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STANDARD MODELS UNDER POLYNOMIAL POSITIVITY CONDITIONS

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Abstract. We develop standard models for commuting tuples of bounded linear operators on a Hilbert space under certain polynomial positivity conditions, generalizing the work of V. Müller and F.-H. Vasilescu in [6], [14].

As a consequence of the model, we prove a von Neumann-type inequality for such tuples. Up to similarity, we obtain the existence of in a certain sense "unitary" dilations.

Keywords: *Multivariable spectral theory, weighted multishifts, standard models, dilations, functional calculus.*

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1. INTRODUCTION

Let $\mathcal{H}$ be a separable Hilbert space and $T = (T_1, \ldots, T_n)$ a commuting tuple of bounded linear operators on $\mathcal{H}$. T is called a *spherical contraction*, if $\sum\limits_{i=1}^{n} {T_i}^* T_i \leqslant \mathbf{1}_{\mathcal{H}}$, and a *spherical unitary*, if $\sum\limits_{i=1}^{n} {T_i}^* T_i = \mathbf{1}_{\mathcal{H}}$ and in addition, all components of T are normal. We say that T has a *spherical dilation* if there is a spherical unitary U which dilates T, i.e. $T^\alpha = P_{\mathcal{H}} U^\alpha | \mathcal{H}$ for all $\alpha \in \mathbb{N}_0^n$. There is no easy generalization of the famous Dilation Theorem for contractions of Sz.-Nagy (see [12]) to spherical contractions: in general, spherical contractions have no spherical dilations, and there is not even a von Neumann-type inequality over the unit ball in $\mathbb{C}^n$ for spherical contractions ([3]). Athavale has shown in [1] that under certain additional positivity conditions a spherical contraction T has a spherical dilation, and Müller and Vasilescu have developed a model for T under these conditions which reproduces this result ([6], [14]). This model consists of a spherical unitary part and a weighted backward multishift part which for suitable order coincides with the adjoint of the tuple of multiplication operators with the coordinates on a Hardy space over the unit ball in $\mathbb{C}^n$. For $n = 1$, this is just the well-known coisometric extension for contractions.

In the current paper, we will develop a model for a commuting tuple T under certain polynomial positivity conditions. We call T a *P-contraction*, where $P = \sum\limits_{\gamma\in\mathbb{N}_0^n} a_\gamma x^\gamma$ is a polynomial with non-negative coefficients of a certain type, if $\sum\limits_{\gamma\in\mathbb{N}_0^n} a_\gamma T^{*\gamma}T^\gamma \leqslant \mathbf{1}_\mathcal{H}$, and a *$P$-unitary* if $\sum\limits_{\gamma\in\mathbb{N}_0^n} a_\gamma T^{*\gamma}T^\gamma = \mathbf{1}_\mathcal{H}$, $T_1, \dots, T_n$ normal. We will show that P-contractions satisfying additional positivity conditions of suitable order have a model consisting of a P-unitary part and a weighted backward multishift part, which may be identified topologically with the adjoint of the multiplication tuple on a Bergman space. In particular, up to topological equivalence, T has a P-unitary dilation and therefore a rich functional calculus.

The crucial tools in identifying the weighted backward multishift with the adjoint Bergman space multiplication tuple are a theorem of A. Cumenge from complex analysis which allows to extend Bergman space functions on a complex submanifold $\mathcal{M}$ to Hardy space functions on a strictly pseudoconvex set containing $\mathcal{M}$ and the simple idea of regarding a P-contraction as a spherical contraction in a higher dimension.

2. PRELIMINARIES AND NOTATION

A commuting tuple $T = (T_1, \dots, T_n)$ of bounded linear operators on the separable Hilbert space $\mathcal{H}$ will be called a *commuting multioperator* or just a *multioperator*. For $A \in \mathcal{L}(\mathcal{H})$, let C_A be the bounded linear map

$$\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), \quad X \mapsto A^*XA, \tag{2.1}$$

and for a commuting tuple $T = (T_1, \dots, T_n) \in \mathcal{L}(\mathcal{H})^n$ let $C_T = (C_{T_1}, \dots, C_{T_n})$. If $P = \sum\limits_{\gamma\in\mathbb{N}_0^n} a_\gamma x^\gamma \in \mathbb{C}[X_1, \dots, X_n]$ is a polynomial, then $P(C_T)$ is the bounded linear map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $X \mapsto \sum\limits_{\gamma\in\mathbb{N}_0^n} a_\gamma T^{*\gamma}XT^\gamma$. This map is well-defined, since $T_1, \dots, T_n$ commute.

If $T = (T_1, \dots, T_n)$ is a commuting multioperator on $\mathcal{H}$, $S = (S_1, \dots, S_n)$ a commuting multioperator on some Hilbert space $\mathcal{H}'$ and $A : \mathcal{H} \to \mathcal{H}'$ is a linear map, then we will write $AT = SA$ for the identity $AT_i = S_iA$, $i = 1, \dots, n$. In this situation, we call T and S *topologically equivalent* or *similar* if A is a topological isomorphism. We will call a commuting multioperator *normal* in case all components are normal.

For $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, we will denote the tuple $(\overline{z}_1 w_1, \ldots, \overline{z}_n w_n)$ by $\overline{z}w$ and the tuple $(|z_1|^2, \ldots, |z_n|^2)$ by $|z|^2$.

Let us introduce the class of polynomials from which our positivity conditions are obtained. A polynomial $P \in \mathbb{C}[X_1, \ldots, X_n]$ is said to be *positive regular*, if

(i) the constant term is 0;

(ii) P has non-negative coefficients;

(iii) the coefficients of the linear terms $X_1, \ldots, X_n$ are all different from 0.

There is a complete Reinhardt domain in $\mathbb{C}^n$ associated to each positive regular polynomial P, namely

$$\mathcal{P} = \{z \in \mathbb{C}^n \mid P(|z|^2) < 1\} \tag{2.2}$$

which we call the *P-ball*. For $P = \sum_{i=1}^{n} x_i$, the P-ball is just the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$.

For a positive regular polynomial $P, X \in \mathcal{L}(\mathcal{H})$ positive and $m \in \mathbb{N}$, we will call a commuting multioperator T *(P, m)-positive for X*, if

$$\Delta_P^{(1)}(X) := (1 - P)(C_T)(X) \geqslant 0 \tag{2.3}$$

and

$$\Delta_P^{(m)}(X) := (1 - P)^m(C_T)(X) \geqslant 0. \tag{2.4}$$

In this case,

$$\Delta_P^{(k)}(X) := (1 - P)^k(C_T)(X) \geqslant 0 \quad \text{for } 1 \leqslant k \leqslant m, \tag{2.5}$$

as one obtains completely analogously to Lemma 2 in [6]. The tuple T is said to be *(P, m)-positive*, if it is (P, m)-positive for $\mathbf{1}_{\mathcal{H}}$. Furthermore, we call T a *P-isometry*, if $\Delta_P^{(1)} := \Delta_P^{(1)}(\mathbf{1}_{\mathcal{H}}) = 0$, and a *$P$-unitary*, if in addition T is normal.

For $P = \sum_{i=1}^{n} x_i$, the $(P, 1)$-positive operators are just the spherical contractions.

3. STANDARD MODELS

We will now develop in analogy to [6] a standard model for (P, m)-positive commuting tuples, consisting of a part which is the adjoint of a multiplication tuple — or, equivalently, a weighted backward multishift — and a P-unitary part.

For $|P(x)| < 1$, we have

$$\frac{1}{(1-P(x))^m} = \Big(\sum_{j=0}^{\infty} P^j(x)\Big)^m. \tag{3.1}$$

Therefore the function $x \mapsto 1/(1-P(x))^m$ has a power series representation which converges compactly on $\{x \,|\, |P(x)| < 1\}$ and coincides with the Taylor series expansion at 0. For positive regular P, all Taylor coefficients are positive.

DEFINITION 3.1. Let P be a positive regular polynomial in n variables and let $m \in \mathbb{N}$. For each $\alpha \in \mathbb{N}_0^n$, let $\rho_P^m(\alpha)$ be the Taylor coefficient at index α of the function $x \mapsto 1/(1-P(x))^m$ at 0.

We will denote the coefficients $\rho_P^m(\alpha)$, $\alpha \in \mathbb{N}_0^n$, as (P, m)-*weights.*

Now let $H^2(\rho_P^m)$ be the linear space of all formal power series $\sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha$ such that $\sum_{\alpha\in\mathbb{N}_0^n} |c_\alpha|^2 1/\rho_P^m(\alpha) < \infty$. The space $H^2(\rho_P^m)$ is obviously a Hilbert space with the inner product

$$\Big\langle \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha\,,\ \sum_{\alpha'\in\mathbb{N}_0^n} b_{\alpha'} z^{\alpha'} \Big\rangle = \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha \bar{b}_\alpha \frac{1}{\rho_P^m(\alpha)}. \tag{3.2}$$

It can be regarded as a space of holomorphic functions on the P-ball $\mathcal{P}$, and there is an obvious reproducing kernel:

LEMMA 3.2. *The elements of $H^2(\rho_P^m)$ define holomorphic functions on the P-ball $\mathcal{P}$. Furthermore, let*

$$\mathfrak{k} : \mathcal{P} \times \mathcal{P} \to \mathbb{C}, \quad \mathfrak{k}(z, w) = \frac{1}{(1-P(\overline{z}w))^m}. \tag{3.3}$$

For each $z \in \mathcal{P}$, the function $\mathfrak{k}_z = \mathfrak{k}(z, \cdot)$ is a holomorphic function on $\mathcal{P}$ and by identification with its Taylor series expansion at 0 *an element of $H^2(\rho_P^m)$ such that*

$$\langle f, \mathfrak{k}_z\rangle = f(z), \quad f \in H^2(\rho_P^m).$$

We have $\|\mathfrak{k}_z\| = \big(1/(1-P(|z|^2))^m\big)^{1/2}$ for $z \in \mathcal{P}$.

