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THE STRUCTURE OF TWISTED CONVOLUTION C^*-ALGEBRAS ON ABELIAN GROUPS

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ABSTRACT. Let $\mathcal{G}$ be a locally compact two step nilpotent group. Each unitary character λ on the closure of the commutator subgroup defines in a canonical fashion a quotient $C^*(\mathcal{G})_\lambda$ of the group C^*-algebra $C^*(\mathcal{G})$. Under a mild extra condition, which is e.g. satisfied if $\mathcal{G}$ is compactly generated, the structure of $C^*(\mathcal{G})_\lambda$ is determined completely. This result is applied to connected Lie groups in order to obtain the structure of certain subquotients of the corresponding group C^*-algebras.

KEYWORDS: *Two step nilpotent groups, Lie groups, group C^*-algebras, non-commutative tori, stable isomorphy, cocycles, imprimitivity theorem.*

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Let $\mathcal{G}$ be a locally compact two step nilpotent group, and let λ be a unitary character on the closure $[\mathcal{G},\mathcal{G}]^-$ of the commutator subgroup of $\mathcal{G}$. This character defines a quotient $C^*(\mathcal{G})_\lambda$ of the group C^*-algebra of $\mathcal{G}$ which is the C^*-completion of the algebra $L^1(\mathcal{G})_\lambda$ consisting of all measurable functions $f:\mathcal{G}\to\mathbb{C}$ such that $f(xz)=\lambda(z)^{-1}f(x)$ for $x\in\mathcal{G}$, $z\in[\mathcal{G},\mathcal{G}]^-$ and $\int\limits_{\mathcal{G}/[\mathcal{G},\mathcal{G}]^-}|f(x)|\,\mathrm{d}\dot{x}<\infty$ where $\mathrm{d}\dot{x}$ denotes the Haar measure on the quotient group. The involution and the product in the latter algebra are given by $f^*(x)=f(x^{-1})^-$ and $(f*g)(x)=\int\limits_{\mathcal{G}/[\mathcal{G},\mathcal{G}]^-}f(xy)g(y^{-1})\,\mathrm{d}\dot{y}$. The representations of $C^*(\mathcal{G})_\lambda$ correspond to the unitary group representations of $\mathcal{G}$ which restrict to λ on $[\mathcal{G},\mathcal{G}]^-$. The whole group C^*-algebra $C^*(\mathcal{G})$ may be viewed as the algebra of sections of a C^*-bundle over the Pontrjagin dual of $[\mathcal{G},\mathcal{G}]^-$, whose fibers are isomorphic to the $C^*(\mathcal{G})_\lambda$; compare [13].

This article contains a single theorem, in which the structure of $C^*(\mathcal{G})_\lambda$ is completely determined if a mild extra condition is satisfied. The extra condition is explained next. The character λ defines a skew-symmetric bicharacter on $\mathcal{G}$ by $(x,y) \mapsto \lambda(xyx^{-1}y^{-1}) = \lambda([x,y])$. This bicharacter factors through $\mathcal{G}/\mathcal{G}_\lambda$ where $\mathcal{G}_\lambda/\ker\lambda$ is by definition the center of $\mathcal{G}/\ker\lambda$, and yields on the abelian group $G = \mathcal{G}/\mathcal{G}_\lambda$ the structure of a quasi-symplectic space in the terminology of [12], i.e., a non-degenerate skew-symmetric bicharacter $\gamma : G \times G \to \mathbf{T}$. With each such quasi-symplectic space there was associated in [12] an invariant $\mathrm{Inv}(G)$ which actually is an equivalence class of discrete abelian groups: Choose a "large compact subgroup" K of G in the sense of (1.4) in [12], i.e., all compact subgroups in G/K are finite, such that K is contained in $K^\perp := \{x \in G \mid \gamma(x,k) = 1 \text{ for all } k \in K\}$. Such K's always exist. The connected component $(G/K)_0$ of the identity is of the form U/K, the group U being open in G and contained in $K^\perp$. While the discrete group $K^\perp/U$ depends on the choice of K, different K's yield equivalent groups for a certain equivalence relation. The equivalence class through $K^\perp/U$ is the invariant $\mathrm{Inv}(G) = \mathrm{Inv}(G,\gamma)$. Here we shall assume that the invariant associated with $\mathcal{G}/\mathcal{G}_\lambda$ contains $\mathbf{Z}^n$ for a certain n. In this case the whole equivalence class is easy to describe: it consists of all groups of the form $\mathbf{Z}^n \times E$ with a finite abelian group E. The assumption implies ([12], (2.4)), that the primitive quotients of $C^*(\mathcal{G})_\lambda$ are stably isomorphic to n-dimensional noncommutative tori.

From these first introductory sentences the reader may already guess that we shall make use of the approach and the results of [12]. Indeed, this article may be viewed as an appendix to [12], where the theorem below was already conjectured.

To gain some more flexibility we shall formulate the theorem for characters λ defined on central subgroups containing the commutator subgroup. The group $\mathcal{G}_\lambda$ and the bicharacter on $\mathcal{G}/\mathcal{G}_\lambda$ can be defined as above.

THEOREM 1. *Let $\mathcal{G}$ be a locally compact two step nilpotent group, let $\mathcal{L}$ be a closed central subgroup containing $[\mathcal{G},\mathcal{G}]$, and let λ be a unitary character on $\mathcal{L}$. Assume further that* $\mathrm{Inv}(\mathcal{G}/\mathcal{G}_\lambda)$ *contains $\mathbf{Z}^n$. Then $C^*(\mathcal{G})_\lambda$ is stably isomorphic to the (C^*-) tensor product of $C_\infty((\mathcal{G}_\lambda/\mathcal{L})^\wedge)$, an n-dimensional noncommutative torus and the algebra of compact operators on a Hilbert space. If $\mathcal{G}/\mathcal{G}_\lambda$ is not discrete then $C^*(\mathcal{G})_\lambda$ is even isomorphic to such a tensor product.*

REMARK 2. In the "type I case" a similar result was obtained in [8].

Proof of Theorem 1. First we observe that it suffices to show that $C^*(\mathcal{G})_\lambda$ is (stably) isomorphic to a tensor product of an n-dimensional noncommutative torus, the algebra of compact operators and an commutative algebra, because the primitive ideal space of $C^*(\mathcal{G})_\lambda$ is homeomorphic to $(\mathcal{G}_\lambda/\mathcal{L})^\wedge$, compare [2], [3] and (2.2) of [12]; the second article ([3]) contains further information on the topology of the whole space Priv $(C^*(\mathcal{G}))$ as well as a discussion of a large variety of examples of two step nilpotent groups.

Our first task will be to reduce to the case of a discrete quotient $\mathcal{G}/\mathcal{G}_\lambda$. This is done in two steps.

Step 1. Reduction to the case that $\mathcal{G}/\mathcal{L}$ is essentially compact-free, i.e., all compact subgroups of $\mathcal{G}/\mathcal{L}$ are finite.

