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FULL PROJECTIONS, EQUIVALENCE BIMODULES AND AUTOMORPHISMS OF STABLE ALGEBRAS OF UNITAL C^*-ALGEBRAS

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ABSTRACT. Let A be a unital C^*-algebra and $\mathbf{K}$ the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. Let $K_0(A)$ and $K_0(A \otimes \mathbf{K})$ be the K_0-groups of A and $A \otimes \mathbf{K}$ respectively. Let β_* be an automorphism of $K_0(A \otimes \mathbf{K})$ induced by an automorphism β of $A \otimes \mathbf{K}$. Since $K_0(A) \cong K_0(A \otimes \mathbf{K})$, we regard β_* as an automorphism of $K_0(A)$. In the present note we will show that there is a bijection between equivalence classes of automorphisms of $A \otimes \mathbf{K}$ and equivalence classes of full projections p of $A \otimes \mathbf{K}$ with $p(A \otimes \mathbf{K})p \cong A$. Furthermore, using this bijection, we give a sufficient and necessary condition that there is an automorphism β of $A \otimes \mathbf{K}$ such that $\beta_* \neq \alpha_*$ on $K_0(A)$ for any automorphism α of A if A has cancellation or A is a purely infinite simple C^*-algebra.

KEYWORDS: *Automorphisms, cancellation, equivalence bimodules, full projections, K_0-groups.*

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0. INTRODUCTION

Let B be a C^*-algebra and $M(B)$ its multiplier algebra. Let $\mathrm{Aut}(B)$ be the group of all automorphisms of B. For each $\alpha \in \mathrm{Aut}(B)$ we can extend it to an automorphism of $M(B)$ by Busby ([4]) and we also denote it by α. For each unitary element $w \in M(B)$ let $\mathrm{Ad}(w)$ denote the automorphism of B defined by $\mathrm{Ad}(w)(b) = wbw^*$ for any $b \in B$. We call $\mathrm{Ad}(w)$ a generalized inner automorphism of B, and we denote by $\mathrm{Int}(B)$ the group of all generalized inner automorphisms of B. It is easily seen that $\mathrm{Int}(B)$ is a normal subgroup of $\mathrm{Aut}(B)$. We denote by

Out(B) the quotient group Aut(B)/Int(B). We note that if B is unital, Int(B) is the group of all inner automorphisms of B since $M(B) = B$. Furthermore, for any $n \in \mathbf{N}$, M_n denotes the $n \times n$-matrix algebra over $\mathbb{C}$ and $M_n(B)$ denotes the $n \times n$-matrix algebra over B. We identify it with $B \otimes M_n$. If B is not unital, we denote by B^+ the unitized C^*-algebra of B.

Let $K_0(B)$ be the K_0-group of B and Aut($K_0(B)$) the group of all automorphisms of $K_0(B)$. Let T_B be the homomorphism of Aut(B) to Aut($K_0(B)$) defined by $T_B(\alpha) = \alpha_*$ for any $\alpha \in$ Aut(A) where α_* is the automorphism of $K_0(B)$ induced by α. Furthermore let range T_B be the range of T_B.

Let H be a group and L a normal subgroup of H. For any $t \in H$ we denote by $[t]$ the corresponding class in H/L.

Let A be a unital C^*-algebra and $\mathbf{K}$ the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. Since $K_0(A) \cong K_0(A \otimes \mathbf{K})$, we identify Aut($K_0(A)$) with Aut($K_0(A \otimes \mathbf{K})$). Hence range T_A is a subgroup of range $T_{A\otimes\mathbf{K}}$.

For some unital C^*-algebras A, range T_A = range $T_{A\otimes\mathbf{K}}$. But for some unital C^*-algebras A, range T_A is a proper subgroup of range $T_{A\otimes\mathbf{K}}$.

For example, let θ be an irrational number and A_θ the corresponding irrational rotation C^*-algebra. If θ is not quadratic, then for any $\beta \in$ Aut($A_\theta \otimes \mathbf{K}$), $\beta_* =$ id on $K_0(A_\theta)$ by Theorem 2, [5]. Hence range T_{A_θ} = range $T_{A_\theta\otimes\mathbf{K}}$. But if θ is quadratic, then there is a $\beta \in$ Aut($A_\theta \otimes \mathbf{K}$) such that $\beta_* \neq$ id on $K_0(A_\theta)$ by Theorem 5, [5], hence range T_{A_θ} is a proper subgroup of range $T_{A_\theta\otimes\mathbf{K}}$ since $\alpha_* =$ id on $K_0(A_\theta)$ for any $\alpha \in$ Aut(A_θ).

Let n be an integer with $n \geqslant 2$ and O_n the corresponding Cuntz algebra. If $n = 2, 3$, for any $\beta \in$ Aut($O_n \otimes \mathbf{K}$), $\beta_* =$ id on $K_0(O_n)$ by Proposition 14, [6]. Hence range T_{O_n} = range $T_{O_n\otimes\mathbf{K}}$. But if n is not a prime number, then there is a $\beta \in$ Aut($O_n \otimes \mathbf{K}$) such that $\beta_* \neq$ id on $K_0(O_n)$ by Theorem 16, [6], hence range T_{O_n} is a proper subgroup of range $T_{O_n\otimes\mathbf{K}}$ since $\alpha_* =$ id on $K_0(O_n)$ for any $\alpha \in$ Aut(O_n).

It is natural to ask why the above facts happen. In this note we attempt to shed some light on this question. Let A and $\mathbf{K}$ be as above and let $\{e_{ij}\}_{ij\in\mathbf{Z}}$ be matrix units of $\mathbf{K}$. Throughout this note we suppose that A is a unital C^*-algebra. We will show that there is a bijection between equivalence classes of Aut($A \otimes \mathbf{K}$) and equivalence classes of full projections p of $A \otimes \mathbf{K}$ with $p(A \otimes \mathbf{K})p \cong A$. In particular, if Out($A$) is a normal subgroup of Out($A \otimes \mathbf{K}$), then the equivalence classes of full projections p of $A \otimes \mathbf{K}$ with $p(A \otimes \mathbf{K})p \cong A$ is a group. Furthermore, using this bijection, we give a sufficient and necessary condition that range T_A is a proper subgroup of range $T_{A\otimes\mathbf{K}}$ if A has cancellation or A is a purely infinite simple C^*-algebra.

