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ON CRISS-CROSS COMMUTATIVITY

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ABSTRACT. Equality of the non zero spectrum of products ab and ba of Banach algebra elements extends to many different kinds of joint spectrum for "criss-cross commuting" pairs of tuples.

KEYWORDS: *Criss-cross commutativity, spectrum, exactness.*

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Recall that if $T : X \to Y$ and $S : Y \to X$ are linear operators then

$$(0.1) \qquad (I - TS)^{-1}(0) \subseteq T(I - ST)^{-1}(0)$$

and hence there is implication

$$(0.2) \qquad I - ST \text{ one-one} \implies I - TS \text{ one-one:}$$

for if $(I - TS)y = 0$ then $y = T(Sy)$ with $(I - ST)Sy = S(I - TS)y = 0$. Dually there is inclusion

$$(0.3) \qquad S^{-1}(I - ST)(X) \subseteq (I - TS)(Y)$$

and hence implication

$$(0.4) \qquad I - ST \text{ onto} \implies I - TS \text{ onto:}$$

for if $Sy = (I - ST)x$ then $y = (I - TS)y + TSy = (I - TS)y + T(I - ST)x = (I - TS)(y + Tx)$. For bounded linear operators between normed spaces we have also implication

$$(0.5) \qquad \forall\, x \in X : \|x\| \leqslant k\|(I - ST)x\| \Longrightarrow \forall\, y \in Y : \|y\| \leqslant h\|(I - TS)y\| :$$

argue

$$\begin{aligned}\|y\| &\leqslant \|(I - TS)y\| + \|T\|\,\|Sy\| \leqslant \|(I - TS)y\| + \|T\|k\|(I - ST)Sy\| \\ &= \|(I - TS)y\| + \|T\|k\|S(I - TS)y\| \leqslant (1 + \|T\|k\|S\|)\|(I - TS)y\|.\end{aligned}$$

If more generally $a, b \in A$ are in an additive category then also

$$(0.6) \qquad c(1 - ba) = 1 \Longrightarrow (1 + acb)(1 - ab) = 1,$$

so that there is implication

$$(0.7) \qquad 1 - ba \in A^{-1}_{\text{left}} \Longrightarrow 1 - ab \in A^{-1}_{\text{left}}.$$

It is familiar that these elementary observations have consequences in spectral theory: for various kinds of "spectrum" ω on linear algebras A there is equality, for arbitrary pairs of elements $a, b \in A$,

$$(0.8) \qquad \omega(ab) \setminus \{0\} = \omega(ba) \setminus \{0\}.$$

These equalities have extensions to "criss-cross commuting" systems of operators or ring elements:

DEFINITION 1. n-tuples $a \in A^n$ and $b \in A^n$ of elements in an additive category A are said to *criss-cross commute* if there is equality, for each i, j, k in $\{1, 2, \ldots, n\}$,

$$(1.1) \qquad a_i b_k a_j = a_j b_k a_i \quad \text{and} \quad b_i a_k b_j = b_j a_k b_i.$$

An immediate consequence is that each of the n-tuples

$$(1.2) \qquad ba = (b_1 a_1, b_2 a_2, \ldots, b_n a_n), \quad ab = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$$

is commutative. By Grimus and Ecker ([1]) there is implication (1.2) $\Rightarrow$ (1.1) when $A = \mathbb{C}^{n \times n}$ and $b = a^*$.

THEOREM 2. *If* $(a, c) \in A^n \times A^m$ *and* $(b, d) \in A^n \times A^m$ *criss-cross commute there is implication*

$$e_1(1 - b_1 a_1) + \sum_{j=2}^{n} e_j(\lambda_j - b_j a_j) + \sum_{j=1}^{m} (\mu_j - d_j c_j) f_j = 1 \tag{2.1}$$

implies

$$(1 + a_1 c_1 b_1)(1 - a_1 b_1) + \sum_{j=2}^{n} (a_1 e_j b_1)(\lambda_j - a_j b_j) + \sum_{j=1}^{m} (\mu_j - c_j d_j) a_1 f_j b_1 = 1. \tag{2.2}$$