To this end, choose a large compact subgroup $\mathcal{K}/\mathcal{L}$ of the locally compact abelian group $\mathcal{G}/\mathcal{L}$ in the sense of (1.4) in [12], and assume further that $\mathcal{K}^\perp := \{x \in \mathcal{G} \mid \lambda([x,y]) = 1 \text{ for all } y \in \mathcal{K}\}$ contains $\mathcal{K}$. It is not hard to see that such a $\mathcal{K}$ exists (if $\mathcal{M}/\mathcal{L}$ is any large compact subgroup of $\mathcal{G}/\mathcal{L}$ then $\mathcal{M} \cap \mathcal{M}^\perp$ is of finite index in $\mathcal{M}$, compare the above discussion of the invariant Inv(G)). The algebra $C^*(\mathcal{G})_\lambda$ is isomorphic to the twisted covariance algebra $C^*(\mathcal{G}, \mathcal{K}, C^*(\mathcal{K})_\lambda, T, \tau)$ where the action T is given by

$$(T_x a)(k) = a^{x^{-1}}(k) = a(x^{-1}kx),$$

and the twist τ is given by

$$(\tau(k)a)(l) = a(k^{-1}l)$$

for $k, l \in \mathcal{K}$, $x \in \mathcal{G}$ and $a \in L^1(\mathcal{K})_\lambda \subset C^*(\mathcal{G})_\lambda$. This twisted covariance algebra is, compare Section 3 of [12], the C^*-completion of the algebra of all measurable functions $f : \mathcal{G} \to C^*(\mathcal{K})_\lambda$ satisfying $f(xk) = \tau(k)^{-1}f(x)$ for $x \in \mathcal{G}$, $k \in \mathcal{K}$ and $\int_{\mathcal{G}/\mathcal{K}} \|f(x)\| \, d\dot{x} < \infty$. The involution and the multiplication in the latter algebra are defined by $f^*(x) = f(x^{-1})^{*x}$ and $(f * g)(x) = \int_{\mathcal{G}/\mathcal{K}} f(xy)^{y^{-1}} g(y^{-1}) \, d\dot{y}$. The isomorphism between $C^*(\mathcal{G})_\lambda$ and $C^*(\mathcal{G}, \mathcal{K}, C^*(\mathcal{K})_\lambda, T, \tau)$ is implemented by assigning to a continuous function $f : \mathcal{G} \to \mathbb{C}$ with $f(xl) = \lambda(l)^{-1}f(x)$ for $x \in \mathcal{G}$, $l \in \mathcal{L}$, whose support projects to a compact subset of $\mathcal{G}/\mathcal{L}$, the function $\widetilde{f} : \mathcal{G} \times \mathcal{K} \to \mathbb{C}$ given by $\widetilde{f}(x,k) = f(xk)$, which in an obvious sense may be viewed as an element of $C^*(\mathcal{G}, \mathcal{K}, C^*(\mathcal{K})_\lambda, T, \tau)$.

For the following computations compare also (2.3) in [12]. As $\mathcal{K}^\perp \supset \mathcal{K}$ the algebra $C^*(\mathcal{K})_\lambda$ is commutative. More explicitly, choose an extension $\nu \in \mathcal{K}^\wedge$ of $\lambda \in \mathcal{L}^\wedge$ and define $\widetilde{a} : (\mathcal{K}/\mathcal{L})^\wedge \to \mathbb{C}$ for $a \in L^1(\mathcal{K})_\lambda$ by

$$\widetilde{a}(\alpha) = \int_{\mathcal{K}/\mathcal{L}} a(k)\nu(k)\alpha(k) \, dk.$$

This map yields an isomorphism from $C^*(\mathcal{K})_\lambda$ onto $C_\infty((\mathcal{K}/\mathcal{L})^\wedge)$; we use the notation C_∞ for the "continuous" functions vanishing at infinity even though $(\mathcal{K}/\mathcal{L})^\wedge$ is a discrete group. In the transformed picture the twist τ and the action T are given by

$$(\tau(k)a)(\alpha) = \nu(k)\alpha(k)a(\alpha),$$
$$a^x(\alpha) = a(\eta(x)\alpha),$$

where $\eta : \mathcal{G} \to (\mathcal{K}/\mathcal{L})^\wedge$ is defined by $\eta(x)(k) = \lambda([k,x])$; note that η factors through $\mathcal{G}/\mathcal{K}^\perp$.

In order to decompose $C^*(\mathcal{G},\mathcal{K},C_\infty((\mathcal{K}/\mathcal{L})^\wedge),T,\tau)$ we choose representatives for the cosets of the subgroup $\operatorname{im}\eta$ of $(\mathcal{K}/\mathcal{L})^\wedge$: Let $(\alpha_j)_{j\in J}$ be an indexed family of elements in $(\mathcal{K}/\mathcal{L})^\wedge$ such that for each $\beta \in (\mathcal{K}/\mathcal{L})^\wedge$ there exist unique elements $j \in J$ and $\alpha \in \operatorname{im}\eta$ with $\beta = \alpha_j\alpha$. Moreover, choose for each $j \in J$ an extension $\rho_j \in (\mathcal{G}/\mathcal{L})^\wedge$ of α_j.

To each continuous function $f : \mathcal{G} \to C_\infty((\mathcal{K}/\mathcal{L})^\wedge)$ satisfying $f(xk) = \tau(k)^{-1}f(x)$ for $x \in \mathcal{G}$, $k \in \mathcal{K}$, whose support projects to a compact subset of $\mathcal{G}/\mathcal{K}$, we associate a family $(f_j)_{j\in J}$ of functions $f_j : \mathcal{G} \to C_\infty(\mathcal{G}/\mathcal{K}^\perp)$, namely

$$f_j(x,\dot{y}) = \rho_j^-(x)f(x,\alpha_j\eta(y)),$$

where $\dot{y} \in \mathcal{G}/\mathcal{K}^\perp$ denotes the coset of $y \in \mathcal{G}$.

Define a twist τ° on $\mathcal{K}$ with values in the unitary group of the multiplier algebra of $C_\infty(\mathcal{G}/\mathcal{K}^\perp)$ by

$$(\tau^\circ(k)a)(\dot{y}) = \nu(k)\eta(y)(k)a(\dot{y}),$$

and define an action T° of $\mathcal{G}$ on $C_\infty(\mathcal{G}/\mathcal{K}^\perp)$ by

$$(T_x^\circ a)(\dot{y}) = a(\dot{x}^{-1}\dot{y}).$$

Then we may form the twisted covariance algebra $C^*(\mathcal{G},\mathcal{K},C_\infty(\mathcal{G}/\mathcal{K}^\perp),T^\circ,\tau^\circ)$. It is easy to see that the functions f_j are members of this twisted covariance algebra. Moreover, one has $(f^*)_j = f_j^*$, and if $g : \mathcal{G} \to C_\infty((\mathcal{K}/\mathcal{L})^\wedge)$ is another function like f then $(f * g)_j = f_j * g_j$ for all $j \in J$. After these observations it is easy to check that the assigment $f \mapsto (f_j)_{j\in J}$ yields an isomorphism from $C^*(\mathcal{G},\mathcal{K},C_\infty((\mathcal{K}/\mathcal{L})^\wedge),T,\tau) \cong C^*(\mathcal{G})_\lambda$ onto $C_\infty(J, C^*(\mathcal{G},\mathcal{K},C_\infty(\mathcal{G}/\mathcal{K}^\perp),T^\circ,\tau^\circ)) \cong C_\infty(J) \otimes C^*(\mathcal{G},\mathcal{K},C_\infty(\mathcal{G}/\mathcal{K}^\perp),T^\circ,\tau^\circ)$.

By the results of [9] the latter tensor factor (note that as $\mathcal{G}/\mathcal{K}^\perp$ is discrete there exists a "measurable" cross section $\mathcal{G}/\mathcal{K}^\perp \to \mathcal{G}$ mapping compact (= finite) sets into relatively compact subsets) is isomorphic to the tensor product of the algebra $\mathfrak{K}(l_2(\mathcal{G}/\mathcal{K}^\perp))$ of compact operators on $l_2(\mathcal{G}/\mathcal{K}^\perp)$ and the twisted covariance algebra $C^*(\mathcal{K}^\perp, \mathcal{K}, \mathbb{C}, \tau')$ with trivial action, where the twist τ' is given by $\tau'(k) = \nu(k)$. We conclude that $C^*(\mathcal{G})_\lambda$ is isomorphic to $C_\infty(J) \otimes \mathfrak{K}(l_2(\mathcal{G}/\mathcal{K}^\perp)) \otimes C^*(\mathcal{K}^\perp)_\nu$.

This means that we may substitute $\mathcal{G}$ by $\mathcal{K}^\perp$ and $\mathcal{L}$ by $\mathcal{K}$. In other words, from now on we may assume that $\mathcal{G}/\mathcal{L}$ is essentially compact free, hence the connected component $(\mathcal{G}/\mathcal{L})_0$ is open in $\mathcal{G}/\mathcal{L}$, and it is a vector group.