1. AN EQUIVALENCE RELATION IN $\mathrm{Out}(A\otimes\mathbf{K})$

We begin this section with simple lemmas.

LEMMA 1.1. *For any unitary element $w \in M(A \otimes \mathbf{K})$, $\mathrm{Ad}(w)_* = \mathrm{id}$ on $K_0(A \otimes \mathbf{K})$.*

Proof. Since A is unital, we have only to show that $[wpw^*] = [p]$ in $K_0(A\otimes\mathbf{K})$ for any projection p in $A \otimes \mathbf{K}$. For any projection $p \in A \otimes \mathbf{K}$,

$$[wpw^*] = [wp(wp)^*] = [(wp)^* wp] = [p]$$

in $K_0(A \otimes \mathbf{K})$ since $wp \in A \otimes \mathbf{K}$. ∎

LEMMA 1.2. *Let p and q be projections in $A \otimes \mathbf{K}$. Then the following conditions are equivalent:*

(i) *There is a partial isometry $z \in A \otimes \mathbf{K}$ such that $z^*z = p$, $zz^* = q$.*

(ii) *There is a unitary element $w \in M(A \otimes \mathbf{K})$ such that $q = wpw^*$.*

Proof. (i) $\Rightarrow$ (ii). By Mingo ([7], p. 401, Lemma), there are partial isometries $z_1, z_2 \in A \underset{\min}{\otimes} M(\mathbf{K}) \subset M(A \otimes \mathbf{K})$ such that

$$z_1^* z_1 = \mathbf{1} \otimes \mathbf{1}, \qquad z_1 z_1^* = \mathbf{1} \otimes \mathbf{1} - p,$$
$$z_2^* z_2 = \mathbf{1} \otimes \mathbf{1}, \qquad z_2 z_2^* = \mathbf{1} \otimes \mathbf{1} - q,$$

where $A \underset{\min}{\otimes} M(\mathbf{K})$ means the minimal tensor product of A and $M(\mathbf{K})$. Let $w = z + z_2 z_1^*$. Then by easy computation we see that w is a unitary element in $M(A \otimes \mathbf{K})$. And

$$\begin{aligned} wpw^* &= (z + z_2 z_1^*)p(z^* + z_1 z_2^*) = (z + z_2 z_1^* z_1 z_1^* p)(z^* + z_1 z_2^*) \\ &= q + zz^* z z_1 z_1^* z_1 z_2^* = q. \end{aligned}$$

Hence we obtain (ii).

(ii) $\Rightarrow$ (i). Put $z = wp \in A\otimes\mathbf{K}$. Then $z^*z = p$, $zz^* = q$. Thus we obtain (i). ∎

LEMMA 1.3. *We suppose that A has cancellation or that A is a purely infinite simple C^*-algebra. Let $\beta \in \mathrm{Aut}(A\otimes\mathbf{K})$ with $\beta_* = \mathrm{id}$ on $K_0(A\otimes\mathbf{K})$. Then there are an $\alpha \in \mathrm{Aut}(A)$ and a unitary element $w \in M(A\otimes\mathbf{K})$ such that $\beta = \mathrm{Ad}(w)\circ\alpha\otimes\mathrm{id}$.*

Proof. If A has cancellation, then by Proposition 6, [6], we obtain the conclusion. If A is a purely infinite simple C^*-algebra, then we can prove this lemma in the same way as in Proposition 14, [6]. ∎

Let Ψ be the homomorphism of $\mathrm{Aut}(A)$ to $\mathrm{Aut}(A\otimes\mathbf{K})$ defined by $\Psi(\alpha)=\alpha\otimes\mathrm{id}$ for any $\alpha\in\mathrm{Aut}(A)$. And let $\widetilde{\Psi}$ be the homomorphism of $\mathrm{Out}(A)$ to $\mathrm{Out}(A\otimes\mathbf{K})$ defined by $\widetilde{\Psi}([\alpha])=[\Psi(\alpha)]=[\alpha\otimes\mathrm{id}]$ for any $\alpha\in\mathrm{Aut}(A)$. By [6] $\widetilde{\Psi}$ is injective. We identify $\mathrm{Out}(A)$ with $\widetilde{\Psi}(\mathrm{Out}(A))$. We define an equivalence relation $\sim$ in $\mathrm{Out}(A\otimes\mathbf{K})$ as follows: for any $\beta_1,\beta_2\in\mathrm{Aut}(A\otimes\mathbf{K})$, $[\beta_1]\sim[\beta_2]$ if there is an $\alpha\in\mathrm{Aut}(A)$ such that $[\beta_1]=[\beta_2][\alpha\otimes\mathrm{id}]$. By an easy computation we see that $\sim$ is an equivalence relation in $\mathrm{Out}(A\otimes\mathbf{K})$. We denote by $[[\beta]]$ the equivalence class of $[\beta]\in\mathrm{Out}(A\otimes\mathbf{K})$ and we denote by $\mathbf{P}$ the quotient set of $\mathrm{Out}(A\otimes\mathbf{K})$ by the above equivalence relation $\sim$. If $\mathrm{Out}(A)$ is a normal subgroup of $\mathrm{Out}(A\otimes\mathbf{K})$, the quotient set $\mathbf{P}$ is the quotient group $\mathrm{Out}(A\otimes\mathbf{K})/\mathrm{Out}(A)$.

REMARK 1.4. We can regard $\mathrm{Out}(A)$ as a group acting on $\mathrm{Out}(A\otimes\mathbf{K})$ by right multiplication. Thus $\mathbf{P}$ is the orbit space of $\mathrm{Out}(A\otimes\mathbf{K})$ on which $\mathrm{Out}(A)$ acts. Furthermore by Brown, Green and Rieffel ([3], Corollary 3.5), $\mathrm{Out}(A\otimes\mathbf{K})$ is isomorphic to $\mathrm{Pic}(A)$, the Picard group of A. Hence we can regard $\mathrm{Out}(A)$ as a group acting on $\mathrm{Pic}(A)$ and $\mathbf{P}$ as the orbit space of $\mathrm{Pic}(A)$ on which $\mathrm{Out}(A)$ acts.

