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COMPOSITION OF SUBFACTORS AND TWISTED BICROSSED PRODUCTS

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ABSTRACT. Subfactors of the form $\mathbf{P}^H \subset \mathbf{P} \rtimes K$, where H, K are finite groups of outer automorphisms of a finite factor $\mathbf{P}$, are studied. The corresponding Jones tower and some relative commutants are explicitly described. Hopf $*$-algebras related to the depth 2 case are calculated. These turn out to have the structure of cocycle twisted bicrossed products. Definitions, properties, and several examples of such twisted bicrossed products are given.

KEYWORDS: *Subfactor, Hopf algebra, twisted bicrossed product.*

AMS SUBJECT CLASSIFICATION: 46L37, 16W30.

1. INTRODUCTION

Since the V. Jones' history making article ([8]) the theory of subfactors has been among the most interesting and rapidly growing areas of research in Functional Analysis. Among the deepest results obtained so far is S. Popa's classification of amenable subfactors ([21]). The basic invariant used in the classification is paragroup (or standard invariant), e.g. cf. [8], [17], [21].

An abstract paragroup is a complicated object, equipped in an intricate algebraic structure. As explained by A. Ocneanu, in the special case of a depth 2 subfactor the corresponding paragroup reduces to a Hopf (or Kac) algebra. This important observation provides an interesting link between subfactors and Hopf algebra actions. By now several rigorous proofs are already available ([5], [11], [24], [26]), covering both finite and infinite factors, as well as subfactors with infinite indices.

Since full understanding of the enormous variety of possible subfactors seems an exceedingly hard and rather hopeless enterprise, much effort has been devoted lately to the investigation of some of their special, more accessible subclasses. Here belong subfactors related to group actions, which have recently attracted attention of a number of researchers. In particular, D. Bisch and U. Haagerup calculated principal graphs for inclusions of the type $\mathbf{P}^H \subset \mathbf{P} \rtimes K$, where $\mathbf{P}$ is a type II_1 factor and H, K finite groups of its outer automorphisms ([2]). In this article we undertake to continue the investigation of such subfactors and their invariants, though from a somewhat different angle.

In Section 3 of the present paper an explicit description of the Jones tower, associated with an inclusion $\mathbf{P}^H \subset \mathbf{P} \rtimes K$, is given. The elements of the tower are obtained as crossed products for suitable H and K actions. This allows us to give a concrete description of the first non-trivial relative commutant. These results are then used in Section 4 in the calculation of Hopf algebras corresponding to the depth 2 case. For this we employ a general method developed in [24] and based on a duality between consecutive relative commutants.

As shown in [2], an inclusion $\mathbf{P}^H \subset \mathbf{P} \rtimes K$ has depth 2 if and only if HK is a group in $\mathrm{Out}(\mathbf{P})$. Thus, H and K form a matched pair, and it turns out that the Hopf algebras have the structure of bicrossed products twisted by unitary 2-cocycles. The bicrossed product is a basic general method for constructing Hopf algebras. Its origins go back to G.I. Kac (e.g. cf. [9]). For a more recent presentation we refer the reader to the fundamental works of S. Majid, e.g. [12], [13].

Cocycle deformations of Hopf algebras have been previously studied by S. Baaj and G. Skandalis ([1]). Their approach is different from ours and involves perturbations of multiplicative unitaries. In our opinion such cocycles for multiplicative unitaries (cf. Definition 8.24, [1]) do not readily lend themselves to a thorough analysis. In particular, the structure of the resulting algebras is not immediately clear. Nor is it easy to construct such objects. Therefore, we feel that an attack on the problem carried from a different angle, as in the present paper, might shed more light on it and increase the understanding of this important and complicated construction.

Reversing the order in which things were discovered, we present our theory of twisted bicrossed products in Section 2, and illustrate it with several examples. We show that a classical construction of G.I. Kac and V. Paljutkin from [10] corresponds to a very special class of twisted bicrossed products, closely related to simple ergodic actions of finite abelian groups (see [18]).

At the closing of Section 4 we show that the two unitary 2-cocycles used in the twisting of a bicrossed product, related to a depth 2 inclusion $\mathbf{P}^H \subset \mathbf{P} \rtimes K$, can be recovered from the numerical 3-cocycle obstruction ([4], [15], [23], [25]). This gives a hint of a deeper connection between deformation theory and third cohomology.

After the work on this paper had been completed we learnt that about the same time some similar results were obtained independently by M. Izumi and H. Kosaki ([7]).

2. TWISTED BICROSSED PRODUCTS

In this section we describe cocycle deformations of finite dimensional bicrossed product Hopf $*$-algebras (cf. [12], [13]) and give several examples. An analogous construction from a different point of view can be found in [1]. Our definitions are motivated by the class of examples calculated in Section 4, and related to depth 2 subfactors.

Let G be a finite group and H, K subgroups of it such that $G = HK$ and $H \cap K = \{e\}$, where e is the neutral element. H acts on K (as on a set) by $\{h \cdot t\} = K \cap Hth^{-1}$, $h \in H$, $t \in K$. Indeed,

$$\begin{aligned}\{g \cdot (h \cdot s)\} &= K \cap H(K \cap Hsh^{-1})g^{-1} \subseteq K \cap H(Hsh^{-1})g^{-1}\\ &= K \cap Hs(gh)^{-1} = \{gh \cdot s\}.\end{aligned}$$

Similarly, K acts on H by $\{t \cdot h\} = H \cap Kht^{-1}$. We have $(h \cdot t)h = (t \cdot h)t$, $(h \cdot t) \cdot h^{-1} = (t \cdot h)^{-1}$, $(t \cdot h) \cdot t^{-1} = (h \cdot t)^{-1}$, and $h \cdot e = t \cdot e = e$.

Let $C(K)$ denote the C^*-algebra of functions on K, with a basis of minimal projections $\{p_t \mid t \in K\}$. There is an action $\gamma : H \to \mathrm{Aut}(C(K))$, $\gamma_h(p_t) = p_{h \cdot t}$. Let $\mu : H \times H \to \mathcal{U}(C(K))$ be a normalized unitary 2-cocycle for γ, i.e., $\gamma_g(\mu(h,f))\mu(g,hf) = \mu(g,h)\mu(gh,f)$ and $\mu(h,e) = I = \mu(e,h)$. We will use the notation $\mu(h,g) = \sum_{t \in K} \mu(h,g)[t]p_t$, where $\mu(h,g)[t] \in \mathbb{T}$ ($\mathbb{T}$ denotes the torus). Let $\mathbf{A}$ be the twisted crossed product $C(K) \rtimes_{\gamma,\mu} H$. $\mathbf{A}$ has a basis $\{p_t v_h \mid h \in H,\ t \in K\}$, where v_h unitary, $p_t v_h p_s v_g = \delta_{t, h\cdot s} p_t \mu(h,g) v_{hg}$, and $(p_t v_h)^* = p_{h^{-1}\cdot t}\mu(h^{-1}, h)^* v_{h^{-1}}$.

Similarly, $C(H)$ has a basis $\{q_h \mid h \in H\}$, and there is an action $\sigma : K \to \mathrm{Aut}(C(H))$, $\sigma_t(q_h) = q_{t\cdot h}$. Let $\nu : K \times K \to \mathcal{U}(C(H))$ be a normalized unitary 2-cocycle for σ, and $\mathbf{B} = C(H) \rtimes_{\sigma,\nu} K$ be the corresponding twisted crossed product, with a basis $\{q_h z_t \mid h \in H,\ t \in K\}$.

We define a bilinear form $\langle \cdot, \cdot \rangle : \mathbf{A} \times \mathbf{B} \to \mathbb{C}$ by

$$\langle p_t v_h, q_g z_s \rangle = \delta_{t, h\cdot s}\delta_{g, s\cdot h}\nu(s^{-1}, s)[h].$$

This form establishes a duality between **A** and **B**. Thus, **B** can be identified through $\langle\cdot,\cdot\rangle$ with $\mathbf{A}^\circ$ — the dual of **A**. Consequently, we can define linear maps $\Delta:\mathbf{A}\to\mathbf{A}\otimes\mathbf{A}$, $\varepsilon:\mathbf{A}\to\mathbb{C}$, and $S:\mathbf{A}\to\mathbf{A}$ by

$$\langle\Delta(a),b_1\otimes b_2\rangle=\langle a,b_1b_2\rangle$$
$$\varepsilon(a)=\langle a,I\rangle$$
$$\langle S(a),b\rangle=\overline{\langle a^*,b^*\rangle}.$$

A tedious but not complicated calculation yields (for any $h\in H$, $t\in K$):

$$\varepsilon(p_tv_h)=\delta_{t,e}$$
$$S(p_tv_h)=\mu(h,h^{-1})[t]\overline{\nu(t^{-1},t)[h^{-1}]}p_{(h^{-1}\cdot t)^{-1}}v_{t\cdot h^{-1}}$$
$$\Delta(p_tv_h)=\sum_{k\in K}\nu((h^{-1}\cdot k)^{-1},(h^{-1}\cdot k)(h^{-1}\cdot t)^{-1})[h]^{-1}p_{tk^{-1}}v_{(k\cdot h^{-1})^{-1}}\otimes p_kv_h.$$

PROPOSITION 2.1. *$\mathbf{A}$ equipped with Δ, S, and ε is a Hopf $*$-algebra if and only if $\nu(s^{-1},s)[h]\in\{\pm1\}$ (i.e. $\nu(s^{-1},s)$ is self-adjoint) and*

$$\frac{\mu((g\cdot s)\cdot h),s\cdot g)[(hg\cdot t)(hg\cdot s)^{-1}]\mu(h,g)[hg\cdot s]}{\mu(h,g)[hg\cdot t]}=\frac{\nu((g\cdot s)^{-1},(g\cdot s)(g\cdot t)^{-1})[h]\nu(s^{-1},st^{-1})[g]}{\nu(s^{-1},st^{-1})[hg]};\tag{2.1}$$

$$\frac{\mu(h^{-1},h)[s]\mu((s\cdot h)^{-1},s\cdot h)[(h^{-1}\cdot t)s^{-1}]}{\mu(h,h^{-1})[t]}=\nu((h\cdot s)^{-1},(h\cdot s)t^{-1})[h^{-1}]\nu(s^{-1},s(h^{-1}\cdot t)^{-1})[h]\tag{2.2}$$

for any $h,g\in H$, $s,t\in K$.

