J. OPERATOR THEORY
37(1997), 379–395

INVARIANT SUBSPACE THEOREMS FOR SUBDIAGONAL ALGEBRAS

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Communicated by William B. Arveson

ABSTRACT. We investigate a certain class of invariant subspaces of subdiagonal algebras which contains both two cases of (extended) weak*-Dirichlet algebras and analytic crossed products. We show a version of the Beurling–Lax–Halmos theorem.

KEYWORDS: *Invariant subspace, subdiagonal algebra.*

AMS SUBJECT CLASSIFICATION: Primary 46K50; Secondary 46J15.

1. INTRODUCTION

Classical Beurling–Lax theorem on invariant subspaces of H^2 in the unit disc (with an easy geometric proof due to Halmos) has several applications and has been extended in many directions. The aim of the paper is to provide another version of Beurling–Lax–Halmos theorem for general subdiagonal algebras.

Srinivasan and Wang ([12]) introduced weak*-Dirichlet algebras as an abstract function theory. It is known that any simply invariant subspace $\mathfrak{M}$ has the form $\mathfrak{M} = qH^2$ for $|q| = 1$ a.e. Most of important theorems for weak*-Dirichlet algebras are generalized to extended weak*-Dirichlet algebras by the first-named author in [9] and [10]. On the other hand Arveson ([1]) introduced the notion of subdiagonal algebras to unify several aspects of non-selfadjoint operator algebras. Subdiagonal algebras are regarded as the noncommutative analogue of weak*-Dirichlet algebras.

After the study of Kawamura–Tomiyama ([5]) and Loeble–Muhly ([6]) on subdiagonal algebras determined by flows on von Neumann algebras, McAsey–Muhly–Saito ([7], [8]) concentrated on the case of analytic crossed products to attack the invariant subspace problem. They finally showed that if the action is trivial on the center, then the Beurling–Lax–Halmos theorem is valid. Furthermore they proved a strong converse: if a (strong) form of the Beurling–Lax–Halmos theorem is valid, then it must be an analytic crossed product. Solel ([11]) compared two invariant subspaces. But their study excludes certain commutative cases of (extended) weak*-Dirichlet algebras.

In this paper we investigate invariant subspaces for general subdiagonal algebras to include both analytic crossed products and (extended) weak*-Dirichlet algebras. To avoid the strong converse mentioned above due to McAsey–Muhly–Saito, we introduce an invariant subspace of $\mathfrak{A}$-type I, which is a generalization of both a simply invariant subspace in a weak*-Dirichlet algebra and a pure invariant subspace in an analytic crossed product. We show a version of Beurling–Lax–Halmos theorem for invariant subspaces of $\mathfrak{A}$-type I. The notion of $\mathfrak{A}$-type I is not so restrictive because we also have a decomposition theorem into a part of $\mathfrak{A}$-type I in Theorem 2.14.

2. DECOMPOSITION

Let B be a finite von Neumann algebra with a (faithful normal normalized) trace τ. We recall the definition of a subdiagonal algebra due to Arveson ([1]). Let $\mathfrak{A}$ be a σ-weakly closed unital subalgebra of B, and let Φ be a faithful and normal conditional expectation from B onto $D = \mathfrak{A} \cap \mathfrak{A}^*$ such that $\tau(\Phi(x)) = \tau(x)$ for $x \in B$. Then $\mathfrak{A}$ is called a *maximal subdiagonal algebra* of B with respect to Φ if the following conditions are satisfied:

(1) $\mathfrak{A} + \mathfrak{A}^*$ is σ-weakly dense in B.

(2) $\Phi(xy) = \Phi(x)\Phi(y)$ for $x, y \in \mathfrak{A}$.

(3) $\mathfrak{A}$ is maximal among those subalgebras of B satisfying (1) and (2).

By a result [2] of Exel, a σ-weakly closed finite subdiagonal algebra is automatically maximal. So we may omit the condition (3) in our setting.

If B is abelian and $B = L^\infty(X, \mu)$, then a subdiagonal algebra $\mathfrak{A}$ is an exactly extended weak*-Dirichlet algebra introduced by the first named author in [10].

Let π (resp. ρ) be a left (resp. right) multiplication of B on $L^2(B, \tau)$ defined by

$$\pi(b)\eta(x) = \eta(bx) \quad \text{and} \quad \rho(b)\eta(x) = \eta(xb) \quad \text{for} \quad b, x \in B$$

where $\eta : B \longrightarrow L^2(B,\tau)$ is the canonical embedding. We sometimes omit the symbol η and π. The closure of $\mathfrak{A}$ in $L^2(B,\tau)$ is denoted by $H^2 = \overline{\eta(\mathfrak{A})}$ and the closure of $\mathfrak{A}_0 = \{a \in \mathfrak{A} \mid \Phi(a) = 0\}$ is denoted by $H_0^2 = \overline{\eta(\mathfrak{A}_0)}$. A closed subspace $\mathfrak{M}$ of $L^2(B,\tau)$ is (left) *$\mathfrak{A}$-invariant* if $\pi(\mathfrak{A})\mathfrak{M} \subset \mathfrak{M}$, *$\mathfrak{A}$-reducing* if $\mathfrak{A}$ and $\mathfrak{A}^*$-invariant, *$\mathfrak{A}$-pure* if $\mathfrak{M}$ contains no non-trivial $\mathfrak{A}$-reducing subspace, *$\mathfrak{A}$-full* if the smallest $\mathfrak{A}$-reducing subspace containing $\mathfrak{M}$ is $L^2(B,\tau)$.

DEFINITION 2.1. Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$. Then $\mathfrak{M}$ is called *simply invariant* if $[\pi(\mathfrak{A}_0)\mathfrak{M}]_2 \subsetneqq \mathfrak{M}$. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$. Then $\mathfrak{M}$ is called of *$\mathfrak{A}$-type* I if $[\pi(B)S]_2 = [\pi(B)\mathfrak{M}]_2$. Let $\mathfrak{M}^\wedge = [\pi(B)\mathfrak{M}]_2 \ominus \mathfrak{M}$. Then $\mathfrak{M}^\wedge$ is an $\mathfrak{A}^*$-invariant subspace. If $\mathfrak{M}^\wedge$ is of $\mathfrak{A}^*$-type I, then $\mathfrak{M}$ is called of *$\mathfrak{A}$-type* II. And $\mathfrak{A}$ is called of *$\mathfrak{A}$-type* III if $\mathfrak{M} = [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ and $\mathfrak{M}^\wedge = [\pi(\mathfrak{A}_0^*)\mathfrak{M}^\wedge]_2$.

EXAMPLE 2.2. In the above situation, H^2 is of $\mathfrak{A}$-type I. Many H_0^2 are of $\mathfrak{A}$-type II. For instance, if H_0^2 is $\mathfrak{A}$-full (in particular, if $D \subset \mathfrak{A}_0^*\mathfrak{A}_0$), then H_0^2 is of $\mathfrak{A}$-type II.

DEFINITION 2.3. Let W be a subspace of $L^2(B,\tau)$. Then W is called *left-wandering* if W and $\pi(\mathfrak{A}_0)W$ are orthogonal. In particular a vector $\xi \in L^2(B,\tau)$ is left-wandering in the sense of [1] if and only if the one dimensional subspace $[\xi]$ spanned by ξ is left-wandering in the above sense. If $\mathfrak{A}$ is the analytic crossed product in $B = D \rtimes_\alpha \mathbb{Z}$ as in [7], then W is left-wandering in our sense if and only if W is left-wandering in the sense of [7] or [11], i.e., $L_\delta^n W$ and $L_\delta^m W$ are orthogonal when $n \neq m$, where L_δ is the unitary in $B = D \rtimes_\alpha \mathbb{Z}$ implementing the automorphism α as in [7] and [11]. A left-wandering subspace W is *complete* if $L^2(B,\tau) = [\pi(B)W]_2$.

