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A CLASS OF OPERATORS ASSOCIATED WITH REPRODUCING KERNELS

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ABSTRACT. For $t > 0$ let A_t be the operator on l^2 whose matrix under the standard basis has as its (i,j) entry $(1-|z_i|^2)^{t/2}(1-|z_j|^2)^{t/2}(1-z_i\bar{z}_j)^{-t}$. Here $\{z_n\}$ is a sequence of points in the open unit disk in the complex plane. The boundedness of the operators A_t, $1 \leqslant t < \infty$, will be characterized in terms of the distribution of the sequence $\{z_n\}$ in the hyperbolic metric.

KEYWORDS: *Reproducing kernel, separated sequences, Carleson measures.*

AMS SUBJECT CLASSIFICATION: 47B37, 46E22, 30C40.

1. INTRODUCTION

Let $\mathbf{D}$ be the open unit disk in the complex plane. For $t > 0$ and $\{a_n\} \subset \mathbf{D}$ we consider the operator on l^2 whose matrix under the standard basis is given by

$$A_t = \left(\frac{(1-|a_i|^2)^{\frac{t}{2}}(1-|a_j|^2)^{\frac{t}{2}}}{(1-a_i\overline{a}_j)^t}\right).$$

Such operators are important in function theory. For example, the operator

$$A_2 = \left(\frac{(1-|a_i|^2)(1-|a_j|^2)}{(1-a_i\overline{a}_j)^2}\right)$$

appears in the proof of Carleson's characterization of interpolating sequences on the unit disk; see [1], where the boundedness of A_2 when $\{a_n\}$ satisfies

$$\prod_{k\geqslant 1, k\neq n}\left|\frac{a_k-a_n}{1-a_k\overline{a}_n}\right| \geqslant \delta, \qquad n = 1,2,\ldots$$

is proved by an ad hoc method.

Let ρ be the pseudo-hyperbolic distance on $\mathbf{D}$. Thus

$$\rho(z,w) = \left|\frac{z-w}{1-\overline{z}w}\right|, \qquad z,w \in \mathbf{D}.$$

Recall that a sequence $\{a_n\} \subset \mathbf{D}$ is called separated if there exists a constant $\delta > 0$ such that $\rho(a_n, a_m) \geqslant \delta$ for all $n, m \geqslant 1$ with $n \neq m$; and the sequence $\{a_n\}$ is called uniformly separated if there exists a constant $\sigma > 0$ such that

$$\prod_{k\geqslant 1, k\neq n} \rho(a_k, a_n) \geqslant \sigma$$

for all $n \geqslant 1$. Carleson's characterization of interpolating sequences states that a sequence $\{a_n\}$ in $\mathbf{D}$ is interpolating for H^∞ if and only if $\{a_n\}$ is uniformly separated; see [1], [2].

The purpose of this note is to prove the following characterization for the boundedness of A_t when $1 \leqslant t < \infty$.

THEOREM. *Let A_t be the infinite matrix defined earlier.*

(i) *If $t > 1$, then A_t is bounded on l^2 if and only if $\{a_n\}$ is the union of finitely many separated sequences.*

(ii) *If $t = 1$, then A_t is bounded on l^2 if and only if $\{a_n\}$ is the union of finitely many uniformly separated sequences.*

We do not have a good answer to the problem of the boundedness of A_t in the case $0 < t < 1$.

2. PROOF OF THE THEOREM

The first part of the proof is quite general; it applies to any reproducing Hilbert space. The second part will then depend on the specific Hilbert spaces. In what follows the letter C will denote a positive constant whose value may change from one occurance to another.

Let H be a reproducing Hilbert space of functions on a domain Ω. We denote by K, or $K(z,w)$, the reproducing kernel of H. For a sequence $\{a_n\}$ in Ω we consider the operator on l^2 whose matrix under the standard basis is

$$A(H) = \left(\frac{K(a_i,a_j)}{\sqrt{K(a_i,a_i)K(a_j,a_j)}}\right).$$

For any finite sequence $\{c_1, \dots, c_n\}$ of complex numbers we have

$$\sum_{i,j=1}^{n} \frac{K(a_i, a_j)}{\sqrt{K(a_i, a_i)K(a_j, a_j)}} c_i \overline{c_j} = \left\| \sum_{k=1}^{n} c_k \frac{K_{a_k}}{\sqrt{K(a_k, a_k)}} \right\|^2,$$

where $K_{a_k}(z) = K(z, a_k)$ and $\|\cdot\|$ is the norm in H. Therefore, the infinite matrix $A(H)$ defines a bounded operator on l^2 if and only if there exists a constant $C > 0$ (independent of n) such that

$$\sup\left\{ \left\| \sum_{k=1}^{n} c_k \frac{K_{a_k}}{\sqrt{K(a_k, a_k)}} \right\| : \sum_{k=1}^{n} |c_k|^2 \leqslant 1 \right\} \leqslant C$$

for all $n \geqslant 1$. Using the usual duality of Hilbert spaces, we see that the above inequality is equivalent to

$$\sup\left\{ \left| \left\langle f, \sum_{k=1}^{n} c_k \frac{K_{a_k}}{\sqrt{K(a_k, a_k)}} \right\rangle \right| : \sum_{k=1}^{n} |c_k|^2 \leqslant 1, \|f\| \leqslant 1 \right\} \leqslant C, \quad n \geqslant 1.$$

By the reproducing property of the kernel functions, the above inequality is equivalent to

$$\sup\left\{ \left| \sum_{k=1}^{n} \frac{c_k f(a_k)}{\sqrt{K(a_k, a_k)}} \right| : \|f\| \leqslant 1, \sum_{k=1}^{n} |c_k|^2 \leqslant 1 \right\} \leqslant C, \quad n \geqslant 1.$$

