J. OPERATOR THEORY
37(1997), 263–279

ON TWO-DIMENSIONAL SINGULAR INTEGRAL OPERATORS WITH CONFORMAL CARLEMAN SHIFT

R. DUDUCHAVA, A. SAGINASHVILI and E. SHARGORODSKY

Communicated by Nikolai K. Nikolskii

ABSTRACT. For the class of singular integral operators with continuous coefficients and with the conformal shift over a two-dimensional bounded domain $G \subset \mathbb{C}$ an explicit Fredholm property criterion is obtained. Operators under consideration have kernels $[(\overline{\zeta} - \overline{z})/(\zeta - z)]^k |\zeta - z|^{-2}$ either with positive or with negative $k \in \mathbb{Z} \setminus \{0\}$; the conformal shift $W\varphi(z) = \varphi(\omega(z))$, $\omega : G \to G$ is of Carleman type: $W^k \neq I$ for $k = 1, 2, \ldots, n-1$ and $W^n = I$. It is proved also that a Fredholm operator A of such type has trivial index $\operatorname{Ind} A = 0$.

KEYWORDS: *Singular integral operators, two-dimensional domains, Carleman shift.*

AMS SUBJECT CLASSIFICATION: Primary 45E99, 45P05; Secondary 35S15, 47B38.

1. INTRODUCTION

Let G be a bounded, simply connected domain in the complex plane with Lyapunov boundary $\partial G \in C^{1+\lambda}, \lambda > 0$ and $S_{G,k}$ denote the following singular integral operator

$$(1.1)\qquad S_{G,k}\varphi(z) = \frac{|k|}{\pi} \iint\limits_G \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^k \frac{1}{|\zeta - z|^2} \varphi(\zeta)\, dx dy,$$
$$z, \zeta \in G \subset \mathbb{C},\ \zeta = x + iy, \quad x, y \in \mathbb{R},\ k \in \mathbb{Z} \setminus \{0\}.$$

Integral equations containing operators $S_{G,k}$ appear in various problems of the theory of generalized analytic functions ([36]), of the theory of quasiconformal mappings and Riemann surfaces ([1], [23], [31]), of the theory of partial differential equations ([9], [27]), etc.

Two-dimensional singular integral equations of mentioned type are intensively studied by many authors. The first investigations of such equations were published by F. Tricomi ([34]). Later I. Vekua ([36]) studied them using the contraction principle (the fixed point theorem of Banach). A. Dzhuraev investigated two-dimensional singular integral equations in Lebesgue spaces $L_p(G)$, $2 < p < \infty$, by reducing them to boundary value problems for generalized analytic functions with further reduction to one-dimensional singular integral equations on the boundary of the domain (see [9] and the references there). To extend the study of such equations to $L_p(G)$ spaces for all $1 < p < \infty$, I. Komyak applied composition formulas for model operators and localization techniques ([21], [22]).

In the papers [4] and [32] there was developed an L_p-theory for multidimensional singular integral equations on manifolds with smooth boundary ($1 < p < \infty$). Invoking these results, one can study Fredholm properties of equations containing $S_{G,k}$ and their combinations. The problem transfers to the problem of factorization of corresponding rational matrix-functions (symbols), namely into the problem of finding partial indices of factorization with respect to the variable in the normal direction to the boundary of the domain ∂G. The most interesting is to find a criterion for the corresponding two-dimensional singular integral operator to be Fredholm, in the form of explicit conditions on the coefficients. This is done in the papers of G. Dzhangibekov ([6], [7], [8]) for sufficiently wide classes of equations.

It turns out that the operators $S_{G,k}$ are pseudodifferential and have the transmission property. Applying the method of L. Boutet de Monvel ([3]), one can investigate corresponding equations in Besov–Triebel–Lizorkin spaces $B^s_{p,q}(G)$, $F^s_{p,q}(G)$ (for definitions see e.g. [35]) provided the coefficients and the boundary ∂G are sufficiently smooth (see [11], [14], [15] and [30], 3.1.1.4). In particular, from the Fredholm property of a singular integral operator with the transmission property in $L_2(G)$ space there follows its Fredholm property in all Besov–Triebel–Lizorkin spaces (we should note that $L_p(G) = F^0_{p,2}(G)$, $C^{m+\nu}(\overline{G}) = B^{m+\nu}_{\infty,\infty}(G)$ for $1 < p < \infty$, $m \in \mathbb{Z}_+ = \mathbb{N} \cup 0$, $\nu \in (0,1)$, cf. Subsections 2.5.6–2.5.7, [35]). Therefore the Fredholm property criterion for the two-dimensional singular integral equations mentioned above is one and the same in all Besov–Triebel–Lizorkin spaces (and in Hölder spaces $C^{m+\nu}(\overline{G})$ among them) and coincides with the Fredholm property criterion in the Hilbert space $L_2(G)$; the latter criterion can be obtained applying results of [10] and [33].

In the present paper we consider two-dimensional singular integral operators with conformal Carleman shift of order n. Some preliminary results were obtained in [5]. The theory of one-dimensional singular integral equations with a shift is far developed. The results obtained in this direction up to the middle 70-ies are reflected in the monograph [24]. More recent results are observed in [18], [19], [20].

There exist several general approaches, which can be applied to both one-dimensional and multidimensional operators with a shift (see e.g. [2], [16], [17]). In Lemmas 4.1 and 4.2 below we will apply the approach suggested in [16] in axiomatic form, which eliminates the shift by increasing the dimension of the system.

2. MAIN RESULT

Let $\omega : G \to G$ be a conformal automorphism of the domain G such that

$$\begin{aligned} &\omega_k(z) \not\equiv z \quad \text{for } k = 1, \dots, n-1, \\ &\omega_n(z) \equiv z, \quad \forall\, z \in \overline{G}, \quad (n \geqslant 2), \end{aligned} \tag{2.1}$$

where ω_k denotes the k-th iteration

$$\omega_k(z) := \omega_{k-1}(\omega(z)), \quad \omega_1(z) := \omega(z). \tag{2.2}$$

Then the operator $W : L_p(G) \to L_p(G)$, $1 < p < \infty$, where

$$W\varphi(z) := (\varphi \circ \omega)(z) := \varphi(\omega(z)), \quad z \in \overline{G} := G \cup \partial G, \tag{2.3}$$

is known as *the Carleman shift* determined by ω and has order n.