PROPOSITION 1.5. *With the above notations, if* $\mathrm{Out}(A)$ *is a normal subgroup of* $\mathrm{Out}(A\otimes\mathbf{K})$, *then* $\mathrm{range}\,T_A$ *is a normal subgroup of* $\mathrm{range}\,T_{A\otimes\mathbf{K}}$. *Furthermore the converse is true if A has cancellation or A is a purely infinite simple C^*-algebra.*

Proof. We suppose that $\mathrm{Out}(A)$ is a normal subgroup of $\mathrm{Out}(A\otimes\mathbf{K})$. Then for any $\beta\in\mathrm{Aut}(A\otimes\mathbf{K})$ and $\alpha\in\mathrm{Aut}(A)$, $[\beta][\alpha\otimes\mathrm{id}][\beta]^{-1}\in\mathrm{Out}(A)$. Hence there is a $\gamma\in\mathrm{Aut}(A)$ such that $[\beta][\alpha\otimes\mathrm{id}][\beta]^{-1}=[\gamma\otimes\mathrm{id}]$. Thus there is a unitary element $w\in M(A\otimes\mathbf{K})$ such that $\beta\circ\alpha\otimes\mathrm{id}\circ\beta^{-1}=\mathrm{Ad}(w)\circ\gamma\otimes\mathrm{id}$. Hence by Lemma 1.1 $(\beta\circ\alpha\otimes\mathrm{id}\circ\beta^{-1})_*=(\gamma\otimes\mathrm{id})_*\in\mathrm{range}\,T_A$. Therefore $\mathrm{range}\,T_A$ is a normal subgroup of $\mathrm{range}\,T_{A\otimes\mathbf{K}}$.

Next we suppose that $\mathrm{range}\,T_A$ is a normal subgroup of $\mathrm{range}\,T_{A\otimes\mathbf{K}}$. We also suppose that A has cancellation or that A is a purely infinite simple C^*-algebra. Then for any $\beta\in\mathrm{Aut}(A\otimes\mathbf{K})$ and $\alpha\in\mathrm{Aut}(A)$, $(\beta\circ\alpha\otimes\mathrm{id}\circ\beta^{-1})_*\in\mathrm{range}\,T_A$. Hence there is a $\gamma\in\mathrm{Aut}(A)$ such that $(\beta\circ\alpha\otimes\mathrm{id}\circ\beta^{-1})_*=(\gamma\otimes\mathrm{id})_*$ on $K_0(A\otimes\mathbf{K})$. Thus $(\beta\circ\alpha\otimes\mathrm{id}\circ\beta^{-1}\circ(\gamma\otimes\mathrm{id})^{-1})_*=\mathrm{id}$ on $K_0(A\otimes\mathbf{K})$. Since A has cancellation or A is a purely infinite simple C^*-algebra, by Lemma 1.3 there are a $\delta\in\mathrm{Aut}(A)$ and a unitary element $w\in M(A\otimes\mathbf{K})$ such that

$$\beta\circ\alpha\otimes\mathrm{id}\circ\beta^{-1}\circ(\gamma\otimes\mathrm{id})^{-1}=\mathrm{Ad}(w)\circ\delta\otimes\mathrm{id}.$$

Hence $[\beta][\alpha\otimes\mathrm{id}][\beta]^{-1}=[(\delta\circ\gamma)\otimes\mathrm{id}]\in\mathrm{Out}(A)$. Therefore $\mathrm{Out}(A)$ is a normal subgroup of $\mathrm{Out}(A\otimes\mathbf{K})$. ∎

2. AN EQUIVALENCE RELATION IN FULL PROJECTIONS

Let p and q be projections in $A \otimes \mathbf{K}$. Then p is *equivalent to* q, *written* $p \sim q$, if there is a partial isometry $z \in A \otimes \mathbf{K}$ such that $p = z^*z$, $q = zz^*$. We denote by (p) the equivalence class of the projection p.

Let FP be the set of all full projections $p \in A \otimes \mathbf{K}$ with $p(A \otimes \mathbf{K})p \cong A$ and FP$/\sim$ the quotient set of FP by the above equivalence relation $\sim$.

REMARK 2.1. For any $n \in \mathbb{N}$ we regard $M_n(A)$ as a C^*-subalgebra of $A \otimes \mathbf{K}$. By Blackadar ([1], Proposition 4.6.6) and Lemma 1.2, for any $p \in$ FP there is a full projection $q \in \bigcup_{n \in \mathbb{N}} M_n(A)$ such that $(p) = (q)$, $q(A \otimes \mathbf{K})q \cong A$.

3. A MAP FROM FP TO P

In this section we will construct a map from FP$/\sim$ to $\mathbf{P}$. Let p be any element in FP. Since $p \in$ FP, there is an isomorphism χ_p of A onto $p(A \otimes \mathbf{K})p$ and by Brown ([2], Lemma 2.5) there is a partial isometry $z \in M(A \otimes \mathbf{K} \otimes \mathbf{K})$ such that $z^*z = p \otimes \mathbf{1}$, $zz^* = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$. Let ψ be an isomorphism of $\mathbf{K} \otimes \mathbf{K}$ onto $\mathbf{K}$ with $\psi_* = \mathrm{id}$ of $K_0(\mathbf{K} \otimes \mathbf{K})$ onto $K_0(\mathbf{K})$. Let $\beta(p, \chi_p)$ be the automorphism of $A \otimes \mathbf{K}$ defined by

$$\beta(p, \chi_p) = \mathrm{id} \otimes \psi \circ \mathrm{Ad}(z) \circ \chi_p \otimes \mathrm{id}.$$

LEMMA 3.1. *With the above notation* $[\beta(p, \chi_p)] \in \mathrm{Out}(A \otimes \mathbf{K})$ *is independent of the choices of* z *and* ψ.