Proof. For $f = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha w^\alpha \in H^2(\rho_P^m)$ and $z \in \mathcal{P}$, we have

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha z^\alpha| &\leqslant \Big(\sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha|^2 \frac{1}{\rho_P^m(\alpha)} \Big)^{1/2} \Big(\sum_{\alpha \in \mathbb{N}_0^n} \rho_P^m(\alpha) |z^\alpha|^2 \Big)^{1/2} \\ &= \frac{1}{(1 - P(|z|^2))^{m/2}} \|f\|. \end{aligned} \tag{3.4}$$

Thus f converges uniformly on compact subsets of $\mathcal{P}$ and defines a holomorphic function on $\mathcal{P}$ (see [9], Corollaries 1.16 and 1.17), which we again call f. Furthermore, one obtains for $z \in \mathcal{P}$

$$\begin{aligned} \|\mathfrak{k}_z\|^2 &= \Big\| \sum_{\alpha \in \mathbb{N}_0^n} \rho_P^m(\alpha) \overline{z}^\alpha w^\alpha \Big\|^2 = \sum_{\alpha \in \mathbb{N}_0^n} |z^\alpha|^2 \rho_P^m(\alpha) \\ &= \frac{1}{(1 - P(|z|^2))^m} < \infty \end{aligned} \tag{3.5}$$

and $\langle f, \mathfrak{k}_z \rangle = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha = f(z)$. ∎

We define multiplication operators M_{z_i}, $i = 1, \dots, n$, on $H^2(\rho_P^m)$ by $M_{z_i} \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^{\alpha + e_i}$.

For the study of the multiplication operators and the construction of the model, we need more information about the (P,m)-weights $(\rho_P^m(\alpha))$. Thus we give a more explicit form and a recursion formula for the weights.

Let us first introduce some notation. For a given positive regular polynomial P, let $I_P = \{\gamma \in \mathbb{N}_0^n \mid a_\gamma > 0\}$ and $\operatorname{mult}(P) = |I_P|$ be the number of nontrivial coefficients in P. We form the vector of the coefficients of P, $A = (a_\gamma)_{\gamma \in I_P} \in \mathbb{C}^{I_P}$. Furthermore, let for $K = (k_\gamma)_{\gamma \in I_P}$, $L = (l_\gamma)_{\gamma \in I_P} \in \mathbb{C}^{I_P}$

$$A^K := \prod_{\gamma \in I_P} a_\gamma^{k_\gamma}, \qquad |K| := \sum_{\gamma \in I_P} k_\gamma, \tag{3.6}$$

$$\binom{|K|}{K} := \frac{|K|!}{\prod_{\gamma \in I_P} k_\gamma!}, \qquad \binom{L}{K} := \prod_{\gamma \in I_P} \binom{l_\gamma}{k_\gamma} \tag{3.7}$$

and

$$[K] := ([K]_1, \dots, [K]_n), \quad \text{where } [K]_i := \sum_{\gamma \in I_P} \gamma_i k_\gamma \text{ for } i \in \{1, \dots, n\}. \tag{3.8}$$

Write $K \leqslant L$ if $k_\gamma \leqslant l_\gamma$ for all $\gamma \in I_P$. We need some combinatorial results:

LEMMA 3.3. *For $L \in \mathbb{N}_0^{I_P}$ and $m \in \mathbb{N}$,*

$$\binom{|L|}{L}\binom{|L|+m}{m} = \sum_{\substack{K\in\mathbb{N}_0^{I_P}\\ K\leqslant L}} \binom{|L-K|}{L-K}\binom{|K|}{K}\binom{|K|+m-1}{m-1}. \tag{3.9}$$

Proof. We obtain the identity

$$\sum_{\substack{K\leqslant L\\ |K|=r}} \binom{L}{K} = \binom{|L|}{r} \qquad \text{for } r = 0, \dots, |L| \tag{3.10}$$

by induction over the number of nontrivial coefficients $|I_P|$ of P and the well-known fact

$$\sum_{q=0}^{r} \binom{|L|-l}{q}\binom{l}{r-q} = \binom{|L|}{r} \qquad \text{for } 0 \leqslant l \leqslant |L|. \tag{3.11}$$

Now, we have

$$\begin{aligned}
&\sum_{\substack{K\in\mathbb{N}_0^{I_P}\\ K\leqslant L}} \binom{|L-K|}{L-K}\binom{|K|}{K}\binom{|K|+m-1}{m-1}\\
&\quad = \sum_{r=0}^{|L|}\left[\sum_{\substack{K\leqslant L\\ |K|=r}} \binom{|L|-r}{L-K}\binom{r}{K}\binom{r+m-1}{m-1}\right]\\
&\quad = \binom{|L|}{L}\sum_{r=0}^{|L|}\left[\frac{(|L|-r)!\,r!}{|L|!}\binom{r+m-1}{m-1}\sum_{\substack{K\leqslant L\\ |K|=r}}\binom{L}{K}\right]\\
&\quad = \binom{|L|}{L}\sum_{r=0}^{|L|}\binom{r+m-1}{m-1}.
\end{aligned} \tag{3.12}$$

It remains to show that $\sum_{r=0}^{|L|} \binom{r+m-1}{m-1} = \binom{|L|+m}{m}$ for $m \in \mathbb{N}$, which is an easy induction. ∎

Furthermore, Equation (3.10) yields the identity

$$\sum_{\substack{K\leqslant L\\ |K|=r}} \binom{r}{K}\binom{|L|-|K|}{L-K} = \frac{r!(|L|-r)!}{|L|!}\binom{|L|}{L}\sum_{\substack{K\leqslant L\\ |K|=r}}\binom{L}{K} = \binom{|L|}{L} \tag{3.13}$$

for $0 \leqslant r \leqslant |L|$. Now we can characterize the (P, m)-weights more explicitly.

Lemma 3.4. *Let P be a positive regular polynomial and $m \in \mathbb{N}$. Then*

$$\rho_P^m(\alpha) = \sum_{\substack{K\in\mathbb{N}_0^{I_P}\\ [K]=\alpha}} A^K \binom{|K|+m-1}{|K|}\binom{|K|}{K} \quad \textit{for } \alpha \in \mathbb{N}_0^n. \tag{3.14}$$

Proof. For $m = 1$ and $|P(x)| < 1$, we have

$$\begin{aligned}\frac{1}{1-P(x)} &= \sum_{j=0}^{\infty} P(x)^j = \sum_{j=0}^{\infty}\Bigg[\sum_{\substack{K\in\mathbb{N}_0^{I_P}\\ |K|=j}} \binom{|K|}{K}\prod_{\gamma\in I_P} a_\gamma^{k_\gamma}(x^\gamma)^{k_\gamma}\Bigg]\\ &= \sum_{j=0}^{\infty}\Bigg[\sum_{\substack{K\in\mathbb{N}_0^{I_P}\\ |K|=j}} A^K\binom{|K|}{K}x^{[K]}\Bigg] = \sum_{\alpha\in\mathbb{N}_0^n} x^\alpha \sum_{\substack{K\in\mathbb{N}_0^{I_P}\\ [K]=\alpha}} A^K\binom{|K|}{K}.\end{aligned} \tag{3.15}$$

So, by uniqueness of the coefficients, (3.14) holds for $m = 1$. Now let (3.14) be valid for an arbitrary $m \in \mathbb{N}$. Then we obtain again by uniqueness and by Lemma 3.3 the identity for $m + 1$:

$$\begin{aligned}\frac{1}{(1-P(x))^{m+1}} &= \Bigg(\sum_{\alpha\in\mathbb{N}_0^n}\rho_P^m(\alpha)x^\alpha\Bigg)\Bigg(\sum_{\alpha\in\mathbb{N}_0^n}\rho_P^1(\alpha)x^\alpha\Bigg)\\ &= \Bigg(\sum_{K\in\mathbb{N}_0^{I_P}}\binom{|K|}{K}\binom{|K|+m-1}{m-1}A^Kx^{[K]}\Bigg)\Bigg(\sum_{J\in\mathbb{N}_0^{I_P}}\binom{|J|}{J}A^Jx^{[J]}\Bigg)\\ &= \sum_{L\in\mathbb{N}_0^{I_P}}\Bigg[A^Lx^{[L]}\sum_{\substack{K\in\mathbb{N}_0^{I_P}\\ K\leqslant L}}\binom{|L-K|}{L-K}\binom{|K|+m-1}{m-1}\binom{|K|}{K}\Bigg]\\ &= \sum_{\alpha\in\mathbb{N}_0^n}x^\alpha\Bigg[\sum_{\substack{L\in\mathbb{N}_0^{I_P}\\ [L]=\alpha}}A^L\binom{|L|}{L}\binom{|L|+m}{m}\Bigg]. \quad ∎\end{aligned} \tag{3.16}$$

Let from now on $\rho_P^m(\alpha) = 0$ for $\alpha \in \mathbb{Z}^n \setminus \mathbb{N}_0^n$. Then we obtain the following recursion formulae for the (P, m)-weights:

Remark 3.5. Let $P = \sum_{\gamma\in\mathbb{N}_0^n} a_\gamma x^\gamma$ be a positive regular polynomial and let $Q = 1 - (1 - P)^m = \sum_{\gamma\in\mathbb{N}_0^n} b_\gamma x^\gamma$. Then

$$\rho_P^m(\alpha) = \sum_{\gamma\in\mathbb{N}_0^n} b_\gamma \rho_P^m(\alpha-\gamma), \quad \alpha \in \mathbb{N}_0^n \tag{3.17}$$

and for $m > 1$,

$$\rho_P^m(\alpha) = \rho_P^{m-1}(\alpha) + \sum_{\gamma \in \mathbb{N}_0^n} a_\gamma \rho_P^m(\alpha - \gamma). \tag{3.18}$$

Proof. For $\alpha \in \mathbb{N}_0^n$, $\sum_{\gamma \in \mathbb{N}_0^n} b_\gamma \rho_P^m(\alpha - \gamma)$ is the coefficient at index α of the product power series $\Big(\sum_{\alpha \in \mathbb{N}_0^n} \rho_P^m(\alpha) x^\alpha\Big)\Big(\sum_{\gamma \in \mathbb{N}_0^n} b_\gamma x^\gamma\Big)$. We obtain Equation (3.17) by comparison of coefficients, since for $|P(x)| < 1$ we have

$$\begin{aligned}\sum_{\alpha \in \mathbb{N}_0^n} x^\alpha \sum_{\gamma \in \mathbb{N}_0^n} b_\gamma \rho_P^m(\alpha - \gamma) =&(1 - P(x))^{-m}(1 - (1 - P(x))^m) \\ =& \sum_{\alpha \in \mathbb{N}_0^n} \rho_P^m(\alpha) x^\alpha - 1.\end{aligned} \tag{3.19}$$

Similarly, $\sum_{\gamma \in \mathbb{N}_0^n} a_\gamma \rho_P^m(\alpha - \gamma)$ is the α-coefficient of the product power series $\Big(\sum_{\alpha \in \mathbb{N}_0^n} \rho_P^m(\alpha) x^\alpha\Big)\Big(\sum_{\gamma \in \mathbb{N}_0^n} a_\gamma x^\gamma\Big)$, and we obtain for $|P(x)| < 1$, $m > 1$

$$\begin{aligned}&\sum_{\alpha \in \mathbb{N}_0^n} x^\alpha \Big(\rho_P^m(\alpha) - \sum_{\gamma \in \mathbb{N}_0^n} a_\gamma \rho_P^m(\alpha - \gamma)\Big) - 1 \\ &\qquad = (1 - P(x))^{-m} - (1 - P(x))^{-m} P(x) - 1 \\ &\qquad = (1 - P(x))^{-m+1} - 1 = \sum_{\alpha \in \mathbb{N}_0^n} \rho_P^{m-1}(\alpha) x^\alpha - 1\end{aligned} \tag{3.20}$$

implying (3.18). ∎

Now we can prove that the multiplication operators are well-defined bounded operators on $H^2(\rho_P^m)$.

LEMMA 3.6. $M_{z_1}, \ldots, M_{z_n} \in \mathcal{L}(H^2(\rho_P^m))$.

Proof. Let e_i be the ith unit vector in $\mathbb{C}^n$, $i = 1, \ldots, n$. It is sufficient to show that for some constant $c > 0$, $\rho_P^m(\alpha + e_i) \geqslant c\rho_P^m(\alpha)$ for all $\alpha \in \mathbb{N}_0^n$. But by Remark 3.5,

$$\rho_P^m(\alpha + e_i) \geqslant \sum_{\gamma \in \mathbb{N}_0^n} a_\gamma \rho_P^m(\alpha + e_i - \gamma) \geqslant a_{e_i} \rho_P^m(\alpha) \tag{3.21}$$

for $\alpha \in \mathbb{N}_0^n$, which proves the lemma. ∎

The multiplication operators are obviously commuting.

For the separable Hilbert space $\mathcal{H}$, we can consider the Hilbert space tensor product $\mathcal{H}\otimes H^2(\rho_P^m) =: H^2_{\mathcal{H}}(\rho_P^m)$. This space can obviously be identified with the space of formal power series with coefficients in $\mathcal{H}$, $\sum_{\alpha\in\mathbb{N}_0^n} h_\alpha z^\alpha$ with $h_\alpha\in\mathcal{H}$ for $\alpha\in\mathbb{N}_0^n$, such that $\sum_{\alpha\in\mathbb{N}_0^n}\|h_\alpha\|^2(1/\rho_P^m(\alpha))<\infty$. The inner product on $H^2_{\mathcal{H}}(\rho_P^m)$ is then given by

$$\Big\langle \sum_{\alpha\in\mathbb{N}_0^n} h_\alpha z^\alpha\,,\ \sum_{\alpha'\in\mathbb{N}_0^n} h'_{\alpha'} z^{\alpha'}\Big\rangle = \sum_{\alpha\in\mathbb{N}_0^n}\langle h_\alpha, h'_\alpha\rangle\frac{1}{\rho_P^m(\alpha)}.$$

We can view $H^2_{\mathcal{H}}(\rho_P^m)$ as a space of $\mathcal{H}$-valued holomorphic functions on $\mathcal{P}$. From now on, we will denote the multiplication operators with the coordinates on $H^2_{\mathcal{H}}(\rho_P^m)$ as well as the ones on $H^2(\rho_P^m)$ by $M_{z_1},\dots,M_{z_n}$. By Lemma 3.6, these operators are also well-defined and bounded on $H^2_{\mathcal{H}}(\rho_P^m)$.

As in the case of spherical contractions, the spectrum of a $(P,1)$-positive multioperator is contained in the closure of the P-ball:

Lemma 3.7. *Let P be a positive regular polynomial and T a $(P,1)$-positive commuting multioperator. Then the Taylor spectrum $\sigma(T)$ of T is contained in the closure $\overline{\mathcal{P}}$ of the P-ball.*

Proof. This lemma is a special case of a more general result ([11], Theorem 1.12). We give a more elementary proof for our situation.

Let $\lambda\in\mathbb{C}^n\setminus\overline{\mathcal{P}}$. We will show that λ is not contained in the joint spectrum of T relative to the closed commutative subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ generated by $T_1,\dots,T_n$, i.e. we will show that the ideal I generated by $\lambda_1\mathbf{1}_{\mathcal{H}}-T_1,\dots,\lambda_n\mathbf{1}_{\mathcal{H}}-T_n$ in $\mathcal{A}$ is equal to $\mathcal{A}$. Since the Taylor spectrum $\sigma(T)$ of T is contained in the joint spectrum of T relative to any closed commutative subalgebra of $\mathcal{L}(\mathcal{H})$ containing T, this means that λ is not in $\sigma(T)$.

Let $Q_\lambda(z)=(1/P(|\lambda|^2))P(\overline{\lambda}z)$. Then $Q_\lambda(\lambda)=1$, and for $h\in\mathcal{H}$, $\|h\|\leqslant 1$,

$$\begin{aligned}\|Q_\lambda(T)h\| &= \frac{1}{P(|\lambda|^2)}\|P(\overline{\lambda}T)h\|\\ &\leqslant \frac{1}{P(|\lambda|^2)}\Big(\sum_{\gamma\in I_P} a_\gamma|\lambda^\gamma|^2\Big)^{1/2}\Big(\sum_{\gamma\in I_P} a_\gamma\|T^\gamma h\|^2\Big)^{1/2}\\ &= \frac{1}{P(|\lambda|^2)^{1/2}}\langle P(C_T)(\mathbf{1}_{\mathcal{H}})h,h\rangle^{1/2}\leqslant\frac{1}{P(|\lambda|^2)^{1/2}}<1\end{aligned}\tag{3.22}$$

by definition of $\mathcal{P}$. Thus $\|Q_\lambda(T)\| < 1$, and $\mathbf{1}_\mathcal{H} - Q_\lambda(T)$ is invertible in $\mathcal{A}$. On the other hand, one easily verifies that

$$\mathbf{1}_\mathcal{H} - Q_\lambda(T) = Q_\lambda(\lambda) - Q_\lambda(T) = \frac{1}{P(|\lambda|^2)} \sum_{\gamma \in I_P} a_\gamma \overline{\lambda}^\gamma (\lambda^\gamma \mathbf{1}_\mathcal{H} - T^\gamma) \in I, \tag{3.23}$$

which finishes the proof. ∎

We are now in the situation to state our model theorem:

THEOREM 3.8. *Let P be a positive regular polynomial in n variables , $T = (T_1, \ldots, T_n)$ a commuting multioperator on the separable Hilbert space $\mathcal{H}$ and $m \in \mathbb{N}$. Then the following are equivalent:*

(i) *T is (P, m)-positive;*

(ii) *there exist a Hilbert space $\mathcal{N}$, a P-unitary operator $N = (N_1, \ldots, N_n) \in \mathcal{L}(\mathcal{N})^n$ and an isometry $V = V_1 \oplus V_2 : \mathcal{H} \to H^2_\mathcal{H}(\rho_P^m) \oplus \mathcal{N}$ such that $VT = (M_z^* \oplus N)V$.*

Proof. First we prove (i) $\Rightarrow$ (ii).

CLAIM 1. Let T be $(P, 1)$-positive for the positive operator $X \in \mathcal{L}(\mathcal{H})$. Then the sequence $(P(C_T)^k(X))_{k \in \mathbb{N}}$ converges to some positive operator $\widetilde{P}_X$ in the strong operator topology (SOT) on $\mathcal{L}(\mathcal{H})$.