LEMMA 2.4. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ and $K = \mathfrak{M} \ominus [\pi(\mathfrak{A})S]_2$. Then we have the following:*

(i) *S is a left-wandering subspace.*

(ii) *$[\pi(B)S]_2$ and K are orthogonal.*

(iii) *The projection p of $L^2(B,\tau)$ onto S is in $\pi(D)'$.*

Proof. (i) Since $S \subset \mathfrak{M}$, it is trivial that $S \perp \pi(\mathfrak{A}_0)S$.

(ii) Since K and $[\pi(\mathfrak{A})S]_2$ are orthogonal and $\mathfrak{A}_0^* + \mathfrak{A}$ is σ-weakly dense in B, it is enough to show K and $\pi(\mathfrak{A}_0^*)S$ are orthogonal. For any $k \in K$, $a \in \mathfrak{A}_0$ and $s \in S$,

$$(k|\pi(a^*)s) = (\pi(a)k|s) = 0$$

since $\pi(a)k \in \pi(\mathfrak{A}_0)K \subset \pi(\mathfrak{A}_0)\mathfrak{M}$.

(iii) For $d \in D$, $s \in S$, $a \in \mathfrak{A}_0$ and $m \in \mathfrak{M}$,

$$(\pi(d)s|\pi(a)m) = (s|\pi(d^* a)m) = 0.$$

Since $\mathfrak{M}$ is D-invariant, S is D-invariant. ∎

DEFINITION 2.5. In the above setting, we shall call that $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ is the wandering subspace of $\mathfrak{M}$.

LEMMA 2.6. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ be the wandering subspace of $\mathfrak{M}$. Then the following conditions are equivalent:*

(i) *$\mathfrak{M}$ is of $\mathfrak{A}$-type* I *(i.e., $[\pi(B)\mathfrak{M}]_2 = [\pi(B)S]_2$);*

(ii) $\mathfrak{M} = [\pi(\mathfrak{A})S]_2$.

Proof. (ii) ⇒ (i): Suppose that $\mathfrak{M} = [\pi(\mathfrak{A})S]_2$. Then

$$[\pi(B)\mathfrak{M}]_2 = [\pi(B)\pi(\mathfrak{A})S]_2 = [\pi(B)S]_2$$

(i) ⇒ (ii): Suppose that $\mathfrak{M}$ is $\mathfrak{A}$-type I and $\mathfrak{M} \neq [\pi(\mathfrak{A})S]_2$. Then $K = \mathfrak{M} \ominus [\pi(\mathfrak{A})S]_2 \neq 0$. By Lemma 2.4, K and $[\pi(B)S]_2$ are orthogonal. Thus $[\pi(B)K]_2$ and $[\pi(B)S]_2$ are also orthogonal. Therefore

$$[\pi(B)\mathfrak{M}]_2 \ominus [\pi(B)S]_2 \supset [\pi(B)K]_2 \neq 0.$$

Thus $[\pi(B)\mathfrak{M}]_2 \neq [\pi(B)S]_2$. This is a contradiction. ∎

COROLLARY 2.7. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$. If $\mathfrak{M}$ is $\mathfrak{A}$-type* I *and non-zero, then $\mathfrak{M}$ is simply invariant.*

Proof. On the contrary, suppose that $\mathfrak{M}$ is not simply invariant. Then the wandering subspace S of $\mathfrak{M}$ is zero. Then by Lemma 2.6, $\mathfrak{M} = [\pi(\mathfrak{A})S]_2 = 0$. This is a contradiction. ∎

PROPOSITION 2.8. *Let D be a finite von Neumann algebra with a trace τ and α an automorphism on D with $\tau \circ \alpha = \tau$. Let $B = D \rtimes_\alpha \mathbf{Z}$ be the crossed product with the canonically extended trace τ and $\mathfrak{A} = D \rtimes_\alpha \mathbf{Z}_+$ be the analytic crossed product. Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$. Let L_δ be the unitary in B which implements α. Then the following conditions are equivalent:*

(i) *$\mathfrak{M}$ is pure.*

(ii) $\bigcap_{n \geqslant 0} {L_\delta}^n \mathfrak{M} = \{0\}$.

(iii) *$\mathfrak{M}$ is of $\mathfrak{A}$-type* I.

Proof. It is shown in [7] that (i) and (ii) are equivalent. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}] = \mathfrak{M} \ominus L_\delta \mathfrak{M}$ be the wandering subspace of $\mathfrak{M}$. Then we have that

$$\mathfrak{M} = \bigcap_{n\geqslant 0} L_\delta{}^n \mathfrak{M} \oplus \sum_{n=0}^{\infty}{}^{\oplus} L_\delta{}^n S.$$

Therefore $\bigcap_{n\geqslant 0} L_\delta{}^n\mathfrak{M} = \{0\}$ if and only if $\mathfrak{M} = \sum_{n=0}^{\infty}{}^{\oplus} L_\delta{}^n S$ if and only if $\mathfrak{M} = [\pi(\mathfrak{A})S]_2$. By Lemma 2.6, $\mathfrak{M} = [\pi(\mathfrak{A})S]_2$ is equivalent to that $\mathfrak{M}$ is $\mathfrak{A}$-type I. ∎

PROPOSITION 2.9. *Let $\mathfrak{A}$ be a weak*-Dirichlet algebra of $L^\infty(X,\mu)$ and $\mathfrak{M}$ an $\mathfrak{A}$-invariant subspace of $L^2(X,\mu)$. Then the following conditions are equivalent:*

(i) *$\mathfrak{M}$ is simply invariant.*

(ii) *$\mathfrak{M}$ is of $\mathfrak{A}$-type* I *and non-zero.*

(iii) *There exists a unimodular function $q \in L^\infty(X,\mu)$ such that*

$$\mathfrak{M} = qH^2.$$

Proof. It is clear that (iii) implies (ii). Corollary 2.7 shows that (ii) implies (i). It is a classical result that (i) implies (iii). ∎

LEMMA 2.10. *Let W be a left-wandering subspace of $L^2(B,\tau)$. Let W_1 and W_2 be the subspaces of W. Suppose that W_1 and W_2 are orthogonal and W_1 is D-invariant, then $\pi(B)W_1$ and $\pi(B)W_2$ are orthogonal.*

Proof. For $d \in D$, $x, y \in \mathfrak{A}_0$, $\xi_1 \in W_1$, $\xi_2 \in W_2$ we have

$$(\pi(x^* + d + y)\xi_1|\xi_2) = (\xi_1|\pi(x)(\xi)_2) + (\pi(d)\xi_1|\xi_2) + (\pi(y)\xi_1|\xi_2) = 0.$$

Since $\mathfrak{A}_0{}^* + D + \mathfrak{A}_0$ is σ-weakly dense in B,

$$(\pi(b)\xi_1|\xi_2) = 0 \quad \text{for} \quad b \in B,\ \xi_1 \in W_1,\ \xi_2 \in W_2.$$