Proof. Equality (2.1) holds if and only if Δ preserves the multiplication. Equality (2.2) holds if and only if Δ preserves the $*$. Both have to be checked only on the basis $\{p_tv_h\mid t\in K,h\in H\}$. Moreover, $\nu(s^{-1},s)[h]\in\{\pm1\}$ is equivalent to $S*S*=\mathrm{id}$. Consequently, the above conditions are necessary for **A** to be a Hopf $*$-algebra.

To see that these conditions are sufficient, we first observe that the very definitions imply that $(\Delta\otimes\mathrm{id})\Delta=(\mathrm{id}\otimes\Delta)\Delta$ and $(\varepsilon\otimes\mathrm{id})\Delta=(\mathrm{id}\otimes\varepsilon)\Delta=\mathrm{id}$. Setting $t=s=e$ in (i) we get $\mu(h,g)[e]=1$, which is equivalent to ε being a $*$-homomorphism. Similarly, setting $h=g=e$ in (i) we get $\nu(s,t)[e]=1$ for any $s,t\in K$. Now setting $g=h^{-1}$ and $t=e$ in (i) we get

$$\mu((s\cdot h^{-1})^{-1},s\cdot h^{-1})[s^{-1}]\mu(h,h^{-1})[s]=\nu((h^{-1}\cdot s)^{-1},h^{-1}\cdot s)[h]\nu(s^{-1},s)[h^{-1}].$$

This means that ($m:\mathbf{A}\otimes\mathbf{A}\to\mathbf{A}$ is the multiplication)

$$m(S\otimes\mathrm{id})\Delta(p_sv_{h^{-1}})=\varepsilon(p_sv_{h^{-1}})I=m(\mathrm{id}\otimes S)\Delta(p_sv_{h^{-1}}),$$

i.e. S is an antipode. ∎

If the conditions of Proposition 2.1 are satisfied then the Hopf $*$-algebras $\mathbf{A}$ and $\mathbf{A}^\circ$ will be called *twisted bicrossed products.*

Suppose that $\mathbf{A}$ is a Hopf $*$-algebra. We denote by $G(\mathbf{A})$ the collection of group-like elements of $\mathbf{A}$. That is $a \in G(\mathbf{A})$ if $a \neq 0$ and $\Delta(a) = a \otimes a$. It is well known that group-like elements are unitary, linearly independent, and form a group under multiplication. Moreover, φ is a group-like element of $\mathbf{A}^\circ$ if and only if φ is a character of $\mathbf{A}$. Groups and their duals provide the simplest examples of Hopf $*$-algebras. Thus, the smaller the collections of group-like elements of a Hopf $*$-algebra and its dual are, the farther away is the algebra from the trivial group case.

Let $t \in K$ and $h \cdot t = t$ for any $h \in H$. Then $\mu(\,\cdot\,,\cdot\,)[t]$ is a numerical 2-cocycle on H. We denote

$$K_0 = \{t \in K \mid h \cdot t = t,\ \forall h \in H,\ \text{and } \mu(\,\cdot\,,\cdot\,)[t] \text{ is a coboundary}\}$$

and $\widehat{K}$ — the group of characters of K. We define H_0 and $\widehat{H}$ analogously.

PROPOSITION 2.2. *If $\mathbf{A}$ is a twisted bicrossed product, then both K_0 and H_0 are subgroups, and there exist short exact sequences:*

$$\langle e\rangle \to \widehat{K} \to G(\mathbf{A}) \to H_0 \to \langle e\rangle$$
$$\langle e\rangle \to \widehat{H} \to G(\mathbf{A}^\circ) \to K_0 \to \langle e\rangle.$$

Proof. For any $t \in K_0$ we fix a function $\lambda_t : H \to \mathbf{T}$ such that $\mu(h,g)[t] = \lambda_t(h)\lambda_t(g)\overline{\lambda_t(hg)}$. Since $\mu(h,g)[e] = 1$ by Proposition 2.1, we may select $\lambda_e \equiv 1$.

Let $\varphi \in G(\mathbf{A}^\circ)$, i.e., φ is a character on $\mathbf{A}$. Since $\varphi|C(K)$ is a homomorphism, there exists a $t \in K$ such that $\varphi(p_s) = \delta_{s,t}$. Since $\mu(h,g)[t]\varphi(v_{hg}) = \varphi(v_h)\varphi(v_g)$, $\mu(\,\cdot\,,\cdot\,)[t]$ is a coboundary and, hence, $t \in K_0$. Therefore, there exists a $\xi \in \widehat{H}$ such that $\varphi(p_s v_h) = \delta_{s,t}\lambda_t(h)\xi(h)$. If $t = e$ then we write $\varphi = \xi'$.

Let $\varphi_1, \varphi_2 \in G(\mathbf{A}^\circ)$ be such that $\varphi_1(p_s v_h) = \delta_{s,t_1}\lambda_{t_1}(h)\xi_1(h)$, $\varphi_2(p_s v_h) = \delta_{s,t_2}\lambda_{t_2}(h)\xi_2(h)$, for $t_1, t_2 \in K_0$, $\xi_1, \xi_2 \in \widehat{H}$. We have

$$(\varphi_1\varphi_2)(p_s v_h) = (\varphi_1 \otimes \varphi_2)\Delta(p_s v_h) = \delta_{s,t_1t_2}\overline{\nu(t_2^{-1}, t_1^{-1})[h]}\lambda_{t_1}(h)\lambda_{t_2}(h)(\xi_1\xi_2)(h).$$

This equality implies that $\xi \mapsto \xi'$ is an imbedding and $\widehat{H}$ is a normal subgroup of $G(\mathbf{A}^\circ)$. Furthermore, K_0 is a group and $G(\mathbf{A}^\circ)/\widehat{H}$ is isomorphic to K_0.

This proves one part of the proposition. The remaining part is established in a similar fashion. ∎

We wonder if there exists a (twisted bicrossed product) finite dimensional Hopf $*$-algebra $\mathbf{A}$ such that neither $\mathbf{A}$ nor $\mathbf{A}^\circ$ possesses non-trivial group-like elements.

EXAMPLE 2.3. *Semi-direct products.* Suppose that K acts trivially on H. Therefore, H acts on K by automorphisms $h \cdot t = hth^{-1}$, and G is isomorphic to a semi-direct product $K \rtimes H$.

In order to formulate the conditions of Proposition 2.1 in a more convenient way, we introduce new cocycles $\widetilde{\mu}$ and $\widetilde{\nu}$. Namely, we define $\widetilde{\mu}(g,h)[t] = \mu(h^{-1}, g^{-1})[(gh)^{-1} \cdot t]$ and $\widetilde{\nu}(t,s)[g] = \nu(s^{-1}, t^{-1})[g^{-1}]$, for any $g, h \in H$, $t, s \in K$. One can easily check that $\widetilde{\mu}$, $\widetilde{\nu}$ are 2-cocycles for γ and σ, respectively. We consider an action of H on $\mathcal{Z}^2(K, \mathcal{U}(C(H)))$, given by $(g\lambda)(s,t) = \lambda(g^{-1} \cdot s, g^{-1} \cdot t)$. Then, condition (i) of Proposition 2.1 is equivalent to the following:

$$\frac{\widetilde{\mu}(g,h)[t]\widetilde{\mu}(g,h)[s]}{\widetilde{\mu}(g,h)[ts]} = \frac{(g\widetilde{\nu})(t,s)[h]\widetilde{\nu}(t,s)[g]}{\widetilde{\nu}(t,s)[gh]}.$$

We can view $\widetilde{\mu}$ and $\widetilde{\nu}$ as elements of $C^1(K, C^2(H, \mathbf{T}))$ and $C^1(H, C^2(K, \mathbf{T}))$, respectively. Denoting by ∂_K and ∂_H the coboundary maps for the related cochain complexes, we can further rewrite it as

$$((\partial_K \widetilde{\mu})(t,s))(g,h) = ((\partial_H \widetilde{\nu})(g,h))(t,s). \tag{2.3}$$

If this equality holds and μ has been selected in such a way that $\mu(h^{-1}, h) = I$ (this is always possible, e.g. see Lemma 4.3), then condition (ii) of Proposition 2.1 becomes $\widetilde{\nu}(e) = I$. Therefore, if $G = K \rtimes H$ and we choose $\mu \equiv I$, then any normalized element $\widetilde{\nu}$ of $\mathcal{Z}^1(H, \mathcal{Z}^2(K, \mathbf{T}))$ determines a twisted bicrossed product.

EXAMPLE 2.4. *Hopf algebras of the Kac-Paljutkin type.* The following construction is due to G. Kac and V. Paljutkin ([10]).

Let K be a finite group of order n^2, and $C(K)$ be the algebra of functions on K with a basis $\{e_s \mid s \in K\}$, where each e_s is a minimal projection. We denote $M = M_n(\mathbb{C})$ and fix a system of matrix units $w_{i,j}$ in M, thus identifying elements of M with matrices. Suppose there exist unitaries T, $\{u_s \mid s \in K\}$ in M, and functions $\zeta, \rho : K \times K \to \mathbf{T}$ such that $T^* = \overline{T}$ ($\overline{T}$ denotes the matrix whose entries are complex conjugates of the entries of T) and

$$u_e = I, \text{ and } \operatorname{tr}(u_s) = 0 \text{ unless } s = e \tag{2.4}$$

$$u_s u_t = \zeta(s,t) u_{st} \tag{2.5}$$

$$u_s T \bar{u}_t T^* = \rho(s,t) T \bar{u}_t T^* u_s \tag{2.6}$$

where tr denotes the normalized trace on M. Then $\mathbf{B} = C(K) \oplus M$ is a Hopf $*$-algebra with a comultiplication given by the following formulae:

$$\Delta(e_s) = \sum_{t\in K} e_{st^{-1}} \otimes e_t + |K|^{-\frac{1}{2}} \sum_{i,j,m,n} (u_sT)_{i,m}\overline{(u_sT)}_{j,n} w_{i,j} \otimes w_{m,n}$$
$$\Delta(x) = \sum_{s\in K} e_{s^{-1}} \otimes u_s x u_s^* + \sum_{s\in K} (T\bar{u}_sT^*)x(T\bar{u}_sT^*)^* \otimes e_s$$

where $x \in M$. Furthermore, K is isomorphic to $G(\mathbf{B}^\circ)$. Conversely, it is shown in Section 8, [10] that any Hopf $*$-algebra $\mathbf{B}$, with underlying C^*-algebra isomorphic to $C(K)\oplus M$ and $G(\mathbf{B}^\circ) \cong K$, must be of this form. A detailed description of the structure of $\mathbf{B}$ and its dual, in the special case when $K = \mathbb{Z}_n \times \mathbb{Z}_n$, can be found in [22].

