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MULTIPLICATION BY FINITE BLASCHKE FACTORS ON DE BRANGES SPACES

DINESH SINGH and VIRENDER THUKRAL

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ABSTRACT. This note characterizes those Hilbert spaces which are algebraically contained in the Hardy space H^2 of scalar valued analytic functions on the open unit disk **D** and on which multiplication by a finite Blaschke product acts as an isometry. A general inner-outer factorization is deduced and some other properties of the operator of multiplication by a finite Blaschke product are described. The main theorem generalizes a recent theorem of de Branges as well as a theorem of Peter Lax.

KEYWORDS: *De Branges spaces, multiplication by a finite Blaschke product, invariant subspaces.*

AMS SUBJECT CLASSIFICATION: Primary 32A35, 46A15; Secondary 30D50.

1. INTRODUCTION

This article describes the solution of the following problem: characterize all Hilbert spaces which are algebraically contained in the Hardy space H^2 of scalar valued analytic functions of the open unit disk **D** and on which multiplication by a finite Blaschke product acts as an isometry. Special cases of this problem have been tackled by L. de Branges ([2], Theorem 15 (scalar version)) and by Peter Lax ([6]) who looked at an equivalent version of a particular case of this problem. Our characterization may be of interest for the following reasons: the Hilbert spaces characterized by de Branges are assumed to be contractively contained in H^2.

We do not make any continuity assumption between the space to be characterized and H^2. All we require is that the Hilbert space is algebraically contained in H^2.

Secondly, our theorem, especially when the Hilbert space which is sought to be characterized is a closed subspace of H^2, is very explicit and intrinsic to the space H^2 and does not rely on vector valued operators à la Lax–Halmos. This leads to some nice and interesting properties of functions in H^2 including a factorization theorem which includes the classical inner-outer factorization as a special case. Also, we get in the H^2 situation a nice description of the commutant and the reducing subspaces of such multiplication operators.

Thirdly, our method enables us to give a fairly explicit description, intrinsic to H^2, of the invariant subspaces, on H^2, of the operator of multiplication by an infinite Blaschke product. (The manuscript of this description is under preparation.) Finally, we are able to perceive a way of describing such characterizations on other Hardy spaces and we give a description of the invariant subspaces on H^1 of the operator of multiplication by z^2.

2. A BRIEF PREVIEW

The rest of this paper is organised as follows: Section 3 deals with the basic terminology and related preliminary results. Section 4 contains the main result of this paper viz. a characterization of those Hilbert spaces which are algebraically contained in H^2 and which are left invariant under the action of T_B, the trasformation of multiplication by B, which acts isometrically. This section also describes some related results. Section 5 describes the factorization of H^2 functions which includes the classical inner-outer factorization as a special case. It also contains properties of these general inner functions. Section 6 contains the characterization of the invariant subspaces of the operator of multiplication by z^2 on the Hardy space H^1.

3. TERMINOLOGY AND PRELIMINARY RESULTS

We shall denote by $\mathbb{D}$ the open unit disk, $\mathbb{T}$ shall stand for its boundary i.e. the unit circle. Normalized Lebesgue measure on $\mathbb{T}$ shall be denoted by dm and L^p will stand for the well known Lebesgue spaces on $\mathbb{T}$. H^p shall denote the class of all L^p functions which have Fourier series of analytic type ($p \geqslant 1$). It is well known that H^p, for each p, can be looked upon as a space of analytic functions on $\mathbb{D}$ which satisfy a growth condition. For this and all other facts to be used about the well known Hardy spaces we refer to any of the numerous standard books in the literature such as [3], [4] or [5]. L^2 is a Hilbert space under the inner product

$$\langle f,g\rangle = \int_{\mathbf{T}} f\overline{g}\,\mathrm{d}m$$

H^2, being a closed subspace of L^2, is a Hilbert space in its own right under the above inner product.

By a finite Blaschke product $B(z)$ we mean

$$B(z) = \prod_{i=1}^{n} \frac{(z-\alpha_i)}{(1-\overline{\alpha}_i z)}$$

where each $\alpha_i \in \mathbb{D}$. The isometric operator of multiplication by B on H^2 shall be denoted by T_B. By an invariant subspace of an operator S on a Banach space X we mean a closed, proper, non-trivial subspace of X which is invariant under the action of S. Due to the conformal invariance of H^2, it is sufficient to characterize the invariant subspaces of the operator T_B where $\alpha_1 = 0$ in the above $B(z)$. Henceforth, (without loss of generality in the case when the invariant subspace under consideration is a closed subspace of H^2), we shall always assume that

$$B(z) = \prod_{i=1}^{n} \frac{(z-\alpha_i)}{(1-\overline{\alpha}_i z)} \qquad (\alpha_1 = 0,\ n \text{ at least } 1). \tag{3.1}$$

Given any f in H^2 with, say, zeros $\beta_1, \beta_2, \ldots, \beta_m$ in $\mathbb{D}$ we shall use the fact that

$$f(z) = I(z)g(z)$$

where $I(z)$ is a finite Blaschke factor whose zeros are precisely $\beta_1, \beta_2, \ldots, \beta_m$ and g is in H^2.

THEOREM 3.1. (Wold–Halmos) *Let H be a Hilbert space and T an isometry on H such that $\bigcap_0^\infty T^n(H) = \{0\}$. Then*

$$H = N \oplus T(N) \oplus T^2(N) \oplus \cdots \quad [\textit{where } N = H \ominus TH].$$

Proof. See [7]. ∎

REMARK 3.2. Such an isometry is called a *shift* and $\dim N$ is called the *multiplicity of the shift.*

Let $M(B)$ be the closed subspace of H^2 given by

$$M(B) = \bigvee_{m=0}^{\infty} \{B^m\}.$$

Note that $\{B^m\}$ is an orthonormal basis of $M(B)$.
Let

$$k_i(z) = \frac{1}{1-\overline{\alpha}_i z}, \quad \widehat{k}_i(z) = \frac{\sqrt{1+|\alpha_i|^2}}{1-\overline{\alpha}_i z}, \quad i = 1, 2, \ldots, n,$$

let

$$B_0(z) = 1,$$

and

$$B_i(z) = \prod_{j=1}^{i} \frac{(z-\alpha_j)}{(1-\overline{\alpha}_j z)}$$

then

$$B_n(z) = B(z).$$

Let

$$e_{jm} = \widehat{k}_{j+1} B_j B^m \quad (0 \leqslant j \leqslant n-1, \ m = 0, 1, 2, \ldots).$$

THEOREM 3.3. *The set $\{e_{jm}\}$ is an orthonormal basis of H^2.*

Proof. We shall first show that $\{e_{jm}\}$ is an orthonormal set. It is rather straightforward to verify that each e_{jm} has norm one. Next choose any two distinct elements from this set say e_{pq} and e_{rs}. Since these are distinct so either $p \neq r$ or $q \neq s$. Suppose $p \neq r$. Assume without loss of generality that $p < r$. Then we have $q > s$ or $q \leqslant s$. Now

$$\langle e_{pq}, e_{rs} \rangle = \int_T \widehat{k}_{p+1} B_p B^q \overline{\widehat{k}_{r+1} B_r B^s}\, \mathrm{d}m = \int_T \widehat{k}_{p+1} \overline{\widehat{k}_{r+1}} B_p B_r B^q \overline{B^s}\, \mathrm{d}m.$$

Now $B_p \overline{B_r} B^q \overline{B^s}$ is either a Blaschke product or the complex conjugate of a Blaschke product. But in the case it is a Blaschke product it will have $(z-\alpha_{r+1})$ as a factor and if it is the complex conjugate of a Blaschke product it will have $\overline{(z-\alpha_{p+1})}$ as a factor.