Therefore for $b_1, b_2 \in B$

$$(\pi(b_1)\xi_1|\pi(b_2)\xi_2) = (\pi(b_2{}^*b_1)\xi_1|\xi_2) = 0.$$

Thus $\pi(B)W_1$ and $\pi(B)W_2$ are orthogonal. ∎

PROPOSITION 2.11. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ and $K = \mathfrak{M} \ominus [\pi(\mathfrak{A})S]_2$. Let p_1 (resp. $p_{\mathfrak{M}}$) be the projection of $L^2(B,\tau)$ onto $[\pi(B)S]$ (resp. $\mathfrak{M}$). Then we have the following:*

(i) $p_1 p_{\mathfrak{M}} = p_{\mathfrak{M}} p_1$ *and* $p_1 \in \pi(B)'$;

(ii) $p_1(\mathfrak{M}) = [\pi(\mathfrak{A})S]_2$;

(iii) $(1-p_1)\mathfrak{M} = K$;

(iv) $\mathfrak{M} = p_1\mathfrak{M} \oplus (1-p_1)\mathfrak{M}$;

(v) $p_1\mathfrak{M}$ *is an $\mathfrak{A}$-invariant subspace and of $\mathfrak{A}$-type* I;

(vi) $(1-p_1)\mathfrak{M} = K$ *is an $\mathfrak{A}$-invariant subspace such that* $K = [\pi(\mathfrak{A}_0)K]_2$.

Proof. Put $C = [\pi(B)S]_2 \ominus [\pi(\mathfrak{A})S]_2$. Then we have $[\pi(B)S]_2 = C \oplus [\pi(\mathfrak{A})S]_2$ and $\mathfrak{M} = K \oplus [\pi(\mathfrak{A})S]_2$. Since K and $C \subset [\pi(B)S]_2$ are orthogonal by Lemma 2.4, we have

$$p_1 p_{\mathfrak{M}} = p_{\mathfrak{M}} p_1, \quad p_1\mathfrak{M} = [\pi(\mathfrak{A})S]_2 \quad \text{and} \quad (1-p_1)\mathfrak{M} = K.$$

Furthermore $\mathfrak{M} = p_1\mathfrak{M} \oplus (1-p_1)\mathfrak{M}$. It is clear that $p_1\mathfrak{M} = [\pi(\mathfrak{A})S]_2$ is $\mathfrak{A}$-invariant and p_1 is in $\pi(B)'$. For $a \in \mathfrak{A}$

$$\pi(\mathfrak{A})K = \pi(a)(1-p_1)\mathfrak{M} = (1-p_1)\pi(a)\mathfrak{M} \subset (1-p_1)\mathfrak{M} = K.$$

Thus K is $\mathfrak{A}$-invariant. In particular $[\pi(\mathfrak{A}_0)K]_2 \subset K$. Since $\mathfrak{M} = [\pi(\mathfrak{A})S]_2 \oplus K = S \oplus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ and $[\pi(\mathfrak{A})S]_2 \supset S$, we have $K \subset [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$. Since $K = (1-p_1)\mathfrak{M}$, we have $(1-p_1)K = (1-p_1)\mathfrak{M}$. Therefore

$$\begin{aligned} K &= (1-p_1)\mathfrak{M} = (1-p_1)K \subset (1-p_1)[\pi(\mathfrak{A}_0)\mathfrak{M}]_2 \\ &\subset [(1-p_1)\pi(\mathfrak{A}_0)\mathfrak{M}]_2 = [\pi(\mathfrak{A}_0)(1-p_1)\mathfrak{M}]_2 = [\pi(\mathfrak{A}_0)K]_2. \end{aligned}$$

Thus $[\pi(\mathfrak{A}_0)K]_2 = K$. Finally we shall show that $p_1\mathfrak{M}$ is of $\mathfrak{A}$-type I. Since

$$p_1\mathfrak{M} = [\pi(\mathfrak{A})S]_2 = [S]_2 \oplus [\pi(\mathfrak{A}_0)S]_2 \qquad (\because S \perp [\pi(\mathfrak{A}_0)S]_2)$$

and

$$[\pi(\mathfrak{A}_0)p_1\mathfrak{M}]_2 = [\pi(\mathfrak{A}_0)\pi(\mathfrak{A})S]_2 = [\pi(\mathfrak{A}_0)S]_2,$$

we have $S^\sim \equiv p_1\mathfrak{M} \ominus [\pi(\mathfrak{A}_0)p_1\mathfrak{M}]_2 = S$. Therefore $[\pi(\mathfrak{A})S^\sim]_2 = [\pi(\mathfrak{A})S]_2 = p_1\mathfrak{M}$. This shows that $p_1\mathfrak{M}$ is of $\mathfrak{A}$-type I by Lemma 2.6. ∎

LEMMA 2.12. *Let $\mathcal{E}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$ and $\mathcal{F}$ a B-invariant subspace of $L^2(B,\tau)$. Let e (resp. f) be the projection of $L^2(B,\tau)$ onto $\mathcal{E}$ (resp. $\mathcal{F}$). If $ef = fe$, then the following hold:*

(i) *If $[\pi(\mathfrak{A}_0)\mathcal{E}]_2 = \mathcal{E}$, then $[\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F})]_2 = \mathcal{E}\cap\mathcal{F}$.*

(ii) *If $\mathcal{E}$ is of $\mathfrak{A}$-type* I, *then $\mathcal{E}\cap\mathcal{F}$ is also of $\mathfrak{A}$-type* I.

(iii) *$\mathcal{E}\cap\mathcal{F} = f\mathcal{E}$ and $(\mathcal{E}\cap\mathcal{F})\ominus[(\pi(\mathfrak{A}_0))(\mathcal{E}\cap\mathcal{F})]_2 = f(\mathcal{E}\ominus[\pi(\mathfrak{A}_0)\mathcal{E}]_2)$.*

Proof. Since $ef = fe$, $\mathcal{E} = (\mathcal{E}\cap\mathcal{F})\oplus(\mathcal{E}\cap\mathcal{F}^\perp)$ and $\mathcal{E}\cap\mathcal{F} = f\mathcal{E}$. By assumption $\mathcal{E}$ is $\mathfrak{A}$-invariant and $\mathcal{F}$ is $\mathfrak{A}$-reducing. Therefore we have

$$[\pi(\mathfrak{A}_0)\mathcal{E}]_2 = [\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F})]_2 \oplus [\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F}^\perp)]_2,$$

$$[\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F})]_2 \subset \mathcal{E}\cap\mathcal{F} \quad \text{and} \quad [\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F}^\perp)]_2 \subset \mathcal{E}\cap\mathcal{F}^\perp.$$

Thus we have

$$\mathcal{E}\ominus[\pi(\mathfrak{A}_0)\mathcal{E}]_2 = ((\mathcal{E}\cap\mathcal{F})\ominus[\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F})]_2)\oplus((\mathcal{E}\cap\mathcal{F}^\perp)\ominus[\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F}^\perp)]_2).$$

Hence we have

$$\begin{aligned}&[\pi(\mathfrak{A})(\mathcal{E}\ominus[\pi(\mathfrak{A}_0)\mathcal{E}]_2)]_2\\ &\quad= [\pi(\mathfrak{A})((\mathcal{E}\cap\mathcal{F})\ominus[\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F})]_2)]_2 \oplus (\pi(\mathfrak{A})((\mathcal{E}\cap\mathcal{F}^\perp)\ominus[\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F}^\perp)]_2)).\end{aligned}$$

By these consideration we have the following:

(i) If $[\pi(\mathfrak{A}_0)\mathcal{E}]_2 = \mathcal{E}$, then $[\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F})]_2 = \mathcal{E}\cap\mathcal{F}$.