Step 2. Reduction to the case that $\mathcal{G}/\mathcal{G}_\lambda$ is discrete.

To this end, let S be the maximal torus in the connected component G_0 of the quasi-symplectic space $G = \mathcal{G}/\mathcal{G}_\lambda$ endowed with the form $\gamma(x,y) = \lambda([x,y])$. By (1.8) of [12], there exist subgroups R and F of G such that G is the γ-orthogonal direct product of R and SF, the product SF is also direct, and γ defines an isomorphism from S onto $\widehat{F}$, and from F onto $\widehat{S}$ as well. Note that abelian groups are also written multiplicatively.

As R is evidently essentially compact free, by (1.16) of [12], there exists a quasi-polarization Q in R which is isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b$ for some $a, b \in \mathbb{N}$. Quasi-polarization means, compare (1.1) of [12], that $Q \subset Q^\perp$, that $Q^\perp/Q$ is discrete, and that $\gamma|R$ induces an isomorphism from $R/Q^\perp$ onto $Q^\wedge$. Then $P := FQ$ is a quasi-polarization in (G,γ). The group P is isomorphic to $\mathbb{R}^a \times \mathbb{Z}^c$, $c = b + \dim S$, in particular it is a projective locally compact abelian group.

Let $\mathcal{P}$ be the preimage of P under the natural map $\mathcal{G} \to \mathcal{G}/\mathcal{G}_\lambda = G$. The exact sequence

$$1 \to \mathcal{G}_\lambda/\mathcal{L} \longrightarrow \mathcal{P}/\mathcal{L} \longrightarrow \mathcal{P}/\mathcal{G}_\lambda \to 1$$

splits, hence there exists a closed subgroup $\mathcal{N}$ of $\mathcal{G}$ with $\mathcal{L} \subset \mathcal{N} \subset \mathcal{P}$ such that the obvious homomorphism $\mathcal{N}/\mathcal{L} \to \mathcal{P}/\mathcal{G}_\lambda$ is an isomorphism.

As above, using $\mathcal{N}$ instead of $\mathcal{K}$, we identify $C^*(\mathcal{G})_\lambda$ with the twisted covariance algebra $C^*(\mathcal{G}, \mathcal{N}, C^*(\mathcal{N})_\lambda, T, \tau)$. Again $C^*(\mathcal{N})_\lambda$ is commutative, its spectrum is homeomorphic to $(\mathcal{N}/\mathcal{L})^\wedge \cong (\mathcal{P}/\mathcal{G}_\lambda)^\wedge \cong \widehat{P} \cong \mathbb{R}^a \times \mathbb{T}^c$. Since P is a quasi-polarization this spectrum can also be identified with $\mathcal{G}/\mathcal{P}^\perp$, and the $\mathcal{G}$-action, derived from T, is just translation. Moreover, it is not hard to show that the natural map $\mathcal{G} \to \mathcal{G}/\mathcal{P}^\perp \cong \mathbb{R}^a \times \mathbb{T}^c$ allows a measurable cross section (with values in the connected component $\mathcal{G}_0$) mapping compact subsets of $\mathcal{G}/\mathcal{P}^\perp$ into relatively compact subsets of $\mathcal{G}_0$.

Hence the results of [9] can be applied. Choose an extension $\nu \in \mathcal{N}^\wedge$ of $\lambda \in \mathcal{L}^\wedge$. The algebra $C^*(\mathcal{G}, \mathcal{N}, C^*(\mathcal{N})_\lambda, T, \tau)$ is isomorphic to the tensor product of the algebra of compact operators on $L^2(\mathcal{G}/\mathcal{P}^\perp)$ and the algebra $C^*(\mathcal{P}^\perp, \mathcal{N}, \mathbb{C}, \tau')$ with trivial action and twist $\tau' = \nu$, i.e., the latter algebra is $C^*(\mathcal{P}^\perp)_\nu$.

Therefore, we may replace $\mathcal{G}$ by $\mathcal{G}' = \mathcal{P}^\perp$, $\mathcal{L}$ by $\mathcal{L}' = \mathcal{N}$ and λ by $\lambda' = \nu$. Then $\mathcal{G}'_{\lambda'}$ is just $\mathcal{P}$, hence $\mathcal{G}'/\mathcal{G}'_{\lambda'}$ is discrete. This finishes Step 2 and solves our first task.

So far we have not used our assumption on $\mathrm{Inv}(\mathcal{G}/\mathcal{G}_\lambda)$. Although it is evident from the considerations in Section 1 of [12], for formal reasons we should remark that the two reduction steps did not change this invariant.

Before continuing with the proof we make two further remarks.

REMARK 3. The above arguments also show that in the proof of (2.3) in [12], one could avoid the use of Kehlet's results on cross sections in a nonseparable context. The reason is the explicit information on the structure of the groups in question.

REMARK 4. Our above two reductions show that the original algebra $C^*(\mathcal{G})_\lambda$ decomposes into a tensor product, one factor being the algebra of compact operators on an infinite-dimensional Hilbert space (and hence it remains unchanged when being tensored with the algebra of compact operators on a separable Hilbert space), except for the case that $\mathcal{K}^\perp$ is of finite index in $\mathcal{G}$ and $\mathcal{P}^\perp$ is of finite index in $\mathcal{K}^\perp$. Since P is a quasi-polarization, i.e., $\mathcal{P}^\perp/\mathcal{P}$ is discrete and $\mathcal{G}/\mathcal{P}^\perp$ is canonically isomorphic to $(\mathcal{P}/\mathcal{G}_\lambda)^\wedge$, this case happens only when $\mathcal{P}/\mathcal{G}_\lambda$ is finite and, consequently, $\mathcal{G}/\mathcal{G}_\lambda$ is discrete. This observation shows that the last sentence of the theorem follows from the preceding statement on stable isomorphy.

If $\mathcal{G}/\mathcal{G}_\lambda$ is discrete, which we may assume from now on, our assumption on $\mathrm{Inv}(\mathcal{G}/\mathcal{G}_\lambda)$ means that $\mathcal{G}/\mathcal{G}_\lambda$ is isomorphic to a product of $\mathbb{Z}^n$ and a finite group. The main difficulty lies in this finite group. If it happens to be trivial the proof is quickly finished: then the exact sequence

$$1 \to \mathcal{G}_\lambda/\mathcal{L} \to \mathcal{G}/\mathcal{L} \to \mathcal{G}/\mathcal{G}_\lambda \to 1$$

splits. Choose a closed subgroup $\mathcal{M}$ of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{L}$ is the direct product of $\mathcal{G}_\lambda/\mathcal{L}$ and $\mathcal{M}/\mathcal{L}$. The group $\mathcal{M}/\mathcal{L}$ is isomorphic to $\mathcal{G}/\mathcal{G}_\lambda$, and this isomorphism is compatible with the bicharacters on those two groups, which are derived from $(x, y) \mapsto \lambda([x, y])$. From the fact that $\mathcal{G}_\lambda$ centralizes $\mathcal{G}$ modulo $\ker\lambda$ one easily concludes that $C^*(\mathcal{G})_\lambda$ is isomorphic to $C^*(\mathcal{G}_\lambda)_\lambda \otimes C^*(\mathcal{M})_\lambda$. The algebra $C^*(\mathcal{G}_\lambda)_\lambda$ is isomorphic to $C_\infty((\mathcal{G}_\lambda/\mathcal{L})^\wedge)$, and $C^*(\mathcal{M})_\lambda$ is an n-dimensional noncommutative torus.

Our final (and most space consuming) task will be to reduce to the case of a free quotient $\mathcal{G}/\mathcal{G}_\lambda$, i.e., to dispose of the potential torsion part of $\mathcal{G}/\mathcal{G}_\lambda$. This will be done in finitely many steps, in each reducing the order of the torsion part. The case $n = 0$ is easier, the reader is invited to look first at Remark 5 below and the comments following it.