[Note: α_1 is to be interpreted as 0]. Hence in either event the above inner product will be zero due to the fact that $\widehat{k}_{r+1}$ and $\widehat{k}_{p+1}$ are scalar multiples of the reproducing kernels at α_{r+1} and α_{p+1}. The other values of p, q, r, s can similarly be tackled.

Next, we show $\{e_{jm}\}$ is a basis of H^2. Let f be an arbitrarily chosen element of H^2 and let f be orthogonal to $\{e_{jm}\}$. Then

$$\begin{aligned} f \perp e_{00} &\Rightarrow f = zf_1 \quad \text{for some } f_1 \text{ in } H^2, \\ f \perp e_{10} &\Rightarrow zf_1 \perp zk_2 \\ &\Rightarrow f_1(\alpha_2) = 0 \\ &\Rightarrow f = B_2 f_2 \quad \text{for some } f_2 \text{ in } H^2. \end{aligned}$$

Next

$$\begin{aligned} f \perp e_{20} &\Rightarrow B_2 f_2 \perp B_2 \widehat{k}_3 \\ &\Rightarrow f_2(\alpha_3) = 0 \\ &\Rightarrow f = B_3 f_3 \quad \text{for some } f_3 \text{ in } H^2; \end{aligned}$$

continuing like this we get $f = B_n f_n$ for some f_n in H^2 and $f = Bf_n$ since $B_n = B$. Then exploit the fact that $f \perp e_{p1}$, $p = 0, \ldots, n$ to get in a fashion similar to above that $f = B^2 f_n^{(1)}$ for some $f_n^{(1)}$ in H^2.

Similarly, $f \perp e_{p2}$, $p = 0, 1, \ldots, n$ will give us $f = B^3 f_n^{(3)}$ for some $f_n^{(2)}$ in H^2 and so continuing this for each e_{pj}, $j = 0, 1, \ldots$ we will find, because z^j divides B_n^j, that z^j is a factor of f for each positive integer j. This forces $f = 0$ and concludes the proof of our theorem. ∎

COROLLARY 3.4. $H^2 = e_{00}M(B) \oplus e_{10}M(B) \oplus \cdots \oplus e_{n-1,0}M(B)$

Proof. Obvious in view of the fact that $\{e_{jm}\}$ is a basis of H^2 and $\{B^m\}_0^\infty$ is a basis of $M(B)$. ∎

REMARK 3.5. In view of the above corollary it follows that for each f in H^2 there exists a set $\{f_0, f_1, \ldots, f_{n-1}\}$ in H^2 (in $M(B)$) such that

$$f = e_{00} f_0 + \cdots + e_{n-1,0} f_{n-1}$$

and

$$\|f\|^2 = \|f_0\|^2 + \cdots + \|f_{n-1}\|^2.$$

REMARK 3.6. If we write $e_{jm} = \overline{\widehat{k}_{j+1} B_j B^m}$ where $0 \leqslant j \leqslant n-1$ and m is any negative integer then using the decomposition $L^2 = H^2 \oplus \overline{zH^2}$ and the fact $\overline{B} = B^{-1}$ we conclude that the set $\{e_{jm} : 0 \leqslant j \leqslant n-1,\ m = 0, \pm 1, \ldots\}$ is an orthonormal basis of L^2.

Hence each f in L^2 can be uniquely written as

$$f = \sum_{j=0}^{n-1} \sum_{m=-\infty}^{\infty} \alpha_{jm} e_{jm}.$$

We shall call α_{jm} the $(j,m)^{\text{th}}$ B-Fourier coefficient of f and above series as the B-Fourier series of f.

We shall now define a few terms which are required for the characterization theorem.

To each r tuple $(\varphi_1, \ldots, \varphi_r)$ of functions in H^2 $(r \leqslant n)$ we associate a matrix called the B-matrix of $(\varphi_1, \ldots, \varphi_r)$ denoted by

$$A = (\varphi_{ij}) \quad (0 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant r)$$

where

$$\varphi_j = \sum_{i=0}^{n-1} e_{i0} \varphi_{ij} \quad 1 \leqslant j \leqslant r$$

is the representation as given in Remark 3.5.

DEFINITION 3.7. Let $(\varphi_1, \ldots, \varphi_r)$ be an r-tuple of H^∞ functions and let $A = (\varphi_{ij})$ be the B-matrix of $(\varphi_1, \ldots, \varphi_r)$. We say A is *B-inner* if

$$(\overline{\varphi_{ji}})(\varphi_{ij}) = (\delta_{st}) \quad \text{almost everywhere}$$

where $1 \leqslant s, t \leqslant r$.

In particular, let φ be an H^∞ function and let it have the representation

$$\varphi = \sum_{j=0}^{n-1} \varphi_j e_{j0}$$

as given by Remark 3.5. Then φ is said to be a B-inner function if

$$\sum_{i=0}^{n-1} |\varphi_i|^2 = 1 \quad \text{almost everywhere.}$$

REMARK 3.8. Clearly when $B(z) = z$, every B-inner function is nothing but an inner function.

LEMMA 3.9. *The B-matrix of the r-tuple $(\varphi_1, \ldots, \varphi_r)$ of H^∞ functions is B-inner if and only if $\{B^m \varphi_i = 1 \leqslant i \leqslant r,\ m = 0, 1, \ldots\}$ is an orthonormal set.*

REMARK 3.10. It follows from the above lemma that an H^∞ function φ is B-inner if an only if $\{B^m\varphi : m = 0, 1, \dots\}$ is an orthonormal set.

Proof of Lemma 3.9. Let $A = (\varphi_{ij})$ be the B-matrix of the given r-tuple.