(ii) Suppose that $\mathcal{E}$ is $\mathfrak{A}$-type I, then by Lemma 2.6 we have

$$\mathcal{E} = [\pi(\mathfrak{A})(\mathcal{E}\ominus[\pi(\mathfrak{A}_0)\mathcal{E}]_2)]_2.$$

Therefore

$$\mathcal{E}\cap\mathcal{F} = [\pi(\mathfrak{A})((\mathcal{E}\cap\mathcal{F})\ominus[\pi(\mathfrak{A}_0)(\mathcal{E}\cap\mathcal{F})]_2)]_2.$$

This shows that $\mathcal{E}\cap\mathcal{F}$ is also $\mathfrak{A}$-type I.

(iii) It is easily observed. ∎

PROPOSITION 2.13. *Let $\mathfrak{A}$ be an extended weak*-Dirichlet algebra of $B = L^\infty(X,\mu)$ and $\mathfrak{M}$ an $\mathfrak{A}$-invariant subspace of $L^2(X,\mu)$. Then the following conditions are equivalent:*

(i) *$\mathfrak{M}$ is of type* I *in the sense of* [10], *i.e., for every nonzero projection $\chi_E \in D$ with $\chi_E\mathfrak{M} \neq 0$*

$$\chi_E\mathfrak{M} \supsetneq \chi_E[\pi(\mathfrak{A}_0)\mathfrak{M}]_2.$$

(ii) *$\mathfrak{M}$ is nonzero and of $\mathfrak{A}$-type* I *in the sense of this paper.*

(iii) *There exist a unimodular function* $q \in B = L^\infty(X,\mu)$ *and a nonzero projection* $\chi_E \in D$ *such that*

$$\mathfrak{M} = \chi_E q H^2.$$

Proof. (iii) $\Rightarrow$ (ii): Since qH^2 is of $\mathfrak{A}$-type I, $\mathfrak{M} = \chi_E q H^2$ is also of $\mathfrak{A}$-type I by applying the above Lemma 2.12 for $\mathfrak{F} = \chi_E L^2(X,\mu)$. Since $\chi_E \in \chi_E H^2$, $\mathfrak{M} = q\chi_E H^2 \neq 0$.

(ii) $\Rightarrow$ (i): Let $\mathfrak{M}$ be of $\mathfrak{A}$-type I. Then for every non-zero projection $\chi_E \in D$ with $\chi_E\mathfrak{M} \neq 0$, $\chi_E\mathfrak{M}$ is of $\mathfrak{A}$-type I by the above Lemma 2.12. Hence $\chi_E\mathfrak{M}$ is simply invariant by Corollary 2.7. Thus

$$\chi_E\mathfrak{M} \supsetneqq [\pi(\mathfrak{A}_0)\chi_E\mathfrak{M}]_2 = \chi_E[\pi(\mathfrak{A}_0)\mathfrak{M}]_2.$$

(i) $\Rightarrow$ (iii): This is shown in [10]. ∎

The following Theorem 2.14 is a generalization of a decomposition theorem of an invariant subspace for (extended) weak*-Dirichlet algebras studied in [9] and [10].

THEOREM 2.14. *Let* B *be a finite von Neumann algebra with a finite trace* τ *and* $\mathfrak{A}$ *a maximal subdiagonal algebra. Let* $\mathfrak{M}$ *be an* $\mathfrak{A}$*-invariant subspace of* $L^2(B,\tau)$ *and* $p_{\mathfrak{M}}$ *the projection of* $L^2(B,\tau)$ *onto* $\mathfrak{M}$*. Then there exist projections* $p_1, p_2, p_3 \in \pi(B)'$ *with* $p_1 + p_2 + p_3 = 1$ *and* $p_i p_{\mathfrak{M}} = p_{\mathfrak{M}} p_i$ *for* $i = 1,2,3$ *such that:*

(i) $\mathfrak{M} = p_1\mathfrak{M} \oplus p_2\mathfrak{M} \oplus p_3\mathfrak{M}$;

(ii) $p_1\mathfrak{M}$ *is an* $\mathfrak{A}$*-invariant subspace of* $\mathfrak{A}$*-type* I;

(iii) $p_2\mathfrak{M}$ *is an* $\mathfrak{A}$*-invariant subspace of* $\mathfrak{A}$*-type* II;

(iv) $p_3\mathfrak{M}$ *is an* $\mathfrak{A}$*-invariant subspace of* $\mathfrak{A}$*-type* III;

(v) $p_2\mathfrak{M}$ *contains no nonzero invariant subspace* $\mathfrak{V}$ *of* $\mathfrak{A}$*-type* I *of the form* $\mathfrak{V} = g\mathfrak{M}$ *for some projection* $g \in \pi(B)'$ *with* $p_{\mathfrak{M}}g = gp_{\mathfrak{M}}$.

Proof. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ and $K = \mathfrak{M} \ominus [\pi(\mathfrak{A})S]_2$. Let p_1 be the projection of $L^2(B,\tau)$ onto $[\pi(B)S]_2$. Then $p_1 \in \pi(B)'$, $p_1 p_{\mathfrak{M}} = p_{\mathfrak{M}} p_1$ and $p_1\mathfrak{M} = [\pi(\mathfrak{A})S]_2$ is an $\mathfrak{A}$-invariant subspace of $\mathfrak{A}$-type I by Proposition 2.11. Let

$$\mathfrak{N} = [\pi(B)\mathfrak{M}]_2 \ominus ([\pi(B)S]_2 \oplus K).$$

Since $K = (1-p_1)\mathfrak{M}$, $[\pi(\mathfrak{A})S]_2 = p_1\mathfrak{M}$ and $\mathfrak{M} = p_1\mathfrak{M} \oplus (1-p_1)\mathfrak{M}$ by Proposition 2.11, we have

$$\mathfrak{N} = ((1-p_1)[\pi(B)\mathfrak{M}]_2) \ominus K = (1-p_1)([\pi(B)\mathfrak{M}]_2 \ominus \mathfrak{M}).$$

Then $\mathfrak{N}$ is $\mathfrak{A}^*$-invariant because $[\pi(B)\mathfrak{M}]_2$ is B-invariant and $[\pi(B)S]_2 \oplus K$ is $\mathfrak{A}$-invariant by Proposition 2.11. Let $T = \mathfrak{N} \ominus [\pi(\mathfrak{A}_0{}^*)\mathfrak{N}]_2$ and $L = \mathfrak{N} \ominus [\pi(\mathfrak{A}^*)T]_2$. See Figure 1.