Proof. Since P is positive regular, $(P(C_T)^k(X))_{k \in \mathbb{N}}$ is a sequence of positive operators and thus bounded below by 0. Moreover, the sequence is decreasing because of

$$P(C_T)^k(X) - P(C_T)^{k+1}(X) = P(C_T)^k(1 - P(C_T))(X) \geqslant 0$$

and consequently converging to some positive operator $\widetilde{P}_X$ in the SOT-topology. ∎

Now define for $X \in \mathcal{L}(\mathcal{H})$, $X \geqslant 0$, and T (P, m)-positive for X the map

$$V_1^X : \mathcal{H} \to H^2_\mathcal{H}(\rho_P^m), \quad h \mapsto \sum_{\alpha \in \mathbb{N}_0^n} \rho_P^m(\alpha) \left((1 - P(C_T))^m(X)\right)^{1/2} T^\alpha h z^\alpha.$$

As one proves by induction completely analogously to [6], Lemmas 4 and 5 (see also [11], 2.1 and 2.8), we have

$$\begin{aligned} &\sum_{j=0}^{k} \binom{j+m-1}{m-1} P(C_T)^j (1 - P(C_T))^m \\ &\qquad = 1 - \sum_{j=0}^{m-1} \binom{k+j}{j} P(C_T)^{k+1} (1 - P(C_T))^j, \quad k \in \mathbb{N} \end{aligned} \tag{3.24}$$

and

$$\lim_{k\to\infty}\binom{k+j}{j}\langle P(C_T)^{k+1}(1-P(C_T))^j(X)h,h\rangle = 0,\quad h\in\mathcal{H}, \tag{3.25}$$

for $j=1,\dots,m-1$. We obtain

$$\|V_1^X h\|^2 = \|h\|^2 - \lim_{k\to\infty}\langle P(C_T)^k(X)h,h\rangle = \|h\|^2 - \langle \widetilde{P}_X h,h\rangle,\quad h\in\mathcal{H} \tag{3.26}$$

by

$$\begin{aligned}
\|V_1^X h\|^2 &= \sum_{\alpha\in\mathbb{N}_0^n}\rho_P^m(\alpha)\,\langle(1-P(C_T))^m(X)T^\alpha h, T^\alpha h\rangle\\
&= \sum_{\alpha\in\mathbb{N}_0^n}\Bigg[\sum_{\substack{K\in\mathbb{N}_0^{I_P}\\ [K]=\alpha}}\binom{|K|+m-1}{m-1}\binom{|K|}{K}A^K\langle C_T^\alpha(1-P(C_T))^m(X)h,h\rangle\Bigg]\\
&= \sum_{j=0}^{\infty}\Bigg[\sum_{\substack{K\in\mathbb{N}_0^{I_P}\\ |K|=j}}\binom{j+m-1}{m-1}\binom{j}{K}A^K\Big\langle C_T^{[K]}(1-P(C_T))^m(X)h,h\Big\rangle\Bigg]\\
&= \sum_{j=0}^{\infty}\binom{j+m-1}{m-1}\langle P(C_T)^j(1-P(C_T))^m(X)h,h\rangle\\
&= \|h\|^2 - \lim_{k\to\infty}\sum_{j=0}^{m-1}\binom{k+j}{j}\langle P(C_T)^{k+1}(1-P(C_T))^j(X)h,h\rangle\\
&= \|h\|^2 - \lim_{k\to\infty}\langle P(C_T)^k(X)h,h\rangle,
\end{aligned} \tag{3.27}$$

according to (3.24) and (3.25), with the limits existing because of Claim 1. For T (P,m)-positive and $V_1 = V_1^{\mathbf{1}_\mathcal{H}}$, one gets

$$\begin{aligned}
V_1\,T_i\,h &= \sum_{\alpha\in\mathbb{N}_0^n}\rho_P^m(\alpha)((1-P(C_T)^m(\mathbf{1}_\mathcal{H}))^{1/2}T^{\alpha+e_i}h\,z^\alpha\\
&= \sum_{\alpha\in\mathbb{N}_0^n}\frac{\rho_P^m(\alpha)}{\rho_P^m(\alpha+e_i)}\rho_P^m(\alpha+e_i)((1-P(C_T))^m(\mathbf{1}_\mathcal{H}))^{1/2}T^{\alpha+e_i}h\,z^\alpha\\
&= M_{z_i}^*\Bigg(\sum_{\alpha\in\mathbb{N}_0^n}\rho_P^m(\alpha+e_i)((1-P(C_T))^m(\mathbf{1}_\mathcal{H}))^{1/2}T^{\alpha+e_i}h\,z^{\alpha+e_i}\Bigg)\\
&= M_{z_i}^*V_1\,h.
\end{aligned} \tag{3.28}$$

So we have constructed the first part of our model. In a second step we construct the P-unitary part, using the fact that $\widetilde{P} = \widetilde{P}_{\mathbf{1}_\mathcal{H}}$ is invariant under $P(C_T)$. In the following, we write s-lim for the limits in the strong operator topology on $\mathcal{L}(\mathcal{H})$.

LEMMA 3.9. *Let T be a $(P,1)$-positive commuting multioperator on $\mathcal{H}$ and $\widetilde{P} = \widetilde{P}_{\mathbf{1}_{\mathcal{H}}} = s\text{-}\lim_{k\to\infty} P(C_T)^k(\mathbf{1}_{\mathcal{H}})$. Then there exist a Hilbert space $\mathcal{N}$, a P-unitary multioperator $N \in \mathcal{L}(\mathcal{N})^n$ and a contractive linear mapping $V_2 : \mathcal{H} \to \mathcal{N}$ such that $\|V_2 h\|^2 = \langle \widetilde{P}h, h\rangle$ for $h \in \mathcal{H}$ and $V_2 T = N V_2$.*

Proof. Let $\mathcal{K} = \overline{\widetilde{P}^{1/2}\mathcal{H}}$ and $V_2 : \mathcal{H} \to \mathcal{K}$, $h \mapsto \widetilde{P}^{1/2}h$. For $i = 1, \dots, n$, the linear map $W_i : \widetilde{P}^{1/2}\mathcal{H} \to \mathcal{K}$,

$$W_i V_2 h = V_2 T_i h \quad \text{for } h \in \mathcal{H}, \tag{3.29}$$

is well-defined and bounded, since

$$\|W_i V_2 h\|^2 = \langle T_i^* \widetilde{P}\, T_i h, h\rangle \leqslant a_{e_i}^{-1}\langle P(C_T)(\widetilde{P})h, h\rangle = a_{e_i}^{-1}\|V_2 h\|^2, \quad h \in \mathcal{H}. \tag{3.30}$$

So we can extend W_i to a bounded linear map $\mathcal{K} \to \mathcal{K}$, which we also call W_i. By (3.29) and continuity, we have $WV_2 = V_2 T$ for $W = (W_1, \dots, W_n)$ and consequently

$$V_2^*(P(C_W)(\mathbf{1}_{\mathcal{K}}))V_2 = P(C_T)(V_2^* V_2) = P(C_T)(\widetilde{P}) = V_2^* V_2 \tag{3.31}$$

because of the SOT-continuity of $P(C_T)$.

Now $P(C_W)(\mathbf{1}_{\mathcal{K}}) = \mathbf{1}_{\mathcal{K}}$, since $V_2\mathcal{H}$ is dense in $\mathcal{K}$. Thus W is a P-isometry. To replace W by a P-unitary tuple, we need the following lemma:

LEMMA 3.10. *Every P-isometry is subnormal, and its minimal normal extension is a P-unitary.*

Proof. Let $W \in \mathcal{L}(\mathcal{W})^n$ be a P-isometry. Then the tuple $(a_\gamma^{1/2} W^\gamma)_{\gamma\in I_P}$ is a spherical isometry and consequently by [1], Proposition 2, a subnormal tuple. Since $a_{e_1}, \dots, a_{e_n}$ are all not 0, in particular the tuple $W = (W_1, \dots, W_n)$ is subnormal. Let $N = (N_1, \dots, N_n)$ be its minimal normal extension on the Hilbert space $\mathcal{N} \supseteq \mathcal{K}$. Then $(a_\gamma^{1/2} N^\gamma)_{\gamma\in I_P}$ is the minimal normal extension of the tuple $(a_\gamma^{1/2} W^\gamma)_{\gamma\in I_P}$ and by [1] also a spherical isometry, which implies that N is a P-unitary. ∎

Now let for a (P,m)-positive multioperator T on $\mathcal{H}$

$$V = V_1 \oplus V_2 : \mathcal{H} \to H^2_{\mathcal{H}}(\rho_P^m) \oplus \mathcal{N}. \tag{3.32}$$

The mapping V is an isometry, and $VT = (M_z^* \oplus N)V$. Note that only the first part of the model depends on m.

For the proof of the reverse direction, we have only to show that $M_z^* \in \mathcal{L}(H^2(\rho_P^m))^n$ is (P,m)-positive for arbitrary m. Then the (P,m)-positivity of M_z^* on $H^2_{\mathcal{H}}(\rho_P^m)$ follows, and we obtain the (P,m)-positivity of T by the fact that any P-unitary is (P,m)-positive for every m and that (P,m)-positivity is preserved under the direct sum $M_z^* \oplus N$, the restriction to the invariant subspace $V\mathcal{H}$ and the unitary transformation $\mathcal{H} \to V\mathcal{H}$.