Proof. For $j = 1, 2$ let z_j be a partial isometry in $M(A \otimes \mathbf{K} \otimes \mathbf{K})$ with $z_j^* z_j = p \otimes \mathbf{1}$, $z_j z_j^* = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$. Let ψ be an isomorphism of $\mathbf{K} \otimes \mathbf{K}$ onto $\mathbf{K}$ with $\psi_* = \mathrm{id}$ of $K_0(\mathbf{K} \otimes \mathbf{K})$ onto $K_0(\mathbf{K})$. For $j = 1, 2$, let $\beta_j(p, \chi_p)$ be the automorphism of $A \otimes \mathbf{K}$ induced by z_j and ψ. Then:

$$\begin{aligned}\beta_2(p, \chi_p) &= \mathrm{id} \otimes \psi \circ \mathrm{Ad}(z_2) \circ \chi_p \otimes \mathrm{id} = \mathrm{id} \otimes \psi \circ \mathrm{Ad}(z_2 z_1^*) \circ \mathrm{Ad}(z_1) \circ \chi_p \otimes \mathrm{id} \\ &= \mathrm{Ad}((\mathrm{id} \otimes \psi)(z_2 z_1^*)) \circ \mathrm{Ad}(z_1) \circ \chi_p \otimes \mathrm{id} \\ &= \mathrm{Ad}((\mathrm{id} \otimes \psi)(z_2 z_1^*)) \circ \beta_1(p, \chi_p).\end{aligned}$$

Hence $[\beta_1(p, \chi_p)] = [\beta_2(p, \chi_p)]$ since $z_2 z_1^*$ is a unitary element in $M(A \otimes \mathbf{K} \otimes \mathbf{K})$.

Next for $j = 1, 2$ let ψ_j be an isomorphism of $\mathbf{K} \otimes \mathbf{K}$ onto $\mathbf{K}$ with $\psi_{j*} = \mathrm{id}$ of $K_0(\mathbf{K} \otimes \mathbf{K})$ onto $K_0(\mathbf{K})$ and let z be a partial isometry in $M(A \otimes \mathbf{K} \otimes \mathbf{K})$ with $z^*z = p \otimes \mathbf{1}$, $zz^* = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$. For $j = 1, 2$, let $\beta_j(p, \chi_p)$ be the automorphism of $A \otimes \mathbf{K}$ induced by z and ψ_j. Then $\psi_2 \circ \psi_1^{-1}$ is an automorphism of $\mathbf{K}$. Hence there is a unitary element $w \in M(\mathbf{K})$ such that $\psi_2 \circ \psi_1^{-1} = \mathrm{Ad}(w)$. Thus

$$\begin{aligned}\beta_2(p, \chi_p) &= \mathrm{id} \otimes \psi_2 \circ \mathrm{Ad}(z) \circ \chi_p \otimes \mathrm{id} = \mathrm{Ad}(\mathbf{1} \otimes w) \circ \mathrm{id} \otimes \psi_1 \circ \mathrm{Ad}(z) \circ \chi_p \otimes \mathrm{id} \\ &= \mathrm{Ad}(\mathbf{1} \otimes w) \circ \beta_1(p, \chi_p).\end{aligned}$$

Hence $[\beta_1(p, \chi_p)] = [\beta_2(p, \chi_p)]$. ∎

We define a map $\mathcal{F}$ from FP/ $\sim$ to $\mathbf{P}$ by $\mathcal{F}((p)) = [[\beta(p, \chi_p)]]$. We show that it is well-defined:

LEMMA 3.2. *$\mathcal{F}$ is well-defined.*

Proof. First we will show that $[[\beta(p, \chi_p)]]$ is independent of the choice of χ_p. Let ρ_p be another isomorphism of A onto $p(A \otimes \mathbf{K})p$. Let α be the automorphism of A defined by $\alpha = \chi_p^{-1} \circ \rho_p$. Let ψ be an isomorphism of $\mathbf{K} \otimes \mathbf{K}$ onto $\mathbf{K}$ with $\psi_* = \mathrm{id}$ of $K_0(\mathbf{K} \otimes \mathbf{K})$ onto $K_0(\mathbf{K})$ and z a partial isometry in $M(A \otimes \mathbf{K} \otimes \mathbf{K})$ with $z^*z = p \otimes \mathbf{1}$, $zz^* = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$. Then

$$\begin{aligned}\beta(p, \rho_p) &= \mathrm{id} \otimes \psi \circ \mathrm{Ad}(z) \circ \rho_p \otimes \mathrm{id} = \mathrm{id} \otimes \psi \circ \mathrm{Ad}(z) \circ (\chi_p \circ \alpha) \otimes \mathrm{id} \\ &= \mathrm{id} \otimes \psi \circ \mathrm{Ad}(z) \circ \chi_p \otimes \mathrm{id} \circ \alpha \otimes \mathrm{id} = \beta(p, \chi_p) \circ \alpha \otimes \mathrm{id}.\end{aligned}$$

Thus $[[\beta(p, \rho_p)]] = [[\beta(p, \chi_p)]]$.

Let $p, q \in$ FP with $(p) = (q)$. Let χ_p be an isomorphism of A onto $p(A \otimes \mathbf{K})p$ and χ_q isomorphism of A onto $q(A \otimes \mathbf{K})q$ respectively. Since $(p) = (q)$, by Lemma 1.2 there is a unitary element $w \in M(A \otimes \mathbf{K})$ such that $p = wqw^*$. Hence $\mathrm{Ad}(w)$ is an isomorphism of $q(A \otimes \mathbf{K})q$ onto $p(A \otimes \mathbf{K})p$. Let γ be the automorphism of A defined by $\gamma = \chi_p^{-1} \circ \mathrm{Ad}(w) \circ \chi_q$. Then

$$\begin{aligned}\beta(p, \chi_p) &= \mathrm{id} \otimes \psi \circ \mathrm{Ad}(z) \circ \chi_p \otimes \mathrm{id} \\ &= \mathrm{id} \otimes \psi \circ \mathrm{Ad}(z(w \otimes \mathbf{1})) \circ \chi_q \otimes \mathrm{id} \circ \gamma^{-1} \otimes \mathrm{id}.\end{aligned}$$

Let $Z = z(w \otimes \mathbf{1})$. Then $Z^*Z = q \otimes \mathbf{1}$, $ZZ^* = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$. Thus by Lemma 2.1 $[\beta(p, \chi_p)] = [\beta(q, \chi_q)][(\gamma \otimes \mathrm{id})^{-1}]$. Therefore $[[\beta(p, \chi_p)]] = [[\beta(q, \chi_q)]]$. ∎

From now on we simply denote $\beta(p, \chi_p)$ by β_p since $[[\beta(p, \chi_p)]]$ is independent of the choice of χ_p.