The asserted property $\partial G \in C^{1+\lambda}$, $\lambda \in (0,1)$ and Kellogg theorem (see e.g. Chapter X, Section 1, [13]) yield $\omega \in C^{1+\lambda}(\overline{G})$ (see Chapter IX, Section 5, Theorem 4, [13]). Moreover, $\omega'(z) \neq 0$ for all $z \in \overline{G}$.

In the next section we shall prove that ω can be represented in the form

$$\omega = \gamma \circ e \circ \gamma^{-1}, \tag{2.4}$$

where γ is a conformal mapping of the unit disk onto G,

$$e(w) := w \cdot \exp\left(\frac{2\pi i}{n}\ell\right), \quad |w| \leqslant 1, \tag{2.5}$$

and $\ell \in \overline{1, n}$ is coprime with n. We can suppose $\ell = 1$ without restricting the generality, since the general case can be easily reduced to the present one by renumerating operators A_k in (2.6) (see below).

Let us consider the operator

$$A = A_0 + \sum_{k=1}^{n-1} W^k A_k : L_p(G) \to L_p(G), \quad 1 < p < \infty, \tag{2.6}$$

where

$$A_k = a_{k0} I + \sum_{j=1}^{N_k} a_{kj} S_{G,j}, \quad a_{kj} \in C(\overline{G}),\ j = 0, 1, \dots, N_k,\ k = 0, 1, \dots, n-1, \tag{2.7}$$

and I denotes the identity operator.

We introduce the symbol of the operator A in (2.6) as follows

$$\mathcal{A}(z, \eta) = \|b_{m\ell}(z, \eta)\|_{n \times n}, \quad z \in \overline{G}, \quad \eta \in \mathbb{C}, \tag{2.8}$$

$$b_{m\ell}(z, \eta) = a_{r(m+\ell-2)0}(\omega_\ell(z)) + \sum_{j=1}^{N_{r(m+\ell-2)}} a_{r(m+\ell-2)j}(\omega_\ell(z)) \left(\frac{\overline{\omega_\ell'(z)}}{\omega_\ell'(z)} \right)^j \eta^j, \tag{2.9}$$

where $r(\tau)$ is the integer remainder of the division $\tau : n$ (i.e. $\tau = mn + r(\tau)$, $m \in \mathbb{N} \cup \{0\}$, $0 \leqslant r(\tau) < n$).

THEOREM 2.1. *The operator* (2.6) *is Fredholm if and only if*

$$\det \mathcal{A}(z, \eta) \neq 0 \quad \text{for all } z \in \overline{G},\ |\eta| \leqslant 1. \tag{2.10}$$

If condition (2.10) *holds, the operator A has trivial index:* $\operatorname{Ind} A = 0$.

We postpone the proof of Theorem 2.1 to Section 5 since we need some preparation for this.

3. PROPERTIES OF CONFORMAL CARLEMAN SHIFTS

For the proof of the main Theorem 2.1 we need information about fixed points of a conformal Carleman shift, exposed in forthcoming Lemma 3.1. Since we can not give a precise reference, we prefer to provide a detailed proof, based on a well-known properties of conformal mappings, available from standard textbooks on complex analysis (cf. e.g. Chapter III, Section 1 and Chapter XII, Section 6, [29]).

Let us consider a conformal mapping α of the unit disk onto G, which exists due to Riemann Theorem. $\alpha^{-1} \circ \omega \circ \alpha$ is, clearly, a conformal automorphism of the unit disk. Therefore $\alpha^{-1} \circ \omega \circ \alpha$ is a linear-fractional mapping and has two fixed points.

LEMMA 3.1. *If $w_1, w_2 \in \mathbb{C}$ are the fixed points of $\alpha^{-1} \circ \omega \circ \alpha$, then they do not belong to the unit circumference and are symmetric with respect to it:*

$$w_2 = \frac{1}{\overline{w_1}}, \quad |w_1| < 1. \tag{3.1}$$

Proof. Let us prove first that the fixed points of $\varepsilon = \alpha^{-1} \circ \omega \circ \alpha$ must be different. In fact, if they coincide then

$$\frac{1}{\varepsilon(w) - w_1} = \frac{1}{w - w_1} + c, \quad c \neq 0,$$

where $w_1 = w_2$ is the fixed point of ε (see Chapter III, Section 1, Subsection 11, [29]). Then clearly

$$\frac{1}{\varepsilon_k(w) - w_1} = \frac{1}{w - w_1} + kc, \quad \forall k \in \mathbf{N},$$

(see (2.2)). This however contradicts the property $\varepsilon_n(w) \equiv w, |w| \leqslant 1$ (see (2.1)).

Next we shall prove that the assumption

$$w_k = \exp(\mathrm{i}\vartheta_k), \quad 0 \leqslant \vartheta_k < 2\pi, \quad k = 1, 2, \quad \vartheta_1 \neq \vartheta_2, \tag{3.2}$$

also leads to a contradiction.

In fact, if β is a conformal automorphism of the unit disk, transforming points 1 and -1 into $\exp(\mathrm{i}\vartheta_1)$ and $\exp(\mathrm{i}\vartheta_2)$ respectively, then $\gamma := \beta^{-1} \circ \alpha^{-1} \circ \omega \circ \alpha \circ \beta$ is a conformal automorphism of the unit disk with the fixed points at ± 1. For γ we have the following representation

$$\gamma(w) = \exp(\mathrm{i}\vartheta)\frac{w - a}{1 - w\overline{a}}, \quad 0 \leqslant \vartheta < 2\pi, \ |a| < 1. \tag{3.3}$$

If we insert $w = 1$ into the equation $\gamma(w) = w$ we obtain $\exp(\mathrm{i}\vartheta) = (1 - \overline{a})/(1 - a)$. Inserting next $w = -1$ and invoking the previous relation we find $a = \overline{a}$, i.e. $a \in (-1, 1)$.