Suppose A is B-inner. This means $(\overline{\varphi_{ji}})(\varphi_{ij}) = (\delta_{st})$, where $1 \leqslant s,\ t \leqslant r$. Hence

$$\sum_{i=0}^{n-1} \overline{\varphi_{ij}}\varphi_{ik} = 0, \quad (j \neq k,\ 1 \leqslant j \leqslant r,\ 1 \leqslant k \leqslant r) \tag{3.2}$$

and

$$\sum_{i=0}^{n-1} |\varphi_{ij}|^2 = 1. \tag{3.3}$$

Now pick any two distinct elements from the set $\{B^m\varphi_i\}$ say $B^m\varphi_s$ and $B^\ell\varphi_t$. Since they are distinct either $m \neq \ell$ or $s \neq t$. If $m = \ell$ so that $s \neq t$ then $\langle B^m\varphi_s, B^m\varphi_t\rangle = \langle \varphi_s, \varphi_t\rangle = 0$ by (3.2) above and Remark 3.5.

If $m \neq \ell$ then assume without loss of generality $m > \ell$ so that

$$\begin{aligned}
\langle B^m\varphi_s, B^\ell\varphi_t\rangle &= \langle B^{m-\ell}\varphi_s, \varphi_t\rangle = \Big\langle B^{m-\ell}\sum_{i=0}^{n-1} e_{i0}\varphi_{is}, \sum_{i=0}^{n-1} e_{i0}\varphi_{it}\Big\rangle \\
&= \sum_{k=0}^{n-1}\langle B^{m-\ell}\varphi_{ks}, \varphi_{kt}\rangle = \sum_{i=0}^{n-1}\int_T B^{m-\ell}\varphi_{is}\overline{\varphi_{it}}\,dm \\
&= \int_T B^{m-\ell}\Big(\sum_{i=0}^{n-1}\varphi_{is}\overline{\varphi_{it}}\Big)dm \\
&= 0
\end{aligned}$$

by (3.2) if $s \neq t$ or by (3.3) if $s = t$. Since it is clear that each element of $\{B^m\varphi_i\}$ has norm one it now follows that it is an orthonormal set.

Conversely, suppose $\{B^m\varphi_i\}$ is an orthonormal set. Then for $1 \leqslant j \leqslant r$, we prove that $\varphi_{ij} \in H^\infty$. Since for $f \in M(B)$

$$\|\varphi_{ij}f\|^2 \leqslant \sum_{k=0}^{n-1}\|\varphi_{kj}f\|^2 = \Big\|\sum_{k=0}^{n-1} e_{k0}\varphi_{kj}f\Big\|^2 = \|\varphi_j f\|^2 < \infty.$$

(Clearly true for f in $M(B) \cap H^\infty$ and the general case follows from a limit argument.) And for $g \in H^2$

$$\begin{aligned}
\|\varphi_{ij}g\|^2 &= \Big\|\varphi_{ij}\sum_{k=0}^{n-1} e_{k0}g_k\Big\|^2, \quad (g_k \in M(B)) \\
&= \sum_{i=0}^{n-1}\|\varphi_{kj}g_k\|^2.
\end{aligned}$$

This gives $\varphi_{ij} \in H^\infty$. So $\sum_{i=0}^{n-1} \varphi_{ij}\overline{\varphi_{ik}}$ is in L^2. Hence for any element e_{pq} of the orthonormal basis of L^2 as given in Remark 3.6, we see that if $j \neq k$

$$\left\langle \sum_{i=0}^{n-1} \varphi_{ij}\overline{\varphi_{ik}}, e_{pq} \right\rangle = \sum_{i=0}^{n-1} \langle \varphi_{ij}\overline{\varphi_{ik}}, e_{pq} \rangle = \sum_{i=0}^{n-1} \langle \varphi_{ij}, \varphi_{ik} e_{pq} \rangle = 0$$

if $p \neq 0$ and if $p = 0$ then the above expression is

$$\left\langle \sum_{i=0}^{n-1} \varphi_{ij}\overline{\varphi_{ik}}, e_{pq} \right\rangle = \langle \varphi_i, \varphi_j e_{0q} \rangle = \langle \varphi_i, \varphi_j B^q \rangle = 0$$

(even if q is negative). For $j = k$,

$$\left\langle \sum_{i=0}^{n-1} |\varphi_i|^2, e_{pq} \right\rangle = \sum_{i=0}^{n-1} \langle \varphi_{ij}, \varphi_{ij} e_{pq} \rangle = 0$$

if $p \neq 0 \neq q$ and for $p = q = 0$

$$\left\langle \sum_{i=0}^{n-1} |\varphi_{ij}|^2, e_{00} \right\rangle = \sum_{i=0}^{n-1} \langle \varphi_{ik} \varphi_{ik} \rangle = \langle \varphi_k, \varphi_k \rangle = \|\varphi_k\|^2 = 1. \quad \blacksquare$$

4. DE BRANGES SPACES INVARIANT UNDER T_B

L. de Branges in Theorem 15, [2], has proved the following theorem (scalar version).

THEOREM. (de Branges) *Let M be a Hilbert space contractively contained in H^2 and let $S(M) \subset M$ (where S is multiplication by z) and let S be an isometry. Then there exists a unique $b(z)$ in the unit ball of H^∞ such that*

$$M = b(z)H^2$$

and

$$\|bf\|_M = \|f\|_{H^2}$$

for all f in H^2.

This theorem clearly generalizes Beurling's invariant subspace theorem ([1]).

Over here we prove a fairly general version of the above theorem of de Branges which also generalizes an equivalent version of the invariant subspace theorem of Lax ([6], [5]).

THEOREM 4.1. *Let M be a Hilbert space such that:*
(i) *M is algebraically contained in H^2;*
(ii) *$T_B(M) \subset M$;*
(iii) *T_B acts isometrically on M.*

Then

$$M = \varphi_1 M(B) \oplus \cdots \oplus \varphi_r M(B)$$

where each φ_j is in H^∞, $1 \leqslant j \leqslant r$, and $r \leqslant n$. Further

$$\|\varphi_1 f_1 + \cdots + \varphi_r f_r\|_M^2 = \|f_1\|_{H^2}^2 + \cdots + \|f_r\|_{H^2}^2, \quad f_j \in M(B)$$

for each

$$f = \varphi_1 f_1 + \varphi_2 f_2 + \cdots + \varphi_r f_r$$

in M.

Before we prove the theorem we state and prove two lemmas which are needed in the proof of the theorem.