We are going to show that any Hopf algebra of the Kac-Paljutkin type is a twisted bicrossed product corresponding to a semi-direct product, as in Example 2.3. In the special case when $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ it is shown in [1] that such an algebra can be constructed from a suitable cocycle deformation of a multiplicative unitary corresponding to a bicrossed product.

We denote $m_s = T\bar{u}_sT^*$. It follows from (2.5) that $m_sm_t = \overline{\zeta(s,t)}m_{st}$. Since $u_su_{s^{-1}}u_s = \zeta(s,s^{-1})u_s = \zeta(s^{-1},s)u_s$, we have $\zeta(s,s^{-1}) = \zeta(s^{-1},s)$. Thus, replacing u_s by $\zeta(s,s^{-1})^{-1/2}u_s$ and m_s by $\zeta(s,s^{-1})^{1/2}m_s$ we may assume that $\zeta(s,s^{-1}) = 1$ and, consequently, $u_s^* = u_{s^{-1}}$ and $m_s^* = m_{s^{-1}}$.

Analyzing the Kac-Paljutkin construction (cf. [10]) from the point of view of the theory of ergodic actions of compact abelian groups developed in [18], one obtains the following.

LEMMA 2.5. *For any* $\mathbf{B}$, *a Hopf $*$-algebra of the Kac-Paljutkin type, as described in Example* 2.4, *the following hold:*

(i) *There exists an abelian group* Γ *such that* K *is isomorphic to* $\widehat{\Gamma} \times \Gamma$.

(ii) $s \mapsto \mathrm{Ad}\, u_s$ *is a faithful ergodic action of* K *on* M, *for which* $\{m_s\}$ *is a complete set of unitary eigenoperators. We denote by* χ *the corresponding symplectic bicharacter (cf.* [18]*). Both* ρ *and* χ *are nondegenerate bicharacters, i.e. they determine isomorphisms between* K *and* $\widehat{K}$.

(iii) *There exists an automorphism* h *of* K *such that* $u_{h(s)} = \xi_s m_s$ *for some* $\xi_s \in \mathbb{T}$. *We have* $h^2 = \mathrm{id}$, $\xi_s = \xi_{h(s)}$, $\xi_s\xi_{s^{-1}} = \xi_e = 1$, $\rho(s,t) = \overline{\chi(s,h(t))}$, *and* $\chi(s,t) = \chi(h(t),h(s))$.

(iv) *The* $M \otimes M$*-component of* $\Delta(e_s)$ *is equal to*

$$|K|^{-1} \sum_{t\in K} \rho(s,t) m_t \otimes u_t.$$

Proof. Formula (2.5) implies that $\mathrm{Ad}\, u$ is an action. By (2.4) $\{u_s\}$ form a v.s. basis of M and, hence, the action is ergodic. As $|K| = \dim M$, it must be faithful. (2.6) means that $(\mathrm{Ad}\, u_s)m_t = \rho(s,t)m_t$ and, hence, $\{m_t\}$ form a complete set of unitary eigenoperators for $\mathrm{Ad}\, u$. This implies that $\mathrm{Ad}\, u_s$ commute and, consequently, K is abelian. If χ is the associated symplectic bicharacter (cf. [18]), then it follows from Theorem 5.8, [18] that χ is nondegenerate, i.e. the map $s \mapsto \rho(s,\cdot)$ is an isomorphism. It is clear that ρ is a nondegenerate bicharacter. Theorem 5.9, [18] implies that there exists an abelian group Γ such that K is isomorphic to $\widehat{\Gamma} \times \Gamma$. In this way parts (i) and (ii) of the lemma are proven.

Since K is abelian, (2.5) implies that $(\mathrm{Ad}\, u_s)u_t$ is a scalar multiple of u_t. Thus, $\{u_s\}$ is another complete set of unitary eigenoperators. Consequently, there exists a bijection $h : K \to K$ and a function $K \ni s \mapsto \xi_s \in \mathbb{T}$ such that $u_{h(s)} = \xi_s m_s$. Since $m_s = T\bar{u}_s T^*$ and $\overline{T} = T^*$, it follows that h is an automorphism and $h^2 = \mathrm{id}$. It is clear that $\xi_s = \xi_{h(s)}$ and $\xi_s \xi_{s^{-1}} = \xi_e = 1$.

According to [18] we have

$$\chi(s,t) = \overline{\zeta(s,t)}\zeta(t,s) = m_s m_t m_s^* m_t^*$$

and, similarly,

$$\overline{\chi(s,t)} = \zeta(s,t)\overline{\zeta(t,s)} = u_s u_t u_s^* u_t^*.$$

Thus, $(\mathrm{Ad}\, u_s)u_t = \overline{\chi(s,t)}u_t$. Consequently,

$$\rho(s,t)m_t = (\mathrm{Ad}\, u_s)m_t = \bar{\xi}_t(\mathrm{Ad}\, u_s)u_{h(t)} = \bar{\xi}_t\overline{\chi(s,h(t))}u_{h(t)} = \overline{\chi(s,h(t))}m_t.$$

Hence, $\rho(s,t) = \overline{\chi(s,h(t))}$. Since

$$\chi(s,t) = m_s m_t m_s^* m_t^* = T\,\overline{u_s u_t u_{s^{-1}} u_{t^{-1}}}\,T^* = \overline{u_s u_t u_{s^{-1}} u_{t^{-1}}}$$

and

$$\chi(h(t),h(s)) = m_{h(t)} m_{h(s)} m_{h(t)}^* m_{h(s)}^* = u_t u_s u_t^* u_s^* = \overline{u_s u_t u_{s^{-1}} u_{t^{-1}}},$$

we have $\chi(s,t) = \chi(h(t),h(s))$. Thus, part (iii) of the lemma is proven.

There exist matrices $x_t \in M$ such that the $M \otimes M$-component of $\Delta(e_s)$ equals $\sum_{t \in K} x_t \otimes u_t$. By (2.4) we have

$$x_t = (\mathrm{id} \otimes \mathrm{tr})(\Delta(e_s)(I \otimes u_{t^{-1}})).$$

Using the expression for $\Delta(e_s)$ from Example 2.4 this can be easily calculated to be equal to $|K|^{-1}\rho(s,t)m_t$. This proves part (iv). ∎

THEOREM 2.6. *A Hopf $*$-algebra* $\mathbf{B}$ *of the Kac-Paljutkin type is isomorphic to a twisted bicrossed product* $\mathbf{A}^\circ = C(H) \rtimes_\nu K$, *where* $H = \mathbb{Z}_2$, $K = \widehat{\Gamma} \times \Gamma$ *for an abelian group* Γ, *and the crossed product is with respect to the trivial action.*

Proof. Let $H = \langle h \rangle$, and define an action of H on K by $h \cdot s = h(s)$, where h is the automorphism of K from Lemma 2.5. We set $G = K \rtimes H$. We also define a unitary 2-cocycle for the trivial action of K on H by setting $\nu(s,t)[e] = 1$, $\nu(s,t)[h] = (\xi_s \xi_t \overline{\xi}_{st})^{-1/2} \zeta(s,t)$, with ξ_s as in Lemma 2.5. By virtue of Lemma 2.3 we have $\nu(s,s^{-1}) = I$ and $\nu(h \cdot s, h \cdot t)[h] = \overline{\nu(s,t)[h]}$. It follows from Example 2.3 that $\mathbf{A}^\circ = C(H) \rtimes_\nu K$ is a twisted bicrossed product.

To show the existence of a Hopf $*$-algebra isomorphism between $\mathbf{A}^\circ$ and $\mathbf{B}$, we define in $\mathbf{B}$ projections $q_e = \sum_{s \in K} e_s$, $q_h = I - q_e$, and unitaries

$$z_s = \sum_{t \in K} \overline{\rho(s,t)} e_t + \xi_s^{-\frac{1}{2}} u_s$$

for $s \in K$. Clearly, we have $z_s z_t = \nu(s,t) z_{st}$. Furthermore, $\Delta(q_e) = q_e \otimes q_e + q_h \otimes q_h$ and $\Delta(q_h) = q_h \otimes q_e + q_e \otimes q_h$.