Thus,

$$\gamma(w) = \frac{w - a}{1 - wa}, \quad a \in (-1, 1). \tag{3.4}$$

After a routine calculation we find out that the composition of γ and of the mapping

$$\delta(w) = \frac{w - b}{1 - wb}, \quad b \in (-1, 1)$$

equals

$$(\gamma \circ \delta)(w) = \frac{w - c}{1 - wc}, \quad c = \frac{a + b}{1 + ab}$$

and leads to the formulae

$$\gamma_m(w) = \frac{w - a_m}{1 - wa_m}, \quad m = 1, 2, \ldots \tag{3.5}$$

(see (2.2)) with

$$a_m := \frac{a + a_{m-1}}{1 + aa_{m-1}}, \quad a_1 = a.$$

If $a \neq 0$, the sequence $\{a_m\}_1^\infty$ is strongly monotone and has the limit sgn a.

Equality (3.5) contradicts the property $\gamma_n(w) = w, |w| \leqslant 1$ (see (2.1)) provided $a \neq 0$.

If $a = 0$, we run into contradiction with (2.1) since from (3.4) there follows $\gamma(w) = w$. ∎

Now let β be a conformal automorphism of the unit disk and $\beta(0) = w_1$. Then $\kappa = \beta^{-1} \circ \alpha^{-1} \circ \omega \circ \alpha \circ \beta$ represents a conformal automorphism of the unit disk with the fixed point at 0. Such an automorphism has the form

$$\kappa(w) = w \cdot \exp(i\vartheta), \quad 0 \leqslant \vartheta < 2\pi. \tag{3.6}$$

From (2.1) we infer

$$\begin{aligned} &\kappa_k(w) \not\equiv w \quad \text{for } k = 1, \ldots, n-1, \\ &\kappa_n(w) \equiv w, \quad \forall z \in \overline{G}. \end{aligned} \tag{3.7}$$

(3.6) and (3.7) yield

$$\kappa(w) = w \cdot \exp\left(\frac{2\pi i}{n}\ell\right), \quad |w| \leqslant 1, \tag{3.8}$$

where $\ell \in \overline{1, n}$ is coprime with n. We can suppose $\ell = 1$ without restricting generality since the general case can be easily reduced to this one by renumerating the operators A_k in (2.6).

Further we shall suppose that

$$\omega = \gamma \circ e \circ \gamma^{-1}, \tag{3.9}$$

where γ is a conformal mapping of the unit disk onto G and

$$e(w) = w \cdot \exp\frac{2\pi i}{n}, \quad |w| \leqslant 1. \tag{3.10}$$

The mapping ω has the unique fixed point $z_1 = \gamma(0) \in G$ in $\overline{G}$. Moreover,

$$\begin{aligned} \omega'(z_1) &= \gamma'(e(\gamma^{-1}(z_1))e'(\gamma^{-1}(z_1))(\gamma^{-1})'(z_1) \\ &= \gamma'(e(0))e'(0)(\gamma'(\gamma^{-1}(z_1)))^{-1} \\ &= \gamma'(0)e'(0)(\gamma'(0))^{-1} = e'(0) = \exp\frac{2\pi i}{n}. \end{aligned} \tag{3.11}$$

4. AUXILIARY RESULTS

LEMMA 4.1. *The operator* (2.6) *is Fredholm if and only if the operator*

$$\tilde{A} = \begin{pmatrix} A_0 & WA_1W^{-1} & \cdots & W^{n-1}A_{n-1}W^{-n+1} \\ A_1 & WA_2W^{-1} & \cdots & W^{n-1}A_0W^{-n+1} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n-1} & WA_0W^{-1} & \cdots & W^{n-1}A_{n-2}W^{-n+1} \end{pmatrix} : L_p(G, \mathbb{C}^n) \to L_p(G, \mathbb{C}^n) \tag{4.1}$$

is Fredholm.

For the proof of the formulated assertion we need some auxiliary results. First of them is based on the local principle of Gohberg–Krupnik (see Section 5.1, [12]). We suppose the knowledge of the basic definitions of this particular local principle, such as *covering system of localizing classes*, *local equivalence* and *local invertibility*. Let $\mathbf{B}$ be a Banach algebra with the identity e and $q \in \mathbf{B}$, $q^n = e$, $n \geqslant 2$, $q^j \neq e$ for $j = 1, \ldots, n-1$.

We should suppose that there exists a covering system of localizing classes $\{M_\delta\}_{\delta\in\Delta}$ with the following properties:

(i) $m_\delta a_k = a_k m_\delta$, $m_\delta q = q m_\delta$, $\forall\, k \in \overline{0, n-1}$, $\forall\, m_\delta \in M_\delta$, $\forall\, \delta \in \Delta$;

(ii) for each $\delta \in \Delta$ there exists an M_δ-invertible element c_δ such that:

(a) $c_\delta m_\delta = m_\delta c_\delta$, $\forall\, m_\delta \in M_\delta$,

(b) $c_\delta a_k \overset{M_\delta}{\sim} a_k c_\delta$, $k = 0, 1, \ldots, n-1$,

(c) $c_\delta q \overset{M_\delta}{\sim} \exp\left(\frac{2\pi i}{n}\right) q c_\delta$.

LEMMA 4.2. *If the conditions* (i) *and* (ii) *hold, then the element*

$$a = a_0 + \sum_{k=1}^{n-1} q^k a_k, \quad a_k \in \mathbf{B}, \quad k = 0, 1, \ldots, n-1, \tag{4.2}$$

is invertible in the algebra $\mathbf{B}$ *if and only if the following matrix element*

$$\tilde{a} = \begin{pmatrix} a_0 & qa_1q^{-1} & \cdots & q^{n-1}a_{n-1}q^{-n+1} \\ a_1 & qa_2q^{-1} & \cdots & q^{n-1}a_0q^{-n+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & qa_0q^{-1} & \cdots & q^{n-1}a_{n-2}q^{-n+1} \end{pmatrix} \tag{4.3}$$

is invertible in the matrix-algebra $\mathbf{B}^{n\times n}$.