Proof. Compute

$$\begin{aligned}(1 + a_1 e_1 b_1)(1 - a_1 b_1) &= 1 - a_1 b_1 + a_1 e_1 (1 - b_1 a_1) b_1 \\ &= 1 - a_1 b_1 + a_1 \Big(1 - \sum_{j=2}^{n} e_j(\lambda_j - b_j a_j) + \sum_{j=1}^{m} (\mu_j - d_j c_j) f_j\Big) b_1 \\ &= 1 - \sum_{j=2}^{n} a_1 e_j (\lambda_j - b_j a_j) b_1 - \sum_{j=1}^{m} a_1 (\mu_j - d_j c_j) b_1,\end{aligned}$$

which by part of the criss-cross condition equals

$$1 - \sum_{j=2}^{n} a_1 e_j b_1 (\lambda_j - a_j b_j) - \sum_{j=1}^{m} (\mu_j - c_j d_j) a_1 f_j b_1. \quad \blacksquare$$

THEOREM 3. *If* $(a, c) \in A^n \times A^m$ *and* $(b, d) \in A^n \times A^m$ *criss-cross commute then there is implication*

$$\begin{aligned}\|uv\| \leqslant k_1 \|u\| \, \|(1 - b_1 a_1) v\| &+ \sum_{j=2}^{n} k_j \|u\| \, \|(\lambda_j - b_j a_j) v\| \\ &+ \sum_{j=1}^{m} h_j \|u(\mu_j - d_j c_j)\| \, \|v\|\end{aligned} \tag{3.1}$$

implies

$$\begin{aligned}\|u'v'\| \leqslant (1 + k_1 \|a_1\| \, \|b_1\|) \|u'\| \, \|(1 - a_1 b_1) v'\| \\ + \sum_{j=2}^{n} k_j \|a_1\| \, \|b_1\| \, \|u'\| \, \|(\lambda_j - a_j b_j) v'\| \\ + \sum_{j=1}^{m} h_j \|a_1\| \, \|b_1\| \, \|u'(\mu_j - c_j d_j)\| \, \|v'\|.\end{aligned} \tag{3.2}$$

There is also implication

$$(3.3)\qquad (1-b_1a_1)v = (\lambda_j - b_ja_j)v = u(\mu_j - d_jc_j) = 0 \Longrightarrow uv = 0$$

implies

$$(3.4)\qquad (1-a_1b_1)v' = (\lambda_j - a_jb_j)v' = u'(\mu_j - c_jd_j) = 0 \Longrightarrow u'v' = 0.$$

Proof. If (3.1) holds then compute, for arbitrary u', v' in A,

$$\begin{aligned}
\|u'v'\| &\leqslant \|u'a_1b_1v'\| + \|u'(1-b_1a_1)v'\| \\
&\leqslant k_1\|u'a_1\|\,\|(1-b_1a_1)b_1v'\| + \sum_{j=2}^{n} k_j\|u'a_1\|\,\|(\lambda_j - b_ja_j)b_1v'\| \\
&\quad + \sum_{j=1}^{m} h_j\|u'a_1(\mu_j - d_jc_j)\|\,\|b_1v'\| + \|u'(1-b_1a_1)v'\| \\
&= k_1\|u'a_1\|\,\|b_1(1-a_1b_1)v'\| + \sum_{j=2}^{n} k_j\|u'a_1\|\,\|b_1(\lambda_j - a_jb_j)v'\| \\
&\quad + \sum_{j=1}^{m} \|u'(\mu_j - c_jd_j)a_1\|\,\|b_1v'\| + \|u'(1-b_1a_1)v'\|. \quad \blacksquare
\end{aligned}$$

