.9. A separable nuclear C^*-algebra algebra is said to satisfy UCT, if for any separable C^*-algebra B the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A,B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0$$

holds.

By [40], every C^*-algebra in the "bootstrap" class has UCT.

PROPOSITION 5.10. *Every nuclear purely infinite simple C^*-algebra with* UCT *and with trivial K_1-group is in Ω.*

Proof. It is clear that $\mathcal{O}_\infty \in \Omega$. It certainly contains all Cuntz-algebras $\mathcal{O}_n$ (even or odd). Therefore, by 3.8 in [29], for any countable abelian group G and an element $g \in G$, there is a purely infinite simple C^*-algebra $A \in \Omega$ such that $(K_0(A), [1_A]) = (G, g)$. (Actually, 3.8 of [29] deals only with the case that G has no even torsion. But that is because that the paper was written before [33] and we did not deal with even Cuntz-algebra. The proof of 3.8 in [29] certainly works for even torsion too.) Furthermore, there is a nonunital purely infinite simple C^*-algebra $B \in \Omega$ such that $K_0(B) = G$. It follows from the recent result of E. Kirchberg ([24]) and N.C. Phillips ([34]) that every nuclear purely infinite simple C^*-algebra with UCT and with trivial K_1-group is isomorphic to one of the purely infinite simple C^*-algebra described above. ∎

We end the paper with a few more words about the invariant $V(A)$. Let $\mathcal{V}$ be the set of direct limits of finite direct sums of semigroup $V(E_n)$, $V(\mathcal{O}_m) \cong \{0\} \sqcup \mathbb{Z}/(m-1)\mathbb{Z}$ and $\mathbb{Z}_+$. Then we have by Theorem 1.8 (and its proof) the following conclusion:

PROPOSITION 5.11. *Let $V \in \mathcal{V}$ be a semigroup and $v \in V$. Then there exists a unital C^*-algebra $A \in \Omega$ such that*

$$(V(A), [1_A]) = (V, v).$$

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REFERENCES

1. W. ARVESON, Notes on extensions of C^*-algebras, *Duke J. Math.* **44**(1977), 329–355.
2. O. BRATTELI, G.A. ELLIOTT, D.E. EVANS, A. KISHIMOTO, On the classification of C^*-algebras of real rank zero. III: The infinite case, in preparation.
3. L.G. BROWN, The universal coefficient theorem for Ext and quasidiagonality, Proc. International Conf. on Operator Algebras and Group Representations, Neptun, Romania, 1980.
4. L.G. BROWN, M. DĂDÂRLAT, Extensions of C^*-algebras and quasidiagonality, preprint.
5. L.G. BROWN, G.K. PEDERSEN, C^*-algebras of real rank zero, *J. Funct. Anal.* **99** (1991), 131–149.
6. J. CUNTZ, K-theory for certain C^*-algebras, *Ann. of Math. (2)* **113**(1981), 181–197.