Proof. It is easy if we invoke the local principle of Gohberg–Krupnik (see Chapter XII, Section 1, [12]) and follow the proof of Theorem 18.1 in [16]. ∎

REMARK 4.3. Invertibility of the element $\tilde{a}$ in (4.3) yields the invertibility of a in (4.2) even if conditions (i) and (ii), formulated before Lemma 4.2, fail to hold (see Remark 18.1, [16])

LEMMA 4.4. *Let $k = 1, 2, \dots, n-1$. The operators*

$$\mathcal{T}_{k,j} := W^k S_{G,j} W^{-k} - \left(\frac{\overline{\omega_k'}}{\omega_k'}\right)^j S_{G,j} \quad \text{for } j \in \mathbb{Z} \setminus \{0\} \tag{4.4}$$

and

$$\mathcal{T}_k := W^k C_G W^{-k} - \frac{\overline{\omega_k'}}{|\omega_k'|} C_G, \tag{4.5}$$

(see (2.2)) with

$$C_G\varphi(z) := \frac{1}{\pi} \int\!\!\int_G \frac{\overline{\zeta} - \overline{z}}{|\zeta - z|^3} \varphi(\zeta)\, dx dy, \quad \zeta = x + iy, \tag{4.6}$$

are compact in the space $L_p(G)$, $1 < p < \infty$.

Proof. Let us recall that ω is a conformal mapping and, therefore, transforms infinitely small disks onto infinitely small disks with small perturbations of higher order. Applying the theorem on change of variables in singular integrals (see e.g. Chapter IX, Section 1, formula (8), [26]) we get the following

$$\begin{aligned} \mathcal{T}_{k,j}\varphi(z) &:= \left(W^k S_{G,j} W^{-k} - \left(\frac{\overline{\omega_k'}}{\omega_k'}\right)^j S_{G,j}\right)\varphi(z) \\ &= \frac{|j|}{\pi} \int\!\!\int_G \left[\left(\frac{\overline{\omega_k(\zeta)} - \overline{\omega_k(z)}}{\omega_k(\zeta) - \omega_k(z)}\right)^j \frac{|\omega_k'(\zeta)|^2}{|\omega_k(\zeta) - \omega_k(z)|^2} \right. \\ &\qquad \left. - \left(\frac{\overline{\omega_k'(z)}}{\omega_k'(z)}\right)^j \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^j \frac{1}{|\zeta - z|^2}\right] \varphi(\zeta)\, dx dy, \qquad z \in G. \end{aligned}$$

Using the mean value theorem we can rewrite the kernel in the form

$$\begin{aligned} &\frac{|j|}{\pi}\left[\left(\frac{\overline{\omega_k'(\xi)}}{\omega_k'(\xi)}\right)^j \left|\frac{\omega_k'(\zeta)}{\omega_k'(\xi)}\right|^2 - \left(\frac{\overline{\omega_k'(z)}}{\omega_k'(z)}\right)^j\right] \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^j \frac{1}{|\zeta - z|^2} \\ &= \frac{|j|}{\pi} \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^j \frac{1}{|\zeta - z|^2} \left\{\left[\left(\frac{\overline{\omega_k'(\xi)}}{\omega_k'(\xi)}\right)^j - \left(\frac{\overline{\omega_k'(z)}}{\omega_k'(z)}\right)^j\right] \left|\frac{\omega_k'(\zeta)}{\omega_k'(\xi)}\right|^2 \right. \\ &\qquad \left. + \left(\frac{\overline{\omega_k'(z)}}{\omega_k'(z)}\right)^j \frac{|\omega_k'(\zeta)|^2 - |\omega_k'(\xi)|^2}{|\omega_k'(\xi)|^2}\right\}, \end{aligned}$$

where $\xi = \vartheta\zeta + (1-\vartheta)z$ for some $\vartheta \in [0,1]$. Recalling that $\omega \in C^{1+\lambda}(\overline{G})$, $\lambda > 0$, one can easily obtain that the integral operator $\mathcal{T}_{k,j}$ has a weakly singular kernel and therefore is a compact operator in $L_p(G)$ (see e.g. Chapter VIII, Section 3, [26]).

Compactness of the operator $\mathcal{T}_k$ is proved analogously. ∎

Proof of Lemma 4.1. Let $\mathbf{B}$ be the quotient algebra of all linear bounded operators in $L_p(G)$ modulo the ideal of all compact operators in $L_p(G)$, known as the Calkin algebra. The quotient class in $\mathbf{B}$ which contains an operator T is denoted by $[T]$.

With any point $z \in \overline{G}$ different from the unique fixed point $z_1 \in G$ of the mapping ω we associate some neighbourhood U_z such that

$$\omega_k(U_z \cap \overline{G}) \cap \omega_\ell(U_z \cap \overline{G}) = \emptyset \qquad \text{for } k \neq \ell,\ k,\ell = 1,\dots,n.$$

In the algebra $\mathbf{B}$ we define the following localizing classes

$$M_z = \left\{ \left[\Big(\sum_{k=1}^{n} \mu_z \circ \omega_k \Big) I \right] \,\Big|\, \mu_z \in C_0^\infty(\mathbb{R}^2),\ \operatorname{supp} \mu_z \subset U_z, \right.$$
$$\left. \mu_z = 1 \text{ in some neighbourhood of } z \right\}, \quad z \in \overline{G},\ z \neq z_1,$$

$$M_{z_1} = \left\{ \left[\frac{1}{n} \Big(\sum_{k=1}^{n} \mu_{z_1} \circ \omega_k \Big) I \right] \,\Big|\, \mu_{z_1} \in C_0^\infty(\mathbb{R}^2), \right.$$
$$\left. \mu_{z_1} = 1 \text{ in some neighbourhood of } z_1 \right\}.$$

The system $\{M_z\}_{z \in \overline{G}}$ is, clearly, a covering system of localizing classes and the quotient classes $[W], [A_k], k = 0, \dots, n-1$, (see (2.7)) commute with all elements of $\bigcup_{z \in \overline{G}} M_z$ in the Calkin algebra $\mathbf{B}$.