In order to complete the proof it sufficies to show that

$$\Delta(z_s) = z_s \otimes q_e z_s + z_{h \cdot s} \otimes q_h z_s .$$

This is easiest seen by cutting both sides of the above equality by the central orthogonal projections $q_e \otimes q_h$, $q_h \otimes q_e$, $q_e \otimes q_e$, and $q_h \otimes q_h$. The first three cases are straightforward, and the fourth is equivalent to

$$\sum_{t \in K} \overline{\rho(s,t)} (q_h \otimes q_h) \Delta(e_t) = (\xi_s \xi_{h \cdot s})^{-\frac{1}{2}} u_{h \cdot s} \otimes u_s .$$

However, applying Lemma 2.5 several times and using the orthogonality relations for group characters we get

$$\begin{aligned}
\sum_{t \in F} \overline{\rho(s,t)} (q_h \otimes q_h) \Delta(e_t) &= |K|^{-1} \sum_{t,k \in K} \overline{\rho(s,t)} \rho(t,k) m_k \otimes u_k \\
&= |K|^{-1} \sum_{k \in K} \Big(\sum_{t \in K} \chi(sk^{-1}, h(t)) \Big) m_k \otimes u_k \\
&= m_s \otimes u_s = (\xi_s \xi_{h \cdot s})^{-\frac{1}{2}} u_{h \cdot s} \otimes u_s ,
\end{aligned}$$

as required. ∎

A concrete realization of such a Hopf $*$-algebra can be described as follows. Let Γ be a finite abelian group, $\widehat{\Gamma}$ its dual, and $\langle\cdot,\cdot\rangle : \Gamma \times \widehat{\Gamma} \to \mathbb{T}$ the pairing. Let $H = \langle h \rangle = \mathbb{Z}_2$ and $K = \widehat{\Gamma} \times \Gamma$. We choose an isomorphism $\theta : \Gamma \to \widehat{\Gamma}$ such that $\langle s, \sigma \rangle = \langle \theta^{-1}(\sigma), \theta(s) \rangle$ (there is always one), and define an action of H on K by $h \cdot (\sigma, s) = (\theta(s), \theta^{-1}(\sigma))$. Finally, we define a unitary 2-cocycle on K (for the trivial action) by setting $\nu(\cdot,\cdot)[e] = 1$, $\nu((\sigma, s), (\tau, t))[h] = \langle s, \tau \rangle \overline{\langle t, \sigma \rangle}$. One can easily check that the conditions of Example 2.3 are satisfied and the resulting twisted bicrossed product $\mathbf{A}^\circ = C(K) \rtimes_\nu H$ is isomorphic to a Hopf $*$-algebra of the Kac-Paljutkin type.

3. SUBFACTORS OF THE FORM $P^H \subseteq P \rtimes K$

In this section we study the class of irreducible, finite index subfactors related to outer actions of finite groups on type II_1 factors. Such inclusions were recently studied by D. Bisch and U. Haagerup ([2]), who among other things determined the corresponding principal graphs. We are going to describe here the related Jones tower, as well as the structure of the first non-trivial relative commutant. This information will be used in Section 4 to determine Hopf $*$-algebras corresponding to the depth 2 case.

Let $\mathbf{P}$ be a type II_1 factor and H, K be finite groups acting outerly on $\mathbf{P}$ by $\alpha : K \to \mathrm{Aut}(\mathbf{P})$ and $\beta : H \to \mathrm{Aut}(\mathbf{P})$. We will often identify H, K with subgroups of $\mathrm{Out}(\mathbf{P})$. We denote $\mathbf{N} = \mathbf{P}^H$ and $\mathbf{M} = \mathbf{P} \rtimes K$, type II_1 factors. The Jones index ([8]) of the subfactor $\mathbf{N} \subseteq \mathbf{M}$ is $[\mathbf{M} : \mathbf{N}] = |H|\,|K|$, where $|G|$ denotes the order of a group G.

We consider a tower $\mathbf{N} \subseteq \mathbf{M} = \mathbf{M}_0 \subseteq \mathbf{M}_1 \subseteq \mathbf{M}_2 \subseteq \cdots$ of Jones extensions ([8]), with the corresponding Jones projections $e_k \in \mathbf{M}_k$. We denote by τ the unique normalized trace on any of the elements of the tower. We also denote by E_k the τ-preserving conditional expectation onto $\mathbf{M}_k$ (from any $\mathbf{M}_n$ with $k \leqslant n$). The index $k = 0$ will be usually omitted.

LEMMA 3.1. *If $\theta \in \mathrm{Aut}(\mathbf{P})$ and $x \in \mathbf{P} \setminus \{0\}$ satisfies $yx = x\theta(y)$ for all $y \in \mathbf{N}$, then there exist $\lambda \in \mathbb{C} \setminus \{0\}$, $u \in \mathcal{U}(\mathbf{P})$, and $h \in H$ such that $x = \lambda u$ and $(\mathrm{Ad}\, u)\theta = \beta_h$.*

Proof. We have $yx = x\theta(y)$ and, consequently, $x^* y = \theta(y) x^*$ for any $y \in \mathbf{N}$. Therefore, $x^* x \theta(y) = x^* y x = \theta(y) x^* x$ and, hence, $x^* x \in \mathbf{N}' \cap \mathbf{P} = \mathbb{C}I$, since β is outer. Thus, there are $\lambda \in \mathbb{C} \setminus \{0\}$ and $u \in \mathcal{U}(\mathbf{P})$ such that $x = \lambda u$. Since $\theta(y) = u^* y u$, $(\mathrm{Ad}\, u)\theta$ is an element of $\mathcal{G}(\mathbf{P}, \mathbf{P}^H)$ — the Galois group of $\mathbf{P}^H \subseteq \mathbf{P}$. By [16] there is an $h \in H$ such that $(\mathrm{Ad}\, u)\theta = \beta_h$, as required. ∎

It is shown in Corollary 3.1, [2] that the inclusion $\mathbf{N} \subseteq \mathbf{M}$ is irreducible (i.e. $\mathbf{N}' \cap \mathbf{M} = \mathbb{C}I$) if and only if $H \cap K = \langle e \rangle$ in $\operatorname{Out}(\mathbf{P})$ (e is the neutral element of a group). In fact, the following holds.

PROPOSITION 3.2. *There exists a 2-cocycle ω on $H \cap K \subset \operatorname{Out}(\mathbf{P})$ (with values in the torus $\mathbb{T}$) such that $\mathbf{N}' \cap \mathbf{M}$ is isomorphic to the twisted group algebra $\mathbb{C}_\omega[H \cap K]$.*

Proof. Let $\sum_{s \in K} x_s s \in \mathbf{N}' \cap \mathbf{M}$, with $x_s \in \mathbf{P}$. If $y \in \mathbf{N}$, then $\sum_{s \in K} y x_s s = \sum_{s \in K} x_s s y = \sum_{s \in K} x_s \alpha_s(y) s$ and, hence, $y x_s = x_s \alpha_s(y)$ for any $s \in K$. If $x_s \neq 0$, then by Lemma 3.1 there are $\lambda_s \in \mathbb{C} \setminus \{0\}$, $u_s \in \mathcal{U}(\mathbf{P})$, and $h(s) \in H$ such that $(\operatorname{Ad} u_s)\alpha_s = \beta_{h(s)}$, i.e. $s = h(s)$ in $\operatorname{Out}(\mathbf{P})$. Consequently,

$$\mathbf{N}' \cap \mathbf{M} = \Big\{ \sum_{s \in H \cap K} \lambda_s u_s s \mid \lambda_s \in \mathbb{C},\ u_s \in \mathcal{U}(\mathbf{P}),\ \operatorname{Ad} u_s = \beta_s \alpha_s^{-1} \Big\}.$$

Since the unitaries $\{u_s\}$ are determined up to scalars, it follows that for any $s, t \in K$ there exists an $\omega(s,t) \in \mathbb{T}$ such that $u_s s u_t t = u_s \alpha_s(u_t) st = \omega(s,t) u_{st} st$. ∎

From now on we assume that the inclusion $\mathbf{N} \subseteq \mathbf{M}$ is irreducible, i.e. $H \cap K = \langle e \rangle$ in $\operatorname{Out}(\mathbf{P})$.

Let K° be the Hopf $*$-algebra dual to K, $\widehat{\alpha} : K^\circ \times \mathbf{M} \to \mathbf{M}$ be the action dual to α, and $\mathbf{P}_1 = \mathbf{M} \times_{\widehat{\alpha}} K^\circ$ be the corresponding crossed product ([19]). We identify K° with a subalgebra K_1° of $\mathbf{P}_1$, whose minimal projections are $\{p_t^1 \mid t \in K\}$. For any $s, t \in K$ we have $s p_t^1 s^{-1} = p_{st}^1$ in $\mathbf{P}_1$. By [14] we have $\mathbf{P}_1 \cong \mathbf{P} \otimes \mathcal{B}(\ell^2(K))$. More specifically, one can easily verify the following.

PROPOSITION 3.3. *Let $\mathbf{D}$ be the $*$-subalgebra of $\mathbf{P}_1$ generated by K and K_1°, and $\mathbf{Q} = \mathbf{D}' \cap \mathbf{P}_1$. We have $\mathbf{D} \cong \mathcal{B}(\ell^2(K))$, $\mathbf{Q} \cong \mathbf{P}$, and $\mathbf{P}_1 \cong \mathbf{Q} \otimes \mathbf{D} \cong \mathbf{P} \otimes \mathcal{B}(\ell^2(K))$. The map $x \mapsto \sum_{s \in K} \alpha_s(x) p_s^1$, $x \in \mathbf{P}$, establishes an isomorphism between $\mathbf{P}$ and $\mathbf{Q}$.*

Since $\mathbf{P}_1 = \mathbf{M} \rtimes_{\widehat{\alpha}} K^\circ$, there is a dual action, denoted α^1, of K on $\mathbf{P}_1$. For this action we have $\mathbf{P}_1^K = \mathbf{M}$. Proposition 3.3 allows us to define an outer action β^1 of H on $\mathbf{P}_1 \cong \mathbf{P} \otimes \mathcal{B}(\ell^2(K))$ as $\beta^1 = \beta \otimes \mathrm{id}$. We have $\beta_h^1(s) = s$, $\beta_h^1(p_s^1) = p_s^1$, and $\beta_h^1\Big(\sum_{s \in K} \alpha_s(x) p_s^1\Big) = \sum_{s \in K} \alpha_s(\beta_h(x)) p_s^1$, for any $h \in H$, $s \in K$, and $x \in \mathbf{P}$. Moreover,

$$\beta_h^1(x) = \beta_h^1\Big(\sum_{s \in K} x p_s^1\Big) = \beta_h^1\Big(\sum_{s \in K} \alpha_s(\alpha_s^{-1}(x)) p_s^1\Big) = \sum_{s \in K} (\alpha_s \beta_h \alpha_s^{-1})(x) p_s^1$$

for any $h \in H$, $x \in \mathbf{P}$. Clearly, $\mathbf{P}_1 \rtimes_{\beta^1} H$ is a type II_1 factor containing $\mathbf{M}$, and $[(\mathbf{P}_1 \rtimes_{\beta^1} H) : \mathbf{M}] = |H||K|$. Moreover, one can easily check that if $\widetilde{\tau}$ is the normalized trace on $\mathbf{P}_1 \rtimes_{\beta^1} H$ and $\widetilde{E} : \mathbf{P}_1 \rtimes_{\beta^1} H \to \mathbf{M}$ the $\widetilde{\tau}$-preserving conditional expectation, then $\widetilde{\tau}(xsp_t^1 h_1) = |K|^{-1}\tau(x)\delta_{s,e}\delta_{h,e}$ and $\widetilde{E}(xsp_t^1 h_1) = |K|^{-1}xs\delta_{h,e}$. Here $H \cong H_1 = \{h_1 \mid h \in H\}$ is a subgroup of unitaries in $\mathbf{P}_1 \rtimes_{\beta^1} H$ implementing β^1. The following proposition follows immediately from Proposition 1.2, [20].

