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THE DUAL THEORY OF WELL-BOUNDED OPERATORS

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ABSTRACT. The adjoint of a well-bounded operator is also well-bounded, but in general the hoped for natural relationship between the corresponding decompositions of the identity do not hold if the underlying Banach space is nonreflexive. In this paper we discuss the conditions under which the adjoint family of a decomposition of the identity forms a decomposition of the identity for the adjoint of a well-bounded operator.

KEYWORDS: *Well-bounded operators, decompositions of the identity.*

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1. INTRODUCTION

Well-bounded operators are those which possess a functional calculus for the absolutely continuous functions on some compact interval $[a, b]$ of the real line. They were introduced by Smart ([15]) to provide a theory which covers operators whose spectral decompositions may converge only conditionally. Smart and Ringrose ([13]) proved that on a reflexive Banach space X, a well-bounded operator can be written as an integral with respect to a suitable family of projections acting on X. Ringrose ([14]) later considered the extension of this theory to the nonreflexive case. The results obtained in this case are less satisfactory because the family of projections acts on the dual space X^* rather than on X, and in general, is no longer uniquely determined. Indeed, it is perhaps more appropriate to say that Ringrose showed that the adjoint of a well-bounded operator (rather than the operator itself) always admits an integral representation with respect to a family of projections. Ringrose called this family of projections a decomposition of the identity (a precise definition will be given is Section 2). It is not too difficult to find examples of well-bounded operators which are not 'decomposable in X', in that the projections in the decomposition of the identity are not formed from the adjoints of projections on X.

It is clear that an operator is well-bounded if and only if its adjoint is. On the other hand, if X is not reflexive, many of the properties that a well-bounded operator may possess do not pass to the adjoint operator. When constructing examples on nonreflexive Banach spaces, one typically has to work quite hard to prove that what one expects to get, actually occurs. One would hope that a suitable family of projections for the adjoint of a well-bounded operator T could be formed by taking the adjoints of a decomposition of the identity associated with T. One of the main aims of this paper is to show that on a wide range of nonreflexive Banach spaces it is not possible to do this. We also show that a necessary (but not sufficient) condition for one to be able to find a decomposition of the identity for T^* by taking the adjoints of a decomposition of the identity for T, is that T have a unique decomposition of the identity. These questions are addressed in Section 4.

In Section 3 we have included some results on quotients and restrictions of well-bounded operators. Some of these results will be needed in Section 4.

2. DEFINITIONS

In this section we shall give some of the basic definitions regarding well-bounded operators. The theory of well-bounded operators is given in more detail in [8].

Throughout X will denote a complex Banach space with dual space X^*. We shall frequently blur the distinction between a Banach space and its canonical image in its second dual. The Banach algebra of all bounded linear operators on X will be denoted by $B(X)$. We use $\mathrm{Lat}(T)$ to denote the invariant subspace lattice of T. If Y is an invariant subspace under T, then $T|Y$ and T^Y are the restriction and the quotient of T on Y, respectively.

DEFINITION 2.1. An operator T in $B(X)$ is said to be *well-bounded* if there exist a constant K and a compact interval $[a,b] \subset \mathbf{R}$ such that for all polynomials p,

$$\|p(T)\| \leqslant K\left\{|p(a)| + \int_a^b |p'(t)|\,dt\right\}.$$

If T is a well-bounded operator, T possesses a bounded functional calculus for $AC[a,b]$, the Banach algebra of all absolutely continuous functions on $[a,b]$.

That is, there exists a Banach algebra homomorphism $f \mapsto f(T)$, extending the natural definition for polynomials, such that

$$\|f(T)\| \leqslant K\left\{|f(a)| + \operatorname*{var}_{[a,b]} f\right\} \equiv K\|f\|_{AC}.$$

DEFINITION 2.2. *A decomposition of the identity for* X (on $[a,b]$) is a family $\{E(s)\}_{s\in\mathbb{R}}$ of projections on X^* such that:

(i) $E(s) = 0$ $(s < a)$, $E(s) = I$ $(s \geqslant b)$;

(ii) $E(s)E(t) = E(t)E(s) = E(s)$ $(s \leqslant t)$;

(iii) there is a real constant K such that $\|E(s)\| \leqslant K$, for $s \in \mathbb{R}$;

(iv) the function $s \mapsto \langle x, E(s)x^*\rangle$ is Lebesgue measurable for $x \in X$ and $x^* \in X^*$;

(v) for each $x \in X$, the map $\gamma_x : X^* \to L^\infty[a,b], x^* \mapsto \langle x, E(s)x^*\rangle$ is continuous when X^* and $L^\infty[a,b]$ are given their weak-$*$ topologies as the duals of X and $L^1[a,b]$ respectively;

(vi) for all $x \in X$, $x^* \in X^*$ and $t_0 \in [a,b)$, if the right derivative of the function $t \mapsto \int_a^t \langle x, E(u)x^*\rangle\, du$ exists at t_0, then its value is $\langle x, E(t_0)x^*\rangle$.