Later we shall need for technical reasons that

(*) there exists an open subgroup $\mathcal{W}$ with $\mathcal{L} \subset \mathcal{W} \subset \mathcal{G}_\lambda$ such that $\mathcal{G}/\mathcal{W}$ is a finitely generated free (abelian) group.

This property is not automatically fulfilled, hence we show how we can reduce to such a situation. Choose a finitely generated free abelian group F and a surjective homomorphism $\kappa : F \to \mathcal{G}/\mathcal{G}_\lambda$. Denote by $\rho : \mathcal{G} \to \mathcal{G}/\mathcal{G}_\lambda$ the natural map, and define $\mathcal{G}'$ to be the subgroup of the direct product $\mathcal{G} \times F$ consisting of all pairs (x, f) with $\rho(x) = \kappa(f)$. There are obvious homomorphisms $\varphi : \mathcal{G}' \to \mathcal{G}$ and $\psi : \mathcal{G}' \to F$, which are easily seen to be onto. The group $\mathcal{L}' := \{(x, e) \mid x \in \mathcal{L}\}$ is contained in $\mathcal{G}'$. Define the unitary character λ' on $\mathcal{L}' \cong \mathcal{L}$ in the most obvious way. Put $\mathcal{W}' := \ker \psi = \{(x, e) \mid x \in \mathcal{G}_\lambda\}$, which is contained in $\mathcal{G}'_{\lambda'}$ as $\mathcal{G}'_{\lambda'} = \mathcal{G}_\lambda \times \ker \kappa$. By construction the quotient $\mathcal{G}'/\mathcal{W}'$ is a free abelian group. The homomorphism φ induces an isomorphism from $\mathcal{G}'/\mathcal{G}'_{\lambda'}$ onto $\mathcal{G}/\mathcal{G}_\lambda$, and a surjection from $C^*(\mathcal{G}')_{\lambda'}$ onto $C^*(\mathcal{G})_\lambda$. Therefore, if we know that $C^*(\mathcal{G}')_{\lambda'}$ has the structure as claimed in the theorem we know it for $C^*(\mathcal{G})_\lambda$ as well. The group $\mathcal{G}'$ possesses the wanted additional property. Now we omit the apostrophe and assume that $\mathcal{G}$ has such a subgroup $\mathcal{W}$.

To reduce the size of the torsion part of $\mathcal{G}/\mathcal{G}_\lambda$ we shall write $C^*(\mathcal{G})_\lambda$ as a quotient of another C^*-algebra. To this end, we choose a non-trivial finite cyclic subgroup $\mathcal{P}/\mathcal{G}_\lambda$ of $\mathcal{G}/\mathcal{G}_\lambda$. Put as usual $\mathcal{P}^\perp = \{x \in \mathcal{G} \mid \lambda([x, y]) = 1 \text{ for all } y \in \mathcal{P}\}$. The groups $\mathcal{G}/\mathcal{P}^\perp$ and $\mathcal{P}/\mathcal{G}_\lambda$ are dual to each other, in particular $\mathcal{G}/\mathcal{P}^\perp$ is cyclic of the same order as $\mathcal{P}/\mathcal{G}_\lambda$. Our goal is to show that $C^*(\mathcal{G})_\lambda$ is stably isomorphic to a quotient of a tensor product of a commutative algebra with the algebra $C^*(\mathcal{P}^\perp)_\lambda$. And clearly the original algebra $C^*(\mathcal{G})_\lambda$ inherits the decisive properties from the newly constructed algebra. The kernel $(\mathcal{P}^\perp)_\lambda$ of the bicharacter on $\mathcal{P}^\perp$ is just $\mathcal{P}$, and the quotient $\mathcal{P}^\perp/\mathcal{P}$ has a smaller torsion subgroup than $\mathcal{G}/\mathcal{G}_\lambda$. Note also that a group $\mathcal{W}$ as in (*) is contained in $\mathcal{P}^\perp$, and that $\mathcal{P}^\perp/\mathcal{W}$ is free; hence the additional property (*) remains valid when replacing $\mathcal{G}$ by $\mathcal{P}^\perp$.

Choose infinite cyclic groups A and B and homomorphisms $\varepsilon_1 : A \to \mathcal{G}$ and $\varepsilon_2 : B \to \mathcal{P}$ such that the composite maps $A \to \mathcal{G} \to \mathcal{G}/\mathcal{P}^\perp$ and $B \to \mathcal{P} \to \mathcal{P}/\mathcal{G}_\lambda$ are onto; denote by $M \subset A$ and $N \subset B$ the kernels of these composite maps. The groups A/M and B/N are cyclic of the same order, say s. On $A \times B$ we define a bicharacter σ by

$$\sigma(a, b; a', b') = \lambda[\varepsilon_2(b), \varepsilon_1(a')].$$

Then we can form the twisted convolution algebra $l^1(A \times B, \sigma)$ of summable functions on $A \times B$ where the multiplication and the involution are given by

$$(f * g)(x) \sum_{y \in A \times B} \sigma(xy, y^{-1}) f(xy) g(y^{-1})$$

and $f^*(x) = \overline{f(x^{-1})}\sigma(x,x)$.

The C^*-completion of $l^1(A \times B, \sigma)$ is denoted by $\mathcal{D} = C^*(A \times B, \sigma)$. Since σ factors through $(A/M) \times (B/N)$ this algebra is indeed a "rational rotation algebra". The group $\mathcal{P}^\perp$ acts on $l^1(A \times B, \sigma)$ by

$$(T_x f)(a,b) = f^{x^{-1}}(a,b) = \lambda([x, \varepsilon_1(a)]) f(a,b)$$

for $x \in \mathcal{P}^\perp$. Clearly, this action extends to $\mathcal{D}$, where it is also denoted by T. Hence we may form the twisted covariance algebra $C^*(\mathcal{P}^\perp, \mathcal{L}, \mathcal{D}, T, \lambda)$ where the twist λ at a point $l \in \mathcal{L}$ is just multiplication by $\lambda(l)$.

Next we claim that $C^*(\mathcal{G})_\lambda$ is a quotient of this twisted covariance algebra. The latter algebra may be considered as the completion of the algebra of measurable functions $f : \mathcal{P}^\perp \times A \times B \to \mathbb{C}$ with $f(xl, a, b) = \lambda(l)^{-1} f(x, a, b)$ for $l \in \mathcal{L}$ which are integrable modulo $\mathcal{L}$. For such an f define $Rf : \mathcal{G} \to \mathbb{C}$ as follows. If $x \in \mathcal{G}$ is written as $x = x'\varepsilon_1(x'')$ with $x' \in \mathcal{P}^\perp$ and $x'' \in A$, then

$$(Rf)(x) = (Rf)(x'\varepsilon_1(x'')) = \sum_{b \in B} \sum_{k \in M} f(x'\varepsilon_1(x'')\varepsilon_2(b)^{-1}\varepsilon_1(x'')^{-1}\varepsilon_1(k)^{-1}, x''k, b).$$

Because of the transformation property of f this sum coincides with

$$\sum_{b \in B} \sum_{k \in M} \lambda([\varepsilon_1(x''), \varepsilon_2(b)]) f(x'\varepsilon_2(b)^{-1}\varepsilon_1(k)^{-1}, x''k, b).$$

First one observes that this definition is independent of the chosen decomposition of x, which is pretty easy. Secondly by a straightforward but lengthy computation one can show that R is a $*$-morphism with values in $L^1(\mathcal{G})_\lambda$. Actually, R maps onto $L^1(\mathcal{G})_\lambda$ and the norm of Rf is bounded by $\sum_{a \in A} \sum_{b \in B} \int_{\mathcal{P}^\perp/\mathcal{L}} |f(x,a,b)| \, dx$; for this estimate one better uses the second expression for $(Rf)(x)$.