PROPOSITION 3.4. *$\mathbf{M}_1$ can be identified with $\mathbf{P}_1 \rtimes_{\beta^1} H$, with the Jones projection $e_1 = |H|^{-1} \sum_{h\in H} p_e^1 h_1$.*

Iterating the above process we obtain the following.

COROLLARY 3.5. *There exist a sequence of type II_1 factors $\mathbf{P}_k$ and outer actions $\alpha^k : K \to \mathrm{Aut}(\mathbf{P}_k)$, $\beta^k : H \to \mathrm{Aut}(\mathbf{P}_k)$ such that*

$$\mathbf{M}_{2k-1} \cong \mathbf{P}_{2k-1} \rtimes_{\beta^{2k-1}} H = \mathbf{P}_{2k}^{K,\alpha^{2k}}$$
$$\mathbf{M}_{2k} \cong \mathbf{P}_{2k} \rtimes_{\alpha^{2k}} K = \mathbf{P}_{2k+1}^{H,\beta^{2k+1}}$$
$$\mathbf{P}_{2k+1} \cong (\mathbf{P}_{2k} \rtimes_{\alpha^{2k}} K) \rtimes_{\hat{\alpha}^{2k}} K^\circ$$
$$\mathbf{P}_{2k} \cong (\mathbf{P}_{2k-1} \rtimes_{\beta^{2k-1}} H) \rtimes_{\hat{\beta}^{2k-1}} H^\circ.$$

Furthermore,

$$e_{2k} = |K|^{-1} \sum_{s\in K} p_e^{2k} s_{2k}$$
$$e_{2k+1} = |H|^{-1} \sum_{h\in H} p_e^{2k+1} h_{2k+1}$$
$$E_k(xa_k p_b^{k+1} c_{k+1}) = \epsilon_k x a_k \delta_{c,e}$$
$$\tau(xa_k p_b^{k+1} c_{k+1}) = \epsilon_k \tau(x)\delta_{a,e}\delta_{c,e}$$

where $x \in \mathbf{P}_k$, and ϵ_k equals $|H|^{-1}$ for k odd and $|K|^{-1}$ for k even.

In the above corollary (and henceforth) we adopt the following notational conventions:

(i) $H_k = \{h_k \mid h \in H\}$, $H \cong H_k \subset \mathbf{M}_k$ for k odd, and $\mathrm{Ad}\, h_k = \beta^k$.

(ii) $K_k = \{t_k \mid t \in K\}$, $K \cong K_k \subset \mathbf{M}_k$ for k even, and $\mathrm{Ad}\, t_k = \alpha^k$.

(iii) H_k° is a span of its minimal projections $\{p_h^k \mid h \in H\}$, $H^\circ \cong H_k^\circ \subset \mathbf{P}_k$ for k even, and $h_{k-1} p_g^k h_{k-1}^{-1} = p_{hg}^k$.

(iv) K_k° is a span of its minimal projections $\{p_t^k \mid t \in K\}$, $K^\circ \cong K_k^\circ \subset \mathbf{P}_k$ for k odd, and $s_{k-1} p_t^k s_{k-1}^{-1} = p_{st}^k$.

$\mathbf{P}_{2k}^{K,\alpha^{2k}}$ and $\mathbf{P}_{2k+1}^{H,\beta^{2k+1}}$ denote the fixed point algebras for the indicated actions.

LEMMA 3.6. *There exists a function $u : K \times H \to \mathcal{U}(\mathbf{P}) \cup \{0\}$ such that the following hold ($u(t,h) = u_{t,h}$).*

(i) *If $u_{t,h} \neq 0$, then there exist unique $s = s(t,h) \in K$ and $g = g(t,h) \in H$ such that*

$$(\operatorname{Ad} u_{t,h})\alpha_{st}\beta_h\alpha_{t^{-1}} = \beta_g .$$

(ii) *$\mathbf{N}' \cap \mathbf{M}_1$ has a vector space basis*

$$\{u_{t,h}s(t,h)p_t^1 h_1 \mid h \in H,\ t \in K,\ u_{t,h} \neq 0\}.$$

(iii) *$u_{t,h} \neq 0$ if and only if $tht^{-1} \in KH$ in* $\operatorname{Out}(\mathbf{P})$.

(iv) *$u_{t,h} = I$ whenever $u_{t,h}$ is a scalar. In particular, $u_{t,e} = I = u_{e,h}$ for any $t \in K$, $h \in H$.*

Proof. Let $\sum_{h\in H, s,t\in K} x_{s,t,h}sp_t^1 h_1 \in \mathbf{N}' \cap \mathbf{M}_1$, for some $x_{s,t,h} \in \mathbf{P}$, and let $y \in \mathbf{N}$. We have

$$\begin{aligned}
\sum_{h\in H,s,t\in K} yx_{s,t,h}sp_t^1 &= \sum_{h\in H,s,t\in K} x_{s,t,h}sp_t^1 h_1 y h_1^{-1} = \sum_{h\in H,s,t\in K} x_{s,t,h}s\beta_h^1(y)p_t^1 \\
&= \sum_{h\in H,s,t\in K} x_{s,t,h}s\Big(\sum_{r\in K}(\alpha_r\beta_h\alpha_{r^{-1}}\Big)(y)p_r^1)p_t^1 \\
&= \sum_{h\in H,s,t\in K} x_{s,t,h}s(\alpha_t\beta_h\alpha_{t^{-1}})(y)p_t^1 \\
&= \sum_{h\in H,s,t\in K} x_{s,t,h}(\alpha_{st}\beta_h\alpha_{t^{-1}})(y)sp_t^1 .
\end{aligned}$$

Thus, $\sum_{h\in H,s,t\in K} yx_{s,t,h}sp_t^1 \in \mathbf{N}' \cap \mathbf{M}_1$ iff $yx_{s,t,h} = x_{s,t,h}(\alpha_{st}\beta_h\alpha_{t^{-1}})(y)$, for any $h \in H$, $s,t \in K$, and $y \in \mathbf{N}$. We fix s,t,h for which $x_{s,t,h} \neq 0$ (if such exists). By Lemma 3.1 there exist $u_{s,t,h} \in \mathcal{U}(\mathbf{P})$ and $g \in H$ such that $(\operatorname{Ad} u_{s,t,h})(\alpha_{st}\beta_h\alpha_{t^{-1}}) = \beta_g$ and $x_{s,t,h} = \lambda u_{s,t,h}$ for some scalar λ.

It is clear from the above reasoning that $u_{s,t,h} \neq 0$ implies $tht^{-1} \in KH \subseteq \operatorname{Out}(\mathbf{P})$, and if $tht^{-1} \in KH$ then there exists an $s \in H$ such that $u_{s,t,h} \neq 0$. Moreover, such s if exists is unique, since by assumption $H \cap K = \langle e \rangle$. Thus, we can write $u_{t,h} = u_{s,t,h}$.

Multiplying by appropriate constants we can assure that $u_{t,h} = I$ whenever $u_{t,h}$ is a scalar. In particular $u_{t,e} = I = u_{e,h}$ for any $t \in K$, $h \in H$. The proof is complete. ∎

Repeating the same argument in higher levels of the tower, we conclude that there are functions $u^{2k+1} : H \times K \to \mathcal{U}(\mathbf{P}_{2k+1}) \cup \{0\}$ and $u^{2k} : K \times H \to \mathcal{U}(\mathbf{P}_{2k}) \cup \{0\}$ with properties analogous to u from Lemma 3.6.

If $u_{t,h} \neq 0$ then we denote $w_{t,h} = u_{t,h} s(t,h) p_t^1 h_1$. Clearly, $w_{t,h}$ is a partial isometry with the domain projection p_t^1 and the range projection p_{st}^1. Since by Lemma 3.6 $\{w_{t,h}\}$ form a basis of $\mathbf{N}' \cap \mathbf{M}_1$, it follows that for any $s, t \in K$ projections p_s^1, p_t^1 are either equivalent in $\mathbf{N}' \cap \mathbf{M}_1$ or their central supports are disjoint. We set $s \sim t$ if p_s^1 is equivalent to p_t^1 in $\mathbf{N}' \cap \mathbf{M}_1$. We fix $\widetilde{K}$, a set of representatives of equivalence classes for $\sim$.

For any $t \in K$ we denote $H_t = H \cap t^{-1}Ht$ (in $\mathrm{Out}(\mathbf{P})$), $S_t = \{s \in K \mid (\exists h \in H)\, s = s(t,h)\}$, and $|S_t|$ the cardinality of S_t. Clearly, $|S_t|$ equals the cardinality of $[t]$, the class of t (with respect to $\sim$).

PROPOSITION 3.7. *There exists an isomorphism*

$$\mathbf{N}' \cap \mathbf{M}_1 \cong \bigoplus_{t \in \widetilde{K}} (M_{|S_t|}(\mathbb{C}) \otimes C_{\omega_t}[H_t])$$

where $C_{\omega_t}[H_t]$ is the twisted group algebra corresponding to a 2-cocycle (with values in the torus $\mathbf{T}$) ω_t on H_t.