Given a decomposition of the identity $\{E(s)\}$, there exists a unique $T \in B(X)$ such that

$$\text{(1.1)} \qquad \langle Tx, x^*\rangle = b\langle x, x^*\rangle - \int_a^b \langle x, E(s)x^*\rangle\, ds, \qquad x \in X,\ x^* \in X^*.$$

Ringrose proved that T must be well-bounded. Conversely, given a well-bounded operator T, there exists a decomposition of the identity such that T has such a representation. In general however, the decomposition of the identity is not uniquely determined by T. If there is only one decomposition of the identity for which equation (1.1) holds we say that T is *uniquely decomposable.*

Condition (vi) in Definition 2.2 is in some sense optional, as any family of projections which satisfies the first five conditions defines a well-bounded operator via equation (1.1). Without condition (vi) (or some alternative; see for example [7]), one does give up any results about uniqueness of decompositions of the identity associated with well-bounded operators.

We recall the following standard definitions.

DEFINITION 2.3. Let T be a well-bounded operator and let $\{E(s)\} \subset B(X^*)$ be a decomposition of the identity for T. If there exists a family of projections $\{F(s)\} \subset B(X)$ such that $F(s)^* = E(s)$ for all $s \in \mathbb{R}$, we will say that T is *decomposable in* X. Further, T is said to be *of type* (A) if the function $s \mapsto F(s)x$ is everywhere continuous on the right for every $x \in X$. We shall say that T is *of type* (B) if T is of type (A) and, in addition, for each real s, $\lim_{t \to s^-} F(t)x$ exists for every $x \in X$.

If T is of type (B), then the projection-valued function $F : \mathbb{R} \to B(X)$ forms a *spectral family* of projections on the Banach space X, concentrated on the interval $[a, b]$. The spectral theorem for well-bounded operators of type (B) states that there is a one-to-one correspondence between well-bounded operators of type (B) and concentrated spectral families given by the integral formula

$$T = \int_{[a,b]}^{\oplus} s \, \mathrm{d}F(s).$$

This happens if and only if the AC-functional calculus of T is weakly compact (see for example [8], [7]).

3. QUOTIENTS AND RESTRICTIONS OF WELL-BOUNDED OPERATORS

It is clear that the restriction of a well-bounded operator to an invariant subspace is again well-bounded. Showing that one gets an appropriate relationship between decompositions of the identity is rather more difficult. The following theorem shows that one can at least restrict to a complemented invariant subspace. If $X = Y \oplus Z$ we shall often denote an element of X by (y, z), where $y \in Y$ and $z \in Z$.

THEOREM 3.1. *Suppose that* $X = Y \oplus Z$ *and that* $P \in B(X)$ *is the projection onto* Y *with kernel* Z. *If* $\{E(\lambda)\} \subset B(X^*)$ *is a decomposition of the identity such that* $P^* E(\lambda) = E(\lambda) P^*$ *for all* $\lambda \in \mathbb{R}$, *then setting* $F(\lambda) y^* = E(\lambda)(y^*, 0)$ *defines a decomposition of the identity on* $Y^* = P^* X^*$. *Furthermore, if* $T \in B(X)$ *is the well-bounded operator associated with* $\{E(\lambda)\}$, *then* Y *is an invariant subspace for* T *and* $T|Y$ *is the well-bounded operator associated with* $\{F(\lambda)\}$.

Proof. In order to avoid having to keep track of where identifications are being made we introduce the following operators: $\widetilde{P} \in B(X^*, Y^*)$, $\widetilde{P}(y^*, z^*) = y^*$, and $J \in B(Y^*, X^*)$, $J(y^*) = (y^*, 0)$. A more precise definition of $F(\lambda)$ then is $F(\lambda) = \widetilde{P} E(\lambda) J$. Note the identities

$$\widetilde{P} J = I \in B(Y^*), \qquad J \widetilde{P} = P^*, \qquad \widetilde{P} P^* = \widetilde{P}.$$

It follows easily from these identities that $\{F(\lambda)\}$ is a uniformly bounded, increasing family of projections. It remains to show that $\{F(\lambda)\}$ satisfies conditions (iv), (v) and (vi) of Definition 2.2.

For all $y \in Y$ and $y^* \in Y^*$, $\langle y, F(\lambda)y^*\rangle = \langle (y,0), E(\lambda)(y^*,0)\rangle$, so (iv) follows immediately from the fact that $\{E(\lambda)\}$ is a decomposition of the identity.