Further details on these computations are omitted; we conclude that $C^*(\mathcal{G})_\lambda$ is a quotient of $C^*(\mathcal{P}^\perp, \mathcal{L}, \mathcal{D}, T, \lambda)$.

Our next step will be to give a more suitable description of $\mathcal{D}$ (together with the action of $\mathcal{P}^\perp$ on $\mathcal{D}$). The following considerations are influenced by the investigations in [4], but our treatment will be selfcontained and differ at some points.

As we remarked above the cocycle σ "lives" on $A/M \times B/N$. The kernel of the associated skew-symmetric bicharacter

$$(a,b;a',b') \mapsto \sigma(a,b;a',b')\sigma(a',b';a,b)^{-1} = \lambda([\varepsilon_2(b),\varepsilon_1(a')])\lambda([\varepsilon_2(b'),\varepsilon_1(a)])^{-1}$$

is just $M \times N$: If for a given pair (a,b) this expression is 1 for all (a',b') then in particular $\lambda[\varepsilon_2(b'),\varepsilon_1(a)] = 1$ for all b'. Because of $\varepsilon_2(B)\mathcal{G}_\lambda = \mathcal{P}$ the element $\varepsilon_1(a)$ has to be in $\mathcal{P}^\perp$, which means that a is in M. Similarly if $\lambda([\varepsilon_2(b),\varepsilon_1(A)]) = 1$ then it follows from $\varepsilon_1(A)\mathcal{P}^\perp = \mathcal{G}$ and $\lambda([\varepsilon_2(b),\mathcal{P}^\perp]) = 1$ that $\lambda([\varepsilon_2(b),\mathcal{G}]) = 1$, whence $\varepsilon_2(b) \in \mathcal{G}_\lambda$ or $b \in N$.

This observation shows that up to equivalence there is a unique irreducible σ-representation of $A \times B$ which is trivial on $M \times N$. We shall write down a particular realization π of such a representation. The space is $l_2(\dot{A})$ where $\dot{A} = A/M$; accordingly for $x \in A$ we denote by $\dot{x} \in \dot{A}$ the corresponding coset. And π is given by

$$(\pi(a,b)h)(\dot{x}) = \lambda([\varepsilon_2(b),\varepsilon_1(a^{-1}x)])h(\dot{a}^{-1}\dot{x})$$

for $(a,b) \in A \times B$ and $h \in l_2(\dot{A})$. It is easy to check that $\pi(x)\pi(y) = \pi(xy)\sigma(x,y)$ is satisfied for all $x,y \in A \times B$.

Taking tensor products of π with unitary characters of $A \times B$ one gets all irreducible σ-representations of $A \times B$ up to equivalence: If ρ is any irreducible σ-representation of $A \times B$ then $\rho|M \times N$ has to be a scalar multiple of the identity, say $\rho(z) = \gamma(z)\mathrm{Id}$ for $z \in M \times N$ with some $\gamma \in (M \times N)^\wedge$. For any extension $\widetilde{\gamma} \in (A \times B)^\wedge$ of γ the tensor product $\widetilde{\gamma}^{-1} \otimes \rho$ is trivial on $M \times N$ and hence equivalent to π. Therefore, ρ is equivalent to $\widetilde{\gamma} \otimes \pi$.

Clearly, not all $\pi \otimes \delta$, $\delta \in \widehat{A} \times \widehat{B}$, are inequivalent. Actually, the representation $\pi \otimes \delta$ is equivalent to $\pi \otimes \delta'$ if and only if the restrictions $\delta|M \times N$ and $\delta'|M \times N$ coincide. We shall write down explicit intertwining operators for π and $\pi \otimes \delta$, $\delta \in (A/M)^\wedge \times (B/N)^\wedge$. For $\xi \in (A/M)^\wedge$ define the unitary operator U_ξ on $l_2(\dot{A})$ by $(U_\xi h)(\dot{x}) = \xi(\dot{x})h(\dot{x})$. For $\eta \in (B/N)^\wedge$ choose a point $a_\eta \in A$ such that $\eta(b) = \lambda[\varepsilon_1(a_\eta),\varepsilon_2(b)]$ holds for all $b \in B$. It is easy to check that such an a_η exists. Actually, the coset $\dot{a}_\eta \in \dot{A}$ is uniquely determined by η, and $\eta \mapsto \dot{a}_\eta$ is an isomorphism from $(B/M)^\wedge$ on $\dot{A}$. Then define the unitary operator V_η on $l_2(\dot{A})$ by $(V_\eta h)(\dot{x}) = h(\dot{a}_\eta^{-1}\dot{x})$. One quickly verifies that the equations

$$\xi(a)\pi(a,b) = U_\xi \pi(a,b) U_\xi^*$$

and

$$\eta(b)\pi(a,b) = V_\eta \pi(a,b) V_\eta^*$$

hold for all $a \in A$, $b \in B$. Moreover the maps $(A/M)^\wedge \ni \xi \mapsto U_\xi$ and $(B/N)^\wedge \ni \eta \mapsto V_\eta$ are homomorphisms into the unitary group of $l_2(\dot{A})$, and the equation $U_\xi V_\eta = \xi(a_\eta)V_\eta U_\xi$ holds.

With each $f \in l^1(A \times B, \sigma)$ we associate a continuous function $\widetilde{f}$ on $\widehat{A} \times \widehat{B}$ with values in the algebra $\mathfrak{K}(l_2(\dot{A}))$ of linear operators $l_2(\dot{A})$ by

$$\widetilde{f}(\alpha, \beta) = \sum_{a\in A}\sum_{b\in B} f(a,b)\alpha(a)\beta(b)\pi(a,b) = ((\alpha,\beta)\otimes\pi)(f).$$

Each map $f \mapsto \widetilde{f}(\alpha,\beta)$ is an involutive representation of $l^1(A\times B,\sigma)$ which extends to $\mathcal{D} = C^*(A \times B, \sigma)$. The whole map $f \mapsto \widetilde{f}$ extends to a C^*-morphism from $\mathcal{D}$ into $C(\widehat{A} \times \widehat{B}, \mathfrak{K}(l_2(\dot{A}))$, the latter algebra being endowed with the pointwise operations. The morphism $\mathcal{D} \to C(\widehat{A} \times \widehat{B}, \mathfrak{K}(l_2(\dot{A}))$ is injective because the set $\{(\alpha,\beta)\otimes\pi \mid \alpha \in \widehat{A}, \beta \in \widehat{B}\}$ contains a representative of each point in $\widehat{\mathcal{D}}$. Clearly, this morphism is not surjective, the image is contained in the subalgebra $\mathcal{E}$ of $C(\widehat{A} \times \widehat{B}, \mathfrak{K}(l_2(\dot{A}))$ consisting of all functions φ with

$$\varphi(\alpha\xi, \beta\eta) = U_\xi V_\eta \varphi(\alpha,\beta)V_\eta^* U_\xi^*$$

for $\alpha \in \widehat{A}$, $\xi \in (A/M)^\wedge$, $\beta \in \widehat{B}$, $\eta \in (B/N)^\wedge$. Indeed, the image of $\mathcal{D}$ coincides with $\mathcal{E}$. This is a particularly easy special case of a C^*-Stone–Weierstrass-situation, see 11.1.6 or 11.1.4 from [5].

Hence in our object of study, namely the algebra $C^*(\mathcal{P}^\perp, \mathcal{L}, \mathcal{D}, T, \lambda)$, we may replace $\mathcal{D}$ by the isomorphic copy $\mathcal{E}$. But we also have to know the action of $\mathcal{P}^\perp$ in the transformed picture $\mathcal{E}$. This transformed action, again denoted by T, is given by

$$(T_x\varphi)(\alpha,\beta) = \varphi(\alpha\alpha_x, \beta)$$

for $\alpha \in \widehat{A}$, $\beta \in \widehat{B}$, $\varphi \in \mathcal{E}$, $x \in \mathcal{P}^\perp$ where $\alpha_x \in \widehat{A}$ is defined by $\alpha_x(a) = \lambda([x, \varepsilon_1(a)])$.