Let us define the following operator

$$C_z : L_p(\Omega) \to L_p(\Omega), \quad C_z := \begin{cases} \overline{\gamma^{-1}} I & \text{for } z \neq z_1, \\ C_G & \text{for } z = z_1 \end{cases} \tag{4.7}$$

(see (2.4), (4.6)). The elements $[C_z]$ are M_z-invertible for all $z \in \overline{G}$. For $z \neq z_1$ this follows from the property $\gamma^{-1}(z) \neq 0$, while for $z = z_1$ we remaind that z_1 is an inner point of the domain G and the operator C_G has an elliptic (non-degenerate) symbol (see Chapter X, Section 2, Chapter XI, Section 7, Chapter XII, Section 3, [26]).

The following relations are easy to be obtained:

(a) $[C_z][gI] = [gI][C_z], \forall\, g \in C(\overline{G})$; in particular $[C_z]m_z = m_z[C_z], \forall\, m_z \in M_z, \forall\, z \in \overline{G}$;

(b) $[C_z][A_k] \overset{M_z}{\sim} [A_k][C_z], k = 0, 1, \dots, n-1, \forall\, z \in \overline{G}$ (see (2.7));

(c) $[C_z][W] \overset{M_z}{\sim} \exp\left(\frac{2\pi i}{n}\right)[W][C_z], \forall\, z \in \overline{G}$ (see (2.3)).

In fact, if $z \neq z_1$ the relations in (a) are evident, the relations in (b) follow from Theorem 7.1 in Chapter XI, [26], while (c) is a consequence of the equality

$$\overline{\gamma^{-1}} \circ \omega = \overline{e \circ \gamma^{-1}} = \exp\frac{2\pi i}{n}\overline{\gamma^{-1}}$$

(see (3.9), (3.10)).

If $z = z_1$ the relations (a), (b) are consequences of Theorem 7.1, Chapter XI in [26]; here it is essential that z_1 is an inner point of the domain G and therefore it is sufficient to consider the integral operators on the entire space $\mathbb{R}^2$.

The relation in (c) for $z = z_1$ is already obtained in Lemma 4.4, since

$$\frac{\overline{\omega'(z_1)}}{|\omega'(z_1)|} = \exp\left(-\frac{2\pi i}{n}\right)$$

(see (3.11)).

Thus, both conditions (i) and (ii) of Lemma 4.2 are justified and the asserted claim follows because $[A] \in \mathbf{B}$ is invertible (because $[\tilde{A}] \in \mathbf{B}^{n\times n}$ is invertible) iff the operator (2.6) is Fredholm (iff the operator (4.1) is Fredholm, respectively; see e.g. Chapter XII, Lemma 1, [26]). ∎

5. PROOF OF THEOREM 2.1

Proof. Let us consider the operator

$$B = \|B_{m\ell}\|_{n\times n} : L_p(G, \mathbb{C}^n) \to L_p(G, \mathbb{C}^n), \quad 1 < p < \infty, \tag{5.1}$$

$$B_{m\ell} = (a_{r(m+\ell-2)0} \circ \omega_\ell)I + \sum_{j=1}^{N_{r(m+\ell-2)}} (a_{r(m+\ell-2)j} \circ \omega_\ell)\left(\frac{\overline{\omega'_\ell}}{\omega'_\ell}\right)^j S_{G,j}, \tag{5.2}$$

where, $r(\tau)$ is known from (2.9). Due to (2.7) and to Lemma 4.4 the operator $\tilde{A} - B$ is compact in $L_p(G, \mathbb{C}^n)$ (see (4.1)). Then the operator (2.6) is Fredholm if and only if the operator (5.1) is Fredholm (see Lemma 4.1). Let us note that

(5.1) is a two-dimensional singular integral operator without shift and the results from [4] can be applied.

For the investigation of the operator (5.1) we apply localization techniques from [4]. With each point $z \in \overline{G}$ we associate the following special local coordinate system: for an inner point $z \in G$ the system coincides with the original one on $\mathbb{C}$, while for $z \in \partial G$ the abscissae axis is directed along the tangent to ∂G and the ordinates axis — along the inner (with respect to G) normal at $z \in \partial G$.

Recalling formulae (8), (9) from Chapter X, Section 2, [26] (see also Chapter X, Section 3, [26]) one can easily find that the symbol of the operator (5.1), written in such a special local coordinate system, has the form

$$(5.3)\quad \sigma_B(z,\xi_1,\xi_2) = \left\| b_{m\ell}\left(z, e^{2i\vartheta(z)}\frac{\xi_2 + i\xi_1}{\xi_2 - i\xi_1}\right)\right\|_{n\times n}, \quad z \in \overline{G},\ (\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\},$$

(see (2.8), (2.9)) where $\vartheta(z) = 0$ for $z \in G$ while for $z \in \partial G$, $\vartheta(z)$ designates the angle between positive direction of the tangent to ∂G at the point z and the abscissae axis of the original coordinate system.

For $\xi_1 > 0$ (for $\xi_1 < 0$) the symbol $\sigma_B(z,\xi_1,\xi_2)$ has the analytic continuation into the lower complex half-plane $\operatorname{Im}\xi_2 \leqslant 0$ (into the upper half-plane $\operatorname{Im}\xi_2 \geqslant 0$, respectively). While ξ_2 ranges over the above-mentioned half-plane the variable

$$\eta = e^{2i\vartheta(z)}\frac{\xi_2 + i\xi_1}{\xi_2 - i\xi_1}$$

covers the unit disk $|\eta| \leqslant 1$.

According to [4], the operator (5.1) is Fredholm if and only if the following conditions hold:

(i) $\det \sigma_B(z,\xi_1,\xi_2) \neq 0$ for all $z \in \overline{G}$ and all $(\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\}$;

(ii) $\det \sigma_B(z,\xi_1,\xi_2) \neq 0$ for all $z \in \partial G$, $\xi_1 \neq 0$, $\xi_1 \cdot \operatorname{Im}\xi_2 \leqslant 0$.