Proof. It follows from Lemma 3.6 and the above discussion that

$$\mathbf{N}' \cap \mathbf{M}_1 \cong \bigoplus_{t \in \widetilde{K}} (M_{|S_t|}(\mathbb{C}) \otimes p_t^1(\mathbf{N}' \cap \mathbf{M}_1)p_t^1)$$

and $p_t^1(\mathbf{N}' \cap \mathbf{M}_1)p_t^1$ has a basis $\{w_{t,h} \mid s(t,h) = e\}$. However, $s(t,h) = e$ if and only if $h \in H_t$.

If $h, g \in H_t$ then $w_{t,h} w_{t,g} = u_{t,h} \beta_h^1(u_{t,g}) p_t^1 h_1 g_1$. On the other hand, $w_{t,h} w_{t,g}$ is a linear combination of $\{w_{t,k} \mid k \in H_t\}$. It follows that there exists an $\omega_t(h,g) \in \mathbf{T}$ such that $w_{t,h} w_{t,g} = \omega_t(h,g) w_{t,hg}$. Clearly, ω_t is a 2-cocycle, and the claim follows. ∎

PROPOSITION 3.8. *For any $t \in K$ we have $|S_t| \leqslant [H : H_t]$. The equality holds for all $t \in K$ if and only if $\mathbf{N} \subset \mathbf{M}$ has depth 2.*

Proof. We work in $\mathrm{Out}(\mathbf{P})$. If $s \in S_t$ then $s^{-1}H \cap tHt^{-1} \neq \emptyset$ by Lemma 3.6. For any two elements x_1, x_2 of $s^{-1}H \cap tHt^{-1}$ we have $x_1^{-1}x_2 \in H_{t^{-1}}$ and, hence, $s^{-1}H \cap tHt^{-1}$ intersects only one of cosets $H_{t^{-1}} \setminus tHt^{-1}$. If $s, w \in S_t$ and both $s^{-1}H$ and $w^{-1}H$ intersect the same coset, then $Hsw^{-1}H \cap H_{t^{-1}} \neq \emptyset$ and, hence, $sw^{-1} \in H$. By our assumption $s = w$. Thus, there is an imbedding $S_t \to H_{t^{-1}} \setminus tHt^{-1}$. Therefore, $|S_t| \leqslant [tHt^{-1} : H_{t^{-1}}] = [H : H_t]$.

$\mathbf{N} \subset \mathbf{M}$ has depth 2 if and only if $\dim(\mathbf{N}' \cap \mathbf{M}_1) = [\mathbf{M} : \mathbf{N}]$ (e.g. see [24]). By Proposition 3.7 we have

$$\dim(\mathbf{N}' \cap \mathbf{M}_1) = |H| \sum_{t \in \widetilde{K}} |S_t|^2 [H : H_t]^{-1} = |H| \sum_{t \in K} |S_t| [H : H_t]^{-1}.$$

Since $|S_t| \leqslant [H : H_t]$, we have $\dim(\mathbf{N}' \cap \mathbf{M}_1) = |H||K| = [\mathbf{M} : \mathbf{N}]$ if and only if $|S_t| = [H : H_t]$ for any $t \in K$. ∎

4. THE ASSOCIATED HOPF ALGEBRAS

In this section we calculate Hopf $*$-algebras corresponding to the depth 2 inclusions. These turn out to have the structure of cocycle twisted bicrossed products. In fact, this was the way we discovered the form of twisted bicrossed products, as presented in Section 2. In our analysis we follow the general method of [24].

Throughout this section we assume that the depth of $\mathbf{N} \subset \mathbf{M}$ is 2. By [2] this means that $\alpha(K)\beta(H)$ is a group in $\mathrm{Out}(\mathbf{P})$, denoted G. We write $\mathbf{A} = \mathbf{N}' \cap \mathbf{M}_1$. As shown in [24], $\mathbf{A}$ is a Hopf $*$-algebra of dimension $|H||K|$, which acts on $\mathbf{M}$ in such a way that $\mathbf{M}_1$ is isomorphic to the crossed product $\mathbf{M} \rtimes \mathbf{A}$. It follows (e.g. cf. [24]) that $\mathbf{M}_2$ is a crossed product of $\mathbf{M}_1$ by an action of $\mathbf{A}^\circ$, the Hopf $*$-algebra dual to $\mathbf{A}$.

For $h \in H$ we define $v_h = \sum_{t \in K} w_{t,h}$, a unitary in $\mathbf{A}$. Clearly, $\{p_t^1 v_h \mid t \in K, h \in H\}$ is a basis of $\mathbf{A}$. Since $u_{t,e} = I$, we have $v_e = I$. H acts on K and K acts on H as in Section 2. We have $h \cdot t = s(t,h)t$ and $(h \cdot t)h = (t \cdot h)t$. Thus, by virtue of Lemma 3.6 we have

$$\mathrm{Ad}\, u_{t,h} = \beta_{t \cdot h} \alpha_t \beta_{h^{-1}} \alpha_{(h \cdot t)^{-1}}. \tag{4.1}$$

If $\gamma : H \to \mathrm{Aut}(C(K))$ is given by $\gamma_h(p_t^1) = p_{h \cdot t}^1$, then $\gamma_h = \mathrm{Ad}\, v_h$. We remark that subgroups H_t considered in Propositions 3.7 and 3.8 are point stabilizers for the action of H on K.

Similarly, under the depth 2 assumption, $\alpha^1(K)\beta^1(H)$ is a group in $\mathrm{Out}(\mathbf{P}_1)$, denoted G_1. The two corresponding actions will be denoted by $\rightharpoonup$. We have $(\mathrm{Ad}\, u_{h,t}^1)\beta_{t \rightharpoonup h}^1 \alpha_t^1 \beta_{h^{-1}}^1 = \alpha_{h \rightharpoonup t}^1$, where $u_{h,t}^1$ is a unitary in $\mathbf{P}_1$ as in Section 3. We define unitaries $\{z_t \mid t \in K\}$ by $z_t = \sum_{h \in H} u_{h,t}^1 (t \rightharpoonup h)_1 h_1^{-1} p_h^2 t_2$. If $\sigma : K \to \mathrm{Aut}(C(H))$ is an action given by $\sigma_s(p_h^2) = p_{s \rightharpoonup h}^2$, then $\sigma_s = \mathrm{Ad}\, z_s$.

LEMMA 4.1. *The actions $\cdot$ and $\rightharpoonup$ of H on K coincide, and so do the actions $\cdot$ and $\rightharpoonup$ of K on H. Consequently, the groups G and G_1 are naturally isomorphic. Furthermore, for any $h \in H$, $t \in K$, the following hold:*

$$u^1_{h,t} = \sum_{k\in K} \alpha_{kt}(u_{t^{-1},t\cdot h})kt(h\cdot t)^{-1}k^{-1}p^1_k \tag{4.2}$$

$$z_t = \sum_{k\in K, h\in H} \alpha_{kt}(u_{t^{-1},t\cdot h})kt(h\cdot t)^{-1}k^{-1}p^1_k(t\cdot h)_1 h_1^{-1}p^2_h t_2. \tag{4.3}$$

Proof. Let $h \in H, t \in K$. Since $(\operatorname{Ad} u^1_{h,t})\beta^1_{t\rightharpoonup h}\alpha^1_t\beta^1_{h^{-1}} = \alpha^1_{h\rightharpoonup t}$, for any $r, k \in K$, $x \in \mathbf{P}$ we have

$$(\beta^1_{t\rightharpoonup h}\alpha^1_t\beta^1_{h^{-1}})(xrp^1_k) = (u^1_{h,t})^*\alpha^1_{h\rightharpoonup t}(xrp^1_k)u^1_{h,t} = (u^1_{h,t})^* xrp^1_{k(h\rightharpoonup t)^{-1}}u^1_{h,t}.$$

The left hand side can be calculated as

$$(\alpha_{rkt^{-1}}\beta_{t\rightharpoonup h}\alpha_{(rkt^{-1})^{-1}})(\alpha_{kr}\beta_{h^{-1}}\alpha_{(kr)^{-1}})(x)rp^1_{kt^{-1}}.$$

Setting $k = r^{-1}$, we get

$$(\alpha_{t^{-1}}\beta_{t\rightharpoonup h}\alpha_t\beta_{h^{-1}})(x) = (u^1_{h,t})^* xrp^1_{r^{-1}(h\rightharpoonup t)^{-1}}u^1_{h,t}.$$

Since $u^1_{h,t} \in \mathbf{P}_1$ can be written as a linear combination of elements from $\mathbf{P}$, $\{w \mid w \in K\}$, and $\{p^1_w \mid w \in K\}$, the above equality implies that there exist $\{ y_k \in \mathcal{U}(\mathbf{P}) \mid k \in K\}$ such that

$$y_k = \alpha_r(y_{r^{-1}k}),\ r, k \in K, \quad\text{and}\quad u^1_{h,t} = \sum_{k\in K} y_k kt(h\rightharpoonup t)^{-1}k^{-1}p^1_k.$$

That is, there is a $\widetilde{u}_{h,t} \in \mathcal{U}(\mathbf{P})$ such that

$$u^1_{h,t} = \sum_{k\in K} \alpha_k(\widetilde{u}_{h,t})kt(h\rightharpoonup t)^{-1}k^{-1}p^1_k.$$

Hence, we have

$$\begin{aligned}
&(\alpha_{t^{-1}}\beta_{t\rightharpoonup h}\alpha_t\beta_{h^{-1}})(x)p^1_{t^{-1}} \\
&\quad = (u^1_{h,t})^* xp^1_{(h\rightharpoonup t)^{-1}}u^1_{h,t} \\
&\quad = \sum_{k,w\in K} p^1_k k(h\rightharpoonup t)t^{-1}k^{-1}\alpha_k(\widetilde{u}^*_{h,t})xp^1_{(h\rightharpoonup t)^{-1}}\alpha_w(\widetilde{u}_{h,t})wt(h\rightharpoonup t)^{-1}w^{-1}p^1_w \\
&\quad = \alpha_{t^{-1}(h\rightharpoonup t)t^{-1}}(\widetilde{u}^*_{h,t})\alpha_{t^{-1}(h\rightharpoonup t)}(x)\alpha_{t^{-1}(h\rightharpoonup t)t^{-1}}(\widetilde{u}_{h,t})p^1_{t^{-1}}.
\end{aligned}$$

Therefore,

$$(\alpha_{t^{-1}}\beta_{t\rightharpoonup h}\alpha_t\beta_{h^{-1}})(x) = \alpha_{t^{-1}(h\rightharpoonup t)t^{-1}}(\widetilde{u}^*_{h,t})\alpha_{t^{-1}(h\rightharpoonup t)}(x)\alpha_{t^{-1}(h\rightharpoonup t)t^{-1}}(\widetilde{u}_{h,t}).$$

Consequently,

$$\operatorname{Ad}\alpha_{t^{-1}}(\widetilde{u}^*_{h,t}) = \alpha_{(h\rightharpoonup t)^{-1}}\beta_{t\rightharpoonup h}\alpha_t\beta_{h^{-1}}.$$

This equality and Formula (4.1) imply that the actions $\cdot$ and $\rightharpoonup$ coincide. From now on we will write $\cdot$ instead of $\rightharpoonup$.