4. A MAP FROM P TO FP/ $\sim$

In this section we will construct a map $\mathcal{J}$ from $\mathbf{P}$ to FP/ $\sim$ and we will show that $\mathcal{J}$ is bijective.

For each $\beta \in \mathrm{Aut}(A \otimes \mathbf{K})$, $\beta(\mathbf{1} \otimes e_{00})$ is a full projection in $A \otimes \mathbf{K}$ since $\mathbf{1} \otimes e_{00}$ is full in $A \otimes \mathbf{K}$. We define a map $\mathcal{J}$ from $\mathbf{P}$ to FP/ $\sim$ by $\mathcal{J}([[\beta]]) = (\beta(\mathbf{1} \otimes e_{00}))$. We can easily show that it is well-defined by Lemma 1.2.

Before we show that $\mathcal{J}$ is injective, we need a lemma.

LEMMA 4.1. *Let* $\beta \in \mathrm{Aut}(A \otimes \mathbf{K})$. *We suppose that* $(\beta(\mathbf{1} \otimes e_{00})) = (\mathbf{1} \otimes e_{00})$. *Then there are an* $\alpha \in \mathrm{Aut}(A)$ *and a unitary element* $u \in M(A \otimes \mathbf{K})$ *such that* $\beta = \mathrm{Ad}(u) \circ \alpha \otimes \mathrm{id}$.

Proof. Since $(\beta(\mathbf{1} \otimes e_{00})) = (\mathbf{1} \otimes e_{00})$, by Lemma 1.2 there is a unitary element $z \in M(A \otimes \mathbf{K})$ such that $\beta(\mathbf{1} \otimes e_{00}) = z(\mathbf{1} \otimes e_{00})z^*$. Since $\mathbf{K}$ has cancellation, there is a unitary element $w_j \in M(\mathbf{K})$ such that $e_{jj} = w_j e_{00} w_j^*$ for any $j \in \mathbb{Z}$. Then for any $j \in \mathbb{Z}$

$$\begin{aligned}\beta(\mathbf{1} \otimes e_{jj}) &= \beta((\mathbf{1} \otimes w_j)(\mathbf{1} \otimes e_{00})(\mathbf{1} \otimes w_j)^*) = \beta(\mathbf{1} \otimes w_j)\beta(\mathbf{1} \otimes e_{00})\beta(\mathbf{1} \otimes w_j)^* \\ &= \beta(\mathbf{1} \otimes w_j)z(\mathbf{1} \otimes w_j)^*(\mathbf{1} \otimes e_{jj})(\mathbf{1} \otimes w_j)z^*\beta(\mathbf{1} \otimes w_j)^*.\end{aligned}$$

Hence by Proposition 6, [6] we obtain the conclusion. ∎

LEMMA 4.2. $\mathcal{J}$ *is injective.*

Proof. Let β_1 and β_2 be in $\mathrm{Aut}(A \otimes \mathbf{K})$ with $\mathcal{J}([[\beta_1]]) = \mathcal{J}([[\beta_2]])$. That is, $(\beta_1(\mathbf{1} \otimes e_{00})) = (\beta_2(\mathbf{1} \otimes e_{00}))$. Hence $(\beta_1^{-1} \circ \beta_2(\mathbf{1} \otimes e_{00})) = (\mathbf{1} \otimes e_{00})$. By Lemma 4.1 there is an $\alpha \in \mathrm{Aut}(A)$ and a unitary element $u \in M(A \otimes \mathbf{K})$ such that $\beta_1^{-1} \circ \beta_2 = \mathrm{Ad}(u) \circ \alpha \otimes \mathrm{id}$. Thus $[[\beta_1]] = [[\beta_2]]$. ∎

LEMMA 4.3. *Let* ψ *be an isomorphism of* $\mathbf{K} \otimes \mathbf{K}$ *onto* $\mathbf{K}$ *with* $\psi_* = \mathrm{id}$ *of* $K_0(\mathbf{K} \otimes \mathbf{K})$ *onto* $K_0(\mathbf{K})$. *Then for any* $n \in \mathbb{N}$ *there is a partial isometry* $v \in \mathbf{K}$ *such that*

$$v^*v = \sum_{j=-n}^{n} e_{jj}, \quad vv^* = \sum_{j=-n}^{n} \psi(e_{jj} \otimes e_{00}),$$
$$ve_{ij}v^* = \psi(e_{ij} \otimes e_{00}) \quad \textit{for } i,j = -n, \dots, 0, \dots, n.$$

Proof. Since $\psi(e_{00} \otimes e_{00})$ is equivalent to e_{00}, there is a partial isometry $w \in \mathbf{K}$ such that $w^*w = e_{00}$, $ww^* = \psi(e_{00} \otimes e_{00})$. Let $v = \sum_{j=-n}^{n} \psi(e_{j0} \otimes e_{00})we_{0j}$. Then for $i,j = -n, \dots, 0, \dots, n$

$$\begin{aligned}ve_{ij}v^* &= \Big(\sum_{k=-n}^{n} \psi(e_{k0} \otimes e_{00})we_{0k}\Big)e_{ij}\Big(\sum_{l=-n}^{n} e_{l0}w^*\psi(e_{0l} \otimes e_{00})\Big) \\ &= \psi(e_{i0} \otimes e_{00})we_{00}w^*\psi(e_{0j} \otimes e_{00}) \\ &= \psi(e_{i0} \otimes e_{00})\psi(e_{00} \otimes e_{00})\psi(e_{0j} \otimes e_{00}) \\ &= \psi(e_{ij} \otimes e_{00}).\end{aligned}$$

Furthermore by the direct computation we see that

$$v^*v = \sum_{j=-n}^{n} e_{jj}, \quad vv^* = \sum_{j=-n}^{n} \psi(e_{jj} \otimes e_{00}). \quad ∎$$

LEMMA 4.4. *For any* $p \in \mathrm{FP}$, $(\mathcal{J} \circ \mathcal{F})((p)) = (p)$ *where* $\mathcal{F}$ *is the map from* $\mathrm{FP}/\sim$ *to* $\mathbf{P}$ *defined in Section* 3.