LEMMA 3.11. *For every $m \in \mathbb{N}$, the commuting multioperator $M_z^* \in \mathcal{L}(H^2(\rho_P^m))^n$ is (P,m)-positive. Moreover, $(1 - P(C_{M_z^*}))^m(\mathbf{1})$ is the orthogonal projection onto the subspace of constants in $H^2(\rho_P^m)$.*

Proof. For $\alpha, \beta \in \mathbb{N}_0^n$, we have

$$M_z^\beta M_z^{*\beta} z^\alpha = \begin{cases} \frac{\rho_P^m(\alpha-\beta)}{\rho_P^m(\alpha)} z^\alpha & \text{if } \beta \leqslant \alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{3.33}$$

So obviously $(1 - P(C_{M_z^*}))^m(\mathbf{1}) z^\alpha = z^\alpha$ for $\alpha = 0$. Let as before $\rho_P^m(\alpha) = 0$ for $\alpha \in \mathbb{Z}^n \setminus \mathbb{N}_0^n$. Since the spaces $\mathbb{C} \cdot z^\alpha$ are invariant under $M_z^\beta M_z^{*\beta}$, thus also invariant under $(1 - P(C_{M_z^*}))(\mathbf{1})$ and $(1 - P(C_{M_z^*}))^m(\mathbf{1})$, it remains to show that

$$\langle (1 - P(C_{M_z^*}))(\mathbf{1}) z^\alpha, z^\alpha \rangle \geqslant 0, \quad \alpha \geqslant 0 \tag{3.34}$$

$$\langle (1 - P(C_{M_z^*}))^m(\mathbf{1}) z^\alpha, z^\alpha \rangle = 0, \quad \alpha \geqslant 0, \, \alpha \neq 0. \tag{3.35}$$

By Equation (3.33), we have

$$\langle (1 - P(C_{M_z^*}))(\mathbf{1}) z^\alpha, z^\alpha \rangle = \frac{1}{\rho_P^m(\alpha)^2} \Big(\rho_P^m(\alpha) - \sum_{\gamma \in I_P} a_\gamma \rho_P^m(\alpha - \gamma) \Big) \tag{3.36}$$

and

$$\langle (1 - P(C_{M_z^*}))^m(\mathbf{1}) z^\alpha, z^\alpha \rangle = \frac{1}{\rho_P^m(\alpha)^2} \Big(\rho_P^m(\alpha) - \sum_{\gamma \in \mathbb{N}_0^n} b_\gamma \rho_P^m(\alpha - \gamma) \Big), \tag{3.37}$$

where $\sum_{\gamma \in \mathbb{N}_0^n} b_\gamma x^\gamma$ is the polynomial $1 - (1-P)^m$. The rest of the proof now results from Remark 3.5. ∎

This finishes the proof of Theorem 3.8. ∎

Via the isometric isomorphism

$$H^2_{\mathcal{H}}(\rho_P^m) \to l^2(\mathbb{N}_0^n, \mathcal{H}), \quad \sum_{\alpha\in\mathbb{N}_0^n} h_\alpha z^\alpha \mapsto \Big(\frac{1}{\rho_P^m(\alpha)^{1/2}} h_\alpha\Big)_{\alpha\in\mathbb{N}_0^n}, \tag{3.38}$$

the multioperator M_z^* may be looked upon as a weighted multi-backward shift. So $V_1\mathcal{H} \subseteq H^2_{\mathcal{H}}(\rho_P^m)$ may be regarded as the shift part of our model, and $V_2\mathcal{H} \subseteq \mathcal{N}$ is the P-unitary part.

In case $m = n = 1$ and $P = x$, the (P, m)-positive operators are just the contractions, and our model is the well-known coisometric extension for contractions.

If P is the polynomial $\sum\limits_{i=1}^{n} x_i$, the P-ball $\mathcal{P} = \{z \in \mathbb{C}^n \mid P(|z|^2) < 1\}$ is just the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, and the P-unitaries are just the spherical unitaries. For this case, Theorem 3.8 was proved by V. Müller and F.-H. Vasilescu in [6]. The positivity conditions $\Delta_P^{(m)} \geqslant 0$, $1 \leqslant m \leqslant n$, were examined earlier by A. Athavale, who showed in [1], Remark 1 to Proposition 4, that the tuple T then has a spherical dilation.

The standard model of Müller and Vasilescu reproduces this result: as one easily verifies, for the above P the space $H^2(\rho_P^m)$ is just the Hardy space

$$H^2(\mathbb{B}^n) = \left\{ f : \mathbb{B}^n \to \mathbb{C} \text{ holomorphic} \,\middle|\, \|f\|^2 := \sup_{0<r<1} \int\limits_{\partial\mathbb{B}^n} |f(rz)|^2 \, \mathrm{d}\sigma < \infty \right\},$$

where σ is the normalized surface measure on $\partial\mathbb{B}^n$, since

$$\int\limits_{\partial\mathbb{B}^n} |z^\alpha|^2 \, \mathrm{d}\sigma = (n-1)!\alpha!/(n-1-|\alpha|)!$$

for $\alpha \in \mathbb{N}_0^n$ (see e.g. [10], Proposition 1.4.9). The adjoint of the multiplication tuple here of course has a spherical dilation, for example the multioperator $M_{\overline{z}} \in \mathcal{L}(L^2(\partial\mathbb{B}^n, \sigma))^n$ via the isometric inclusion $H^2(\mathbb{B}^n) \hookrightarrow L^2(\partial\mathbb{B}^n, \sigma)$. Thus $M_z^* \oplus N$, where N is a spherical unitary, has a spherical dilation, and T, being unitarily equivalent to the restriction of $M_z^* \oplus N$ to an invariant subspace, has a spherical dilation, too.

The existence of a spherical dilation implies a von Neumann-type inequality over $\mathbb{B}^n$ and consequently the existence of a contractive $\mathcal{A}(\mathbb{B}^n)$-functional calculus for T, where $\mathcal{A}(\mathbb{B}^n) = \{f : \overline{\mathbb{B}}^n \to \mathbb{C} \text{ continuous} \,|\, f|\mathbb{B}^n \text{ holomorphic}\}$.

But since the multioperator $M_z^* \in \mathcal{L}(H^2_{\mathcal{H}}(\rho_P^m))^n = \mathcal{L}(H^2_{\mathcal{H}}(\mathbb{B}^n))^n$ has an obvious $H^\infty(\mathbb{B}^n)$-functional calculus defined by

$$f(M_z^*) = (M_{\check{f}})^*, \quad f \in H^\infty(\mathbb{B}^n) \tag{3.39}$$

with $\check{f}(z) = \overline{f(\overline{z})}$, every (P, n)-positive operator for which the model given by Theorem 3.8 consists only of the first part has even an $H^\infty(\mathbb{B}^n)$-functional calculus. So, according to Lemma 3.9 in the proof of Theorem 3.8, every (P, n)-positive multioperator T with $s\text{-}\lim_{k\to\infty} P(C_T)^k(\mathbf{1}_\mathcal{H}) = 0$ has a $H^\infty(\mathbb{B}^n)$-functional calculus. This result is contained in [6] and may also be obtained by means of an operator-valued Poisson integral formula ([14]).

So for general positive regular polynomials P, a natural question to ask is whether $H^2(\rho_P^m)$ may be identified for suitable m with a well-known Hilbert space of holomorphic functions on the P-ball $\mathcal{P}$ and thus one can obtain a rich functional calculus for $M_z^* \in \mathcal{L}(H^2(\rho_P^m))^n$ (and consequently for (P, m)-positive T) by this identification.

In the next section, we will show that such an identification is possible by passing to an equivalent norm.

4. THE FUNCTIONAL MODEL

THEOREM 4.1. *Let P be a positive regular polynomial and $m = \mathrm{mult}(P) > n$. Furthermore, let μ be the normalization of the positive measure $(1 - P(|z|^2))^{m-n-1}\,\mathrm{d}\lambda$ on $\mathcal{P}$, where $\mathrm{d}\lambda$ denotes Lebesgue measure. Then the space $H^2(\rho_P^m)$ and the Bergman space*

$$B^2(\mathcal{P}, \mu) = \Big\{ f : \mathcal{P} \to \mathbb{C} \text{ holomorphic } \Big| \int_{\mathcal{P}} |f(z)|^2\,\mathrm{d}\mu < \infty \Big\}$$

coincide as sets of functions on $\mathcal{P}$, and the identifying map id $: B^2(\mathcal{P}, \mu) \to H^2(\rho_P^m)$ *is a topological isomorphism.*

Proof. Let us first introduce some notations. With $P = \sum_{\gamma\in\mathbb{N}_0^n} a_\gamma x^\gamma$, $I_P = \{\gamma \in \mathbb{N}_0^n \mid a_\gamma > 0\}$ and $|I_P| = \mathrm{mult}(P) = m$, identify $\mathbb{C}^m$ with $\mathbb{C}^{I_P}$ and denote the elements of $\mathbb{C}^m$ by $w = (w_\gamma)_{\gamma\in I_P}$. Let $\tau : \mathbb{C}^m \to \mathbb{C}^n$, $w = (w_\gamma)_{\gamma\in I_P} \mapsto (w_{e_1}, \dots, w_{e_n})$, and $\kappa : \mathbb{C}^m \to \mathbb{C}^n$, $w = (w_\gamma)_{\gamma\in I_P} \mapsto (a_{e_1}^{-1/2} w_{e_1}, \dots, a_{e_n}^{-1/2} w_{e_n})$. Now define the holomorphic map

$$\varphi : \mathbb{C}^m \to \mathbb{C}^m, \quad \varphi(w)_\gamma = \begin{cases} a_\gamma^{1/2} w_\gamma & \text{if } \gamma \in e_1, \dots, e_n, \\ w_\gamma + a_\gamma^{1/2} \tau(w)^\gamma & \text{otherwise.} \end{cases} \tag{4.1}$$

The map φ is biholomorphic, since

$$\varphi^{-1} : \mathbb{C}^m \to \mathbb{C}^m, \quad \varphi^{-1}(w)_\gamma = \begin{cases} a_\gamma^{-1/2} w_\gamma & \text{if } \gamma \in e_1, \dots, e_n, \\ w_\gamma - a_\gamma^{1/2} \kappa(w)^\gamma & \text{otherwise;} \end{cases} \tag{4.2}$$

is obviously a holomorphic inverse map. Let $D = \varphi^{-1}(\mathbb{B}^m)$. Then D is strictly pseudoconvex, since $\mathbb{B}^m$ is strictly pseudoconvex (see e.g. [9], II.2.7), and we have

$$\begin{aligned} &D \cap (\mathbb{C}^n \times \{0\} \times \cdots \times \{0\}) \\ &= \left\{ w \in \mathbb{C}^m \,\middle|\, w_\gamma = 0 \text{ for } \gamma \notin \{e_1, \dots, e_n\},\ \sum_{\gamma \in I_P} a_\gamma |\tau(w)^\gamma|^2 < 1 \right\} \\ &= \mathcal{P} \times \{0\} \times \cdots \times \{0\}. \end{aligned} \tag{4.3}$$

Moreover, $\mathcal{M} = \varphi(\mathcal{P})$ is a complex submanifold of $\mathbb{B}^m$ such that $\mathcal{M} = \{w \in \mathbb{B}^m \mid w_\gamma = a_\gamma^{1/2} \kappa(w)^\gamma\}$.