Fix $y_0 \in Y$. Suppose that for all $y \in Y$, $\langle y, y_\alpha^*\rangle \to \langle y, y_0^*\rangle$. Then, for all $(y,z) \in X$, $\langle (y,z), Jy_\alpha^*\rangle = \langle y, y_\alpha^*\rangle \to \langle y, y_0^*\rangle = \langle (y,z), Jy_0^*\rangle$. Since $\{E(\lambda)\}$ is a decomposition of the identity, this means that $\langle (y,z), E(\cdot)Jy_\alpha^*\rangle \to \langle (y,z), E(\cdot)Jy_0^*\rangle$ in the weak-$*$ topology of L^∞. That is,

$$\langle y_0, F(\cdot)y_\alpha^*\rangle = \langle (y_0,0), E(\cdot)Jy_\alpha^*\rangle \to \langle (y_0,0), E(\cdot)Jy_0^*\rangle = \langle y_0, F(\cdot)y_0^*\rangle,$$

and so condition (v) is satisfied.

Showing that condition (vi) is satisfied is similar.

Let T denote the well-bounded operator associated with $\{E(\lambda)\}$. To show that Y is an invariant subspace for T it suffices to show that for all $y \in Y$ and $z^* \in Z^*$, $\langle T(y,0), (0,z^*)\rangle = 0$. Now

$$\begin{aligned}\langle T(y,0),(0,z^*)\rangle &= b\langle (y,0),(0,z^*)\rangle - \int_a^b \langle (y,0), E(\lambda)(0,z^*)\rangle \, \mathrm{d}\lambda \\ &= 0 - \int_a^b \langle P(y,0), E(\lambda)(0,z^*)\rangle \, \mathrm{d}\lambda = -\int_a^b \langle (y,0), P^*E(\lambda)(0,z^*)\rangle \, \mathrm{d}\lambda \\ &= -\int_a^b \langle (y,0), E(\lambda)P^*(0,z^*)\rangle \, \mathrm{d}\lambda = 0.\end{aligned}$$

We leave it to the reader to verify that the well-bounded operator associated with $\{F(\lambda)\}$ is $T|Y$. ∎

THEOREM 3.2. *Suppose that T is a well-bounded operator on a Banach space X and that Y is a subspace of X invariant under T. Then the quotient operator T^Y of T is a well-bounded operator on the quotient space X/Y. Furthermore, if T is of type* (B), *then T^Y is also of type* (B).

Proof. Since T is well-bounded, we can find constants a, b and K in $\mathbb{R}$ such that

$$\|p(T)\| \leqslant K\|p\|_{AC[a,b]}$$

for any polynomial p on $[a, b]$. Let $[x]$ be a arbitrary element of the quotient space X/Y , then

$$\begin{aligned}\|p(T^Y)[x]\|_{X/Y} &= \|[p(T)x]\|_{X/Y} = \inf_{y\in Y}\|p(T)x+y\|\\ &\leqslant \inf_{y\in Y}\|p(T)x+p(T)y\| \leqslant \inf_{y\in Y}\|p(T)\|\cdot\|x+y\|\\ &= \|p(T)\|\cdot\|[x]\|_{X/Y} \leqslant K\|p\|_{AC[a,b]}\|[x]\|_{X/Y},\end{aligned}$$

and hence T^Y is well-bounded on X/Y.

Suppose now that T is well-bounded of type (B). Then for any $x \in X$, the map $\varphi : AC[a,b] \to X, \varphi(f) = f(T)x$ is weakly compact. Since the quotient map is continuous, it follows that $\Phi : AC[a,b] \to X/Y$, $\Phi(f) = [\varphi(f)]$ is also weakly compact. Hence T^Y is of type (B). ∎

Let T and Y be as in Theorem 3.2, and let $\{E(s)\}$ be a decomposition of the identity for T. One could hope that each decomposition of the identity for T would give rise to a decomposition of the identity for T^Y. The projections in a decomposition of the identity for T^Y act of course on $(X/Y)^*$ which we shall identify in the usual way with $Y^\perp$. For $s \in \mathbb{R}$ set $G(s) = E(s)|Y^\perp$. In order to show that $G(s)$ is a projection we need to check that $Y^\perp$ is invariant under $E(s)$ for all $s \in \mathbb{R}$.

PROPOSITION 3.3. *Let T be a well-bounded operator and let $\{E(s)\}$ form a decomposition of the identity for T. If Y is an invariant subspace for T then*

$$Y^\perp \in \bigcap_{s\in\mathbb{R}} \operatorname{Lat} E(s),$$

and $\{E(s)|Y^\perp\}$ forms a decomposition of the identity for T^Y.