Proof. Let p be any element in FP. Then

$$(\mathcal{J} \circ \mathcal{F})((p)) = \mathcal{J}([[\beta_p]]) = (\beta_p(\mathbf{1} \otimes e_{00})),$$

where β_p is an automorphism of $A \otimes \mathbf{K}$ induced by the projection p. Let ψ be an isomorphism of $\mathbf{K} \otimes \mathbf{K}$ onto $\mathbf{K}$ with $\psi_* = \mathrm{id}$ of $K_0(\mathbf{K} \otimes \mathbf{K})$ onto $K_0(\mathbf{K})$ and z a partial isometry in $M(A \otimes \mathbf{K} \otimes \mathbf{K})$ with $z^*z = p \otimes \mathbf{1}$, $zz^* = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$. Furthermore let χ_p be an isomorphism of A onto $p(A \otimes \mathbf{K})p$. By the definition of β_p, $\beta_p = \mathrm{id} \otimes \psi \circ \mathrm{Ad}(z) \circ \chi_p \otimes \mathrm{id}$. Thus

$$\beta_p(\mathbf{1} \otimes e_{00}) = (\mathrm{id} \otimes \psi)(z)(\mathrm{id} \otimes \psi)(p \otimes e_{00})(\mathrm{id} \otimes \psi)(z)^*.$$

By Remark 2.1 we may assume that there is an $n \in \mathbb{N}$ such that p is in $M_{2n+1}(A) \subset A \otimes \mathbf{K}$. Since $p \in M_{2n+1}(A)$, we can write $p = \sum_{i,j=-n}^{n} a_{ij} \otimes e_{ij}$, where $a_{ij} \in A$. Hence

$$(\mathrm{id} \otimes \psi)(p \otimes e_{00}) = \sum_{i,j=-n}^{n} a_{ij} \otimes \psi(e_{ij} \otimes e_{00}).$$

By Lemma 4.3 there is a partial isometry $v \in \mathbf{K}$ such that

$$v^*v = \sum_{j=-n}^{n} e_{jj}, \quad vv^* = \sum_{j=-n}^{n} \psi(e_{jj} \otimes e_{00}),$$
$$ve_{ij}v^* = \psi(e_{ij} \otimes e_{00}) \quad \text{for } i,j = -n, \dots, 0, \dots, n.$$

Then

$$\begin{aligned}(\mathbf{1} \otimes v)p(\mathbf{1} \otimes v)^* &= \sum_{i,j=-n}^{n} a_{ij} \otimes ve_{ij}v^* = \sum_{i,j=-n}^{n} a_{ij} \otimes \psi(e_{ij} \otimes e_{00}) \\ &= (\mathrm{id} \otimes \psi)(p \otimes e_{00}).\end{aligned}$$

Hence

$$\beta_p(\mathbf{1} \otimes e_{00}) = (\mathrm{id} \otimes \psi)(z)(\mathbf{1} \otimes v)p(\mathbf{1} \otimes v)^*(\mathrm{id} \otimes \psi)(z^*).$$

Furthermore

$$\begin{aligned}&[p(\mathbf{1} \otimes v)^*(\mathrm{id} \otimes \psi)(z^*)][p(\mathbf{1} \otimes v)^*(\mathrm{id} \otimes \psi)(z^*)]^* \\ &\quad = p(\mathbf{1} \otimes v)^*(\mathrm{id} \otimes \psi)(p \otimes \mathbf{1})(\mathbf{1} \otimes v)p = p(\mathbf{1} \otimes v)^*(\mathrm{id} \otimes \psi)(p \otimes e_{00})(\mathbf{1} \otimes v)p \\ &\quad = p(\mathbf{1} \otimes v)^*(\mathbf{1} \otimes v)p(\mathbf{1} \otimes v)^*(\mathbf{1} \otimes v)p = p\Big(\mathbf{1} \otimes \sum_{j=-n}^{n} e_{jj}\Big)p\Big(\mathbf{1} \otimes \sum_{j=-n}^{n} e_{jj}\Big)p \\ &\quad = p.\end{aligned}$$

Thus $(\beta_p(\mathbf{1} \otimes e_{00})) = (p)$. ∎

THEOREM 4.5. *Let A be a unital C^*-algebra. Let $\mathcal{F}$, $\mathcal{J}$ and* $\mathbf{P}$, $\mathrm{FP}/\sim$ *be as above. Then $\mathcal{J}$ is a bijection from* $\mathbf{P}$ *onto* $\mathrm{FP}/\sim$. *Furthermore $\mathcal{F}$ is the inverse map of $\mathcal{J}$ from* $\mathrm{FP}/\sim$ *onto* $\mathbf{P}$.

Proof. This is immediate by Lemmas 4.2 and 4.4. ∎

By Theorem 4.5 we can easily obtain the following corollaries.

COROLLARY 4.6. *With the above notations the following conditions are equivalent:*

(i) *There is a* $\beta \in \mathrm{Aut}(A \otimes \mathbf{K})$ *such that* $[\beta] \notin \mathrm{Out}(A)$;

(ii) *There is a full projection* $p \in A \otimes \mathbf{K}$ *such that*

$$p(A \otimes \mathbf{K})p \cong A, \quad (p) \neq (\mathbf{1} \otimes e_{00}).$$

COROLLARY 4.7. *With the above notations we suppose that A has cancellation or that A is a purely infinite simple C^*-algebra. Then the following conditions are equivalent:*

(i) *There is a* $\beta \in \mathrm{Aut}(A \otimes \mathbf{K})$ *such that* $\beta_* \neq \alpha_*$ *on* $K_0(A)$ *for any* $\alpha \in \mathrm{Aut}(A)$;

(ii) *There is a full projection* $p \in A \otimes \mathbf{K}$ *such that*

$$p(A \otimes \mathbf{K})p \cong A, \quad [p] \neq [\mathbf{1} \otimes e_{00}] \quad \textit{in } K_0(A \otimes \mathbf{K}).$$

COROLLARY 4.8. *We suppose that* $\mathrm{Out}(A)$ *is a normal subgroup of* $\mathrm{Out}(A \otimes \mathbf{K})$. *Then* $\mathrm{FP}/\sim$ *is a group and we have a short exact sequence*

$$1 \to \mathrm{Out}(A) \to \mathrm{Out}(A \otimes \mathbf{K}) \to \mathrm{FP}/\sim \to 1.$$

REMARK 4.9. (i) With the same assumptions as in Corollary 4.8, the product of the group $\mathrm{FP}/\sim$ is the following: for any $p, q \in \mathrm{FP}$, $(p)(q) = (\beta_p(q))$ where β_p is the automorphism of $A \otimes \mathbf{K}$ induced by $p \in \mathrm{FP}$.