Let Q be the polynomial in m variables that corresponds to the unit ball, $Q \in \mathbb{C}[(X_\gamma)_{\gamma \in I_P}]$, $Q = \sum_{\gamma \in I_P} x_\gamma$.

We will now construct the identifying map $B^2(\mathcal{P}, \mu) \to H^2(\rho_P^m)$ in several steps.

Step 1. THE RESTRICTION. As in (3.8), let $[\,\cdot\,] : \mathbb{N}_0^m = \mathbb{N}_0^{I_P} \to \mathbb{N}_0^n$, $[\beta]_i = \sum_{\gamma \in I_P} \gamma_i \beta_\gamma$.

LEMMA 4.2. *With $A = (a_\gamma)_{\gamma \in I_P}$ and the notation in* (3.6)*, the map*

$$\pi : H^2(\mathbb{B}^m) \to H^2(\rho_P^m), \quad \sum_{\beta \in \mathbb{N}_0^m} c_\beta w^\beta \mapsto \sum_{\beta \in \mathbb{N}_0^m} c_\beta A^{\beta/2} z^{[\beta]} \tag{4.4}$$

is well-defined, surjective, linear and has norm 1.

Proof. First notice that the (P, m)-weights may be expressed in terms of (Q, m)-weights: For $\alpha \in \mathbb{N}_0^n$, we have

$$\rho_P^m(\alpha) = \sum_{\substack{\beta \in \mathbb{N}_0^m \\ [\beta] = \alpha}} A^\beta \binom{|\beta| + m - 1}{m - 1} \binom{|\beta|}{\beta} = \sum_{\substack{\beta \in \mathbb{N}_0^m \\ [\beta] = \alpha}} A^\beta \rho_Q^m(\beta). \tag{4.5}$$

As one shows easily by induction over r, for any $a_1, \dots, a_r$, $b_1, \dots, b_r \in \mathbb{R}$ with $a_1, \dots, a_r \geqslant 0$ and $b_1, \dots, b_r > 0$ one has

$$\frac{\left(\sum_{i=1}^r a_i \right)^2}{\sum_{i=1}^r b_i} \leqslant \sum_{i=1}^r \frac{a_i^2}{b_i}. \tag{4.6}$$

Consequently we obtain for arbitrary $f = \sum_{\beta\in\mathbb{N}_0^m} c_\beta w^\beta \in H^2(\mathbb{B}^m)$, $\alpha \in \mathbb{N}_0^n$

$$\frac{\Big|\sum\limits_{\substack{\beta\in\mathbb{N}_0^m\\ [\beta]=\alpha}} A^{\beta/2} c_\beta\Big|^2}{\rho_P^m(\alpha)} \leqslant \frac{\Big(\sum\limits_{\substack{\beta\in\mathbb{N}_0^m\\ [\beta]=\alpha}} A^{\beta/2}|c_\beta|\Big)^2}{\sum\limits_{\substack{\beta\in\mathbb{N}_0^m\\ [\beta]=\alpha}} A^\beta \rho_Q^m(\beta)} \leqslant \sum_{\substack{\beta\in\mathbb{N}_0^m\\ [\beta]=\alpha}} \frac{|c_\beta|^2}{\rho_Q^m(\beta)} \tag{4.7}$$

and

$$\|\pi(f)\|^2 = \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\rho_P^m(\alpha)}\Big|\sum_{\substack{\beta\in\mathbb{N}_0^m\\ [\beta]=\alpha}} A^{\beta/2} c_\beta\Big|^2 \leqslant \|f\|^2. \tag{4.8}$$

To show the surjectivity of π, consider the map $\iota : H^2(\rho_P^m) \to H^2(\mathbb{B}^m)$, $g = \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha \mapsto \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha \sum_{\substack{\beta\in\mathbb{N}_0^m\\ [\beta]=\alpha}} A^{\beta/2}(\rho_Q^m(\beta)/\rho_P^m(\alpha))w^\beta$. Then ι is well-defined and isometric, since $\iota(g) \in H^2(\mathbb{B}^m)$ with $\|\iota(g)\|^2 = \sum_{\alpha\in\mathbb{N}_0^n} |c_\alpha|^2 \sum_{\substack{\beta\in\mathbb{N}_0^m\\ [\beta]=\alpha}} A^\beta(\rho_Q^m(\beta)/\rho_P^m(\alpha)^2) = \|g\|^2$ by Equation (4.5), and $\pi \circ \iota = \mathbf{1}$. ∎

Thus the map π can be regarded as the orthogonal projection from $H^2(\mathbb{B}^m)$ onto the closed subspace $H^2(\rho_P^m)$. This close relationship between $H^2(\rho_P^m)$ and $H^2(\mathbb{B}^m)$ and the definitions of φ and π become clearer by considering the following idea:

Let $T = (T_1, \dots, T_n)$ be a (P, m)-positive multioperator on $\mathcal{H}$ and let $V_1 : \mathcal{H} \to H^2_{\mathcal{H}}(\rho_P^m)$ be the map constructed in Theorem 3.8. Let W be the commuting m-tuple $(W_\gamma)_{\gamma\in I_P}$, $W_\gamma = a_\gamma^{1/2} T^\gamma$. Then

$$(1 - P)(C_T) = (1 - Q)(C_W) \tag{4.9}$$

and thus W is (Q, m)-positive. Again by Theorem 3.8, now applied to the m-tuple W, we obtain the map $\widetilde{V}_1 : \mathcal{H} \to H^2_{\mathcal{H}}(\mathbb{B}^m)$ as first part of the model for the tuple W. Therefore

$$\begin{aligned}(\mathbf{1}_{\mathcal{H}}\otimes\pi) \circ \widetilde{V}_1(h) &= (\mathbf{1}_{\mathcal{H}}\otimes\pi)\Big(\sum_{\beta\in\mathbb{N}_0^m} \rho_Q^m(\beta)((1-Q)^m(C_W)(\mathbf{1}_{\mathcal{H}}))^{1/2} W^\beta h w^\beta\Big)\\ &= \sum_{\alpha\in\mathbb{N}_0^n} \sum_{\substack{\beta\in\mathbb{N}_0^m\\ [\beta]=\alpha}} \rho_Q^m(\beta) A^\beta((1-P)^m(C_T)(\mathbf{1}_{\mathcal{H}}))^{1/2}\, T^{[\beta]} h z^\alpha\\ &= \sum_{\alpha\in\mathbb{N}_0^n} \rho_P^m(\alpha)((1-P)^m(C_T)(\mathbf{1}_{\mathcal{H}}))^{1/2}\, T^\alpha h z^\alpha = V_1(h)\end{aligned} \tag{4.10}$$

for $h \in \mathcal{H}$, and we have

$$(\mathbf{1}_{\mathcal{H}} \otimes \pi) \circ \widetilde{V}_1 = V_1. \tag{4.11}$$

In particular, the map $\mathbf{1}_{\mathcal{H}} \circ \pi$ is isometric on $\widetilde{V}_1\mathcal{H}$, since

$$\|V_1 h\|^2 = \lim_{k\to\infty} \langle P(C_T)^k(\mathbf{1}_{\mathcal{H}})h, h\rangle = \lim_{k\to\infty} \langle Q(C_W)^k(\mathbf{1}_{\mathcal{H}})h, h\rangle = \|\widetilde{V}_1 h\|^2. \tag{4.12}$$

The submanifold $\mathcal{M} = \{w \in \mathbb{B}^m \mid w_\gamma = a_\gamma^{1/2}\kappa(w)^\gamma\}$ corresponds to the identities $W_\gamma = a_\gamma^{1/2} T^\gamma$. The map π may be regarded as the restriction of functions in $H^2(\mathbb{B}^m)$ to the submanifold $\mathcal{M}$, up to the biholomorphic map φ. For $z \in \mathcal{P}$ and $f = \sum_{\beta\in\mathbb{N}_0^m} c_\beta w^\beta \in H^2(\mathbb{B}^m)$, we have

$$\begin{aligned} f\circ\varphi(z) &= \sum_{\beta\in\mathbb{N}_0^m} c_\beta(\varphi(z))^\beta = \sum_{\beta\in\mathbb{N}_0^m} c_\beta \prod_{\gamma\in I_P} a_\gamma^{\beta_\gamma/2}(z^\gamma)^{\beta_\gamma} \\ &= \sum_{\beta\in\mathbb{N}_0^m} c_\beta A^{\beta/2} z^{[\beta]} = \pi(f)(z). \end{aligned} \tag{4.13}$$