Proof. Since Y is invariant under T, $Y^\perp$ is invariant under T^*, and hence under $p(T^*)$ for every polynomial p. It follows that $Y^\perp$ is also invariant under $f(T^*)$ for all $f \in AC[a,b]$. Thus, for $x \in Y$ and $x^* \in Y^\perp$ we have

$$\langle f(T)x, x^*\rangle = \langle x, f(T^*)x^*\rangle = 0$$

and

$$\langle x, x^*\rangle = 0.$$

Thus

$$\int_a^b \langle x, E(s)x^*\rangle f'(s)\,ds = b\langle x, x^*\rangle - \langle f(T)x, x^*\rangle = 0.$$

This implies that $\langle x, E(s)x^*\rangle = 0$ for almost for all $s \in \mathbb{R}$. Condition (vi) for a decomposition of the identity ensures that $\langle x, E(s)x^*\rangle = 0$ for all $s \in \mathbb{R}$. That is $E(s)x^* \in Y^\perp$, for any $s \in \mathbb{R}$.

Thus setting $G(s) = E(s)|Y^\perp$ gives a uniformly bounded family of projections on $(X/Y)^*$. Showing that this family satisfies conditions (i)–(vi) of Definition 2.2 is not too difficult since if $[x] \in X/Y$ and $x^* \in Y^\perp$ then

$$\langle [x], G(s)x^*\rangle = \langle x, E(s)x^*\rangle.$$

We leave the details to the reader. ∎

If T is of type (B), then the spectral families of T and T^Y match up. The following proposition is due to Berkson ([1], Theorem 3.1).

PROPOSITION 3.4. *Let T be well-bounded operator of type* (B) *and let $\{F(s)\}$ form a spectral family for T. Then*

$$\mathrm{Lat}\,(T) = \bigcap_{s\in\mathbb{R}} \mathrm{Lat}\,F(s).$$

COROLLARY 3.5. *Let T be a well-bounded operator of type* (B) *and let $\{F(s)\}$ be the spectral family for T. Suppose that Y is an invariant subspace for T. Then $\{F(s)^Y\}$ forms a spectral family for T^Y. In this case,*

$$T^Y = \int_{[a,b]}^{\oplus} s \,\mathrm{d}F(s)^Y.$$

Again the details are easy to check.

4. DUAL PROPERTIES OF WELL-BOUNDED OPERATORS

It is clear that $T \in B(X)$ is well-bounded if and only if the adjoint of T is. The main questions that we want to consider concern the relationships between decompositions of the identity for T and those for T^*. In particular, we shall examine the question of when the adjoints of a decomposition of the identity for T form a decomposition of the identity for T^*.

To place this in some context we shall recall some of the corresponding results for scalar-type spectral operators. We refer the reader to [8] for the appropriate definitions.

THEOREM 4.1. ([8], Theorem 6.9) *Let $T \in B(X)$ be a scalar-type spectral operator with resolution of the identity $\mathcal{E}(\cdot)$. Then T^* is a scalar-type prespectral operator with resolution of the identity $\mathcal{E}(\cdot)^*$ of class X.*

Theorems 4.2 and 4.3 are due to Jiang and Zou ([11]).

THEOREM 4.2. *Suppose that $c_0 \not\subset X^*$. If $T \in B(X)$ is a scalar-type spectral operator with resolution of the identity $\mathcal{E}(\cdot)$, then T^* is a scalar-type spectral operator with resolution of the identity $\mathcal{E}(\cdot)^*$.*

Proof. By Theorem 4.1, T^* is scalar-type prespectral. Clearly X^* can not contain a copy of ℓ^∞. Theorem 1 of [10] shows that T^* is therefore a scalar-type spectral operator. Theorem 4.1 and Theorem 6.7 of [8] show that the resolution of the identity for T^* is $\mathcal{E}(\cdot)^*$. ∎

THEOREM 4.3. *Suppose that $c_0 \subset X^*$. Then there exists a scalar-type spectral operator $T \in B(X)$ such that T^* is not scalar-type spectral.*

Proof. If $c_0 \subset X^*$ then $X = \ell^1 \oplus Y$ for some Banach space Y ([12], Proposition 2.e.8). Define $T_0 \in B(\ell^1)$ by $T_0(x_1, x_2, \ldots) = (x_1, x_2/2, x_3/3, \ldots)$, and $T \in B(X)$ by $T = T_0 \oplus \mathbf{0}$ on $\ell^1 \oplus Y$. It is easy to check that T is scalar-type spectral, but T^* is not. ∎

We now return to our study of well-bounded operators.

DEFINITION 4.4. Suppose that $\{E(\lambda)\} \subset B(X^*)$ is a decomposition of the identity. We shall say that $\{E(\lambda)\}$ is *transposable* if the family of projections $\{E(\lambda)^*\} \subset B(X^{**})$ is also a decomposition of the identity.

Of course if $\{E(\lambda)\}$ is a decomposition of the identity then $\{E(\lambda)^*\}$ is necessarily a uniformly bounded increasing family of projections on X^{**}. Verifying conditions (iv)–(vi) of Definition 2.2 is usually more difficult however. The next result shows that one does at least always gets the expected well-bounded operator from a transposable decomposition of the identity. The straightforward proof is left for the reader.