Since $\beta_{t\cdot h}\alpha_t\beta_{h^{-1}} = \alpha_{h\cdot t}(\operatorname{Ad}\alpha_{t^{-1}}(\widetilde{u}^*_{h,t}))$, and $\beta_{t\cdot h}\alpha_t\beta_{h^{-1}} = (\operatorname{Ad}u_{t,h})\alpha_{h\cdot t}$ by Lemma 3.6, we have $\operatorname{Ad}\alpha_{t^{-1}}(\widetilde{u}^*_{h,t}) = \alpha^{-1}_{h\cdot t}(\operatorname{Ad}u_{t,h})\alpha_{h\cdot t}$. Hence,

$$\operatorname{Ad}u_{t^{-1},t\cdot h} = \beta_h\alpha_{t^{-1}}\beta_{(t\cdot h)^{-1}}\alpha_{h\cdot t}.$$

Now we have

$$\begin{aligned}\operatorname{Ad}\widetilde{u}_{h,t} &= \alpha_{t(h\cdot t)^{-1}}(\operatorname{Ad}\widetilde{u}^*_{t,h})\alpha_{(h\cdot t)t^{-1}} = \alpha_{t(h\cdot t)^{-1}}(\alpha_{h\cdot t}\beta_h\alpha_{t^{-1}}\beta_{(t\cdot h)^{-1}})\alpha_{(h\cdot t)t^{-1}}\\ &= \alpha_t(\beta_h\alpha_{t^{-1}}\beta_{(t\cdot h)^{-1}}\alpha_{h\cdot t})\alpha_{t^{-1}} = \alpha_t(\operatorname{Ad}u_{t^{-1},t\cdot h})\alpha_{t^{-1}} = \operatorname{Ad}\alpha_t(u_{t^{-1},t\cdot h}).\end{aligned}$$

Thus, multiplying the unitaries $u^1_{h,t}$ by appropriate scalars we get $\widetilde{u}_{h,t} = \alpha_t(u_{t^{-1},t\cdot h})$. This proves (4.2), and (4.3) is its immediate consquence. ∎

According to Section 5, [24], the action of $\mathbf{A}$ on $\mathbf{M}$ is given by $a\cdot y = [\mathbf{M}:\mathbf{N}]E(aye_1)$, $a\in\mathbf{A}$, $y\in\mathbf{M}$. Corollary 3.5 implies that

$$p^1_t v_h\cdot xk = \delta_{h\cdot k,t}u_{k,h}(\alpha_t\beta_h\alpha_{k^{-1}})(x)t \tag{4.4}$$

for any $h\in H$, $t,k\in K$, $x\in\mathbf{P}$. Similarly, for the action of $\mathbf{A}^\circ$ on $\mathbf{M}_1$ we have

$$p^2_g z_s\cdot yh_1 = \delta_{s\cdot h,g}u^1_{h,s}(\beta^1_g\alpha^1_s\beta^1_{h^{-1}})(y)g_1 \tag{4.5}$$

for any $h,g\in H$, $s\in K$, $y\in\mathbf{P}_1$.

LEMMA 4.2. *There exist normalized 2-cocycles $\mu: H\times H\to\mathcal{U}(C(K))$ for γ and $\nu: K\times K\to\mathcal{U}(C(H))$ for σ, such that for any $h,g\in H$, $s,t\in K$*

$$v_hv_g = \mu(h,g)v_{hg} \quad \text{and} \quad z_sz_t = \nu(s,t)z_st.$$

Therefore,

$$\mathbf{A}\cong C(K)\rtimes_{\gamma,\mu}H \quad \text{and} \quad \mathbf{A}^\circ\cong C(H)\rtimes_{\sigma,\nu}K.$$

Proof. At first we observe that

$$\mathbf{N}'\cap\mathbf{P}_1 = \operatorname{span}\{p^1_t\mid t\in K\}\cong C(K).$$

Indeed, we have $\mathbf{N}'\cap\mathbf{P}_1 = E^{M_1}_{P_1}(\mathbf{N}'\cap\mathbf{M}_1)$, and by virtue of Lemma 3.6 this set equals $\operatorname{span}\{u_{t,e}s(t,e)p^1_t\mid t\in K\} = \operatorname{span}\{p^1_t\mid t\in K\}$.

However, it is immediate from the definition that for any $h,g\in H$ we have $v_hv_gv^*_{hg}\in\mathbf{N}'\cap\mathbf{P}_1$. Thus, there exist $\mu(h,g)\in\mathcal{U}(C(K))$ such that $v_hv_g = \mu(h,g)v_{hg}$. Since v_h implements action γ of H on $C(K)$, it follows by a standard argument (associativity of the multiplication) that μ is a 2-cocycle for γ. Since $v_e = I$, μ is normalized. This proves one part of the lemma. The other one is established in a similar fashion. ∎

LEMMA 4.3. *For any $h, g \in H$, $s, t \in K$ we have*

$$\mu(h,g)[t] = \beta_{(t\cdot h^{-1})^{-1}}(u_{(hg)^{-1}\cdot t,g})u_{h^{-1}\cdot t,h}u^*_{(hg)^{-1}\cdot t,hg}$$
$$\nu(s,t)[h] = u_{t^{-1},s^{-1}\cdot h}\alpha_{(s^{-1}\cdot h)\cdot t^{-1}}(u_{s^{-1},h})u^*_{(st)^{-1},h}$$
$$u^*_{t,h} = \overline{\mu(h^{-1},h)[t]}\beta_{t\cdot h}(u_{h\cdot t,h^{-1}}) = \overline{\nu(t^{-1},t)[h]}\alpha_{h\cdot t}(u_{t^{-1},t\cdot h}).$$

Furthermore, unitaries $\{u_{t,h}\}$ can be selected in such a way that $\mu(h^{-1},h) = I$ for any $h \in H$.

Proof. The definitions of μ and ν from Lemma 4.2, formula (4.1), and Lemma 4.1 lead through a straightforward calculation to the first two equalities. Replacing h by g^{-1} in the first and s by t^{-1} in the second we get the third one. The map $f : (t,h) \mapsto (h\cdot t, h^{-1})$ is a bijection of $K \times H$ satisfying $f^2 = \mathrm{id}$. Thus, replacing $u_{t,h}$ by $\mu(h^{-1},h)[t]^{-1/2}$ we get $\mu(h^{-1},h)[t] = 1$, as desired. ∎

From now on we will assume that, according to the above lemma, the unitaries $\{u_{t,h}\}$ were selected in such a way that $\mu(h^{-1},h) = I$.

According to [24], there is a duality between $\mathbf{A} = \mathbf{N}'\cap\mathbf{M}_1$ and $\mathbf{A}^\circ = \mathbf{M}'\cap\mathbf{M}_2$, given by a bilinear form

$$\langle a,b\rangle = [\mathbf{M}:\mathbf{N}]^2\tau(ae_2e_1b)$$

where $a \in \mathbf{A}$, $b \in \mathbf{A}^\circ$.

LEMMA 4.4. *For any $h, g \in H$, $s, t \in K$, we have*

$$\langle p^1_t v_h, p^2_g z_s\rangle = \delta_{t,h\cdot s}\delta_{g,s\cdot h}\nu(s^{-1},s)[h].$$

Proof. For any $k \in K$, $f \in H$, we have

$$p^2_e k_2 p^1_e f_1 = p^2_e\alpha^2_k(p^1_e)f_1k_2 = p^2_e p^1_{k^{-1}}f_1k_2 = p^1_{k^{-1}}f_1(f_1^{-1}p^2_e f_1)k_2 = p^1_{k^{-1}}f_1p^2_{f^{-1}}k_2.$$

Hence,

$$\begin{aligned}\langle p^1_t v_h, p^2_g z_s\rangle &= |H|^2|K|^2\tau(p^1_t v_h e_2 e_1 p^2_g z_s) = |H|\,|K|\sum_{f\in H,k\in K}\tau(p^1_t v_h p^2_e k_2 p^1_e f_1 p^2_g z_s)\\ &= |H|\,|K|\sum_{f\in H,k\in K}\tau(p^1_t v_h p^1_{k^{-1}}f_1 p^2_{f^{-1}}k_2 p^2_g z_s)\\ &= |H|\,|K|\tau(p^1_t v_h g_1^{-1}p^2_g(h^{-1}\cdot t)_2^{-1}z_s)\\ &= |H|\,|K|\tau(p^1_t v_h g_1^{-1}E_1(p^2_g(h^{-1}\cdot t)_2^{-1}z_s).\end{aligned}$$