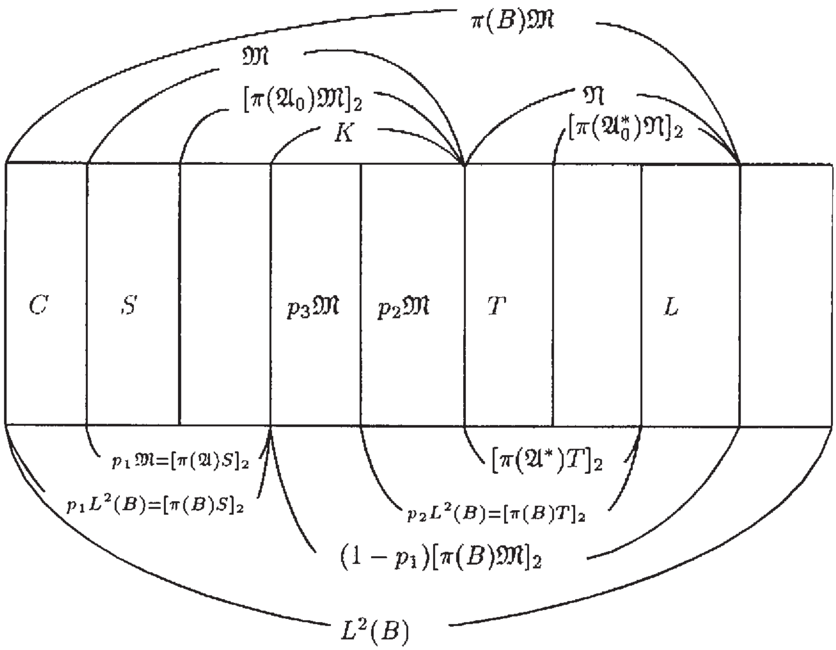

Figure 1.

We shall show that the subspaces L, $[\pi(B)T]_2$ and $[\pi(B)S]_2$ are orthogonal each other. Applying Lemma 2.4 to $\mathfrak{N}$ and $\mathfrak{A}^*$ instead of $\mathfrak{M}$ and $\mathfrak{A}$, we have that $[\pi(B)T]_2$ and L are orthogonal. Since $\mathfrak{N}$ and $[\pi(B)S]_2$ are orthogonal and $L \subset \mathfrak{N}$, L and $[\pi(B)S]_2$ are orthogonal. For $x \in S$, $t \in T$, $x \in B$ and $y \in B$, we have

$$(\pi(x)s|\pi(y)t) = (\pi(y^*x)s|t) = 0$$

because $\pi(y^*x)s \in [\pi(B)S]_2$, $t \in T \subset \mathfrak{N}$, $[\pi(B)S]$ and T are orthogonal. Thus $[\pi(B)S]_2$ and $[\pi(B)T]_2$ are orthogonal.

Let p_2 (resp. $p_{\mathfrak{N}}$) be the projection of $L^2(B,\tau)$ onto $[\pi(B)T]_2$ (resp. $\mathfrak{N}$). Then applying Proposition 2.11 to $\mathfrak{N}$ and $\mathfrak{A}^*$ instead of $\mathfrak{M}$ and $\mathfrak{A}$, we have that $p_2 \in \pi(B)'$,

$$p_2 p_{\mathfrak{N}} = p_{\mathfrak{N}} p_2, \quad p_2\mathfrak{N} = [\pi(\mathfrak{A}^*)T]_2 = [\pi(B)T] \cap \mathfrak{N},$$

$(1-p_2)\mathfrak{N} = L$, $[\pi(\mathfrak{A}^*)T]_2$ is an $\mathfrak{A}^*$-invariant subspace of $\mathfrak{A}^*$-type I, and L is an $\mathfrak{A}^*$-invariant subspace such that $L = [\pi(\mathfrak{A}_0^*)L]_2$. Put $C = [\pi(B)S]_2 \ominus [\pi(\mathfrak{A})S]_2$. Let p_C (resp. r) be the projection of $L^2(B,\tau)$ onto C (resp. $[\pi(B)\mathfrak{M}]_2$). Since

$$[\pi(B)\mathfrak{M}]_2 = [\pi(B)S]_2 \oplus K \oplus \mathfrak{N} = C \oplus [\pi(\mathfrak{A})S]_2 \oplus K \oplus \mathfrak{N} = C \oplus \mathfrak{M} \oplus \mathfrak{N}$$

we have $r = p_C + p_{\mathfrak{M}} + p_{\mathfrak{N}}$. Since $[\pi(B)T]_2 \subset [\pi(B)\mathfrak{M}]_2$, $p_2 r = r p_2 = p_2$. Because $[\pi(B)S]_2$ and $[\pi(B)T]_2$ are orthogonal and $C \subset [\pi(B)S]_2$, C and $[\pi(B)T]_2$ are orthogonal and $p_C p_2 = p_2 p_C = 0$. Therefore p_2 commutes with $p_{\mathfrak{M}} = r - p_C - p_{\mathfrak{N}}$. Thus

$$p_2\mathfrak{M} = \mathfrak{M} \cap [\pi(B)T]_2.$$

Since $\mathfrak{M} = [\pi(\mathfrak{A})S]_2 \oplus K$ and that $[\pi(B)S]_2$ ($\supset [\pi(\mathfrak{A})S]_2$) and $[\pi(B)T]_2$ are orthogonal, we have $p_2\mathfrak{M} \subset K$. It is clear that $p_2\mathfrak{M} = \mathfrak{M} \cap [\pi(B)T]_2$ is an $\mathfrak{A}$-invariant subspace. We shall show that $p_2\mathfrak{M}$ is of $\mathfrak{A}$-type II. Put $\mathfrak{M}_2 = p_2\mathfrak{M} \subset [\pi(B)T]_2$ and $\mathfrak{M}_2^\wedge = [\pi(B)\mathfrak{M}_2] \ominus \mathfrak{M}_2$. We have

$$p_2 = p_2 r = p_2(p_C + p_{\mathfrak{M}} + p_{\mathfrak{N}}) = p_2 p_{\mathfrak{M}} + p_2 p_{\mathfrak{N}}.$$

Therefore $[\pi(B)T]_2 = \mathfrak{M}_2 \oplus [\pi(\mathfrak{A}^*)T]_2$. Thus $[\pi(B)\mathfrak{M}_2]_2 \subset [\pi(B)T]_2$. Then

$$\begin{aligned}\mathfrak{M}_2^\wedge &= [\pi(B)\mathfrak{M}_2]_2 \ominus \mathfrak{M}_2 = [\pi(B)\mathfrak{M}_2]_2 \cap \mathfrak{M}_2^\perp \\ &= [\pi(B)\mathfrak{M}_2]_2 \cap ([\pi(B)T]_2 \cap \mathfrak{M}_2^\perp) = [\pi(B)\mathfrak{M}_2]_2 \cap [\pi(\mathfrak{A}^*)T]_2.\end{aligned}$$

Thus

$$\mathfrak{M}_2^\wedge = [\pi(B)\mathfrak{M}_2]_2 \cap [\pi(\mathfrak{A}^*)T]_2$$

and

$$[\pi(B)\mathfrak{M}_2]_2 = \mathfrak{M}_2 \oplus \mathfrak{M}_2^\wedge \subset \mathfrak{M}_2 \oplus [\pi(\mathfrak{A}^*)T]_2.$$

Recall that $[\pi(\mathfrak{A}^*)T]_2$ is of $\mathfrak{A}^*$-type I. The projection of $L^2(B,\tau)$ onto $[\pi(\mathfrak{A}^*)T]_2$ is $p_2 p_{\mathfrak{N}}$. Let $\mathfrak{F} = [\pi(B)\mathfrak{M}_2]_2$ and f be the projection of $L^2(B,\tau)$ onto $\mathfrak{F}$. Then $(p_2 p_{\mathfrak{N}})f = f(p_2 p_{\mathfrak{N}})$. Applying Lemma 2.12, we have that $\mathfrak{M}_2^\wedge = [\pi(\mathfrak{A}^*)T]_2 \cap [\pi(B)\mathfrak{M}_2]$ is $\mathfrak{A}^*$-type I.