LEMMA 4.2. *Let M satisfy the hypothesis of Theorem 4.1. Let φ be an element of M such that $\{\varphi B^k\}_{k=0}^{\infty}$ is an orthonormal set in the inner product of M. Then:*
(i) *$\varphi M(B) \subset M$;*
(ii) *$\|\varphi f\|_M = \|f\|_{H^2}$, for all f in $M(B)$;*
(iii) *$\varphi \in H^\infty$.*

Proof. Let f be any element of $M(B)$. Then $f(z) = \sum_{m=0}^{\infty} \alpha_m B^m$. Put $f_k(z) = \sum_{m=0}^{k} \alpha_m B^m$. Then $\|f_k - f\|_{H^2} \to 0$ as $k \to \infty$ and

$$\begin{aligned}\|\varphi f_k\|_M^2 &= \left\|\varphi(z) \sum_{m=0}^{k} \alpha_m B^m\right\|_M^2 = \left\|\sum_{m=0}^{k} \alpha_m \varphi(z) B^m\right\|_M^2 \\ &= \sum_{m=0}^{k} |\alpha_m|^2 \quad (\text{as } \{\varphi(z)B^m\} \text{ is orthonormal in } M) \\ &= \|f_k\|_{H^2}^2.\end{aligned}$$

This proves that $\{\varphi f_k\}$ is a Cauchy sequence in M because $\{f_k\}$ is Cauchy in H^2. Hence there exists g in M such that φf_k converges in the norm of M to g. We prove $g = \varphi f$.

Let $\varphi(z) = \sum_{k=0}^{n} \sum_{m=0}^{\infty} \beta_{mk} e_{mk}$. Now from what has been done above the series

$$\alpha_0 \varphi(z) + \alpha_1 B \varphi(z) + \alpha_2 B^2 \varphi(z) + \cdots$$

converges, in the norm of M, to g.

Substituting the B-Fourier series of φ in g, we see that the $(i,j)^{\text{th}}$ B-Fourier coefficient of g is

$$\alpha_0 \beta_{ij} + \alpha_1 \beta_{i-1,j} + \cdots + \alpha_i \beta_{0j}$$

which is the same as $(i,j)^{\text{th}}$ B-Fourier coefficient of φf.

So $g = \varphi f$ and $\|\varphi f\|_M = \|f\|_{H^2}$.

Now we prove that $\varphi \in H^\infty$.

Let $g \in H^2$. Using the decomposition in Corollary 3.4

$$g = \sum_{m=0}^{n-1} e_{m0} g_m, \qquad g_m \in M(B)$$

$$\varphi g = \sum_{m=0}^{n-1} e_{m0} g_m \varphi.$$

But $\varphi f \in H^2$ for all f in $M(B)$ as proved above, we get $\varphi g \in H^2$ for all $g \in H^2$. So $\varphi \in H^\infty$. This completes the proof. ∎

LEMMA 4.3. *Let M satisfy the hypothesis of Theorem* 4.1. *Suppose there is an r-tuple $(\varphi_1, \ldots, \varphi_r)$ of H^∞ functions which satisfies:*

(i) *$\varphi_i M(B) \subset M$, $i = 1, 2, \ldots, r$;*

(ii) *$\varphi_i M(B) \perp \varphi_j M(B)$, in the inner product of M when $i \neq j$ then $r \leqslant n$.*

Proof. For the sake of notational simplicity we prove the result for $n = 2$. The proof for a general n is identical to the case $n = 2$. Suppose there are three elements $\varphi_1, \varphi_2, \varphi_3$ in M, which satisfy the hypothesis of the lemma. Using the decomposition of Corollary 3.4, we have

$$\varphi_j = e_{00} \varphi_{0j} + e_{10} \varphi_{2j} \qquad (\varphi_{ij} \in M(B),\ i = 0, 1,\ j = 1, 2, 3). \tag{4.1}$$

Define the matrix

$$A = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_{01} & \varphi_{02} & \varphi_{03} \\ \varphi_{11} & \varphi_{12} & \varphi_{13} \end{pmatrix}.$$

Then

$$\begin{aligned} \operatorname{Det} A = {} & \varphi_1(\varphi_{02}\varphi_{13} - \varphi_{12}\varphi_{03}) - \varphi_2(\varphi_{01}\varphi_{13} - \varphi_{11}\varphi_{03}) \\ & + \varphi_3(\varphi_{01}\varphi_{12} - \varphi_{11}\varphi_{02}) \\ = {} & 0 \end{aligned} \tag{4.2}$$

(by substituting the values of φ_j, $j = 1,2,3$ from (4.1)). Let

$$\begin{aligned} \lambda_1 &= \varphi_{02}\varphi_{13} - \varphi_{03}\varphi_{12} \\ \lambda_2 &= \varphi_{01}\varphi_{13} - \varphi_{03}\varphi_{11} \\ \lambda_3 &= \varphi_{01}\varphi_{12} - \varphi_{11}\varphi_{02}. \end{aligned} \tag{4.3}$$

We claim that λ_j is in $M(B)$ for $j = 1,2,3$. We see this as follows. Since $\varphi_j \in H^\infty$ we get from the proof of Lemma 3.9 that $\varphi_{ij} \in H^\infty$. Therefore $\lambda_j \in M(B)$, for $j = 1,2,3$, and the equation (4.2) can be written as

$$\varphi_1\lambda_1 - \varphi_2\lambda_2 + \varphi_3\lambda_3 = 0. \tag{4.4}$$

But $\varphi_i M(B) \perp \varphi_j M(B)$, so that from (4.4) we conclude that $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Now

$$\varphi_1 - \varphi_{02} - \varphi_2\varphi_{01} = (e_{00}\varphi_{01} + e_{01}\varphi_{11})\varphi_{02} - (e_{00}\varphi_{02} + e_{10}\varphi_{12})\varphi_{01} = 0.$$

This gives $\varphi_1\varphi_{02} = \varphi_2\varphi_{01}$. But $\varphi_1 M(B) \perp \varphi_2 M(B)$. So $\varphi_{02} = \varphi_{01} = 0$.

Similarly by considering the other minors of A, we get $\varphi_{ij} = 0$ for all $i = 0,1$, $j = 1,2,3$. So $\varphi_j = 0$, $j = 1,2,3$. This proves the lemma. ∎

Proof of Theorem 4.1. Since T_B is a pure isometry on M, using Theorem 3.1

$$\begin{aligned} M &= N \oplus BN \oplus B^2N \oplus \cdots \\ N &= M \ominus BM. \end{aligned}$$

Choose an element φ in N of unit norm, then $\{B^m\varphi\}_{m=0}^{\infty}$ is an orthonormal set in M. So by Lemma 4.2:

(i) $\varphi M(B) \subset M$;

(ii) $\|\varphi f\|_M = \|f\|_{H^2}$ for all f in $M(B)$;

(iii) $\varphi \in H^\infty$.