Altogether, we have the following commutative diagram.

$$\begin{array}{ccccc} & & \widetilde{V}_1\mathcal{H} & \hookrightarrow & H^2_{\mathcal{H}}(\mathbb{B}^m) \\ & \overset{\widetilde{V}_1}{\nearrow} & \downarrow\wr & & \mathbf{1}\otimes\iota \uparrow\downarrow \mathbf{1}_{\mathcal{H}}\otimes\pi = \cdot\circ\varphi|_{\mathcal{P}} \\ \mathcal{H} & \xrightarrow{V_1} & V_1\mathcal{H} & \hookrightarrow & H^2_{\mathcal{H}}(\rho_P^m). \end{array} \tag{4.14}$$

Step 2. THE TRANSFORMATION. Recall that the Hardy space $H^p(\Omega)$, $1 < p < \infty$, over a bounded strictly pseudoconvex set $\Omega \subseteq \mathbb{C}^n$ with C^2-boundary can be obtained in the following way (see e.g. [5], Section 8.3):

Let $\varrho : U \to \mathbb{R}$ be a strictly plurisubharmonic defining C^2-function for Ω, defined on some region $U \supset \overline{\Omega}$. That means,

$$\Omega = \{z \in U \mid \varrho(z) < 0\}. \tag{4.15}$$

Now for $\varepsilon > 0$ let $\Omega_\varepsilon = \{z \in U \mid \varrho(z) < \varepsilon\}$. For sufficiently small ε_0, $\partial\Omega_\varepsilon$ is a real C^2-manifold for each ε with $0 < \varepsilon < \varepsilon_0$. Let σ_ε be the surface measure on $\partial\Omega_\varepsilon$ and define

$$H^p(\Omega) = \left\{ f : \Omega \to \mathbb{C} \text{ holomorphic} \;\middle|\; \|f\|_p = \left(\sup_{\varepsilon_0>\varepsilon>0} \int_{\partial\Omega_\varepsilon} |f(z)|^p \, \mathrm{d}\sigma_\varepsilon \right)^{1/p} < \infty \right\}. \tag{4.16}$$

Then $H^p(\Omega, \|\cdot\|_p)$ is a Banach space. The space $H^p(\Omega)$ is independent of the choice of the defining function ϱ in the sense that any two plurisubharmonic defining C^2-functions for Ω induce equivalent norms on $H^p(\Omega)$. Furthermore, by passing to nontangential boundary values $H^p(\Omega)$ may be embedded topologically into $L^p(\partial\Omega, \sigma)$, where σ is the surface measure on $\partial\Omega$.

Our aim is to show that the biholomorphic map $\varphi : D \to \mathbb{B}^m$ induces a topological isomorphism

$$U_\varphi : H^2(\mathbb{B}^m) \to H^2(D), \quad f \mapsto f \circ \varphi. \tag{4.17}$$

This can be done by using the transformation formula and looking at the Jacobimatrix for φ on ∂D, but an alternative characterization of $H^p(\Omega)$ and an equivalent norm to $\|\cdot\|_p$ give a much shorter and less technical proof. We have

$$H^p(\Omega) = \big\{ f : \Omega \to \mathbb{C} \text{ holomorphic} \,\big|\, |f|^p \text{ has a harmonic majorant on } \Omega \big\}, \tag{4.18}$$

and if Ω is connected, for any $z \in \Omega$

$$\|f\|_{p,z} = \big(\inf\{ g(z) \mid g : \Omega \to \mathbb{R} \text{ harmonic}, \ g \geqslant |f|^p \} \big)^{1/p} \tag{4.19}$$

defines an equivalent norm to $\|\cdot\|_p$ on $H^p(\Omega)$ (see e.g. [15], Section 2.2).

Since composition with the biholomorphic map φ maps the class of real-valued harmonic functions on $\mathbb{B}^m$ bijectively onto the class of real-valued harmonic functions on D, for any fixed $z_0 \in D$ and any $f \in H^2(\mathbb{B}^m)$ we have

$$\begin{aligned} \|f \circ \varphi\|_{2,z_0}^2 &= \inf\{ g(z_0) \mid g : D \to \mathbb{R} \text{ harmonic}, \ g \geqslant |f \circ \varphi|^2 \} \\ &= \inf\{ g(\varphi(z_0)) \mid g : \mathbb{B}^m \to \mathbb{R} \text{ harmonic}, \ g \geqslant |f|^2 \} \\ &= \|f\|_{2,\varphi(z_0)}^2, \end{aligned} \tag{4.20}$$

and U_φ in (4.17) is thus a topological isomorphism with inverse $U_{\varphi^{-1}}$.

Step 3. THE EXTENSION. Now we come to the main step of our construction of the identification $B^2(\mathcal{P}, \mu) \to H^2(\rho_P^m)$, using a theorem of A. Cumenge.

We will show that for a measure $\widetilde{\mu}$ equivalent to μ, there is a bounded linear extension operator $E : B^2(\mathcal{P}, \widetilde{\mu}) \to H^2(D)$ and that the restriction $R : H^2(D) \to B^2(\mathcal{P}, \widetilde{\mu})$ is well-defined, bounded and surjective. To apply the theorem of Cumenge, we first have to show that $\mathcal{P}$ may be extended to a complex manifold transverse to ∂D, i.e. that there is a complex submanifold $\widetilde{\mathcal{P}}$ of $\mathbb{C}^m$ intersecting ∂D transversally such that $\mathcal{P} = D \cap \widetilde{\mathcal{P}}$.

Let $\widetilde{\mathcal{P}} = \mathbb{C}^n \times \{0\} \times \cdots \times \{0\}$. Then $\mathcal{P} = D \cap \widetilde{\mathcal{P}}$ by (4.3). The function $r : \mathbb{C}^m \to \mathbb{R}$, $r(z) = \sum_{\gamma \in I_P} |z_\gamma|^2 - 1$, is a strictly plurisubharmonic defining C^∞-function for $\mathbb{B}^m$. Thus $\varrho = \varphi \circ r$ is a strictly plurisubharmonic defining C^∞-function for D.

To prove that $\widetilde{\mathcal{P}}$ intersects ∂D transversally, we have to show that

$$\mathrm{d}\varrho(z) \wedge \left(\bigwedge_{\gamma \in I_P \setminus \{e_1, \ldots, e_n\}} \mathrm{d}z_\gamma \right) \neq 0 \quad \text{for all } z \in \widetilde{\mathcal{P}} \cap \partial D \tag{4.21}$$

(see e.g. [9], p. 118). So it suffices to prove that for every $z \in \widetilde{\mathcal{P}} \cap \partial D$, there is an $i \in \{1, \ldots, n\}$ such that $\partial\varrho / \partial z_{e_i}(z) \neq 0$. On $\widetilde{\mathcal{P}}$, identify z with $\widetilde{z} = \tau(z) \in \mathbb{C}^n$ to obtain $\varrho(z) = \sum_{\gamma \in I_P} a_\gamma |z^\gamma|^2$. Now let $z \in \widetilde{\mathcal{P}} \cap \partial D$. Since $0 \notin \partial D$, there is an i with $\tau(z)_i \neq 0$, and we obtain

$$\begin{aligned} \frac{\partial \varrho}{\partial z_{e_i}}(z) = \frac{\partial \varrho}{\partial \widetilde{z}_i}(\widetilde{z}) &= a_{e_1} \overline{\tau(z)}_i + \sum_{\substack{\gamma \in I_P \setminus \{e_1, \ldots, e_n\} \\ \gamma_i \neq 0}} \gamma_i a_\gamma \overline{\tau(z)^\gamma} \tau(z)^{\gamma - e_i} \\ &= \overline{\tau(z)}_i \left(a_{e_i} + \sum_{\substack{\gamma \in I_P \setminus \{e_1, \ldots, e_n\} \\ \gamma_i \neq 0}} \gamma_i a_\gamma |\tau(z)^{\gamma - e_i}|^2 \right) \neq 0, \end{aligned} \tag{4.22}$$

since the second factor is strictly positive.

Now $\mathcal{P}$ is a complex submanifold of codimension $m - n$ of the smoothly bounded strictly pseudoconvex set D. Thus we are in the situation of Theorem 0.1 in [2]: let $\widetilde{\mu}$ be the measure $\mathrm{dist}(z, \partial D)\mathrm{d}\lambda$ on $\mathcal{P}$. Then $f|\mathcal{P} \in B^2(\mathcal{P}, \widetilde{\mu})$ for every $f \in H^2(\partial D)$, and there exists a bounded linear extension operator $E : B^2(\mathcal{P}, \widetilde{\mu}) \to H^2(D)$, $Eg|\mathcal{P} = g$ for $g \in B^2(\mathcal{P}, \widetilde{\mu})$.

Moreover, the restriction operator $R : H^2(D) \to B^2(\mathcal{P}, \widetilde{\mu})$ is bounded since $\widetilde{\mu}$ is a Carleson measure on D by Hörmander's formulation of Carleson's Theorem and by Lemme II.1.1 in [2] (see [2], Section II.1, and [4], Theorem 4.3). It is surjective since $R \circ E = \mathbf{1}_{B^2(\mathcal{P}, \tilde{\mu})}$. The map $\pi \circ U_{\varphi^{-1}} \circ E : B^2(\mathcal{P}, \widetilde{\mu}) \to H^2(\rho_P^m)$ now maps each function $g \in B^2(\mathcal{P}, \widetilde{\mu})$ onto itself. It is bounded by construction and has the bounded inverse $R \circ U_\varphi \circ \iota$. Altogether, we have the following commutative diagram:

$$\begin{array}{ccc} H^2(D) & \xrightarrow{U_{\varphi^{-1}}} & H^2(\mathbb{B})^m \\ E \Big\uparrow \Big\downarrow R & & \Big\uparrow \Big\downarrow \\ B^2(\mathcal{P}, \widetilde{\mu}) & \underset{\mathrm{id}}{\xrightarrow{\sim}} & H^2(\rho_P^m) \end{array}$$

It remains to compare μ and $\widetilde{\mu}$.