Finally put $p_3 = I - p_1 - p_2$. Then $p_3 \in \pi(B)'$ and $p_3 p_{\mathfrak{M}} = p_{\mathfrak{M}} p_3$. Put $\mathfrak{M}_3 = p_3\mathfrak{M} = \mathfrak{M} \cap p_3 L^2(B,\tau)$. Then

$$\mathfrak{M}_3 = \mathfrak{M} \cap (p_1\mathfrak{M})^\perp \cap (p_2\mathfrak{M})^\perp = \mathfrak{M} \cap [\pi(\mathfrak{A})S]_2^\perp \cap \mathfrak{M}_2^\perp = K \cap \mathfrak{M}_2^\perp.$$

Since $\mathfrak{M}_2 \subset K$, we have $\mathfrak{M}_3 = K \ominus \mathfrak{M}_2$. Let p_K be the projection of $L^2(B,\tau)$ onto K. Then $p_K = p_2 + p_3 p_{\mathfrak{M}}$. Let $\mathfrak{F}' = [\pi(B)\mathfrak{M}]_2 \ominus [\pi(B)T]_2$ and f' be the projection of $L^2(B,\tau)$ onto $\mathfrak{F}'$. Let ℓ be the projection of $L^2(B,\tau)$ onto L. Then by Figure 1 we have

$$\pi(B)' \ni f' = r - p_2 = p_1 + p_2 + p_3 p_{\mathfrak{M}} + \ell - p_2 = p_1 + p_3 p_{\mathfrak{M}} + \ell,$$

$p_K f' = f' p_K = p_3 p_{\mathfrak{M}}$ and $\mathfrak{M}_3 = K \cap \mathfrak{F}'$. Since $K = [\pi(\mathfrak{A}_0)K]_2$ by Proposition 2.11, we have

$$\mathfrak{M}_3 = [\pi(\mathfrak{A}_0)\mathfrak{M}_3]_2$$

by Lemma 2.12. Let $\mathfrak{M}_3^\wedge = [\pi(B)\mathfrak{M}_3]_2 \ominus \mathfrak{M}_3$. Let $\mathfrak{F}^\flat = [\pi(B)\mathfrak{M}_3]_2$ and $f^\flat$ be the projection of $L^2(B,\tau)$ onto $\mathfrak{F}^\flat$. Recall that $L = \mathfrak{N} \ominus [\pi(\mathfrak{A}^*)T]_2$ and $L = [\pi(\mathfrak{A}_0^*)L]_2$. We have that $[\pi(B)\mathfrak{M}_3]_2 \subset \mathfrak{M}_3 \oplus L$, because $[\pi(B)\mathfrak{M}]_2 = [\pi(B)S] \oplus \mathfrak{M}_3 \oplus [\pi(B)T]_2 \oplus L$. Thus

$$\mathfrak{M}_3^\wedge = [\pi(B)\mathfrak{M}_3]_2 \ominus \mathfrak{M}_3 = L \cap [\pi(B)\mathfrak{M}_3]_2 = L \cap \mathfrak{F}^\flat.$$

Since $f^\flat \in \pi(B)'$ and $f^\flat \ell = \ell f^\flat$, we have

$$\mathfrak{M}_3^\wedge = [\pi(\mathfrak{A}_0^*)\mathfrak{M}_3^\wedge]_2$$

by Lemma 2.12. Thus we have shown that $\mathfrak{M}_3 = p_3\mathfrak{M}$ is of $\mathfrak{A}$-type III.

Let $\mathfrak{V}$ be an invariant subspace of $p_2\mathfrak{M}$ of $\mathfrak{A}$-type I of the form $\mathfrak{V} = g\mathfrak{M}$ for some projection $g \in \pi(B)'$ with $p_{\mathfrak{M}} g = g p_{\mathfrak{M}}$. Let $\mathfrak{G}$ be the range of g. By Lemma 2.12, $\mathfrak{V} = g\mathfrak{M} = \mathfrak{M} \cap \mathfrak{G}$ and

$$\mathfrak{V} \ominus [\pi(\mathfrak{A}_0)\mathfrak{V}]_2 = g(\mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2) \doteq gS.$$

Since $g\mathfrak{M} \subset \mathfrak{M}$ and $g\pi(\mathfrak{A}_0)\mathfrak{M} \subset \pi(\mathfrak{A}_0)g\mathfrak{M} \subset \pi(\mathfrak{A}_0)\mathfrak{M}$, we have $gS \subset S$. Therefore

$$\mathfrak{V} = [\pi(\mathfrak{A})gS]_2 \subset [\pi(\mathfrak{A})S]_2 = p_1\mathfrak{M}.$$

Hence $\mathfrak{V} \subset p_1\mathfrak{M} \cap p_2\mathfrak{M}$ and $\mathfrak{V} = 0$. ∎

EXAMPLE 2.15. Let D be a finite von Neumann algebra with a trace τ and α an automorphism on D with $\tau \circ \alpha = \tau$. Let $B = D \rtimes_\alpha \mathbb{Z}$ be the crossed product with the canonically extended trace τ and $\mathfrak{A} = D \rtimes_\alpha \mathbb{Z}_+$ be the analytic crossed product. Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$. Let L_δ be the unitary in B which implements α. Then we have that

$$\mathfrak{M} = \bigcap_{n \geqslant 0} L_\delta{}^n \mathfrak{M} \oplus \sum_{n=0}^{\infty}{}^{\oplus} L_\delta{}^n S.$$

Using the notation of the above decomposition theorem, we have that $p_1\mathfrak{M} = \sum_{n=0}^{\infty}{}^{\oplus} L_\delta{}^n S$, $p_2\mathfrak{M} = 0$ and $p_3\mathfrak{M} = \bigcap_{n \geqslant 0} L_\delta{}^n S$.

EXAMPLE 2.16. Let D be a type II_1 factor and $\alpha : \mathbb{Z}^2 \to \mathrm{Aut}\, D$ an outer action. Let $B = D \rtimes_\alpha \mathbb{Z}^2$ be the crossed product. Fix an irrational positive number θ and consider the positive cones

$$P_\theta = \{(m,n) \in \mathbb{Z}^2; \theta m + n \geqq 0\}.$$

Let λ_g be the unitary B which implements α_g for $g \in \mathbb{Z}^2$. Let $\mathfrak{A}$ be the σ-weak closure of the set of all finite sums $\sum_{g \in P_\theta} x_g \lambda_g$, where $x_g \in D$ and $x_g = 0$ except for finitely many $g \in P_\theta$. Let $\mathfrak{M} = H_0^2$. Then H_0^2 is of $\mathfrak{A}$-type II. Since the wandering subspace $S = 0$, we have $p_1\mathfrak{M} = 0$, $p_2\mathfrak{M} = H_0^2$ and $p_3\mathfrak{M} = 0$.

EXAMPLE 2.17. We can also construct several examples such that none of $p_i\mathfrak{M}$ $(i = 1,2,3)$ are zero in the decomposition of the above Theorem 2.14. For example consider the direct sum of the subdiagonal algebras and invariant subspaces of the just above two examples. Helson–Lowdenslanger ([4]) give an $\mathfrak{A}$-invariant subspace of $\mathfrak{A}$-type III which is not $\mathfrak{A}$-reducing.