To complete the proof we must show that

$$\dim N \leqslant n.$$

Suppose there are r elements $\varphi_1, \varphi_2, \ldots, \varphi_r$ in N. We choose them in such a way that each element is of unit norm in M, and any two of them taken in pair are orthogonal in M. Then by Lemma 4.2 and Theorem 3.1:

(i) $\varphi_i M(B) \subset M$;

(ii) $\varphi_i M(B) \perp \varphi_j M(B)$, $i \neq j$;

so that the hypothesis of Lemma 4.3 are satisfied and we get $r \leqslant n$. So, once again by Theorem 3.1

$$M = \varphi_1 M(B) \oplus \cdots \oplus \varphi_r M(B)$$

where $r \leqslant n$ and $\varphi_j \in H^\infty$; $1 \leqslant j \leqslant r$. For $f \in M$

$$f = \varphi_1 f_1 + \cdots + \varphi_r f_r, \quad f_j \in M(B)$$

and

$$\begin{aligned} \|f\|_M^2 &= \|\varphi_1 f_1 + \cdots + \varphi_r f_r\|_M^2 \\ &= \|\varphi_1 f_1\|_M^2 + \cdots + \|\varphi_r f_r\|_M^2 \quad (\varphi_j M(B) \perp \varphi_k M(B);\ i \leqslant j,\ k \leqslant r;\ j \neq k) \\ &= \|f_1\|_{H^2}^2 + \cdots + \|f_r\|_{H^2}^2 \quad \text{(by Lemma 4.2).} \ \blacksquare \end{aligned}$$

The following are immediate from our Theorem 4.1.

COROLLARY 4.4. ([8], Theorem) *Let M be a Hilbert space contained in H^2 as a vector subspace such that:*

(i) *$S(M) \subset M$;*

(ii) *S acts as an isometry on M.*

Then $M = bH^2$ where b is an H^∞ functions and $\|bf\|_M = \|f\|_2$, $\forall f$ in H^2.

COROLLARY 4.5. ([2], Theorem 15 (scalar case)) *Let M be a Hilbert space contractively contained in H^2, such that:*

(i) *$S(M) \subset M$;*

(ii) *S acts as an isometry on M.*

Then $M = bH^2$ where b is in the unit ball of H^∞ and $\|bf\|_M = \|f\|_2$, $\forall f$ in H^2.

We are now in a position to describe the invariant and reducing subspaces of T_B as well as its commutant.

We first observe that for each B-inner function φ in H^∞, the space

$$\varphi M(B) = \{\varphi f : f \text{ is in } M(B)\}$$

is a closed subspace of H^2 invariant under T_B. We prove this as follows: $\varphi M(B)$ is clearly invariant under T_B. We show it is closed.

Let f be any element of $M(B)$ and let φ have the decomposition $\varphi = \sum_{j=0}^{n-1} e_{j0}\varphi_j$ as given in Remark 3.5. Then

$$\|\varphi f\|^2 = \Big\|\Big(\sum_{j=0}^{n-1} e_{j0}\varphi_j\Big) f\Big\|^2 = \Big\|\sum_{j=0}^{n-1} e_{j0}\varphi_j f\Big\|^2 = \sum_{j=0}^{n-1} \|e_{j0}\varphi_j f\|^2,$$

(by the fact that $\varphi_j f$ is in $M(B)$ and by the orthogonal decomposition of H^2 as given by Corollary 3.4),

$$\begin{aligned}\|\varphi f\|^2 &= \sum_{j=0}^{n-1} \|\varphi_j f\|^2 = \sum_{j=0}^{n-1} \int_T |\varphi_j|^2 |f|^2 \, dm = \int_T \Big(\sum_{j=0}^{n-1} |\varphi_j|^2\Big) |f|^2 \, dm \\ &= \int_T |f|^2 \, dm \quad (\text{since } \varphi \text{ is } B\text{-inner}) \\ &= \|f\|^2.\end{aligned}$$

Hence multiplication by φ is an isometry on $M(B)$ so that $\varphi M(B)$ is closed.

We can now state the invariant subspace theorem.

THEOREM 4.6. *Let M be a T_B-invariant subspace. Then there are B-inner functions $\varphi_1, \dots, \varphi_r$ $(r \leqslant n)$ such that*

$$M = \varphi_1 M(B) \oplus \cdots \oplus \varphi_r M(B)$$

and the B-matrix of $(\varphi_1, \dots, \varphi_r)$ is B-inner. Further, the above representation is unique in the sense that if

$$M = \psi_1 M(B) \oplus \cdots \oplus \psi_s M(B)$$

then $r = s$, $\varphi_i = \sum_{j=1}^{r} \alpha_{ij} \psi_j$ for scalar α_{ij} and the matrix (α_{ij}) is unitary.

Proof. The proof of this theorem is more or less a direct consequence of Theorem 4.1. We just remark that the B-matrix of $(\varphi_1, \dots, \varphi_r)$ is B-inner, since $\{\varphi_i B^n\}$ is an orthonormal set.

Now suppose M has another representation

$$M = \psi_1 M(B) \oplus \cdots \oplus \psi_s M(B); \tag{4.5}$$

then by looking at the statement of Theorem 4.6 we conclude that the multiplicity of T^B on M is r. On the other hand by looking at the second representation in (4.5) we conclude that the multiplicity is s. Thus $r = s$.

Next from (4.5), for each i, $1 \leqslant i \leqslant r$

$$\varphi_i = \sum_{j=0}^{r} f_{ji} \psi_j, \quad f_{ji} \in M(B) \tag{4.6}$$

and similarly for $1 \leqslant j \leqslant r$

$$\psi_j = \sum_{i=1}^{r} g_{ij}\varphi_i, \quad g_{ij} \in M(B). \tag{4.7}$$

So from (4.6) and (4.7) above we conclude that

$$(g_{ij})(f_{ij}) = (\delta_{ks}) \quad (1 \leqslant k,\ s \leqslant r). \tag{4.8}$$

Also, letting (φ_{kj}) and (ψ_{kj}) be the B-matrices of (φ_j) and (ψ_j) respectively we get from (4.6)

$$(\varphi_{kj}) = (\psi_{kj})(f_{ji}) \quad (0 \leqslant k \leqslant n-1), \tag{4.9}$$

and from the first part of the theorem each φ_i and ψ_i are B-inner so their B-matrices are B-inner and hence

$$(\overline{\varphi_{jk}})(\varphi_{kj}) = (\delta_{st}) \tag{4.10}$$

$$(\overline{\psi_{jk}})(\psi_{kj}) = (\delta_{st}). \tag{4.11}$$

Putting the values from (4.9) in (4.10), we get that

$$(\overline{f_{ij}})(\overline{\psi_{jk}})(\psi_{kj})(f_{ji}) = (\delta_{st})$$

and so

$$(\overline{f_{ij}})(f_{ji}) = (\delta_{st}). \tag{4.12}$$

This gives from (4.8) and (4.12) that

$$(\overline{f_{ij}}) = (g_{ji})$$

so that each $\overline{f_{ij}}$ is in H^2. Since each f_{ij} is already in H^2 we conclude that f_{ij} is a constant for each i and j. This completes the proof of the theorem in view of (4.6). ∎

COROLLARY 4.7. ([1], Beurling's Theorem) *Let M be a closed subspace of H^2, which is invariant under the multiplication by z. Then there exists an inner function φ which is unique up to a constant of absolute value 1, such that $M = \varphi H^2$.*

We are now in a position to describe the reducing subspaces of T_B, for $n > 1$, since for $n = 1$, T_B is irreducible.