Using the duality of the usual l^2 spaces we conclude that the boundedness of $A(H)$ on l^2 is equivalent to

$$\sup\left\{ \sum_{k=1}^{n} \frac{|f(a_k)|^2}{K(a_k, a_k)} : \|f\| \leqslant 1 \right\} \leqslant C^2,$$

or

$$\sum_{k=1}^{n} \frac{|f(a_k)|^2}{K(a_k, a_k)} \leqslant C^2 \|f\|^2$$

for all n and f. This is clearly equivalent to

$$\sum_{k=1}^{\infty} \frac{|f(a_k)|^2}{K(a_k, a_k)} \leqslant C^2 \|f\|^2$$

for all $f \in H$. This finishes the general part of the proof.

We now specialize to the operators A_t.

If $t = 1$, then it is clear that $A_1 = A(H^2)$, where H^2 is the Hardy space on the unit disk $\mathbb{D}$, whose reproducing kernel is given by

$$K(z,w) = \frac{1}{1 - z\overline{w}}.$$

By the first part of this proof, A_1 is bounded on l^2 if and only if

$$\sum_{k=1}^{\infty}(1 - |a_k|^2)|f(a_k)|^2 \leqslant C\|f\|^2_{H^2}$$

for some constant $C > 0$ and all $f \in H^2$. It is well known that this holds if and only if the sequence $\{a_n\}$ is a finite union of uniformly separated sequences; see [3] for example.

When $1 < t < \infty$, it is easy to check that $A_t = A(H_t)$, where H_t is the weighted Bergman space consisting of analytic functions f in the unit disk $\mathbb{D}$ with

$$\|f\|^2 = (t-1)\int_{\mathbb{D}} |f(z)|^2(1 - |z|^2)^{t-2}\,\mathrm{d}A(z) < \infty.$$

The reproducing kernel of H_t is

$$K(z,w) = \frac{1}{(1 - z\overline{w})^t}.$$

See [6] for further information on weighted Bergman spaces. By the first part of the proof, the matrix A_t, $1 < t < \infty$, defines a bounded operator on l^2 if and only if

$$\sum_{k=1}^{\infty}(1 - |a_k|^2)^t|f(a_k)|^2 \leqslant C\int_{\mathbb{D}} |f(z)|^2(1 - |z|^2)^{t-2}\,\mathrm{d}A(z)$$

for all $f \in H_t$. Let μ_t be the atomic measure on $\mathbb{D}$ defined by

$$\mu_t(E) = \sum\left\{(1 - |a_k|^2)^t : a_k \in E\right\}, E \subset \mathbb{D}.$$

For $a \in \mathbb{D}$ and $0 < r < 1$ let $E(a,r)$ be the pseudo-hyperbolic disk centered at a with radius r, namely,

$$E(a,r) = \{z \in \mathbb{D} : \rho(z,a) < r\}.$$

Then combining the above with a well-known result from the theory of Carleson measures for Bergman spaces (see [5] for example) we conclude that A_t is bounded on l^2 if and only if there exists a constant $C > 0$ (depending on r only) such that

$$\mu_t\left(E(a,r)\right) \leqslant C(1 - |a|^2)^t,$$

or

$$\sum\left\{(1-|a_k|^2)^t : a_k \in E(a,r)\right\} \leqslant C(1-|a|^2)^t$$

for all $a \in \mathbb{D}$. Now for any fixed $0 < r < 1$ it is easy to find a constant $C > 0$ such that

$$1-|z|^2 \leqslant C(1-|a|^2) \leqslant C^2(1-|z|^2)$$

for all z and a in $\mathbb{D}$ with $\rho(z,a) < r$. In particular, for every $a \in \mathbb{D}$ we have

$$1-|a_k|^2 \leqslant C(1-|a|^2) \leqslant C^2(1-|a_k|^2), \qquad a_k \in E(a,r).$$

Thus the operator A_t, $1 < t < \infty$, is bounded on l^2 if and only if there exists a constant $C > 0$ (depending on r and t) such that

$$\sum\left\{(1-|a_k|^2)^2 : a_k \in E(a,r)\right\} \leqslant C(1-|a|^2)^2$$

for all $a \in \mathbb{D}$, which in turn is equivalent to

$$\sum_{k=1}^{\infty}(1-|a_k|^2)^2|f(a_k)|^2 \leqslant C\int_{\mathbb{D}}|f(z)|^2\,\mathrm{d}A(z)$$

for all f in the regular Bergman space. It is well-known (see [7] for example) that the above holds if and only if the sequence is the union of finitely many separated sequences. This completes the proof of the theorem. ∎

3. FURTHER REMARKS

As was mentioned earlier, we do not know when A_t is bounded on l^2 in the case $0 < t < 1$. To answer this question, we need to know when the atomic measure μ_t is a Carleson measure on certain Dirichlet type spaces. Such measures are characterized in [4] in terms of certain capacities on the unit circle. We feel that in the case of the atomic measures μ_t one should be able to determine more concretely when μ_t is a Carleson measure on these Dirichlet type spaces.

Since our proof of the main theorem only depends on duality and Carleson measures, one sees that A_t, $1 \leqslant t < \infty$, is bounded on l^2 if and only if A_t is bounded on l^p for some (or all) p with $1 < p < \infty$. We omit the details.

Finally, note that for $1 < t < \infty$, the boundedness of A_t on l^2 is independent of t, and depends only on the sequence $\{a_n\}$.

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