3. INVARIANT SUBSPACES

Loebl–Muhly ([6]) and Kawamura–Tomiyama ([5]) investigate maximal subdiagonal algebras determined by flows in von Neumann algebras. Later the study on invariant subspaces is focused on analytic crossed product $\mathfrak{A} = D \rtimes_\alpha \mathbb{Z}_+$ and Mcasey, Muhly and Saito ([7]) determined when a version of the Beurling–Lax–Halmos theorem (abbreviated the BLH theorem) is valid for $\mathfrak{A} = D \rtimes_\alpha \mathbb{Z}_+$. But their situation excludes certain commutative cases as in [10] and [12].

The aim of the section is to unify both the cases of analytic crossed products and (extended) weak*-Dirichlet algebras.

The following Lemma 3.1 is a generalization of Lemma 4.2.2 in Arveson ([1]) on wandering vectors and is a key lemma to prove our main theorem.

LEMMA 3.1. *For $i = 1, 2$, let K_i be a D-invariant subspace of $L^2(B,\tau)$ and q_i be the projection of $L^2(B,\tau)$ onto K_i. Assume that K_1 and K_2 are left-wandering. Suppose that there exists a partial isometry $w \in \pi(D)'$ such that $w^*w = q_1$ and $ww^* = q_2$. Then there exists a partial isometry $v \in \pi(B)'$ satisfying the following:*

(i) *v^*v is the projection onto $[\pi(B)K_1]_2$ and vv^* is a projection onto $[\pi(B)K_2]_2$;*

(ii) *$v\pi(b)\xi = \pi(b)w\xi$ for all $\xi \in K_1$ and $b \in B$;*

(iii) $v([\pi(\mathfrak{A})K_1]_2) = [\pi(\mathfrak{A})K_2]_2$.

Furthermore there exists a unitary $u \in \pi(B)'$ *such that* $u\xi = v\xi$ *for* $\xi \in [\pi(B)K_1]_2$.

Proof. First we shall show that

$$(3.1) \qquad (\pi(b)\xi_1|\xi_2) = (\pi(b)w\xi_1|w\xi_2) \quad \text{for} \quad b \in B,\ \xi_1, \xi_2 \in K_1.$$

Since $\mathfrak{A}_0^* + D + \mathfrak{A}_0$ is σ-weakly dense in B, it is enough to consider three cases: (I) $b \in D$, (II) $b \in \mathfrak{A}_0$, and (III) $b \in \mathfrak{A}_0^*$.

(I) Suppose that $b \in D$. Since $\pi(b)K_1 \subset K_1$, $w \in \pi(D)'$ and w is a partial isometry whose support is K_1, we have

$$(\pi(b)w\xi_1|w\xi_2) = (w\pi(b)\xi_1|w\xi_2) = (\pi(b)\xi_1|\xi_2).$$

(II) Suppose that $b \in \mathfrak{A}_0$. Since K_1 and K_2 are left-wandering, K_i and $\pi(\mathfrak{A}_0)K_i$ are orthogonal for $i = 1, 2$. We have $\xi_1, \xi_2 \in K_1$ and $w\xi_1, w\xi_2 \in K_2$. Therefore

$$(\pi(b)\xi_1|\xi_2) = (\pi(b)w\xi_1|w\xi_2).$$

(III) Suppose that $b \in \mathfrak{A}_0^*$. Since K_i and $\pi(\mathfrak{A}_0)K_i$ are orthogonal, $\pi(\mathfrak{A}_0^*)K_i$ and K_i are orthogonal for $i = 1, 2$. Therefore

$$(\pi(b)\xi_1|\xi_2) = 0 = (\pi(b)w\xi_1|w\xi_2).$$

Thus we have proved (3.1).

For $b_i \in B$ and $\xi_i \in K_1$ $(i = 1, 2, \ldots, n)$, using (3.1), we have

$$\begin{aligned}\left\|\sum_{i=1}^{n} \pi(b_i)\xi_i\right\|_2^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}(\pi(b_i)\xi_i|\pi(b_j)\xi_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\pi(b_j^*b_i)\xi_i|\xi_j)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}(\pi(b_j^*b_i)w\xi_i|w\xi_j) = \left\|\sum_{i=1}^{n}\pi(b_i)w\xi_i\right\|_2^2.\end{aligned}$$

Therefore there exists a partial isometry $v \in \mathcal{L}(L^2(B), \tau)$ such that

$$v\pi(b)\xi = \pi(b)w\xi \quad \text{for} \quad b \in B,\ \xi \in K_1$$

and $v\eta = 0$ for $\eta \in [\pi(B)K_1]_2^{\perp}$. Thus (i) and (ii) are proved.

We shall show that $v \in \pi(B)'$. Take $x \in B$. For $b \in B$, $\xi \in K_1$,

$$v\pi(x)\pi(b)\xi = v\pi(xb)\xi = \pi(xb)w\xi = \pi(x)\pi(b)w\xi = \pi(x)v\pi(b)\xi.$$

For $b \in B$, $\eta \in [\pi(B)K_1]_2^{\perp}$, we have $\pi(b)\eta \in [\pi(B)K_1]_2^{\perp}$. Therefore

$$v\pi(x)\eta = 0 = \pi(x)v\eta.$$

Thus $v\pi(x) = \pi(x)v$ for any $x \in B$, that is, $v \in \pi(B)'$. Since $wK_1 = K_2$, we have

$$v[\pi(\mathfrak{A})K_1]_2 = [\pi(\mathfrak{A})wK_1]_2 = [\pi(\mathfrak{A})K_2]_2.$$

Furthermore, since $\pi(B)'$ is also a finite von Neumann algebra, $v \in \pi(B)'$ can be extended to a unitary u in $\pi(B)'$. ∎

The following Lemma 3.2 is an extension of Theorem 3.2 in [7].

LEMMA 3.2. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be $\mathfrak{A}$-invariant subspaces of $\mathfrak{A}$-type* I. *Let $S_i = \mathfrak{M}_i \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}_i]_2$ be the left-wandering subspaces of $\mathfrak{M}_i$ and $p_i \in \pi(D)'$ be the projections of $L^2(B,\tau)$ onto S_i for $i = 1, 2$. If $p_2 \preceq p_1$ in $\pi(D)'$, then there exists a partial isometry $v \in \pi(B)'$ such that $v\mathfrak{M}_1 = \mathfrak{M}_2$.*

Proof. Since $p_2 \preceq p_1$, there exists a partial isometry $w \in \pi(D)'$ such that $ww^* = p_2$ and $w^*w \leqq p_1$. Put $p_1' = w^*w \in \pi(D)'$ and let S_1' be the range of p_1'. Then

$$\pi(D)S_1' = \pi(D)p_1'S_1 = p_1'\pi(D)S_1 \subset p_1'S_1 = S_1'.$$

Since $\pi(\mathfrak{A}_0)S_1' \subset \pi(\mathfrak{A}_0)S_1$, $S_1' \subset S_1$ and $\pi(\mathfrak{A}_0)S_1$ is orthogonal to S_1, we have that $\pi(\mathfrak{A}_0)S_1'$ and S_1' are orthogonal. Thus S_1' is also D-invariant and left-wandering. By Lemma 3.1, there exists a partial isometry $v \in \pi(B)'$ such that $v([\pi(\mathfrak{A})S_1']_2) = [\pi(\mathfrak{A})S_2]_2$. Furthermore the support of v is $[\pi(B)S_1']_2$ and the range of v is $[\pi(B)S_2]_2$. Let $S_1^\circ = S_1 \ominus S_1'$. Then S_1° is also a D-invariant and left-wandering. Therefore $\pi(B)S_1'$ and $\pi(B)S_1^\circ$ are orthogonal by Lemma 2.10. Thus $[\pi(B)S_1]_2 = [\pi(B)S_1']_2 \oplus [\pi(B)S_1^\circ]$. Hence

$$v[(\pi(\mathfrak{A})S_1)]_2 = v[\pi(\mathfrak{A})S_1']_2.$$

Since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are of $\mathfrak{A}$-type I,

$$v\mathfrak{M}_1 = v[\pi(\mathfrak{A})S_1]_2 = v[\pi(\mathfrak{A})S_1']_2 = [\pi(\mathfrak{A})S_2]_2 = \mathfrak{M}_2. \quad ∎$$