The argument of Theorem 2 extends to the situation, for a normed linear category A, in which the tuples $e \in A^n$ and $f \in A^m$ are replaced by bounded sequences; the reader can check that if

$$(3.5)\qquad \left\| e_{1k}(1-b_1a_1) + \sum_{j=2}^{n} e_{jk}(\lambda_j - b_ja_j) + \sum_{j=1}^{m}(\mu_j - d_jc_j)f_{jk} - 1 \right\| \to 0$$

with

$$(3.6)\qquad \sup_k \left(\|e_{1k}\| + \sum_{j=2}^{n}\|e_{jk}\| + \sum_{j=1}^{m}\|f_{jk}\|\right) < \infty$$

then also

$$(3.7)\quad \left\|(1+a_1e_{1k}b_1)(1-a_1b_1) + \sum_{j=2}^{n} e_{jk}(\lambda_j - a_jb_j) + \sum_{j=1}^{m}(\mu_j - c_jd_j)a_1f_{jk}b_1 - 1\right\| \to 0.$$

There is also an operator analogue of Theorem 3, based on "almost exactness" ([2], Definition 1.1; [3], Definition 10.3.1):

THEOREM 4. *If* $(T_1, T_2) \in BL(X,Y)^2$ *and* $(S_1, S_2) \in BL(Y,X)^2$ *criss-cross commute, and if there are* $k > 0$ *and* $h > 0$ *for which for arbitrary* $w \in X$ *there is* $x \in X$ *for which*

$$(4.1) \qquad \|w - (I - S_1T_1)x\| \leqslant k\|(\lambda_2 I - S_2T_2)w\| \quad \textit{with } \|x\| \leqslant h\|w\|$$

then for arbitrary $z \in Y$ *there is* $y \in Y$ *for which*

$$(4.2) \qquad \begin{aligned} \|z - (I - T_1S_1)y\| &\leqslant k\|T_1\|\,\|S_1\|\,\|(\lambda_2 I - T_2S_2)z\| \\ &\quad \textit{with } \|y\| \leqslant (1 + h\|T_1\|\,\|S_1\|)\|z\|. \end{aligned}$$

Proof. Take $x \in X$ from (4.1) with $z = S_1w$ and find $y = z + T_1x$:

$$\begin{aligned} \|z - (I - T_1S_1)(z + T_1x)\| &= \|T_1S_1z - (I - T_1S_1)T_1x\| \\ &\leqslant \|T_1\|\,\|S_1z - (I - S_1T_1)x\| \\ &\leqslant k\|T_1\|\,\|(\lambda_2 I - S_2T_2)S_1z\| \end{aligned}$$

which by criss-cross commutativity is

$$k\|T_1\|\,\|S_1(\lambda_2 I - T_2S_2)z\| \leqslant k\|T_1\|\,\|S_1\|\,\|(\lambda_2 I - T_2S_2)z\|;$$

also

$$\|z + T_1x\| \leqslant \|z\| + \|T_1\|\,\|x\| \leqslant (1 + h\|T_1\|\,\|S_1\|)\|z\|. \quad \blacksquare$$

We offer the following hybridizations of the spectrum, approximate point and point spectrum for systems of algebra elements:

DEFINITION 5. If $a \in A^n$ and $c \in A^m$ for a complex (normed) linear algebra A (with identity 1), then

$$(5.1) \quad \sigma_A^{\mathrm{left,right}}(a,c) = \Big\{(\lambda,\mu) \in \mathbb{C}^n \times \mathbb{C}^m : 1 \notin \sum_{j=1}^{n} A(\lambda_j - a_j) + \sum_{j=1}^{m} (\mu_j - c_j)A\Big\};$$

$$(5.2) \quad \begin{aligned} \tau_A^{\mathrm{left,right}}(a,c) = \Big\{&(\lambda,\mu) \in \mathbb{C}^n \times \mathbb{C}^m : \\ &\inf_{\|uv\| \geqslant 1} \sum_{j=1}^{n} \|u\|\,\|(\lambda_j - a_j)v\| + \sum_{j=1}^{m} \|u(\mu_j - c_j)\|\,\|v\| = 0\Big\}; \end{aligned}$$

$$(5.3) \quad \begin{aligned} \pi_A^{\mathrm{left,right}}(a,c) = \Big\{&(\lambda,\mu) \in \mathbb{C}^n \times \mathbb{C}^m : \exists\, uv \neq 0 \in A \text{ such} \\ &((\lambda - a)v, u(\mu - c)) = (0,0) \in A^n \times A^m\Big\}. \end{aligned}$$

With this notation, we can state a theorem about joint spectra:

THEOREM 6. *If $(a,c) \in A^n \times A^m$ and $(b,d) \in A^n \times A^m$ criss-cross commute there is equality*

$$\omega_A^{\text{left,right}}(ab, cd) \setminus \{(0,0)\} = \omega_A^{\text{left,right}}(ba, dc) \setminus \{(0,0)\} \tag{6.1}$$

for each ω of $\{\sigma, \tau, \pi\}$.