PROPOSITION 4.5. *Suppose that $\{E(\lambda)\} \subset B(X^*)$ is a decomposition of the identity with associated well-bounded operator T. If $\{E(\lambda)^*\}$ is also a decomposition of the identity, then its associated well-bounded operator is T^*, and hence T^* is decomposable in X^*.*

COROLLARY 4.6. *If T^* is not decomposable in X^* then no decomposition of the identity for T is transposable.*

If T^* is decomposable in X^*, then it is uniquely decomposable. On the other hand, it is well-known that a well-bounded operator may have more than one decomposition of the identity. It follows that if T is such an operator, then at most one of the decompositions of the identity for T can be transposable. As we shall see below, in fact none of the decompositions of the identity for such an operator can be transposable.

We shall now show that the converse to Proposition 4.5 holds. For $s \in [a,b]$, let $\mathcal{F}_s = \{f \in AC[a,b] : f(t) = 0 \text{ for } t \in [a,s]\}$. The following lemma is a corollary of Theorem 3.3 (ii) of [17] and Theorem 5.7 of [2].

LEMMA 4.7. *Let T be a well-bounded operator which is decomposable in X and $\{F(s)\}$ be a family of projections on X whose adjoints form a decomposition of the identity for T. Then*

$$F(s)X = \{x \in X : f(T)x = 0 \text{ for all } f \in \mathcal{F}_s\}.$$

THEOREM 4.8. *Let $T \in B(X)$ be well-bounded. If T^* is decomposable in X^* then T is uniquely decomposable, and the decomposition of the identity for T is transposable.*

Proof. By Theorem 15.19 of [8], there exists a decomposition of the identity $\{E(s)\}$ for T such that if $S \in B(X)$ commutes with T, then S^* commutes with each $E(s)$, $s \in \mathbf{R}$. In particular, T^* commutes with each $E(s)$, $s \in \mathbf{R}$.

Let $\{E_1(s)\}$ be a family of projections on X^* such that $\{E_1^*(s)\}$ forms a decomposition of the identity for T^*. It is sufficient then for us to prove that $E(s) = E_1(s)$, for all $s \in \mathbf{R}$. By Theorem 16.3 of [8],

$$E(s)E_1(s) = E_1(s)E(s),$$

for $s \in \mathbf{R}$. By Lemma 4.7 and Theorem 15.18 of [8], the ranges of each $E(s)$ and $E_1(s)$ are equal, so $E(s) = E_1(s)$, for $s \in \mathbf{R}$.

Now, if T is not uniquely decomposable, then by Corollary 15.23 of [8] there exist two distinct associated decompositions of the identity, say $\{E(s)\}$ and $\{F(s)\}$ having property (iii) of Theorem 15.19 of [8]. But the argument above proves that $E(s) = F(s)$, for all $s \in \mathbf{R}$. The theorem is thus proved. ∎

COROLLARY 4.9. *Suppose that T is not uniquely decomposable. Then no decomposition of the identity for T is transposable.*

EXAMPLE 4.10. It follows from the above that the well-bounded operator

$$Tf(x) = xf(x) + \int_x^1 f(t)\,\mathrm{d}t, \qquad x \in [0,1], \quad f \in L^\infty[0,1]$$

is not decomposable in $L^\infty[0,1]$. This is because $T = S^*$ where S is the operator on $L^1[0,1]$ given by

$$Sf(x) = xf(x) + \int_0^x f(t)\,\mathrm{d}t, \qquad x \in [0,1]$$

which is known to have many decomposition of the identity (see [8], Example 15.25).

A natural problem is to try to identify those Banach spaces on which every decomposition of the identity is transposable. If X is reflexive then everything behaves just as one would hope. The following proposition is a simple consequence of Theorem 17.17, [8] and the proof of Theorem 3.5, [5].

PROPOSITION 4.11. *Suppose that X is reflexive, and that $T \in B(X)$ is a well-bounded operator with (unique) decomposition of the identity $\{E(\lambda)\}$. Then $\{E(\lambda)\}$ is transposable.*

Beyond Theorem 4.8 there seems to be very little that can be said in a positive direction once one leaves reflexive spaces. As the examples below will illustrate, even if a well-bounded operator has a decomposition of the identity with good properties, its adjoint operator need not be so well-behaved. In what follows, if a well-bounded operator S has a unique decomposition of the identity we shall sometimes denote this by $\{E_S(\lambda)\}$. As usual, we shall let ba denote the dual of ℓ^∞.

EXAMPLE 4.12. Define $T \in B(c_0)$ by $T(x_1, x_2, \ldots) = (x_1, x_2/2, x_3/3, \ldots)$. It is easy to check that T and T^* are both well-bounded operators of type (B). Let $\{E(\lambda)\} \subset B(c_0)$ denote the spectral family for T. Then T and T^* both have unique decompositions of the identity, given by $\{E(\lambda)^*\}$ and $\{E(\lambda)^{**}\}$ respectively. However, as we shall see, $\{E(\lambda)^{***}\}$ is not a decomposition of the identity.