Thus, the operator (2.6) is Fredholm if and only if:

(i′) $\det \mathcal{A}(z,\eta) \neq 0$ for all $z \in \overline{G}$, $|\eta| = 1$;

(ii′) $\det \mathcal{A}(z,\eta) \neq 0$ for all $z \in \partial G$, $|\eta| \leqslant 1$.

From the argument principle (see e.g. Chapter IV, Section 3, [25]) and with (ii′), the argument

$$(5.4)\qquad \arg\det \mathcal{A}(z,\,\cdot\,)$$

has no increment with respect to the second variable along the unit circumference $\Gamma = \{\eta \in \mathbb{C} : |\eta| = 1\}$ for all $z \in \partial G$. On the other hand from (i′) and from the continuity of $\det \mathcal{A}(z,\,\cdot\,)$ with respect to $z \in \overline{G}$ we obtain that the integer

$$\operatorname{ind}\det \mathcal{A}(z,\,\cdot\,) = \frac{1}{2\pi}\,\Delta\,[\arg\det \mathcal{A}(z,\,\cdot\,)]_\Gamma$$

is independent of $z \in \overline{G}$; therefore $\operatorname{ind}\det \mathcal{A}(z,\,\cdot\,) \equiv 0$ for all $z \in \overline{G}$. Again invoking the argument principle we find that conditions (i′), (ii′) are equivalent to (2.10).

The first half of Theorem 2.1 is proved and there remains to find the index of the operator (2.6).

Let us introduce the following parameter-dependent operator

$$\text{(5.5)}\quad A(t) := A_0(t) + \sum_{k=1}^{n-1} W^k A_k(t) : L_p(G) \to L_p(G), \quad 1 < p < \infty, \quad 0 \leqslant t \leqslant 1,$$

where

$$\text{(5.6)}\qquad A_k(t) = a_{k0} I + \sum_{j=1}^{N_k} a_{kj} t^j S_{G,j}.$$

The symbol $\mathcal{A}_t$ of this operator (cf. (2.8), (2.9)) has the form

$$\mathcal{A}_t(z,\eta) = \mathcal{A}(z, t\eta), \quad z \in \overline{G}, \quad \eta \in \mathbb{C}, \quad t \in [0,1].$$

Due to condition (2.10) $A(t)$ are Fredholm operators for all $t \in [0,1]$ (see (5.1)–(5.4)). Moreover, $A(t)$ depends continuously on t and

$$A(1) = A, \quad A(0) = a_{00} I + \sum_{k=1}^{n-1} W^k a_{k0} I.$$

It is well-known that under stated conditions the indices of $A(1)$ and $A(0)$ coincide (see e.g. Chapter I, Theorem 3.11, [26]).

According to Remark 4.3 the operator $A(0)$ is invertible in $L_p(G)$ provided the operator

$$\begin{pmatrix} a_{00}I & W a_{10} W^{-1} & \cdots & W^{n-1} a_{(n-1)0} W^{-n+1} \\ a_{10}I & W a_{20} W^{-1} & \cdots & W^{n-1} a_{n0} W^{-n+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{(n-1)0}I & W a_{00} W^{-1} & \cdots & W^{n-1} a_{(n-2)0} W^{-n+1} \end{pmatrix} : L_p(G,\mathbb{C}^n) \to L_p(G,\mathbb{C}^n)$$

is invertible, i.e. the symbol matrix $\mathcal{A}(z,0)$ is invertible for all $z \in \overline{G}$. The latter condition is implied by (2.10). Therefore $\operatorname{Ind} A = \operatorname{Ind} A(0) = 0$. ∎

REMARK 5.1. If $n = 2$ and

$$\text{(5.7)}\qquad A = a_{00} I + a_{0j} S_{G,j} + W(a_{10} I + a_{1j} S_{G,j})$$

(cf. (2.6), (2.7)), the function $\det \mathcal{A}(z,\eta)$ is a polynomial of order 2 with respect to the variable η^j; then condition (2.10) can be written explicitely in the form of inequalities for the coefficients a_{00}, a_{0j}, a_{10} and a_{1j} in (5.7) (see [5] for details).

6. SOME GENERALIZATIONS

The norm of the operator

$$S_{G,j} : L_p(G) \to L_p(G), \quad 1 < p < \infty,$$

can be estimated as follows

$$\|S_{G,j}\| \leqslant C_p|j|, \quad \forall j \in \mathbb{Z} \setminus \{0\},$$

(see e.g. Chapter XI, Section 3, [26]). Thus, the operator

$$A_k = a_{k0}I + \sum_{j=1}^{\infty} a_{kj}S_{G,j}, \quad a_{kj} \in C(\overline{G}),$$

(cf. (2.7)) exists and is bounded in $L_p(G)$ provided

$$\sum_{j=0}^{\infty}(1+j) \sup_{z\in\overline{G}} |a_{kj}(z)| < +\infty. \tag{6.1}$$

Let us suppose (6.1) holds for $k = 0, 1, \ldots, n-1$ (see (2.7)).

The operator A, defined by formulae (2.6), (2.7) with $N_{r(m+\ell-2)} = \infty$, is obviously bounded in $L_p(G, \mathbb{C}^n)$, $1 < p < \infty$. On defining the symbol matrix function $\mathcal{A}(z,\eta)$ by formulae (2.8), (2.9) with $N_{r(m+\ell-2)} = \infty$, we can prove Theorem 2.1 for A under asserted assumptions. The proof is analogous to that from Section 5 and is based on the following variant of the Wiener theorem.