LEMMA 3.3. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant subspace and $u \in \pi(B)'$ a partial isometry such that $\mathfrak{M} = uH^2$. Then the following are equivalent:*

(i) *$\mathfrak{M}$ is full;*

(ii) *u is a unitary.*

Proof. (i) ⇒ (ii): Suppose that $\mathfrak{M}$ is full. Then

$$uL^2(B) = u[\pi(B)H^2]_2 = [\pi(B)uH^2]_2 = [\pi(B)\mathfrak{M}]_2 = L^2(B).$$

Hence u is a co-isometry in $\pi(B)'$. Since $\pi(B)'$ is finite, u is in fact a unitary.

(ii) ⇒ (i): Suppose that u is a unitary. Then

$$[\pi(B)\mathfrak{M}]_2 = [\pi(B)uH^2]_2 = [u\pi(B)H^2]_2 = u[\pi(B)H^2]_2 = uL^2(B) = L^2(B).$$

Thus $\mathfrak{M}$ is full. ∎

We have a version of the Beurling–Lax–Halmos theorem for subdiagonal algebras.

THEOREM 3.4. *Let B be a finite von Neumann algebra with a trace τ, D a von Neumann subalgebra of B and $\Phi : B \to D$ a (faithful normal) conditional expectation with $\tau \circ \Phi = \tau$. Let $\mathfrak{A}$ be a maximal subdiagonal algebras with respect to Φ. Suppose that the center $Z(B)$ of B contains the center $Z(D)$ of D. Let $\mathfrak{M}$ be a $\mathfrak{A}$-invariant subspace of $L^2(B,\tau)$ and of $\mathfrak{A}$-type* I. *Then there exists a partial isometry $v \in \pi(B)'$ such that*

$$\mathfrak{M} = vH^2.$$

Proof. Let $S = \mathfrak{M} \ominus [\pi(\mathfrak{A}_0)\mathfrak{M}]_2$ be the left-wandering subspace of $\mathfrak{M}$. Recall that $L^2(D)$ is the left-wandering subspace of H^2. Let e_D (resp. e_S) be the projection of $L^2(B)$ onto $L^2(D)$ (resp. S). Then e_D and e_S are in $\pi(D)'$. By the comparison theorem, there exists a projection $q \in Z(D)$ such that

$$\pi(q)e_D \preceq \pi(q)e_S \quad \text{and} \quad \pi(1-q)e_S \preceq \pi(1-q)e_D.$$

By the assumption, we have $q \in Z(D) \subset Z(B)$. Since $\mathfrak{M}$ and H^2 are of $\mathfrak{A}$-type I, $\pi(q)\mathfrak{M} = \mathfrak{M} \cap (\pi(q)L^2(B))$ and $\pi(q)H^2 = H^2 \cap (\pi(q)L^2(B))$ are $\mathfrak{A}$-invariant subspaces of $\mathfrak{A}$-type I by Lemma 2.12. And the projection of $L^2(B)$ onto the left-wandering subspace of $\pi(q)\mathfrak{M}$ (resp. $\pi(q)H^2$) is $\pi(q)e_S$ (resp. $\pi(q)e_D$) by Lemma 2.12. Since $\pi(q)e_D \preceq \pi(q)e_S$ in $\pi(D)'$, there exists a partial isometry $v_1 \in \pi(B)'$ such that $\pi(q)H^2 = v_1\pi(q)\mathfrak{M}$ by Lemma 3.2. Similarly there exists a partial isometry $v_2 \in \pi(B)'$ such that $\pi(1-q)\mathfrak{M} = v_2\pi(1-q)H^2$. We shall show that $v_1\pi(q)$ is a co-isometry on $\pi(q)L^2(B)$. In fact

$$\begin{aligned} v_1\pi(q)(\pi(q)L^2(B)) \supset [v_1\pi(q)\pi(B)\mathfrak{M}]_2 &= [\pi(B)v_1\pi(q)\mathfrak{M}]_2 \\ &= [\pi(q)\pi(B)H^2]_2 = \pi(q)L^2(B). \end{aligned}$$

Thus $v_1\pi(q)$ is a co-isometry in a finite von Neumann algebra $\pi(q)\pi(B)'$, so that $v_1\pi(q)$ is a unitary on $\pi(q)L^2(B)$. Therefore $(v_1\pi(q))(\pi(q)\mathfrak{M}) = \pi(q)H^2$ implies that

$$\pi(q)\mathfrak{M} = \pi(q)v_1^*\pi(q)H^2 = \pi(q)v_1^*H^2.$$

We also have $\pi(1-q)\mathfrak{M} = \pi(1-q)v_2H^2$. Let

$$v = \pi(q)v_1^* + \pi(1-q)v_2.$$

Then v is a partial isometry in $\pi(B)'$ and we have

$$\mathfrak{M} = \pi(q)\mathfrak{M} \oplus \pi(1-q)\mathfrak{M} = \pi(q)v_1^*H^2 \oplus \pi(1-q)v_2H^2 = vH^2. \quad \blacksquare$$

REMARK 3.5. The above proof itself is a generalization of that of Halmos ([3]) and McAsey, Muhly and Saito ([7]).

COROLLARY 3.6. (McAsey, Muhly and Saito [7]) *Let D be a finite von Neumann algebra with a finite trace τ and α an automorphism of D with $\tau \circ \alpha = \tau$. Let $B = D \rtimes_\alpha \mathbb{Z}$ be the crossed product and $\mathfrak{A} = D \rtimes_\alpha \mathbb{Z}_+$ the analytic crossed product. Suppose that α fixes the center $Z(D)$ of D elementwise. Then every pure invariant subspace $\mathfrak{M}$ of $L^2(B)$ has the form vH^2 for some partial isometry $v \in \pi(B)'$.*

Proof. The condition that α fixes $Z(D)$ elementwise is equivalent to that $Z(D) \subset Z(B)$. Proposition 2.8 shows that $\mathfrak{M}$ is pure if and only if $\mathfrak{M}$ is $\mathfrak{A}$-type I. Therefore we can apply Theorem 3.3. ∎

REMARK 3.7. The assumption that $Z(D) \subset Z(B)$ in Theorem 3.3 is necessary. For example let $B = M_2(\mathbb{C})$ be the algebra of 2 by 2 matrices and D the algebra of diagonal matrices. Then the set $\mathfrak{A}$ of upper triangular matrices is a maximal subdiagonal algebra. We note that $Z(D) = D \not\subset Z(B) = \mathbb{C}$. Let $\mathfrak{M} = L^2(B)$. Then $\mathfrak{M}$ is of $\mathfrak{A}$-type I. But there exists no partial isometry $v \in \pi(B)'$ such that $\mathfrak{M} = vH^2$, because $\dim \mathfrak{M} = 4$ and $\dim vH^2 \leqq \dim H^2 = 3$.

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Received April 19, 1996.