Next we extend the representation $(A/M)^\wedge \ni \xi \mapsto U_\xi$ to a continuous unitary representation $\widetilde{U}$ of $\widehat{A}$ in $l_2(\dot{A})$, for instance also by multiplication operators. Then for $\varphi \in \mathcal{E}$ define $\varphi' : \widehat{A} \times \widehat{B} \to \mathfrak{K}(l_2(\dot{A}))$ by

$$\varphi'(\alpha,\beta) = \widetilde{U}_\alpha^* \varphi(\alpha,\beta)\widetilde{U}_\alpha.$$

The image $\mathcal{E}' := \{\varphi' \mid \varphi \in \mathcal{E}\}$ consists of all continuous functions φ' which are constant on $(A/M)^\wedge$-cosets (i.e., they are functions on $\widehat{M} \times \widehat{B}$), and which satisfy the equation

$$\begin{aligned}\varphi'(\alpha,\beta\eta) &= \widetilde{U}_\alpha^* V_\eta \widetilde{U}_\alpha \varphi'(\alpha,\beta)\widetilde{U}_\alpha^* V_\eta^* \widetilde{U}_\alpha \\ &= \alpha(a_\eta)\widetilde{U}_\alpha^* V_\eta \widetilde{U}_\alpha \varphi'(\alpha,\beta)\widetilde{U}_\alpha^* V_\eta^* \widetilde{U}_\alpha \alpha(a_\eta)^{-1}\end{aligned}$$

for $\alpha \in \widehat{A}$, $\beta \in \widehat{B}$ and $\eta \in (B/N)^\wedge$. The factor $\alpha(a_\eta)$ causes that $\alpha \mapsto \alpha(a_\eta)\widetilde{U}_\alpha^* V_\eta \widetilde{U}_\alpha$ is constant on $(A/M)^\wedge$-cosets (due to the commutation relation between U_ξ and V_η).

Now we choose an isomorphism $j : \mathbf{T} \to \widehat{B}$. If, as above, s denotes the order of the cyclic group B/N (= order of A/M) then $\eta_0 := j(\mathrm{e}^{2\pi i \frac{1}{s}})$ is a generator of $(B/N)^\wedge$. For short let $V_0 = V_{\eta_0}$ and $a_0 = a_{\eta_0}$. With each $\varphi \in \mathcal{E}$ associate the function $\varphi'' : \widehat{A} \times [0, \frac{1}{s}] \to \mathfrak{K}(l_2(\dot{A}))$ by

$$\varphi''(\alpha, r) = \varphi'(\alpha, j(\mathrm{e}^{2\pi i r})) = \widetilde{U}_\alpha^* \varphi(\alpha, j(\mathrm{e}^{2\pi i r}))\widetilde{U}_\alpha.$$

The (isomorphic) image $\mathcal{F} := \{\varphi'' \mid \varphi \in \mathcal{E}\}$ of $\mathcal{E}$ consists of all continuous functions ψ on $\widehat{A} \times [0, \frac{1}{s}]$ which satisfy

$$\psi\left(\alpha, \frac{1}{s}\right) = \alpha(a_0)\widetilde{U}_\alpha^* V_0 \widetilde{U}_\alpha \psi(\alpha, 0)\widetilde{U}_\alpha^* V_0^* \widetilde{U}_\alpha \alpha(a_0)^{-1}$$

for all $\alpha \in \widehat{A}$, and which are constant on $(A/M)^\wedge$-cosets with respect to the first variable.

The action T_x, $x \in \mathcal{P}^\perp$, in the transformed picture $\mathcal{F}$ is given by

$$(T_x\psi)(\alpha, r) = \widetilde{U}_{\alpha_x}\psi(\alpha\alpha_x, r)\widetilde{U}_{\alpha_x}^*.$$

Now we consider the tensor product $C^*(\mathcal{P}^\perp, \mathcal{L}, \mathcal{F}, T, \lambda) \otimes \mathfrak{K}(\mathfrak{H})$ where $\mathfrak{K}(\mathfrak{H})$ denotes the algebra of compact operators on a chosen separable Hilbert space $\mathfrak{H}$. This tensor product can be identified with the twisted covariance algebra $C^*(\mathcal{P}^\perp, \mathcal{L}, \mathcal{F} \otimes \mathfrak{K}(\mathfrak{H}), T, \lambda)$ where the action T and the twist λ are defined in the obvious way. The tensor product $\mathcal{F} \otimes \mathfrak{K}(\mathfrak{H})$ can be identified with the algebra of continuous functions $\psi : \widehat{M} \times [0, \frac{1}{s}] \to \mathfrak{K}(l_2(\dot{A}) \otimes \mathfrak{H})$ satisfying

$$\psi\left(\gamma, \frac{1}{s}\right) = C(\gamma)\psi(\gamma, 0)C(\gamma)^*$$

where we put $C(\gamma) = \alpha(a_0)\widetilde{U}_\alpha^* V_0 \widetilde{U}_\alpha \otimes \mathrm{Id}_\mathfrak{H}$ if $\alpha \in \widehat{A}$ is any extension of $\gamma \in \widehat{M}$.

In this picture, the action of $x \in \mathcal{P}^\perp$ is given by

$$(T_x\psi)(\gamma, r) = U'_x \psi(\gamma\alpha_x|M, r){U'_x}^*,$$

where we put

$$U'_x := \widetilde{U}_{\alpha_x} \otimes \mathrm{Id}_\mathfrak{H}.$$

The unitary group of the separable Hilbert space $l_2(\dot{A}) \otimes \mathfrak{H}$ endowed with the strong operator topology is contractible, see Lemma 3, p. 251, [7]. Hence there exists a continuous map H from $\widehat{M} \times [0, \frac{1}{s}]$ into this unitary group such that $H(\gamma, \frac{1}{s}) = C(\gamma)H(\gamma, 0)$ for all $\gamma \in \widehat{M}$. One could for instance choose $H(\gamma, 0)$ to be the identity, but this does not matter.

We just remark that a result in [11] says that unitary groups on separable Hilbert spaces are even contractible in the norm topology.

For $\psi \in \mathcal{F} \otimes \mathfrak{K}(\mathfrak{H})$ viewed as above as a function on $\widehat{M} \times [0, \frac{1}{s}]$ with values in $\mathfrak{K}(l_2(\dot{A}) \otimes \mathfrak{H})$, we define $\widetilde{\psi} : \widehat{M} \times [0, \frac{1}{s}] \to \mathfrak{K}(l_2(\dot{A}) \otimes \mathfrak{H})$ by

$$\widetilde{\psi}(\gamma, r) = H(\gamma, r)^* \psi(\gamma, r) H(\gamma, r).$$

The (isomorphic) image $\widetilde{\mathcal{F}}$ of $\mathcal{F} \otimes \mathfrak{K}(\mathfrak{H})$ consists of all continuous functions $\widetilde{\psi}$ on $\widehat{M} \times [0, \frac{1}{s}]$ with values in $\mathfrak{K}(l_2(\dot{A}) \otimes \mathfrak{H})$ such that $\widetilde{\psi}(\gamma, \frac{1}{s}) = \widetilde{\psi}(\gamma, 0)$, i.e., $\widetilde{\mathcal{F}}$ is the tensor product of the algebra of complex-valued continuous functions on a two-torus with $\mathfrak{H}(l_2(\dot{A}) \otimes \mathfrak{H})$.