THEOREM 6.1. *Let $\mathcal{W}_1^+$ be the Banach algebra of all functions h,*

$$h(z,\eta) = \sum_{j=0}^{\infty} h_j(z)\eta^j, \quad z \in \overline{G},\ |\eta| \leqslant 1,\ h_j \in C(\overline{G}), \tag{6.2}$$

such that

$$\|h|\mathcal{W}_1^+\| := \sum_{j=0}^{\infty}(1+j) \sup_{z\in\overline{G}} |h_j(z)| < +\infty.$$

Then $h \in \mathcal{W}_1^+$ is invertible in $\mathcal{W}_1^+$ iff

$$h(z,\eta) \neq 0 \quad \text{for all} \quad z \in \overline{G}, \quad |\eta| \leqslant 1. \tag{6.3}$$

For the proof we refer to Chapter III, Section 11, Subsection 2, Proposition IV and Example 2; the last example of Subsection 3, [28].

REMARK 6.2. Since the operators

$$S_{G,j}S_{G,m} - S_{G,j+m} : L_p(G) \to L_p(G), \quad 1 < p < \infty, \quad \forall j, m \in \mathbb{N}$$

are compact, (see Chapter 2, Section 10, [9]) one can easily verify that if $h \in \mathcal{W}_1^+$ satisfies condition (6.3), the corresponding operator

$$H = h_0 I + \sum_{j=1}^{\infty} h_j S_{G,j} : L_p(G) \to L_p(G), \quad 1 < p < \infty,$$

(see (6.2)) has a regularizer

$$R = r_0 I + \sum_{j=1}^{\infty} r_j S_{G,j},$$

where

$$\sum_{j=0}^{\infty} r_j(z)\eta^j = \frac{1}{h(z,\eta)}, \quad z \in \overline{G}, \quad |\eta| \leqslant 1.$$

For the operator which is obtained from (2.6), (2.7) by replacing $S_{G,j}$ with $S_{G,-j}$ there holds an analogue of Theorem 2.1. The proof is the same.

Complications appear if we try to replace operators A_k in (2.6) with

$$A'_k := a_{k0} I + \sum_{j=-M_k}^{-1} a_{kj} S_{G,j} + \sum_{j=1}^{N_k} a_{kj} S_{G,j}, \quad M_k, N_k \in \mathbb{N}.$$

The Fredholm properties of such an operator depend on the Fredholm properties of matrix singular integral operator without shift (cf. Lemma 4.1). But the symbol in this case is a general rational matrix-function with respect to the variables (ξ_1, ξ_2). Threre are known only few explicit conditions which can ensure the partial indices of factorization of a matrix-function to be trivial; one of such conditions is the strong ellipticity, i.e. when $\operatorname{Re} \mathcal{A}(z, \xi_1, \xi_2)$ is positive definite. However there exist algorithms which can be applied in each concrete case.

7. THE CASE OF BESOV–TRIEBEL–LIZORKIN SPACES

THEOREM 7.1. *Let* $0 < p \leqslant \infty$ ($0 < p < \infty$ *if we consider* $F^s_{p,q}$), $0 < q \leqslant \infty$, $\max(n/p - n, 1/p - 1) < s < \infty$; *let* $G \subset \mathbb{C}$ *be a bounded, simply connected domain with an infinitely smooth boundary* ∂G, $a_{kj} \in C^\infty(\overline{G})$, $j = 0, 1, \dots, N_k$, $k = 0, 1, \dots, n-1$.

If condition (2.10) *holds, the operator* A *in* (2.6), (2.7) *is Fredholm in the spaces* $B^s_{p,q}(G), F^s_{p,q}(G)$ *and* $\operatorname{Ind} A = 0$.

Proof. Let us note that $\partial G \in C^\infty$ yields $\omega \in C^\infty(\overline{G})$. To prove this it suffices to apply the Lindelöf formula (see e.g. Chapter X, Section 1, Theorem 4, [13]) and recall that a function which is analytic in G and has $C^\infty(\overline{G})$-smooth real part belongs to $C^\infty(\overline{G})$ itself. Thus, the operator W is continuous in the spaces $B^s_{p,q}(G), F^s_{p,q}(G)$ (see 2.10.2, [35]) and ensures the continuity of A in $B^s_{p,q}(G), F^s_{p,q}(G)$ for $s > \max(n/p - n, 1/p - 1)$ (see [11], [15]). Due to Remark 4.3 the operator A is Fredholm in $B^s_{p,q}(G), F^s_{p,q}(G)$ if the operator $\tilde{A}$ (see (4.1)) is Fredholm in $B^s_{p,q}(G, \mathbb{C}^n), F^s_{p,q}(G, \mathbb{C}^n)$.

From the theorem on change of variables in pseudodifferential operators (see e.g. 1.2.3.4, [30]) it follows that $\tilde{A}$ is pseudodifferential and has the transmission property (see 2.3.2.2 and 2.3.3.1, [30]). The difference $T = \tilde{A} - B$ (see (5.1), (5.2)) has order -1 and posseses the transmission property. T is compact in $B^s_{p,q}(G, \mathbb{C}^n)$, $F^s_{p,q}(G, \mathbb{C}^n)$ (see [11], [15] and Remark 4.3.2-1, [35]). Now the asserted claim follows from [11], [14], [15] and 3.1.1.4, [30]. ∎

This research was supported by the ISF *under grant* MXG000.

REFERENCES

1. L.V. AHLFORS, *Lectures on Quasiconformal Mappings*, Van Nostrand, New York, 1966.
2. A.B. ANTONEVICH, *Linear Functional Equations. An Operator Approach*, [Russian], Izdatel'stvo "Universitetskoe", Minsk, 1988.
3. L. BOUTET DE MONVEL, Boundary problems for pseudo-differential operators, *Acta Math.* **126**(1971), 11–51.
4. R. DUDUCHAVA, On multidimensional singular integral operators, I, II, *J. Operator Theory* **11**(1984), 41–76, 199–214.
5. R.V. DUDUCHAVA, A.I. SAGINASHVILI, E.M. SHARGORODSKY, On two-dimensional singular integral operators with shift, [Russian], *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR* **103**(1995), 3–13.
6. G. DZHANGIBEKOV, On some two-dimensional singular integral operators, [Russian], *Mat. Zametki* **46**(1989), 91–93.