It follows from Corollary 3.5 and Lemma 4.1 that $E_1(p_g^2(h^{-1}\cdot t)_2^{-1}z_s)=0$ unless $t=h\cdot s$. For notational convenience we write

$$\theta(c)=\delta_{t,h\cdot s}|K|(\beta_c^1\alpha_s^1\beta_{c^{-1}}^1)(u_{r,h}(c\cdot r)r^{-1}p_r^1)^*$$

for $c\in H$. Then

$$\begin{aligned}
\langle p_t^1 v_h, p_g^2 z_s\rangle &= \delta_{t,h\cdot s}\langle p_{h\cdot s}^1 v_h, p_g^2 z_s\rangle = \delta_{t,h\cdot s}\langle v_h, p_g^2 z_s\rangle \\
&= \delta_{t,h\cdot s}|H|\,|K|\tau(v_h g_1^{-1}p_g^2 s_2^{-1}z_s) = \delta_{t,h\cdot s}|H|\,|K|\tau(\alpha_s^2(v_h)g_1^{-1}p_g^2 z_s s_2^{-1}) \\
&= |H| \sum_{b,f\in H, r,k\in K} \tau(\theta(b)p_b^2 h_1 g_1^{-1}p_g^2\alpha_{ks}(u_{s^{-1},s\cdot f})ks(f\cdot s)^{-1}k^{-1}p_k^1(s\cdot f)_1 f_1^{-1}p_f^2) \\
&= |H| \sum_{r,k\in K} \tau(\theta(h)h_1 g_1^{-1}\alpha_{ks}(u_{s^{-1},g})ks(g\cdot s^{-1})k^{-1}p_k^1 g_1(s^{-1}\cdot g)_1^{-1}p_{s^{-1}\cdot g}^2) \\
&= \sum_{r,k\in K} \tau(\theta(h)h_1 g_1^{-1}\alpha_{ks}(u_{s^{-1},g})ks(g\cdot s^{-1})k^{-1}p_k^1 g_1(s^{-1}\cdot g)_1^{-1}) \\
&= \sum_{r,k\in K} \tau(\theta(h)\beta_{hg^{-1}}^1(\alpha_{ks}(u_{s^{-1},g}))ks(g\cdot s^{-1})k^{-1}p_k^1 h_1(s^{-1}\cdot g)_1^{-1})
\end{aligned}$$

equals 0 unless $g=s\cdot h$. Consequently,

$$\langle p_t^1 v_h, p_g^2 z_s\rangle = \delta_{t,h\cdot s}\delta_{g,s\cdot h}\langle p_{h\cdot s}^1 v_h, p_{s\cdot h}^2 z_s\rangle = \delta_{t,h\cdot s}\delta_{g,s\cdot h}\langle v_h, z_s\rangle.$$

Since $e_2=e_2p_e^2z_s^*$ for any $s\in K$, a tedious but straightforward calculation (using Corollary 3.5, formula (4.1), and Lemma 4.1) yields

$$\begin{aligned}
\langle v_h, z_s\rangle &= |H|^2|K|^2\tau(v_h e_2 e_1 z_s) = |H|^2|K|^2\tau(v_h e_2(p_e^2 z_s^* e_1 z_s)) \\
&= |H|\,|K|^2 \sum_{f,g\in H} \tau(v_h e_2 f_1\alpha_{s^{-1}}^2(\alpha_{g\cdot s}(u_{s^{-1},s\cdot g})s(g\cdot s)^{-1}p_{(g\cdot s)s^{-1}}^1) \\
&\qquad \cdot (s\cdot g)_1 g_1^{-1}p_g^2) \\
&= |H|\,|K|^2 \sum_{f,g\in H} \tau(v_h e_2 f_1(\beta_{s\cdot g}^1\alpha_{s^{-1}}^1\beta_{(s\cdot g)^{-1}}^1)(\alpha_{g\cdot s}(u_{s^{-1},s\cdot g})s(g\cdot s)^{-1}p_{(g\cdot s)s^{-1}}^1) \\
&\qquad \cdot (s\cdot g)_1 g_1^{-1}p_g^2) \\
&= |H|\,|K|^2 \sum_{f,g\in H} \tau(v_h e_2 f_1(\alpha_s\beta_{s\cdot g}\alpha_{s^{-1}}\beta_{(s\cdot g)^{-1}}\alpha_{g\cdot s})(u_{s^{-1},s\cdot g}) \\
&\qquad \cdot s(g\cdot s)^{-1}p_{g\cdot s}^1(s\cdot g)_1 g_1^{-1}p_g^2) \\
&= |H|\,|K|^2 \sum_{f,g\in H} \tau(v_h e_2 f_1(\alpha_s\beta_{(s\cdot g)g^{-1}})(u_{s^{-1},s\cdot g})s(g\cdot s)^{-1}p_{g\cdot s}^1(s\cdot g)_1 g_1^{-1}p_g^2) \\
&= |K| \sum_{t\in K} \tau(u_{t,h}(h\cdot t)t^{-1}p_t^1\beta_{h(s\cdot h)^{-1}}^1((\alpha_s\beta_{(s\cdot h)h^{-1}})(u_{s^{-1},s\cdot h}))s(h\cdot s)^{-1}p_{h\cdot s}^1) \\
&= \tau(u_{s,h}\alpha_{h\cdot s}(u_{s^{-1},s\cdot h})),
\end{aligned}$$

which equals $\nu(s^{-1},s)[h]$ by Lemma 4.3. ∎

THEOREM 4.5. **A** *and* $\mathbf{A}^0$ *have the structure of twisted bicrossed products dual to one another, as described in Section* 2.

Proof. This follows from Lemmas 4.1, 4.2, 4.3, and 4.4. ∎

Since $\alpha(K)\beta(H)$ is a group in $\mathrm{Out}(\mathbf{P})$, denoted G, it determines a G-kernel (cf. Section 3.1, [23]). We denote by $\eta : G \times G \to \mathcal{U}(\mathbf{P})$ the unitary 2-cocycle defined by C. Sutherland in Section 3.1, [23]. Our identity (cf. formula (4.1))

$$\alpha_t\beta_h\alpha_s\beta_g = (\mathrm{Ad}\,\alpha_t(u_{s,s^{-1}\cdot h}))\alpha_{t(h\cdot s^{-1})^{-1}}\beta_{(s^{-1}\cdot h)g}$$

gives

$$\eta(th, sg) = \alpha_t(u_{s,s^{-1}\cdot h})$$

for any $h, g \in H$, $t, s \in K$. Observe that η does not depend on g. We denote by ω the normalized numerical 3-cocycle on G (cf. Section 3.1, [23]). In our setting it is determined by the identity

$$(4.6)\quad \beta_g(\eta(th,s))\eta(g, t(h\cdot s^{-1})^{-1}) = \omega(rg, th, sf)\eta(g,t)\eta((g\cdot t^{-1})^{-1}(t^{-1}\cdot g)h, s)$$

$g, h, f \in H$, $r, t, s \in K$. Observe that ω does not depend on r, f. In particular, its restriction to either H or K is trivial. We recall the 3-cocycle identity satisfied by ω

$$(4.7)\qquad \omega(a,b,c)\omega(b,c,d)\omega(a,bc,d) = \omega(ab,c,d)\omega(a,b,cd)$$

for $a, b, c, d \in G$. The normalization of ω means that $\omega(a_1, a_2, a_3) = 1$ if $a_i = e$ for some i.

PROPOSITION 4.6. *The unitary* 2*-cocycles* μ *and* ν *are determined by the numerical* 3*-cocycle* ω*, according to the following formulae:*

$$\mu(h,g)[t] = \omega(h, g((hg)^{-1}\cdot t)^{-1}, (hg)^{-1}\cdot t)\omega(h^{-1}, h(h^{-1}\cdot t)^{-1}, h^{-1}\cdot t)$$
$$\nu(s,t)[g] = \omega(((st)^{-1}\cdot g)^{-1}, ((st)^{-1}\cdot g)t^{-1}, s^{-1}).$$

Proof. Replacing in (4.6) g by $t\cdot h^{-1}$ and taking into account that $\eta(th, sg) = \alpha_t(\eta(h,s))$ we get

$$\omega(t\cdot h^{-1}, th, s)\eta(t\cdot h^{-1}, t) = \beta_{t\cdot h^{-1}}(\eta(th,s))\eta(t\cdot h^{-1}, t(h\cdot s^{-1})^{-1}).$$

On the other hand, since $\mu(h^{-1}, h) = I$ for any $h \in H$, Lemma 4.3 leads to

$$\nu(s^{-1}, (h^{-1}\cdot t)^{-1})[s^{-1}\cdot h]\eta(t\cdot h^{-1}, t) = \beta_{t\cdot h^{-1}}(\eta(th,s))\eta(t\cdot h^{-1}, t(h\cdot s^{-1})^{-1}).$$

Consequently,

$$\omega(t \cdot h^{-1}, th, s) = \nu(s^{-1}, (h^{-1} \cdot t)^{-1})[s^{-1} \cdot h].$$

This gives the second equality of the proposition.

Replacing in (4.6) s by $(h^{-1} \cdot t)^{-1}$ we get

$$\omega(g, th, ((h^{-1} \cdot t)^{-1})\eta(g,t)\eta((g \cdot t^{-1})^{-1}(t^{-1} \cdot g)h, (h^{-1} \cdot t)^{-1})$$

$$= \beta_g(\eta(th, (h^{-1} \cdot t)^{-1})).$$

On the other hand, Lemma 4.3 combined with the expression for η above leads to

$$\mu(g, (t \cdot h^{-1})^{-1})[g \cdot t^{-1}]\eta(g,t)\eta((g \cdot t^{-1})^{-1}(t^{-1} \cdot g)h, (h^{-1} \cdot t)^{-1})$$

$$= \nu(t, t^{-1})[g]\beta_g(\eta(th, (h^{-1} \cdot t)^{-1})).$$

Consequently,

$$\omega(g, th, ((h^{-1} \cdot t)^{-1}) = \mu(g, (t \cdot h^{-1})^{-1})[g \cdot t^{-1}]\nu(t, t^{-1})[g].$$

From this and the second equality of the proposition we infer that the first one holds true as well. ∎

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