Proof. Without loss of generality suppose that among all the λ_j and μ_j of a point $(\lambda, \mu) \in \omega^{\text{left,right}}$ it is $\lambda_1 \neq 0$ and then normalise by scalar multiplication to $\lambda_1 = 1$; now Theorem 2 and Theorem 3 give the argument. ∎

These hybrid results can be used in the argument of Li Shaukuan ([4]) in establishing the analogue of Theorem 5 for the Taylor spectrum. We offer only a fragment:

THEOREM 7. *If $(T_1, T_2) \in BL(X,Y)^2$ and $(S_1, S_2) \in BL(Y,X)^2$ criss-cross commute then*

$$(I - S_1T_1)x_2 = (\lambda_2 I - S_2T_2)x_1 \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} I - S_1T_1 \\ \lambda_2 I - S_2T_2 \end{pmatrix} x_0 \tag{7.1}$$

implies

$$(I - T_1S_1)y_2 = (\lambda_2 I - T_2S_2)y_1 \Longrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} I - T_1S_1 \\ \lambda_2 I - T_2S_2 \end{pmatrix} y_0 , \tag{7.2}$$

and also

$$\begin{pmatrix} I - S_1T_1 \\ \lambda_2 I - S_2T_2 \end{pmatrix} \begin{pmatrix} R_1' & R_2' \end{pmatrix} + \begin{pmatrix} -R_2'' \\ R_1'' \end{pmatrix} \begin{pmatrix} -\lambda_2 I + S_2T_2 & I - S_1T_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{7.3}$$

implies

$$\begin{aligned} &\begin{pmatrix} I - T_1S_1 \\ \lambda_2 I - T_2S_2 \end{pmatrix} \begin{pmatrix} I + T_1R_1'S_1 & T_1R_2'S_1 \end{pmatrix} \\ &\quad + \begin{pmatrix} -T_1R_2''S_1 \\ I + T_1R_1''S_1 \end{pmatrix} \begin{pmatrix} -\lambda_2 I + T_2S_2 & I - T_1S_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \tag{7.4}$$

Proof. The left hand side of (7.2) and criss-cross commutativity gives $(I - S_1T_1)S_1y_2 = (\lambda_2 I - S_2T_2)S_1y_1$ and hence by (7.1) there is y_0 for which

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} I - T_1S_1 & 0 \\ 0 & I - T_1S_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix} \begin{pmatrix} I - S_1T_1 \\ \lambda_2 I - S_2T_2 \end{pmatrix} y_0$$

which by more criss-cross commutativity is $\begin{pmatrix} I - T_1S_1 \\ \lambda_2 I - T_2S_2 \end{pmatrix}(y_1 + T_1y_0)$. If (7.3) holds then the left hand side of (7.4) reduces to

$$\begin{pmatrix} I & (I - T_1S_1)T_1R_2'S_1 - T_1R_2''S_1(I - T_1S_1) \\ (\lambda_2 I - S_2T_2)R_1' - R_1''(\lambda_2 I - S_2T_2)S_1 & I \end{pmatrix}$$

$$= \begin{pmatrix} I & T_1(I - T_1S_1)R_2' - R_2''(I - S_1T_1)S_1 \\ T_1(\lambda_2 I - S_2T_2)R_1' - R_1''(\lambda_2 I - S_2T_2)S_1 & I \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \quad \blacksquare$$

In general we are unable to settle whether the analogue of (1.2) is sufficient for Theorem 2, Theorem 3, Theorem 4 or Theorem 7.

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