(ii) We suppose that A has cancellation or that A is a purely infinite simple C^*-algebra. If $\mathrm{range}\, T_A$ is a normal subgroup of $\mathrm{range}\, T_{A \otimes \mathbf{K}}$, we have the same thing as in Corollary 4.8 by Proposition 1.5. Furthermore since we can easily see that $\mathrm{range}\, T_{A \otimes \mathbf{K}}/\mathrm{range}\, T_A \cong \mathrm{Out}(A \otimes \mathbf{K})/\mathrm{Out}(A)$ in the same way as in Proposition 4, [6], $\mathrm{FP}/\sim$ is isomorphic to $\mathrm{range}\, T_{A \otimes \mathbf{K}}/\mathrm{range}\, T_A$.

5. APPLICATION

In this section we apply our results to Heisenberg C^*-algebras of class 2 and class 3. Following Packer ([8], [9]) we will define Heisenberg C^*-algebras. For any θ, $\eta \in \mathbb{R}$ let $H(\theta, \eta)$ be the universal C^*-algebra generated by unitary elements u, v and w satisfying

$$uv = e^{2\pi i\theta} vu, \quad wv = e^{2\pi i\eta} vw, \quad uw = vwu.$$

If τ is any normalized faithful trace on $H(\theta, \eta)$, $H(\theta, \eta)$ will be said to be of *class* 1, 2 or 3 if $\tau_*(K_0(H(\theta, \eta))) = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\eta$ is generated by 1, 2 but not 1 or 3 but not 2 elements in $\mathbb{R}$ respectively.

We suppose that θ and η are irrational numbers and 1, θ and η are linearly independent. Then $H(\theta, \eta)$ is of class 3. By Packer ([8]), $H(\theta, \eta)$ is simple and has the unique tracial state τ. For each $n \in \mathbb{N}$ we extend it to the unnormalized finite trace on $M_n(H(\theta, \eta))$. We also denote it by τ. By Packer ([9], Section 2) there are projections p and q in some $M_n(H(\theta, \eta))$ such that

$$K_0(H(\theta, \eta)) = \mathbb{Z}[1] \oplus \mathbb{Z}[p] \oplus \mathbb{Z}[q], \quad \tau(p) = \theta, \quad \tau(q) = \eta.$$

In the same way as in the proof of Theorem 2, [5], for any $\beta \in \mathrm{Aut}(H(\theta, \eta) \otimes \mathbf{K})$ $\beta_* = \mathrm{id}$ on $K_0(H(\theta, \eta))$ if the numbers 1, θ, θ^2, η, η^2 and $\theta\eta$ are linearly independent. Hence we obtain the following proposition.

PROPOSITION 5.1. *Let θ and η be irrational numbers. We suppose that* 1, *θ, θ^2, η, η^2 and $\theta\eta$ are linearly independent. Then there is no projection f in $H(\theta, \eta) \otimes \mathbf{K}$ such that*

$$f(H(\theta, \eta) \otimes \mathbf{K})f \cong H(\theta, \eta), \quad [f] \neq [1 \otimes e_{00}] \quad \text{in } K_0(H(\theta, \eta) \otimes \mathbf{K}).$$

Proof. By the above discussion for any $\beta \in \mathrm{Aut}(H(\theta, \eta) \otimes \mathbf{K})$, $\beta_* = \mathrm{id}$ on $K_0(H(\theta, \eta) \otimes \mathbf{K})$. And by Packer ([9], Proposition 2.1) $H(\theta, \eta)$ has cancellation. Thus by Corollary 4.7 we obtain the conclusion. ∎

Next we suppose that θ is irrational and that $\eta = 0$. Then the corresponding Heisenberg C^*-algebra $H(\theta, \eta)$ is of class 2. We denote it by H_θ. By Packer ([8]), H_θ is simple and has the unique tracial state τ. For each $n \in \mathbb{N}$ we extend it to the unnormalized finite trace on $M_n(H_\theta)$. We also denote it by τ. Let p be a projection in some $M_n(H_\theta)$ with $\tau(p) = \theta$. Let $p(1,1)$ be the projection in $M_2(H_\theta)$ defined in Rieffel ([12]) or Packer ([9]) which is called to have trace 1 and twist -1. Then by Packer ([9])

$$K_0(H_\theta) = \mathbb{Z}[p] \oplus \mathbb{Z}[1] \oplus \mathbb{Z}([1] - [p(1,1)]).$$

Furthermore by Packer ([9], Lemma 2.9) there is a non-zero projection q in some $M_n(H_\theta)$ with

$$[q] = a[p] + b[1] + c([1] - [p(1,1)]) \quad \text{in } K_0(H_\theta)$$

if and only if $a\theta + b > 0$, where a, b and $c \in \mathbb{Z}$. Let q be such a projection. Let d be the greatest (positive) common divisor of a, b and c and write $(a,b,c) = d(l,m,k)$ where l,m,k have no common factor. Let f be the greatest (positive) common divisor of l,m and write

$$(a,b,c) = d(fg, fh, k) \quad \text{where} \quad (g,h) = 1.$$

We note that $g\theta + h > 0$ since $a\theta + b = dfg\theta + dfh > 0$ and that $(f,h) = 1$. Hence there are $r, s \in \mathbb{Z}$ with $rf - sk = 1$. Since $(g,h) = 1$, there are $x, y \in \mathbb{Z}$ such that $xh - yg = 1$. By Packer ([9], the proof of Lemma 2.9),

$$qM_n(H_\theta)q \cong M_d\left(H\left(\frac{\eta}{f}, \frac{s}{f}\right)\right)$$

where $\eta = (x\theta + y)/(g\theta + h)$.