The action T_x in the transformed picture $\widetilde{\mathcal{F}}$ is given by

$$(T_x \widetilde{\psi})(\gamma, r) = H(\gamma, r)^* U'_x H(\gamma\alpha_x|M, r) \widetilde{\psi}(\gamma\alpha_x|M, r) H(\gamma\alpha_x|M, r)^* U'^*_x H(\gamma, r).$$

We conclude that $C^*(\mathcal{P}^\perp, \mathcal{L}, \mathcal{D}, T, \lambda) \otimes \mathfrak{K}(\mathfrak{H})$ is isomorphic to $C^*(\mathcal{P}^\perp, \mathcal{L}, \widetilde{\mathcal{F}}, T, \lambda)$. The algebra $\widetilde{\mathcal{F}}$ is simpler than $\mathcal{D}$ (or $\mathcal{D} \otimes \mathfrak{K}(\mathfrak{H})$), but the action has become more complicated. To repair this evil, we finally choose a continuous curve $\zeta : [0, \frac{1}{s}] \to (\mathcal{G}/\mathcal{W})^\wedge$, $\mathcal{W}$ as in (*), such that $\zeta(0) = 1$ and $\zeta(\frac{1}{s})(x) = \alpha_x(a_0) = \lambda([x, \varepsilon_1(a_0)])$. Clearly, the homomorphism $x \mapsto \alpha_x(a_0)$ factors through $\mathcal{G} \to \mathcal{G}/\mathcal{G}_\lambda$ and hence through $\mathcal{G} \to \mathcal{G}/\mathcal{W}$. As $\mathcal{G}/\mathcal{W}$ is free, the group $(\mathcal{G}/\mathcal{W})^\wedge$ is connected. Then for appropriate functions $f : \mathcal{P}^\perp \times \widehat{M} \times [0, \frac{1}{s}] \to \mathfrak{K}(l_2(\dot{A}) \otimes \mathfrak{H})$ in $C^*(\mathcal{P}^\perp, \mathcal{L}, \widetilde{\mathcal{F}}, T, \lambda)$ we define $f^\natural : \mathcal{P}^\perp \times \widehat{M} \times [0, \frac{1}{s}] \to \mathfrak{K}(l_2(\dot{A}) \otimes \mathfrak{H})$ by

$$f^\natural(x, \gamma, r) = \zeta(r)(x) H(\gamma\alpha_x|M, r)^* U'^*_x H(\gamma, r) f(x, \gamma, r).$$

The function $f^\natural$ also satisfies

$$f^\natural(xl, \gamma, r) = \lambda(l)^{-1} f^\natural(x, \gamma, r)$$

and

$$f^\natural\left(x, \gamma, \frac{1}{s}\right) = f^\natural(x, \gamma, 0)$$

for $x \in \mathcal{P}^\perp$, $l \in \mathcal{L}$, $\gamma \in \widehat{M}$. The latter equation follows from a straightforward computation using the properties of $\zeta, U'_x, H, \widetilde{U}_\alpha$ and $C(\gamma)$.

But if $g : \mathcal{P}^{\perp} \times \widehat{M} \times [0, \frac{1}{s}] \to \mathfrak{K}(l_2(\dot{A}) \otimes \mathfrak{H})$ is another such function then

$$(f * g)^{\natural}(x, \gamma, r) = \int\limits_{\mathcal{P}^{\perp}/\mathcal{L}} f^{\natural}(xy, \gamma, r) g^{\natural}(y^{-1}, \gamma, r)\, \mathrm{d}y.$$

We conclude that the assigment $f \mapsto f^{\natural}$ leads to an isomorphism from $C^*(\mathcal{P}^{\perp}, \mathcal{L}, \widetilde{\mathcal{F}}, T, \lambda)$ onto $C^*(\mathcal{P}^{\perp})_{\lambda} \otimes \widetilde{\mathcal{F}}$, where $\widetilde{\mathcal{F}}$ is a tensor product of $C(\mathrm{T}^2)$ with $\mathfrak{K}(l_2(\dot{A}) \otimes \mathfrak{H})$.

Consequently, $C^*(\mathcal{G})_{\lambda} \otimes \mathfrak{K}(\mathfrak{H})$ is isomorphic to the tensor product of $\mathfrak{K}(l_2(\dot{A}) \otimes \mathfrak{H})$ with a quotient of $C^*(\mathcal{P}^{\perp})_{\lambda} \otimes C(\mathrm{T}^2)$. As we remarked earlier the torsion group of $\mathcal{P}^{\perp}/(\mathcal{P}^{\perp})_{\lambda}$ is smaller than the torsion group of $\mathcal{G}/\mathcal{G}_{\lambda}$. Repeating the argument finitely often we find that $C^*(\mathcal{G})_{\lambda} \otimes \mathfrak{K}(\mathfrak{H})$ is isomorphic to the tensor product of $\mathfrak{K}(\mathfrak{H})$ with a quotient of $C^*(\mathcal{Q})_{\lambda} \otimes \mathcal{A}$, where $\mathcal{A}$ is a commutative algebra and $\mathcal{Q}$ is a two step nilpotent group containing $\mathcal{L}$ such that $\mathcal{Q}/\mathcal{Q}_{\lambda}$ is isomorphic to $\mathbb{Z}^n$. As we have seen earlier, in this case $C^*(\mathcal{Q})_{\lambda}$ is the tensor product of a commutative algebra $\mathcal{A}'$ with an n-dimensional noncommutative torus $\mathcal{T}$. Hence $C^*(\mathcal{G})_{\lambda} \otimes \mathfrak{K}(\mathfrak{H})$ is isomorphic to the tensor product of $\mathfrak{K}(\mathfrak{H})$ with a quotient of $\mathcal{T} \otimes \mathcal{A} \otimes \mathcal{A}'$. But each quotient of $\mathcal{T} \otimes \mathcal{A} \otimes \mathcal{A}'$ is again the tensor product of $\mathcal{T}$ with a commutative algebra. ∎

REMARK 5. *Let still $\mathcal{G}, \mathcal{L}, \lambda$ be as in Theorem 1. The final part of the proof simplifies considerably if $C^*(\mathcal{G})_{\lambda}$ is of type I which means $n = 0$; for further conditions equivalent to $n = 0$ see* (1.23), (2.6) *in* [12], *compare also* [1].

Outline of the proof in the type I case. As above one reduces to $\mathcal{G}/\mathcal{G}_{\lambda}$ being discrete. But then $n = 0$ implies that $\mathcal{G}/\mathcal{G}_{\lambda}$ is finite. By (1.18) in [12] one knows very precisely the structure of the canonical bicharacter on $\mathcal{G}/\mathcal{G}_{\lambda}$. There exist a direct decomposition $\mathcal{G}/\mathcal{G}_{\lambda} = AB$ and an isomorphism $j : A \to \widehat{B}$ with the following property. If $\rho : \mathcal{G} \to \mathcal{G}/\mathcal{G}_{\lambda}$ denotes the natural map, $\rho(x) = \rho^A(x)\rho^B(x)$ with $\rho^A(x) \in A$ and $\rho^B(x) \in B$, then $\lambda([x,y]) = j(\rho^A(x))(\rho^B(y)) j(\rho^A(y))(\rho^B(x))^{-1}$ for $x, y \in \mathcal{G}$.

Decompose A into a direct product of cyclic groups, $A = A_1 \cdot \dots \cdot A_r$. The isomorphism $j : A \to \widehat{B}$ gives a corresponding decomposition of $\widehat{B}$, whence a decomposition $B = B_1 \cdot \dots \cdot B_r$ and isomorphisms $j_k : A_k \to \widehat{B}_k$, $1 \leqslant k \leqslant r$. Denote by $p_k : \mathcal{G}/\mathcal{G}_{\lambda} \to A_k$ and $q_k : \mathcal{G}/\mathcal{G}_{\lambda} \to B_k$ the projections corresponding to the constructed direct decomposition of $\mathcal{G}/\mathcal{G}_{\lambda}$. Then one has

$$\lambda([x,y]) = \prod_{k=1}^{r} j_k p_k \rho(x)(q_k \rho(y))(j_k p_k \rho(y))(q_k \rho(x))^{-1}$$

for $x, y \in \mathcal{G}$.