7. M. DĂDÂRLAT, Reduction to dimension three of local spectra of real rank zero C^*-algebras, preprint.
8. M. DĂDÂRLAT, Approximately unitarily equivalent morphisms and inductive limit C^*-algebras, preprint.
9. M. DĂDÂRLAT, T. LORING, Extensions of certain real rank zero C^*-algebras, preprint.
10. M. DĂDÂRLAT, T. LORING, Classifying C^*-algebras via ordered, mod-p K-theory, preprint.
11. G.A. ELLIOTT, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, *J. Algebra* **38**(1876), 29–44.
12. G.A. ELLIOTT, On the classification of C^*-algebras of real rank zero, *J. Reine Angew. Math.* **443**(1993), 179–219.
13. G.A. ELLIOTT, A classification of certain simple C^*-algebras, in *Quantum and Non-Commutative Analysis* (edited by H. Araki et. al), Kluwer, Dordrecht, 1993, pp. 373–385.
14. G.A. ELLIOTT, A classification of certain simple C^*-algebras. II, preprint.
15. G.A. ELLIOTT, Are amenable C^*-algebras classifiable?, Representation Theory of Groups and Algebras, Contemporary Math., vol 145, Amer. Math. Soc., Providence RI., 1993.
16. G.A. ELLIOTT, D.E. EVANS, The structure of irrational rotation C^*-algebras, *Ann. of Math. (2)* **138**(1993), 477–501.
17. G.A. ELLIOTT, G. GONG, On inductive limits of matrix algebras over two-tori, preprint.
18. G.A. ELLIOTT, G. GONG, On the classification of C^*-algebras of real rank zero. II, preprint.
19. G.A. ELLIOTT, G. GONG, H. LIN, C. PASNICU, Abelian C^*-subalgebras of C^*-algebras of real rank zero and inductive limit C^*-algebras, *Duke J. Math.*, to appear.
20. G.A. ELLIOTT, M. RØRDAM, Classification of certain infinite simple C^*-algebras. II, preprint.
21. G. GONG, Classification of C^*-algebras of real rank zero and unsuspended E-equivalent types, preprint.
22. G. GONG, On inductive limits of matrix algebras over higher dimensional spaces, part I, preprint.
23. G. GONG, On inductive limits of matrix algebras over higher dimensional spaces, part II, preprint.
24. E. KIRCHBERG, in preparation.
25. H. LIN, Approximation by normal elements with finite spectra in C^*-algebras of real rank zero, *Pacific J. Math.*, to appear.
26. H. LIN, Almost commuting unitaries and classification of purely infinite simple C^*-algebras, preprint.
27. H. LIN, Almost commuting unitaries in purely infinite simple C^*-algebras, *Math. Ann.* **303**(1995), 599–616.
28. H. LIN, N.C. PHILLIPS, Classification of direct limits of even Cuntz-circle algebras, *Mem. Amer. Math. Soc.* **118**(1995).
29. H. LIN, N.C. PHILLIPS, Approximate unitarily equivalence of homomorphisms from $\mathcal{O}_\infty$, *J. Reine Angew. Math.* **464**(1995), 173–186.
30. H. LIN, N.C. PHILLIPS, Almost multiplicative morphisms and Cuntz-algebra $\mathcal{O}_2$, *J. International Math.* **6**(1995), 625–643.

31. T.A. LORING, C^*-algebras generated by stable relations, *J. Funct. Anal.* **112**(1993), 159–203.
32. G.K. PEDERSEN, *C^*-Algebras and Their Automorphism Groups*, Academic Press, London, New York, San Francisco, 1979.
33. N.C. PHILLIPS, Approximate unitarily equivalence of homomorphisms from odd Cuntz-algebras, preprint.
34. N.C. PHILLIPS, A classification theorem for nuclear purely infinite simple C^*-algebras, preprint.
35. M. RØRDAM, Classification of inductive limits of Cuntz algebras, *J. Reine Angew. Math.* **440**(1993), 175–200.
36. M. RØRDAM, A short proof of Elliott's Theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, *C.R. Math. Rep. Acad. Sci. Canada* **16**(1994), 31–36.
37. M. RØRDAM, Classification of Cuntz-Krieger algebras, *K-Theory*, to appear.
38. M. RØRDAM, Classification of certain infinite simple C^*-algebras, *J. Funct. Anal.*, preprint.
39. M. RØRDAM, Classification of certain infinite simple C^*-algebras. III, preprint.
40. J. ROSENBERG, C. SCHOCHET, The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor, *Duke Math. J.* **55**(1987), 431–474.
41. H. SU, K-theoretic classifications for certain real rank zero C^*-algebras with torsion K_1, in *Quantum and Non-Commutative Analysis* (edited by H. Araki et. al), Kluwer, Dordrecht, 1993, 387–397.
42. H. SU, On the classification of C^*-algebras of real rank zero: Inductive limits of matrix algebras over non-Hausdorff graphs, *Mem. Amer. Math. Soc.* **114**(1995).
43. H. SU, Classification for certain real rank zero C^*-algebras, preprint.
44. K. THOMSEN, On isomorphisms of inductive limits of C^*-algebras, *Proc. Amer. Math. Soc.* **113** (1991), 947-953.
45. D. VOICULESCU, A non-commutative Weyl-von Neumann Theorem, *Rev. Roumaine Math. Pures Appl.* **21**(1976), 97–113.

HUAXIN LIN
Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
U.S.A.

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