THEOREM 4.8. *Let M be a reducing subspace of T_B. Then*

$$M = \varphi_1 M(B) \oplus \cdots \oplus \varphi_r M(B)$$

with $r < n$ and each φ_i is given by

$$\varphi_j = \sum_{i=0}^{n-1} \alpha_{ij} e_{i0}, \quad \alpha_{ij} \in \mathbb{C}$$

and

$$\sum_{i=0}^{n-1} |\alpha_{ij}|^2 = 1.$$

Conversely every such subspace is a reducing subspace.

Proof. Since M is an invariant subspace of T_B it follows by Theorem 4.6 that

$$M = \varphi_1 M(B) \oplus \cdots \oplus \varphi_r M(B), \quad r \leqslant n,$$

where the φ_i's are B-inner function. We shall first show that $r < n$. So suppose $r = n$. Now, since M is a reducing subspace we get

$$T_B(M^\perp) \subset M^\perp$$

and hence, once again by Theorem 4.6

$$M^\perp = \psi_1 M(B) \oplus \cdots \oplus \psi_k M(B).$$

Since $M^\perp \neq \{0\}$ and H^2 we conclude that $k \geqslant 1$. But $k + r$ is the multiplicity of T_B on H^2. So $k + r = n$ and hence $r < n$.

It is also obvious that the kernel of T_B^* on H^2 is the span of $\{e_{j0} : 0 \leqslant j \leqslant n-1\}$. Thus to complete the proof of the necessary part of the theorem all we need to show is that each φ_i is in the kernel of T_B^*. To do this we shall show that $T_B^* \varphi_i$ is orthogonal to both M and $M^\perp$. Now, let f be any element of M so that

$$f = \varphi_1 f_1 + \cdots + \varphi_r f_r$$

where each f_i is in $M(B)$. Then

$$\begin{aligned}\langle T_B^* \varphi_i, f\rangle &= \Big\langle T_B^* \varphi_i, \sum_{j=1}^{r} \varphi_j f_j \Big\rangle = \Big\langle \varphi_i, T_B \sum_{j=1}^{r} \varphi_j f_j \Big\rangle = \Big\langle \varphi_i, \sum_{j=1}^{r} \varphi_j B f_j \Big\rangle \\ &= \sum_{j=1}^{r} \langle \varphi_i, \varphi_j B f_j\rangle = \langle \varphi_i, \varphi_i B f_i\rangle\end{aligned}$$

(because $\varphi_i \perp \varphi_j M(B)$ for $j \neq i$ and since $Bf_j \in M(B)$);

$$\langle T_B^* \varphi_i, f\rangle = \langle \mathbf{1}, Bf_i\rangle$$

(because by the observation immediately preceding the statement of Theorem 4.6, multiplication by each such φ_i is an isometry on $M(B)$);

$$\langle T_B^* \varphi_i, f\rangle = 0$$

(because z devides B).

Now suppose f is in $M^\perp$. Then $\langle T_B^* \varphi_i, f\rangle = \langle \varphi_i, Bf\rangle = 0$ (because $T_B(M^\perp) \subset M^\perp$ and $\varphi_i \in M$). Hence each φ_i is in the kernel of T_B^*.

The statement about $\sum_{i=0}^{n-1} |\alpha_{ij}|^2 = 1$ follows from the fact that each φ_i is a B-inner function. The converse is easy and we omit the proof. ∎

We now define a particular partial isometry on H^2 and use it to give another characterization of the invariant subspaces of T^B on H^2.

Let $(\varphi_1, \ldots, \varphi_r)$ be an r-tuple of H^∞ functions.

Recall from Corollary 3.4 that $H^2 = e_{00}M(B) \oplus \cdots \oplus e_{n-1,0}M(B)$.

Define $s_\varphi : H^2 \to H^2$ as follows: for each $f = e_{00}f_0 + \cdots + e_{n-1,0}f_n$ in H^2 $S_\varphi f = \varphi_1 f_0 + \cdots + \varphi_r f_{r-1}$ where $r \leqslant n$. S_φ is clearly a partial isometry when the B-matrix of $(\varphi_1, \ldots, \varphi_r)$ is B-inner (with initial space $e_{00}M(B) \oplus \cdots \oplus e_{r-1,0}M(B)$ and range space $\varphi_1 M(B) \oplus \cdots \oplus \varphi_r M(B)$), and S_φ becomes an isometry when $r = n$. Hence Theorem 4.6 can be rewritten as

THEOREM 4.9. *Let M be an invariant subspace of T^B. Then*

$$M = S_\varphi(H^2)$$

where S_φ is as described above, and the B-matrix of $(\varphi_1, \ldots, \varphi_r)$ is B-inner.

Now we describe the commutant of T_B.

THEOREM 4.10. *The commutant $\{T_B\}'$ pf T^B is the set of operators*

$$\{S_\varphi : \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n),\ \varphi_k \in H^\infty,\ q \leqslant k \leqslant n\}.$$

Proof. Suppose $A \in \{T_B\}'$. Then

$$AT_B = T_B A.$$

Let $A(e_{k-1,0}) = \varphi_k$, $1 \leqslant k \leqslant n$. Then, for a fixed i, $0 \leqslant i \leqslant n-1$ and $j = 0, 1, \ldots,$

$$A(e_{ij}) = A(B^j e_{i0}) = AT_B^j(e_{i0}) = T_B^j A(e_{i0}) = T_B^j(\varphi_{i+1}) = B^j \varphi_{i+1}.$$

So, if p_m is a polynomial in B, then $A(p_m e_{i0}) = p_m \varphi_{i+1}$. Since polynomials in B are dense in $M(B)$, we get $A(e_{i0}g) = g\varphi_{i+1}$ for all $g \in M(B)$. This gives $\varphi_{i+1}M(B) \subset H^2$. So $\varphi_{i+1} \in H^\infty$; $0 \leqslant i \leqslant n-1$. And for $f \in H^2$

$$A(f) = A\Big(\sum_{i=0}^{n-1} e_{i0} f_i\Big) = \sum_{i=0}^{n-1} A(e_{i0} f_i) = \sum_{i=0}^{n-1} f_i \varphi_{i+1}.$$

This gives $A = S_\varphi$ where $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$.