For $1 \leqslant k \leqslant r$ choose infinite cyclic groups C_k, D_k and surjective homomorphisms $\varepsilon_k : C_k \to A_k$, $\eta_k : D_k \to B_k$. In an obvious way these homomorphisms define a surjection κ from $F := C_1 \times \cdots \times C_r \times D_1 \times \cdots \times D_r$ onto $\mathcal{G}/\mathcal{G}_\lambda$. As in the proof of Theorem 1 let $\mathcal{G}' \leqslant \mathcal{G} \times F$ be the pullback of ρ and κ, let $\varphi : \mathcal{G}' \to \mathcal{G}$ and $\psi : \mathcal{G}' \to F$ be the restrictions of the projections, let $\mathcal{W}' := \ker \psi$, let $\mathcal{L}' := \{(x, e) \mid x \in \mathcal{L}\}$, and let $\lambda' \in (\mathcal{L}')^\wedge$ be the character corresponding to $\lambda \in \mathcal{L}^\wedge$. The algebra $C^*(\mathcal{G})_\lambda$ is a quotient of $C^*(\mathcal{G}')_{\lambda'}$, and we shall investigate the latter.

Since $\mathcal{G}'/\mathcal{W}' \cong F$ is free, the sequence

$$1 \to \mathcal{W}'/\mathcal{L}' \to \mathcal{G}'/\mathcal{L}' \to \mathcal{G}'/\mathcal{W}' \to 1$$

splits, hence there exists an open subgroup $\mathcal{N}'$ of $\mathcal{G}'$ such that $\mathcal{W}' \cap \mathcal{N}' = \mathcal{L}'$ and $\mathcal{G}' = \mathcal{W}'\mathcal{N}'$. From the fact that $\mathcal{W}'$ centralizes $\mathcal{G}'$ modulo $\ker \lambda'$ one readily concludes that $C^*(\mathcal{G}')_{\lambda'}$ is isomorphic to the tensor product of the commutative algebra $C^*(\mathcal{W}')_{\lambda'}$ with $C^*(\mathcal{N}')_{\lambda'}$. The homomorphism ψ induces a surjection from $\mathcal{N}'$ onto F with kernel $\mathcal{L}'$. Choose any cross section $\iota : F \to \mathcal{N}'$ to this surjection with $\iota(e) = e$. This cross section yields an isomorphism from $C^*(\mathcal{N}')_{\lambda'}$ onto the twisted convolution algebra $C^*(F, m)$ for some cocycle m on F. The explicit form of m does not matter, we only note that the antisymmetrization $(x, y) \mapsto m(x, y)m(y, x)^{-1}$ of m, which determines (see [10]) the cohomology class of m, is given by

$$m(x, y)m(y, x)^{-1} = \lambda'([\iota(x), \iota(y)]).$$

For $x, y \in \mathcal{G}'$ one has

$$\begin{aligned}\lambda'([x, y]) &= \lambda([\varphi(x), \varphi(y)]) \\ &= \prod_{k=1}^{r} j_k p_k \rho\varphi(x)(q_k \rho\varphi(y)) j_k p_k \rho\varphi(y)(q_k \rho\varphi(x))^{-1} \\ &= \prod_{k=1}^{r} j_k p_k \kappa\psi(x)(q_k \kappa\psi(y)) j_k p_k \kappa\psi(y)(q_k \kappa\psi(x))^{-1}.\end{aligned}$$

Therefore, the antisymmetrization of m is given by

$$m(x, y)m(y, x)^{-1} = \prod_{k=1}^{r} j_k p_k \kappa(x)(q_k \kappa(y)) j_k p_k \kappa(y)(q_k \kappa(x))^{-1}$$

for $x, y \in F$. Define cocycles σ_k on $C_k \times D_k$ by $\sigma_k(c_k, d_k; c_k', d_k') = j_k \varepsilon_k(c_k)(\eta_k(d_k'))$, and define σ on $F = C_1 \times \cdots \times C_r \times D_1 \times \cdots \times D_r$ by

$$\sigma(c_1, \ldots, c_r, d_1, \ldots, d_r; c_1', \ldots, c_r', d_1', \ldots, d_r') = \prod_{k=1}^{r} \sigma_k(c_k, d_k; c_k', d_k').$$

Then the antisymmetrizations of m and σ coincide, hence m and σ are cohomologous, and the twisted convolution algebras $C^*(F, m)$ and $C^*(F, \sigma)$ are isomorphic.

It follows that $C^*(\mathcal{N}')_{\lambda'}$ is isomorphic to $C^*(F, \sigma)$, which is isomorphic to $\bigotimes_{k=1}^{r} C^*(C_k \times D_k, \sigma_k)$. But the $C^*(C_k \times D_k, \sigma_k)$ are rational rotation algebras, which are stably isomorphic to commutative algebras, either by [4] or by the considerations in the proof of Theorem 1. Therefore, $C^*(\mathcal{G}')_{\lambda'}$ is stably isomorphic to a commutative algebra, which implies that the same is true for $C^*(\mathcal{G})_\lambda$. ∎

From Theorem 1 we shall draw two consequences. The first one is included to justify the title of the article and also because the algebras $C^*(\mathcal{G})_\lambda$ appear frequently in this form.

COROLLARY 6. *Let H be a locally compact abelian group, and let m be a measurable cocycle on H. The antisymmetrization $(x, y) \mapsto m(x, y)m(y, x)^{-1}$ of m induces the structure of a quasi-symplectic space on H/Z_m in the terminology of* [12], *where $Z_m = \{x \in H \mid m(x, y) = m(y, x), \forall y \in H\}$. Suppose that the invariant $\mathrm{Inv}(H/Z_m)$ of this space contains $\mathbb{Z}^n$ for a certain n. Then the twisted convolution C^*-algebra $C^*(H, m)$ is stably isomorphic to the tensor product of $C_\infty((Z_m)^\wedge)$, an n-dimensional noncommutative torus and the algebra of compact operators on a Hilbert space.*

Proof. This corollary follows immediately from Theorem 1 by observing that $C^*(H, m)$ is isomorphic to $C^*(\mathcal{G})_\lambda$ where $\mathcal{G}$ is the central group extension

$$1 \to \mathbf{T} \to \mathcal{G} \to H \to 1$$

corresponding to m, and $\lambda(z) = z$ for $z \in \mathcal{L} := \mathbf{T}$. ∎

For the next (and final) corollary let G be a simply connected Lie group with commutator subgroup N, and let $\mathfrak{X}$ be a G-quasi-orbit in $\mathrm{Priv}\, C^*(N) = \widehat{N}$. Observe that N is locally isomorphic to an algebraic group, and hence it is a type I group, see [6]. From the results in [14] it follows that $\mathfrak{X}$ is a locally closed subset of $\widehat{N}$, compare also the discussion in front of (3.9) from [12]. For subsets $\mathfrak{A}$ of $\widehat{N}$ we denote by $k(\mathfrak{A}) = \bigcap_{\alpha \in \mathfrak{A}} \ker \alpha$ the kernel of $\mathfrak{A}$ in $C^*(N)$, and by $\overline{\mathfrak{A}}$ the closure of $\mathfrak{A}$ in $\widehat{N}$.

COROLLARY 7. *Let G, N and $\mathfrak{X}$ be a above. Then either $\mathfrak{X}$ is a closed point, $\mathfrak{X} = \{\rho\}$, ρ is finite-dimensional and extendible to G, and $C^*(G)/C^*(G) * k(\mathfrak{X})$ is isomorphic to the tensor product of a commutative algebra and a matrix algebra in $\dim \rho$ dimensions, or there exists an integer $n \geqslant 0$ such that the subquotient $C^*(G) * k(\overline{\mathfrak{X}} \setminus \mathfrak{X})/C^*(G) * k(\overline{\mathfrak{X}})$ of $C^*(G)$ is isomorphic to the tensor product of* *an n-dimensional noncommutative torus, a commutative algebra, and the algebra of compact operators on an infinite-dimensional separable Hilbert space.*

Proof. This corollary follows from (3.12) in [12] and Theorem 1. The calculation of the above integer n is the main issue in [12]. ∎

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