Let $\{F(\lambda)\} \subset B(ba)$ denote a decomposition of the identity for T^{**}. Then by Theorem 15.8 (iii), [8]

$$\{\varphi \in ba \,:\, T^{***}\varphi = 0\} \subset F(0)(ba).$$

However, if φ is a Banach limit on ℓ^∞ then, for all $x \in \ell^\infty$,

$$\langle x, T^{***}\varphi\rangle = \langle T^{**}x, \varphi\rangle = 0$$

since $T^{**}x \in c_0$. Thus $F(0) \neq 0$. But it is easy to check that $E(0) = 0$. Thus $F(0) \neq E(0)^{***} = E_{T^*}(0)^*$. It follows that $\{E(\lambda)^{***}\}$ is not a decomposition of the identity since if it were, the corresponding well-bounded operator would have to be T^{**}.

In this example then, the decomposition of the identity for T is transposable, whilst that for T^* is not. In particular, this gives an example of a well-bounded operator on ℓ^1 whose (only) decomposition of the identity is not transposable.

We can actually say a little more about this example. It is clear that T, and hence all its adjoints, have a $C[0,1]$ functional calculus. Since ba does not contain a subspace isomorphic to c_0, this means that $T^{***} \in B(ba)$ is scalar-type spectral ([6], Theorem 3.1). In particular then, T^{***} is a well-bounded operator of type (B) ([8], Theorem 16.17), and hence is decomposable in ba. By Theorem 4.8, this means that the T^{**} is uniquely decomposable, and that the decomposition of the identity for T^{**} *is* transposable.

Constructing examples of well-bounded operators on c_0 or ℓ^∞ which do not possess decompositions of the identity which are transposable is more difficult.

EXAMPLE 4.13. For $n = 1, 2, \ldots$ define projections $Q_n \in B(c_0)$ by

$$Q_n(x_1, x_2, \ldots) = (\underbrace{x_n, \ldots, x_n}_{n \text{ times}}, x_{n+1}, x_{n+2}, \ldots).$$

For $n \geqslant 1$, let $P_n = Q_n - Q_{n+1}$. Then $\{P_n\}$ forms a sequence of disjoint finite-rank projections so Theorem 3.2, [4] shows that $T = \sum_{n=1}^{\infty} \frac{1}{n} P_n$ is well-bounded. Indeed, since $Q_n \to 0$ in the strong operator topology, it is not too hard to see that T is of type (B). The unique decomposition of the identity for T is given by

$$E_T(\lambda) = \begin{cases} 0 & \text{if } \lambda \leqslant 0, \\ Q_n^* & \text{if } \lambda \in [\frac{1}{n}, \frac{1}{n-1}) \text{ for } n \geqslant 2, \\ I & \text{if } \lambda \geqslant 1. \end{cases}$$

We claim that $\{E_T(\lambda)\}$ is not transposable, or equivalently, that T^* is not decomposable in X^*. Let $\{E(\lambda)\} \subset B(\ell^\infty)$ denote any decomposition of the identity for T^*. Standard operator theoretic arguments show that $E(\lambda) = E_T(\lambda)^*$ for all $\lambda \neq 0$. Our aim then is to show that there does not exist $S \in B(\ell^1)$ such that $S^* = E(0)$.

We claim that if $y = (y_n) \in c$ then $E(0)y = L(y)u$ where $u = (1, 1, 1, \ldots) \in c$ and $L(y) = \lim_n y_n$. Fix $x = (x_n) \in \ell^1$ and $y = (y_n) \in c \subset \ell^\infty$. Define

$$G(t) = \int_0^t \langle x, E(\lambda)y \rangle \, d\lambda.$$

For $t \in (0, 1)$, let N_t be the unique integer such that $1 - t < tN_t \leqslant 1$. Then

$$\begin{aligned} \frac{G(t) - G(0)}{t} &= \frac{1}{t} \int_0^t \langle x, E(\lambda)y \rangle \, d\lambda \\ &= \frac{1}{t} \int_0^{\frac{1}{N_t}} \langle x, E(\lambda)y \rangle \, d\lambda + \frac{1}{t} \int_{\frac{1}{N_t}}^t \langle x, E(\lambda)y \rangle \, d\lambda. \end{aligned}$$

Now

$$\left|\frac{1}{t}\int_{\frac{1}{N_t}}^{t}\langle x, E(\lambda)y\rangle\,\mathrm{d}\lambda\right| \leqslant \frac{t-\frac{1}{N_t}}{t} K\,\|x\|\,\|y\| \to 0,$$

as $t \to 0^+$. On the other hand

$$\begin{aligned}\frac{1}{t}\int_0^{\frac{1}{N_t}}\langle x, E(\lambda)y\rangle\,\mathrm{d}\lambda &= \frac{1}{t}\sum_{n=N_t+1}^{\infty}\frac{1}{n(n-1)}\langle Q_n^* x, y\rangle \\ &= \frac{1}{t}\sum_{n=N_t+1}^{\infty}\frac{1}{n(n-1)}\Big(y_n\sum_{i=1}^{n}x_i + \sum_{i=n+1}^{\infty}x_i y_i\Big).\end{aligned}$$