Conversely, for some n-tuple $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ and $f \in H^2$

$$\begin{aligned} T_B S_\varphi(f) &= T_B\Big(\sum_{i=0}^{n-1} \varphi_{i+1} f_i\Big) = B\Big(\sum_{i=0}^{n-1} \varphi_{i+1} f_i\Big) \\ &= \sum_{i=0}^{n-1} \varphi_{i+1}(Bf_i) = S_\varphi(B_f) = S_\varphi T_B(f). \end{aligned}$$

5. A GENERAL INNER-OUTER FACTORIZATION ON H^2

In this section we shall assume that $B(z) = z^n$ where n is any natural number. Thus all our results so far are valid when we assume $B(z) = z^n$ where n is any natural number. We would also like to state that what we are going to say for $B(z) = z^n$ is also true for an arbitrary $B(z)$. The results for an arbitrary $B(z)$ will be that much more technical. Thus for the sake of simplicity we shall take $B(z) = z^n$.

DEFINITION 5.1. An n-inner function in H^2 is a B-inner function as defined in 3.7 when $B(z) = z^n$.

DEFINITION 5.2. An n-outer function in H^2 is a function f such that

$$\bigvee_{k=0}^{\infty} \{f z^{nk}\} = p(z) M(z^n)$$

where p is a polynomial of degree less than n.

REMARK 5.3. By virtue of

$$H^2 = M(z^n) \oplus z M(z^n) \oplus \cdots \oplus z^{n-1} M(z^n)$$

every $f(z)$ in H^2 is of the type

$$f(z) = f_0(z) + z f_1(z) + \cdots + z^{n-1} f_{n-1}(z).$$

Hence, by virtue of Definition 3.7, if $f(z)$ is n-inner then

$$|f_0|^2 + \cdots + |f_{n-1}|^2 = 1$$

a.e. on $\mathbb{T}$. The converse is also true.

REMARK 5.4. Every inner function is n-inner for an arbitrary n. (The converse is obviously not true.) To see this, let $f(z)$ be an inner function. Then $\{f(z)z^k\}^{\infty}$ is an orthonormal set. It then follows that $\{f(z)z^{kn}\}_{k=0}^{\infty}$ is an orthonormal set. Hence, by Remark 3.10, $f(z)$ is an n-inner function.

THEOREM 5.5. (i) *Every n-outer function in $M(z^n)$ is an outer function.*

(ii) *Every n-outer function $f(z)$ in H^2 is of the type*

$$f(z) = p(z)O(z)$$

where $p(z)$ is a polynomial of degree less than n and $O(z)$ is an outer function in H^2 and conversely.

Proof. (i) Let $f(z)$ be an n-outer function in $M(z^n)$. Since f is in $M(z^n)$, we conclude that

$$\bigvee_{k=0}^{\infty} \{f(z)z^{nk}\} = M(z^n). \tag{5.1}$$

Now, by Remark 3.5, or even directly every polynomial $p(z)$ in H^2, can be written as

$$p(z) = p_0(z) \oplus zp_1(z) \oplus \cdots \oplus z^{n-1}p_{n-1}(z)$$

where each p_i is a polynomial in $M(z^n)$.

Hence, using the fact that $f(z) \in M(z^n)$, so $p_i f \in M(z^n)$, we get

$$\begin{aligned}\bigvee\{fp\} &= \bigvee\{fp_0\} \oplus \cdots \oplus \bigvee\{fz^{n-1}p_{n-1}\} \\ &= \bigvee\{fp_0\} \oplus \cdots \oplus z^{n-1}\bigvee\{fp_{n-1}\} \\ &= M(z^n) \oplus \cdots \oplus z^{n-1}M(z^n) \\ &= H^2\end{aligned}$$

where the span on the left is over all polynomials p in H^2 and the span on the right is over all polynomials p_i in $M(z^n)$.

Hence f is outer in H^2.

(ii) Let f be n-outer and in H^2. Then there exists a polynomial $p(z)$ of degree less than n such that

$$\bigvee_{k=0}^{\infty} \{fz^{nk}\} = p(z)M(z^n). \tag{5.2}$$

Thus

$$f(z) = p(z)O(z) \tag{5.3}$$

for some $O(z)$ in $M(z^n)$. We shall first show $O(z)$ is n-outer. By (5.2) and (5.3) above, and the facts that degree $p(z)$ less than n and $O(z) \in M(z^n)$, we get that

$$\bigvee_{k=0}^{\infty} \{p(z)O(z)z^{nk}\} = p(z) \bigvee_{k=0}^{\infty} \{O(z)z^{nk}\} \tag{5.4}$$

(because if $p(z) = \alpha_0 + \cdots + \alpha_{n-1}z^{n-1}$ then $p(z)O(z)z^{nk} = \alpha_0 O(z)z^{nk} \oplus \cdots \oplus \alpha_{n-1}z^{n-1}O(z)z^{nk}$). So by (5.2), (5.3) and (5.4)

$$M(z^n) = \bigvee_{k=0}^{\infty} \{O(z)z^{nk}\}.$$

This means $O(z)$ is n-outer. But $O(z)$ is in $M(z^n)$, so by (i) $O(z)$ is outer. This proves the assertion.

The converse of statement (ii) is quite obvious in view of the above arguments which proved (ii) and hence we omit its proof. ∎

THEOREM 5.6. *A function f of unit norm in H^2 is n-inner as well as n-outer if and only if f is a polynomial of degree less than n.*

Proof. If $f(z)$ is a polynomial of unit norm and of degree less than n then obviously $\{f(z)z^{nk} : k = 0, 1, \ldots\}$ is an orthonormal set and so it is n-inner. On the other hand

$$\bigvee_{k=0}^{\infty} \{f(z)z^{nk}\} = f(z)M(z^n)$$

(because of the orthonormality of $\{f(z)z^{nk}\}$) so that $f(z)$ is n-outer.

Conversely, suppose $f(z)$ is n-inner as well as n-outer. Then since $f(z)$ is n-outer

$$f(z) = p(z)O(z)$$

and

$$\|f(z)\| = \|p(z)\| = 1,$$

where $p(z)$ is a polynomial of degree less than n and $O(z)$ is an outer function in $M(z^n)$ which is also n-outer (by Theorem 5.5). Now letting $p(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_r z^r$ such that $r < n$ we have

$$f(z) = \alpha_0 O(z) + \cdots + \alpha_r z^r O(z)$$

and since $O(z)$ is in $M(z^n)$ and f is n-inner we conclude by Remark 5.3 that

$$\Big(\sum_{j=0}^{r} |\alpha_j|^2\Big) |O(z)|^2 = 1$$

a.e. on $\mathbb{T}$, i.e. $|O(z)| = 1$, a.e. on $\mathbb{T}$ (since $\|p(z)\| = 1$), which means $O(z)$ is an inner function. But we know $O(z)$ is an outer function also. Thus $O(z)$ is a constant and hence

$$f(z) = p(z)$$

where $p(z)$ is a polynomial of degree less than n. ∎

Finally, we present the factorization of H^2 functions into sums of products of n-inner and n-outer functions.

