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BOOK REVIEW:

Pseudo-Differential Operators, Singularities, Applications, **Yuri V. Egorov, Bert-Wolfgang Schulze**, Operator Theory: Advances and Applications, vol. 93, Birkhäuser Verlag, Basel – Boston – Berlin, 1997, IX + 349 pag., ISBN 3–7643–5484–4.

KEYWORDS: *Pseudo-differential operators, elliptic pseudo-differential operators, elliptic boundary value proplems, Fourier integral operators, cone algebra, weighted Sobolev spaces, conical and edge singularities.*

AMS SUBJECT CLASSIFICATION: 35S30, 47G30, 58G15.

The pseudo-differential operators are already one of the most used tools in the study of partial differential equations. There are a lot of books about this topic and there are results which can be considered as classical. But the theory of pseudo-differential operators and its range of applications are still developing. Both these aspects, the classical and the modern one, are illustrated in the book of Yu.V. Egorov and B.-W. Schulze. The first six chapters can be considered as a course on pseudo-differential operators. The theory of pseudo-differential operators in $\mathbb{R}^n$ and on smooth manifolds and its applications to elliptic boundary value problems as well as elements of Fourier integral operators and propagation of singularities are presented. Also exposed is the theory of Kondratiev of boundary value problems on cones. A first frame of the problems which arise and of the methods employed to solve them is thus obtained. This allows to better understand the second part of the book, Chapters 7–9, where the modern theory of pseudo-differential operators on manifolds with conical and edge singularities is exposed. This theory is applied to the study of elliptic problems on manifolds with singularities.

Below is a short description of the content of each chapter.

Chapter 1: *Sobolev spaces.* Sobolev spaces in $\mathbb{R}^n$ and in a domain are introduced. Basic results concerning embedding properties, traces on the boundary and invariance under diffeomorphisms are given.

Chapter 2: *Pseudo-differential operators.* After introducing differential operators on a manifold it is proved that a local operator is a differential operator on any compact set. Next the construction of the fundamental solution of a partial differential operator with constant coefficients and, as particular cases, of the wave operator, of the heat operator and of the laplacian is described. Pseudo-differential operators on $\mathbf{R}^n$ are a natural generalization of differential operators. The theorems on the composition of two pseudo-differential operators and on the adjoint are proved. It is also proved that pseudo-differential operators are pseudo-local. The theorem on the transformation of pseudo-differential operators under a change of coordinates allows to define such operators and their principal symbol on a smooth manifold. The chapter ends with proofs of Gøarding and sharp Gøarding inequalities.

Chapter 3: *Elliptic pseudo-differential operators.* For elliptic operators with variable coefficients and for elliptic pseudo-differential operators one cannot have in general a fundamental solution. But it is possible to construct a parametrix which is again a pseudo-differential operator. Once the existence of the parametrix is proved one can show that an elliptic operator on a compact manifold is Fredholm if it is considered as an operator between Sobolev spaces. If the manifold is also simply connected and of dimension greater than two then the index of this operator is equal to zero. All these results are proved in Chapter 3.

Chapter 4: *Elliptic boundary value problems.* The elliptic boundary problem in a bounded domain with smooth boundary is defined. Using a parametrix it is demonstrated that the problem is well posed. In order to make the ideas more clear the case of an elliptic problem in a semispace is considered first.

Chapter 5: *Kondratiev's theory.* Elliptic boundary value problems in a bounded domain with a conical singularity of the form $x_n^{2p} = \sum_{|\alpha|=2p} h_\alpha n^\alpha$ $(p > 0)$ are solved. The asymptotics of the solutions in the neighbourhood of the singularity are also studied.

Chapter 6: *Non-elliptic operators; propagation of singularities.* For the proof of the main theorems of this chapter Fourier integral operators are used. Their definition and some of their principal properties are given. For example it is proved that the conjugate of a pseudo-differential operator with a Fourier integral operator with symbol of order zero is again a pseudo-differential operator. The wave front of a distribution and its connections with pseudo-differential operators and Fourier integral operators are also described. These notions are next applied to prove a propagation of singularity theorem, local and semi-local solvability of real principal type operators and to solve the Cauchy problem for strongly hyperbolic equations.

Chapter 7: *Pseudo-differential operators on manifolds with conical and edge singularities; motivation and technical preparations.* First of all the main steps of the development of a theory of pseudo-differential operators on manifolds with singularities appropriate for the study of elliptic problems are pointed out. Besides an algebra of pseudo-differential operators, adequate symbolic structures, Sobolev spaces and subspaces of distributions with asymptotics are needed. Next an index theory must be developed. These ideas are illustrated for the cases of Fuchs type differential operators on manifolds with conical singularities and edge degenerate operators on manifolds with edges. The main results say, roughly speaking, that ellipticity is equivalent with fredholmicity. In the next two chapters only the implication from left to right is proved. Parameter-dependent pseudo-differential operators and Mellin pseudo-differential operators with operator valued symbols are the main subjects of the second part of the chapter.

Chapter 8: *Pseudo-differential operators on manifolds with conical singularities.* The fact that elliptic pseudo-differential operators on bounded manifolds with conical singularities or on an infinite cone are Fredholm is proved. The ingredients for the proof are: weighted Sobolev spaces with asymptotics, smoothing operators (Green operators and smoothing Mellin operators), the cone algebra of pseudo-differential operators, the notion of ellipticity for operators in this algebra defined in terms of interior symbols and conormal symbols, the construction of a parametrix for elliptic operators. Regularity results are also obtained. In the case of the infinite cone the behaviour at infinity is controlled by the exit symbol and this symbol intervenes also in the definition of ellipticity. These infinite cones will be the model cones of the wedges from the next chapter.

Chapter 9: *Pseudo-differential operators on manifolds with edges.* The main steps of the previous chapter are taken again for manifolds with edges. In this case the operators can be regarded as pseudo-differential operators along the edge with cone operator-valued symbols, the cone being the model cone of the wedge. The edge operator associated to an edge problem for a pseudo-differential operator is a matrix whose principal entries are the pseudo-differential operator, the edge trace operator and the edge potential operator. The trace operator corresponds to the boundary conditions. Again a parametrix is constructed for elliptic edge operators and as a consequence it is proved that these operators are Fredholm operators between weighted Sobolev spaces. The pseudo-differential calculus employed in this chapter is for the first time exposed here. In the final section it is pointed out that boundary value problems on manifolds with smooth boundary can be interpreted as edge problems where the edge is the boundary and the model cone is the inner normal.

In conclusion this book combines a rapid introduction into the theory of pseudo-differential operators with current research in the field; it is therefore adressed to a large spectrum of readers.

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