Let

$$\varepsilon_n = \Big(y_n\sum_{i=1}^{n}x_i + \sum_{i=n+1}^{\infty}x_i y_i\Big) - L(y)\sum_{i=1}^{\infty}x_i.$$

Now, given any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $n > N_{t_\varepsilon}$, $|\varepsilon_n| < \varepsilon$. Thus,

$$\begin{aligned}\frac{1}{t}\int_0^{\frac{1}{N_t}}\langle x, E(\lambda)y\rangle\,\mathrm{d}\lambda &= \frac{1}{t}\sum_{n=N_t+1}^{\infty}\frac{1}{n(n-1)}\Big(L(y)\sum_{i=1}^{\infty}x_i + \varepsilon_n\Big) \\ &= \frac{1}{tN_t}L(y)\sum_{i=1}^{\infty}x_i + \frac{1}{t}\sum_{n=N_t+1}^{\infty}\frac{\varepsilon_n}{n(n-1)} \\ &= I + II,\end{aligned}$$

say. For any $t < t_\varepsilon$,

$$|II| < \frac{\varepsilon}{t}\sum_{n=N_t+1}^{\infty}\frac{1}{n(n-1)} \leqslant \varepsilon.$$

On the other hand, $(tN_t)^{-1} \to 1$ as $t \to 0^+$. It follows then that G is right differentiable at 0 and that

$$\lim_{t\to 0+}\frac{G(t)-G(0)}{t} = L(y)\sum_{i=1}^{\infty}x_i = \langle x, L(y)u\rangle.$$

By condition (vi) for a decomposition of the identity then, we must have that $E(0)y = L(y)u$.

If $y = (y_i) \in \ell^\infty$, then y is the weak-$*$ limit of the sequence $\{w_n\} \subset c$ where $w_n = (y_1, y_2, \ldots, y_n, y_n, \ldots)$. Suppose that $S \in B(\ell^1)$ satisfies $S^* = E(0)$. Then, for any $x \in \ell^1$,

$$\begin{aligned}\langle x, E(0)y\rangle = \langle Sx, y\rangle &= \lim_n\langle Sx, w_n\rangle = \lim_n\langle x, E(0)w_n\rangle \\ &= \lim_n\langle x, L(w_n)u\rangle = \lim_n y_n\sum_{i=1}^{\infty}x_i.\end{aligned}$$

But this last limit may not exist. It follows that no such operator S can exist.

Thus T^* is not decomposable in ℓ^1 and the decomposition of the identity for T is not transposable. ∎

For the example on ℓ^∞ we shall need a lemma due to Ringrose (see [14] or [8], Theorem 15.22).

LEMMA 4.14. *Let $T \in B(X)$ be a well-bounded operator with decomposition of the identity $\{E(\lambda)\} \subset B(X^*)$ concentrated on $[a, b]$. Then T is uniquely decomposable if and only if for all $x \in X$ and $x^* \in X^*$, the map*

$$G(t) = \int_a^t \langle x, E(\lambda)x^* \rangle \, \mathrm{d}\lambda$$

is right differentiable for all $t \in [a, b)$.

EXAMPLE 4.15. Let T be the well-bounded defined in Example 4.13. A slight alteration of the calculation in Example 4.13 can be used to show that T^* is not uniquely decomposable. By Lemma 4.14, it suffices to show that if $\{E(\lambda)\}$ is any decomposition of the identity for T^*, then there exist $x \in \ell^1$ and $y \in \ell^\infty$ such that the map

$$G(t) = \int_a^t \langle x, E(\lambda)y \rangle \, \mathrm{d}\lambda$$

is not right differentiable at 0. Taking $x = (1, 1/4, 1/9, \ldots)$ and $y = (0, 1, 0, 1, \ldots)$ is sufficient. We leave the calculation to the reader.

Let $U = T^{**} \in B(\ell^\infty)$. Suppose that U has a decomposition of the identity which is transposable. By Proposition 4.5, U^* is decomposable in ba and hence is uniquely decomposable with decomposition of the identity $\{E_{U^*}(\lambda)\}$.