7. G. DZHANGIBEKOV, Noetherity and index for certain class of two-dimensional singular integral operators, [Russian], *Dokl. Akad. Nauk* **313**(1990), 541–545; English transl. *Soviet Math. Dokl.* **42**(1990).
8. G. DZHANGIBEKOV, On Noetherity conditions and index of certain two-dimensional singular integral operators, [Russian], *Dokl. Akad. Nauk* **319** (1991), 811–815; English transl. *Soviet Math. Dokl.* **48**(1991).
9. A.D. DZHURAEV, *The Method of Singular Integral Equations*, [Russian], Nauka, Moscow 1987.
10. G.I. ESKIN, *Boundary Value Problems for Elliptic Pseudodifferential Equations*, Transl. Math. Monographs, vol. 52, Amer. Math. Soc., Providence, R.I., 1981.
11. J. FRANKE, Besov–Triebel–Lizorkin spaces and boundary value problems, *Semin. Anal. 1984/85, K. Weierstrass Inst. Math. Mechanik*, Berlin, 1985, 89–104.
12. I.T. GOHBERG, N.JA. KRUPNIK, *One Dimensional Linear Singular Integral Equations*, vol. I–II, Oper. Theory Adv. Appl., vol. 53–54, Birkhäuser. Verlag, Basel, 1979.
13. G.M. GOLUZIN, *Geometric Theory of Functions of Complex Variable*, [Russian], Nauka, Moscow 1966.
14. G. GRUBB, Pseudo-differential boundary problems in L_p spaces, *Comm. Partial Differential Equations* **15**(1990), 289–340.
15. J. JOHNSEN, Elliptic boundary problems and the Boutet de Monvel calculus in Besov and Triebel–Lizorkin spaces, *Copenh. Univ. Math. Inst. Prepr. Series*, No. 25, 1994.
16. N.K. KARAPETJANTZ, S.G. SAMKO, *Equations with Involutory Operators and their Applications*, [Russian], Izdatelstvo Rostovskogo Universiteta, Rostov–on–Don, 1988.
17. JU.I. KARLOVICH, Local trajectory method of invertibility research in C^*-algebras of operators with discrete shift groups, [Russian], *Dokl. Akad. Nauk* **299**(1988), 546–550.
18. JU.I. KARLOVICH, On algebra of singular integral operators with discrete groups of shifts, [Russian], *Dokl. Akad. Nauk* **304**(1989), 274–280.
19. JU.I. KARLOVICH, V.G. KRAVCHENKO, Algebra of singular integral operators with piecewise smooth shift on a composite contour, [Russian], *Izv. Akad. Nauk SSSR, Ser. Mat.* **47**(1983), pp. 1030–1077.
20. JU.I. KARLOVICH, V.G. KRAVCHENKO, G.S. LITVINCHUK, Noether theory of the singular integral operators with a shift, *Izv. Vyssh. Ucheb. Zaved., Mat.* **251**(1983), 3–27.
21. I.I. KOMYAK, On a certain class of two-dimensional singular integral operators over a circular domain, [Russian], *Dokl. Akad. Nauk Belarusi* **23**(1979), 972-975.
22. I.I. KOMYAK, Noetherity conditions and index formula for a certain class of singular integral equations over a circular domain, [Russian], *Differentsial'nye Uravneniya* **16**(1980), 328–343; English transl. *Differential Equations* **16**(1980), 215–226.
23. I. KRA, *Automorphic Forms and Kleinian Groups*, W.A. Benjamin, Inc., Massachusetts, 1972.
24. G.S. LITVINCHUK, *Boundary Value Problems and Singular Integral Equations with Shift*, [Russian], Nauka, Moscow, 1977.
25. A.U. MARKUSHEVICH, *Theory of Analytic Functions*, [Russian], Gostekhizdat, Moscow, 1950.

26. S.G. MIKHLIN, S. PRÖSSDORF, *Singular Integral Operators*, Springer-Verlag, Heidelberg 1986.
27. V.N. MONAKHOV, *Boundary Value Problems with Free Boundaries for Elliptic Systems of Equations*, [Russian], Nauka, Novosibirsk, 1987.
28. M.A. NAIMARK, *Normed Rings*, [Russian], Nauka, Moscow, 1968.
29. I.I. PRIVALOV, *Introduction to the Theory of Functions of Complex Variable*, [Russian], Nauka, Moscow, 1967.
30. S. REMPEL, B.-W.SCHULZE, *Index Theory of Elliptic Boundary Problems*, Akademie-Verlag, Berlin, 1982.
31. M. SCHIFFER, D.C. SPENCER, *Functionals of Finite Riemann Surfaces*, Princeton University Press, Princeton, New Jersey, 1954.
32. E. SHAMIR, Elliptic systems of singular integral operators. I. The half-space case, *Trans. Amer. Math. Soc.* **127**(1967), 107–124.
33. I.B. SIMONENKO, A new general method for the investigation of linear operator equations of singular integral equations type. I–II, [Russian], *Izv. Akad. Nauk SSSR, Ser. Matem.* **29**(1965), pp. 567–586, pp. 757–782.
34. F. TRICOMI, Equazioni integrali contenti il valor principale di un integrale doppio, *Math. Z.* **128**(1928), 87–133.
35. H. TRIEBEL, *Theory of Function Spaces*, Birkhäuser-Verlag, Boston, 1983.
36. I.N. VEKUA, *Verallgemeinerte Analytische Functionen*, Akademie-Verlag, Berlin, 1963.

R. DUDUCHAVA
A. Razmadze Mathematical Institute
Academy of Sciences of Georgia
1, M. Aleksidze str.
Tbilisi 93
GEORGIA

A. SAGINASHVILI
A. Razmadze Mathematical Institute
Academy of Sciences of Georgia
1, M. Aleksidze str.
Tbilisi 93
GEORGIA

E. SHARGORODSKY
A. Razmadze Mathematical Institute
Academy of Sciences of Georgia
1, M. Aleksidze str.
Tbilisi 93
GEORGIA

Current address:

Department of Mathematics
Kings College London
Strand, London
WC2R 2LS
U.K.

Received November 24, 1995; revised October 20, 1996.