THEOREM 5.2. *Let θ be an irrational number and H_θ the corresponding Heisenberg C^*-algebra. Then there is a $\beta \in \mathrm{Aut}(H_\theta \otimes \mathbf{K})$ such that $\beta_* \neq \alpha_*$ on $K_0(H_\theta)$ for any $\alpha \in \mathrm{Aut}(H_\theta)$.*

Proof. We use the same notations as in the above discussion. Put $a = 0$, $b = 1$ and $c = 1$. By Packer ([9], Lemma 2.9) there is a projection q in some $M_n(H_\theta)$ such that $[q] = [1] + ([1] - [p(1,1)])$ in $K_0(H_\theta)$. Then $d = f = h = k = 1$ and $g = 0$. Put $r = 1$, $s = 0$ and $x = 1$, $y = 0$. Then $rf - sk = 1$ and $xh - yg = 1$. Hence

$$qM_n(H_\theta)q \cong H(\theta, 0) = H_\theta.$$

Since H_θ is simple, q is full in $M_n(H_\theta)$ and since $[q] = [1] + ([1] - [p(1,1)])$ in $K_0(H_\theta)$, $[q] \neq [1]$ in $K_0(H_\theta)$. Therefore by Corollary 4.7 we obtain the conclusion. ∎

6. A MAP FROM Pic(A) TO FP/ $\sim$

Let Pic(A) be the Picard group of A which is defined in Brown, Green and Rieffel ([3]). In this section we will construct a map $\overline{\mathcal{J}}$ from Pic(A) to FP/ $\sim$ by modifying the method stated in Rieffel ([11], Propostion 2.1).

Let X be an $A-A$-equivalence bimodule. Then we can find $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ in X such that

$$\sum_{i=1}^{n} \langle x_i, y_i \rangle_A = 1.$$

Let $E = M_n(A)$ and we consider X^n as an $E-A$-equivalence bimodule in the evident way. Let $x = \{x_1, \ldots, x_n\}$ and $y = \{y_1, \ldots, y_n\}$ in X^n. Let $z = {}_E\langle y, y \rangle^{1/2} x$ and $p = {}_E\langle z, z \rangle$. Then by Rieffel ([11], Propostion 2.1) p is a full projection in E and $pEp \cong A$. We can regard p as a full projection in $A \otimes \mathbf{K}$. Hence we can define an element $(p) \in$ FP/ $\sim$. Then the element $(p) \in$ FP/ $\sim$ is independent of the choices of $x, y \in X^n$.

LEMMA 6.1. *The element $(p) \in$ FP/ $\sim$ defined in the above is independent of the choices of $x, y \in X^n$.*

Proof. Let X be an $A-A$-equivalence bimodule and p the projection in E defined in the above. We will show that X is isomorphic to $(\mathbf{1} \otimes f_{11})Ep$ as left Hilbert A-modules, where $\{f_{ij}\}_{ij=1}^{n}$ is matrix units of M_n. For any $t \in X$ we regard t as an element $\{t, 0, \ldots, 0\} \in X^n$. Let ρ be the map of X to $(\mathbf{1} \otimes f_{11})Ep$ defined by $\rho(t) = {}_E\langle t, z \rangle$ for any $t \in X$. By using $\langle z, z \rangle_A = 1$ and routine computation, we can see that ρ is an isomorphism of X onto $(\mathbf{1} \otimes f_{11})Ep$. Therefore (p) is independent of the choices of $x, y \in X^n$. ∎

By the above lemma we can define a map $\overline{\mathcal{J}}$ from Pic(A) to FP/ $\sim$ by $\overline{\mathcal{J}}([X]) = (p)$ where p is a full projection in $A \otimes \mathbf{K}$ defined in the above way.

For any $\beta \in \mathrm{Aut}(A \otimes \mathbf{K})$ we can construct an $A-A$-equivalence bimodule X_β as follows: let X_β be the vector space defined by $X_\beta = (\mathbf{1} \otimes e_{00})(A \otimes \mathbf{K})\beta(\mathbf{1} \otimes e_{00})$. We define the obvious left action of A on X_β and the obvious A-valued inner product, but we define the right action of A on X_β defined by $x \cdot a = x\beta(a \otimes e_{00})$ for any $a \in A$ and $x \in X_\beta$ and the A-valued inner product by $\langle x, y \rangle_A = \beta^{-1}(x^* y)$ for any $x, y \in X_\beta$. By Brown, Green and Rieffel ([3]) the map $[\beta] \to [X_\beta]$ is an isomorphism of $\mathrm{Out}(A \otimes \mathbf{K})$ onto Pic(A). We denote by φ the map $[\beta] \to [X_\beta]$. And we define the map $\widetilde{\mathcal{J}}$ from $\mathrm{Out}(A \otimes \mathbf{K})$ onto FP/ $\sim$ by $\widetilde{\mathcal{J}}([\beta]) = \mathcal{J}([\beta])$ for any $\beta \in \mathrm{Aut}(A \otimes \mathbf{K})$.

PROPOSITION 6.2. *With the above notations* $\widetilde{\mathcal{J}} = \overline{\mathcal{J}} \circ \varphi$.

Proof. Let $\beta \in \mathrm{Aut}(A \otimes \mathbf{K})$. By the definition of $\widetilde{\mathcal{J}}$, $\widetilde{\mathcal{J}}([\beta]) = (\beta(\mathbf{1} \otimes e_{00}))$. And by the definition of $\overline{\mathcal{J}}$, $(\overline{\mathcal{J}} \circ \varphi)([\beta]) = \overline{\mathcal{J}}([X_\beta]) = (p)$ where p is a full projection in $A \otimes \mathbf{K}$ defined in the same way as in Rieffel ([11], Proposition 2.1). Then by Lemma 6.1, X_β is isomorphic to $(\mathbf{1} \otimes e_{00})(A \otimes \mathbf{K})p$ as left Hilbert A-modules. Thus $(\mathbf{1} \otimes e_{00})(A \otimes \mathbf{K})\beta(\mathbf{1} \otimes e_{00})$ is isomorphic to $(\mathbf{1} \otimes e_{00})(A \otimes \mathbf{K})p$ as left Hilbert A-modules. Hence $(p) = (\beta(\mathbf{1} \otimes e_{00}))$. Therefore we obtain the conclusion. ∎

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