THEOREM 5.7. *If f is in H^2 then*

$$f = \varphi_1 f_1 \oplus \cdots \oplus \varphi_r f_r \quad (r \leqslant n)$$

where for each i, f_i is in $M(z^n)$ and is n-outer and φ_i is n-inner.

Proof. Let $M = \bigvee_{k=0}^{\infty} \{f z^{nk}\}$. Then M is invariant under the action of z^n and so by Theorem 4.6

$$M = \varphi_1 M(z^n) \oplus \cdots \oplus \varphi_r M(z^n) \quad (r \leqslant n)$$

where each φ_i is an n-inner. Since f is in M we conclude that

$$f = \varphi_1 f_1 \oplus \cdots \oplus \varphi_r f_r$$

where each f_i is in $M(z^n)$. We claim each f_i is n-outer. If this is not so, then for at least one i say $i = j$,

$$\bigvee_{k=0}^{\infty} \{f_j z^{nk}\} \subsetneqq M(z^n).$$

This means

$$\varphi_j \bigvee_{k=0}^{\infty} \{f_j z^{nk}\} \subsetneqq \varphi_j M(z^n)$$

i.e.

$$\bigvee_{k=0}^{\infty} \{\varphi_j f_j z^{nk}\} \subsetneqq \varphi_j M(z^n)$$

(because $\varphi_j \bigvee_{k=0}^{\infty} \{f_j z^{nk}\} = \bigvee_{k=0}^{\infty} \{\varphi_j f_j z^{nk}\}$). Hence

$$\begin{aligned} M &= \bigvee_{k=0}^{\infty} \{f z^{nk}\} \\ &= \bigvee_{k=0}^{\infty} \{\varphi_1 f_1 z^{nk}\} \oplus \cdots \oplus \bigvee_{k=0}^{\infty} \{\varphi_j f_j z^{nk}\} \oplus \cdots \oplus \bigvee_{k=0}^{\infty} \{\varphi_r f_r z^{nk}\} \\ &\subsetneqq \varphi_1 M(z^n) \oplus \cdots \oplus \varphi_j M(z^n) \oplus \cdots \oplus \varphi_r M(z^n) \\ &= M \end{aligned}$$

i.e. $M \subsetneqq M$. This contradiction implies that each f_j is n-outer and hence the theorem is proved. ∎

6. INVARIANT SUBSPACES IN H^1

In this section, we describe the invariant subspaces of S^2 on the Hardy space H^1, where S^2 is multiplication by z^2.

THEOREM 6.1. *Let M be a closed subspace of H^1 invariant under S^2. Then*

$$M = \varphi N(z^2) \oplus \psi N(z^2)$$

where φ, ψ are 2-inner function in H^2 and $N(z^2)$ is the closure of the span of $\{z^{2n}\}$ in H^1.

Proof. We first claim that

$$M \cap H^2 \neq \{0\}.$$

This is proved as follows: fix any f in M. Then $|f|^{1/2}$ is in L^2. Put

$$g(z) = \frac{(|f(z)|^{\frac{1}{2}} + |f(-z)|^{\frac{1}{2}})}{2}.$$

Clearly $g(z)$ is in L^2 and its Fourier series consists of even powers of z only. So let

$$g(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^{2n}.$$

Let $h(z)$ denote the harmonic conjugate of $g(z)$. Clearly $h(z)$ has Fourier series in which only the even powers of z occur. Hence $g + ih$ is in $M(z^2) = \bigvee_{n=0}^{\infty} \{z^{2n}\}$ in L^2. Put

$$k(z) = \exp[-(g + ih)].$$

Then k is an outer function in $M(z^2)$. Further by its very construction it can be seen that kf is in H^∞ and hence in H^2. We show kf is in M. Let k_n be the n^{th} Cesaro means of k. Then each k_n is a polynomial in $M(z^2)$ and so $k_n f$ is in M for every n. Further $\|k_n\|_\infty \leqslant \|k\|_\infty$ and $k_n \to k$ almost everywhere. Thus

$$\|k_n f - kf\|_1 \to 0 \quad \text{as} \quad n \to \infty.$$

This means kf is in M.

We have thus shown that $M \cap H^2$ is a non-trivial subspace of H^2. Further $M \cap H^2$ is closed in H^2 because M is closed in H^1. Also $M \cap H^2$ is invariant under S^2 so by the invariant subspace theorem (Theorem 4.6) for the case $B(z) = z^2$ we conclude that

$$M \cap H^2 = \varphi M(z^2) \oplus \psi M(z^2)$$

where φ and ψ are 2-inner function in H^∞.

Our theorem will be proved if we show that the arbitrary f that we started with is in $\varphi N(z^2) \oplus \psi N(z^2)$. Now

$$kf = \varphi h + \psi g$$

(for some g, h in $M(z^2)$). Also, it can be verified that

$$\frac{(|k(z)f(z) + k(-z)f(-z)|^2 + |k(z)f(z) - k(-z)f(-z)|^2)}{4} = |h|^2 + |g|^2.$$

But $k(z) = k(-z)$, so that

$$|h|^2 + |g|^2 = \frac{|k(z)|^2}{4}(|f(z) + f(-z)|^2 + |f(z) - f(-z)|^2)$$

so that

$$\left|\frac{h}{k}\right|^2 \leqslant \left|\frac{h}{k}\right|^2 + \left|\frac{g}{k}\right|^2 = \frac{(|f(z)|^2 + |f(-z)|^2)}{2} \leqslant \frac{(|f(z) + |f(-z)|)^2}{2}.$$

Hence

$$\left|\frac{h}{k}\right| \leqslant \frac{(|f(z)| + |f(-z)|)}{2^{\frac{1}{2}}}$$

so that h/k is in L^1 because f is in H^1. But h is in H^2 and k is an outer function, so h/k is in H^1. Similarly, g/k is in H^1. But

$$\frac{h(z)}{k(z)} = \frac{h(-z)}{k(-z)}$$

and

$$\frac{g(z)}{k(z)} = \frac{g(-z)}{k(-z)}$$

so h/k and g/k are in $N(z^2)$. Since

$$f = \varphi\frac{h}{k} + \psi\frac{g}{k}$$

our theorem is proved. ∎

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DINESH SINGH
Department of Mathematics
University of Delhi
Delhi 110007
INDIA

Current address:

Indian Statistical Institute
7 S.J.S. Sansanwal Marg
New Delhi 110016
INDIA

VIRENDER THUKRAL
Department of Mathematics
University of Delhi
Delhi 110007
INDIA

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