It is clear that any two distinct decompositions of the identity for T^* can only differ by their value at $\lambda = 0$. For $\delta \in (0, 1]$, let g_δ denote the AC function which takes on the value 1 at 0, is 0 on the interval $[\delta, 1]$ and which is linear on the interval $[0, \delta]$. From the proof of Theorem 15.22 of [8], as well as Lemma 15.17 of [8], there exist subnets $\{g_\alpha^{(1)}\}_{\alpha \in A}$ and $\{g_\beta^{(2)}\}_{\beta \in B}$ of $\{g_\delta\}_{\delta > 0}$, and decompositions of the identity $\{E^{(1)}(\lambda)\}$ and $\{E^{(2)}(\lambda)\}$ such that $E^{(1)}(0) \neq E^{(2)}(0)$ and such that for all $x \in \ell^1$ and $y \in \ell^\infty$,

$$\lim_{\alpha \in A} \langle g_\alpha^{(1)}(T^*)x, y \rangle = \langle x, E^{(1)}(0)y \rangle$$
$$\lim_{\beta \in B} \langle g_\beta^{(2)}(T^*)x, y \rangle = \langle x, E^{(2)}(0)y \rangle.$$

Since $U^* = T^{***}$ is uniquely decomposable, if $x \in ba$ and $y \in ba^*$ then

$$\lim_{\delta\to 0}\langle g_\delta(U^*)x, y\rangle = \lim_{\delta\to 0}\langle x, g_\delta(U^{**})y\rangle = \langle x, E_{U^*}(0)y\rangle.$$

Taking subnets of $\{g_\delta(U^{**})\}_{\delta>0}$, it follows that

$$\lim_{\alpha\in A}\langle g_\alpha^{(1)}(U^*)x, y\rangle = \lim_{\beta\in B}\langle g_\beta^{(2)}(U^*)x, y\rangle = \langle x, E_{U^*}(0)y\rangle.$$

In particular, if $x \in \ell^1 \subset ba$ and $y \in \ell^\infty \subset ba^*$, then

$$\langle x, E^{(1)}(0)y\rangle = \langle x, E^{(2)}(0)y\rangle = \langle x, E_{U^*}(0)y\rangle.$$

That is, $E^{(1)}(0)y = E^{(2)}(0)y = E_{U^*}(0)y|\ell^1$, and so $E^{(1)}(0) = E^{(2)}(0)$. But this contradicts the fact that T^* is not uniquely decomposable. It follows that U cannot have a transposable decomposition of the identity. ∎

LEMMA 4.16. *Suppose that $X = Y \oplus Z$, that $T_1 \in B(Y)$ and $T_2 \in B(Z)$ are well-bounded, and that $T = T_1 \oplus T_2$. Then T has a transposable decomposition of the identity if and only if T_1 and T_2 both have transposable decompositions of the identity.*

Proof. Let P, $\widetilde{P}$ and J be as in the proof of Theorem 3.1. Suppose that T has a transposable decomposition of the identity $\{E(\lambda)\}$. Then T is uniquely decomposable (Theorem 4.8) and so by Theorem 15.19, [8]

$$P^*E(\lambda) = E(\lambda)P^*, \qquad \lambda \in \mathbb{R}.$$

It follows from Theorem 3.1 that setting $F_1(\lambda) = \widetilde{P}E(\lambda)J$ defines a decomposition of the identity for T_1. Note that $\widetilde{P}^* \in B(Y^{**}, X^{**})$ satisfies $\widetilde{P}^*(y^{**}) = (y^{**}, 0)$, whilst $J^* \in B(X^{**}, Y^{**})$ satisfies $J(y^{**}, z^{**}) = y^{**}$. Since $P^{**}E(\lambda)^* = E(\lambda)^*P^{**}$ it follows from Theorem 3.1 that if we define $F_2(\lambda) = J^*E(\lambda)^*\widetilde{P}^*$, $\lambda \in \mathbb{R}$, then $\{F_2(\lambda)\}$ is a decomposition of the identity for $T^*|Y^* = T_1^*$. But $J^*E(\lambda)^*\widetilde{P}^* = (\widetilde{P}E(\lambda)J)^* = F_1(\lambda)^*$ and so $\{F_1(\lambda)\}$ is transposable. The same argument obviously works for T_2.

The converse implication is completely straightforward, if a little tedious. ∎

THEOREM 4.17. *If X satisfies any of the following conditions then there is a well-bounded operator on X none of whose decompositions of the identity are transposable.*

(i) *X contains a complemented subspace isomorphic to ℓ^1.*

(ii) *X contains a complemented subspace isomorphic to c_0.*

(iii) *X contains a subspace isomorphic to ℓ^∞.*

Proof. (i) and (ii) follow immediately from Examples 4.12 and 4.13 via the usual extension of an operator from a complemented subspace to the whole Banach space. That the required properties pass to the extended operators follows from Lemma 4.16. Case (iii) follows similarly from Example 4.15 by noting any subspace isomorphic to ℓ^∞ is necessarily complemented. ∎

An obvious question is whether the conclusions of Theorem 4.17 hold on every nonreflexive Banach space. This is related to the still open question of whether there are any nonreflexive Banach spaces on which every well-bounded operator is of type (B) (see [3]).

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