Step 4. The equivalence of the measures. It suffices to show that there are constants $c_1, c_2 > 0$ such that

$$c_1 \mathrm{dist}(z, \partial D) \leqslant 1 - P(|z_1|^2, \dots, |z_n|^2) \leqslant c_2 \mathrm{dist}(z, \partial D), \quad z \in \partial\mathcal{P}. \tag{4.23}$$

Then $B^2(\mathcal{P}, \mu)$ and $B^2(\mathcal{P}, \widetilde{\mu})$ coincide as sets and carry equivalent norms.

The second inequality just follows by the Lipschitz continuity of the map $z \mapsto P(|z_1|^2, \dots, |z_n|^2)$ on the compact set $\overline{\mathcal{P}}$. For the first inequality, choose for $z \in \mathcal{P}$ some $w \in \partial\mathcal{P}$ such that $z = \lambda w$ for a suitable $\lambda \in [0, 1)$. Then

$$\begin{aligned} 1 - P(|z_1|^2, \dots, |z_n|^2) &= \sum_{\gamma \in I_P} a_\gamma (|w^\gamma|^2 - |z^\gamma|^2) \\ &\geqslant (1 - \lambda^2) \sum_{i=1}^{n} a_{e_i} |w_i|^2 \geqslant c(1 - \lambda)\|w\|^2 \\ &\geqslant c_1(1 - \lambda)\|w\| = c_1\|w - z\| \geqslant c_1 \mathrm{dist}(z, \partial\mathcal{P}) \end{aligned} \tag{4.24}$$

for suitable constants $c, c_1 > 0$, since $\partial\mathcal{P}$ is bounded away from 0. Thus we obtain (4.23), which finishes the proof of the theorem. ∎

5. DILATIONS

The identifying map $B^2(\mathcal{P}, \mu) \to H^2(\rho_P^m)$ obviously intertwines the multiplication operators with the coordinate functions on $B^2(\mathcal{P}, \mu)$ and $H^2(\rho_P^m)$. So its adjoint intertwines the adjoints of the multiplication operators, and we obtain the following easy consequence of Theorem 3.8 and Theorem 4.1. Let as before P be a positive regular polynomial with $m = \mathrm{mult}(P) > n$, μ the normalization of the measure $(1 - P(|z_1|^2, \dots, |z_n|^2))^{m-n-1}\,\mathrm{d}\lambda$ on $\mathcal{P}$ and let $M = (M_1, \dots, M_n)$ be the tuple of multiplication operators with the coordinate functions on $B^2_{\mathcal{H}}(\mathcal{P}, \mu)$.

Corollaty 5.1. *The following are equivalent:*

(i) *T is topologically equivalent to a (P, m)-positive multioperator;*

(ii) *T is topologically equivalent to the restriction of $M^* \oplus N \in \mathcal{L}(B^2_{\mathcal{H}}(\mathcal{P}, \mu) \oplus \mathcal{N})^n$ to an invariant subspace, where N is a P-unitary operator on some separable Hilbert space $\mathcal{N}$.*

Moreover, the functional model for a (P, m)-positive multioperator T implies — up to topological equivalence — the existence of a P-unitary dilation for T. Unlike the situation of the unit ball, we cannot obtain a P-unitary dilation directly. We have to check the complete boundedness of the map $q \mapsto q(T)$ on the algebra of polynomials, equipped with the supremum norm on $\mathcal{P}$.

THEOREM 5.2. *Let T be a (P,m)-positive commuting multioperator. Then T is topologically equivalent to a multioperator S which has a P-unitary dilation.*

Proof. By Corollary 5.1, T is topologically equivalent to the restriction of $M^* \oplus N$ to an invariant subspace. Thus it is sufficient to show that M^* has a P-unitary dilation.

The algebra $\mathbb{C}[X_1,\dots,X_n]$ carries an operator algebra structure as a subalgebra of the commutative C^*-algebra $\mathcal{C}(\partial\mathcal{P})$ of continuous functions on $\partial\mathcal{P}$. We denote this operator algebra by $\mathrm{Pol}(\mathcal{P})$.

REMARK 5.3. The algebra homomorphism

$$\Phi : \mathrm{Pol}(\mathcal{P}) \to \mathcal{L}(B^2_{\mathcal{H}}(\mathcal{P},\mu)), \quad q \mapsto q(M^*) \tag{5.1}$$

is completely contractive.

Proof. Let $M_n(\mathcal{L}(B^2_{\mathcal{H}}(\mathcal{P},\mu)))$ be the C^*-algebra of $n\times n$-matrices over $\mathcal{L}(B^2_{\mathcal{H}}(\mathcal{P},\mu))$ and let $M_n(\mathrm{Pol}(\mathcal{P}))$ be the algebra of $n\times n$-matrices over $\mathrm{Pol}(\mathcal{P})$, carrying the norm $\|(q_{i,j})\|_n = \sup\{\|(q_{i,j}(z))\| \mid z\in\mathcal{P}\}$, where $\|(q_{i,j}(z))\|$ denotes the usual operator norm of the complex $n\times n$-matrix $(q_{i,j}(z))$. We have to show that for each n, the map

$$\Phi^{(n)} : M_n(\mathrm{Pol}(\mathcal{P})) \to M_n(\mathcal{L}(B^2_{\mathcal{H}}(\mathcal{P},\mu))), \quad (q_{i,j}) \mapsto (q_{i,j}(M^*)) \tag{5.2}$$

is a contraction.

For $q\in\mathbb{C}[X_1,\dots,X_n]$, let $\check{q}$ be the polynomial obtained by complex conjugation of the coefficients of q. Then for $(q_{i,j})\in M_n(\mathrm{Pol}(\mathcal{P}))$, $\|\Phi^{(n)}((q_{i,j}))\| = \|(q_{i,j}(M^*))\| = \|(\check{q}_{j,i}(M))\|$, and for $f=(f_1,\dots,f_n)\in B^2_{\mathcal{H}}(\mathcal{P},\mu)^n = B^2_{\mathcal{H}^n}(\mathcal{P},\mu)$ we have

$$\begin{aligned}\|(\check{q}_{j,i}(M))f\|^2 &= \int_{\mathcal{P}} \|((\check{q}_{j,i}(M))f)(z)\|^2\,\mathrm{d}\mu = \int_{\mathcal{P}} \big\|(\check{q}_{j,i}(z)\mathbf{1}_{B^2_{\mathcal{H}}(\mathcal{P},\mu)})f(z)\big\|^2\,\mathrm{d}\mu\\ &\leqslant \int_{\mathcal{P}} \|(\check{q}_{j,i}(z))\|^2\|f(z)\|^2\,\mathrm{d}\mu \leqslant \|(\check{q}_{j,i})\|_n^2\,\|f\|^2 = \|(q_{i,j})\|_n^2\,\|f\|^2.\end{aligned} \tag{5.3}$$

Thus $\Phi^{(n)}$ is a contraction, and the remark is proved. ∎

To finish the proof of the theorem, note that by a corollary to Arveson's Extension Theorem (see [7], Corollary 6.7) the map Φ dilates to a homomorphism $\Psi : \mathcal{C}(\mathcal{P}) \to \mathcal{L}(\mathcal{K})$ with some Hilbert space $\mathcal{K} \supseteq B^2_{\mathcal{H}}(\mathcal{P},\mu)$. Then the tuple $K = (\Psi(z_1),\dots,\Psi(z_n))$ is a normal multioperator dilating M^*, and the Taylor spectrum of K is contained in $\partial\mathcal{P}$. By the Spectral Theorem for normal multioperators (see [13], Theorem 7.26), we have

$$P(C_K)(\mathbf{1}_{\mathcal{K}}) = \int\limits_{\partial\mathcal{P}} P(|z|^2)\,\mathrm{d}E = \mathbf{1}_{\mathcal{K}}, \tag{5.4}$$

where E is the spectral measure for the tuple K on $\mathcal{K}$. ∎

In particular, Theorem 5.2 implies that each (P,m)-positive multioperator satisfies a von Neumann-type inequality with respect to the P-ball $\mathcal{P}$. Let $\mathcal{A}(\mathcal{P})$ be the Banach algebra of complex-valued continuous functions on $\overline{\mathcal{P}}$ which are holomorphic on $\mathcal{P}$, together with the supremum norm on $\mathcal{P}$.

Corollary 5.4. *Let T be a (P,m)-positive multioperator. Then T has a continuous $\mathcal{A}(\mathcal{P})$-functional calculus. In particular, there is a constant $c>0$ such that*

$$\|q(T)\| \leqslant c\,\sup\big\{|q(z)|\;\big|\; z\in\mathcal{P}\big\} \quad \textit{for } q\in\mathbb{C}[X_1,\dots,X_n]. \tag{5.5}$$

Proof. As one easily sees by the Spectral Theorem for normal multioperators (see [13], Theorem 7.26) and by Lemma 3.7, a P-unitary multioperator U satisfies the von Neumann-inequality

$$\|q(U)\| \leqslant \sup\big\{|q(z)|\;\big|\; z\in\mathcal{P}\big\} \quad \text{for } q\in\mathbb{C}[X_1,\dots,X_n]. \tag{5.6}$$

The corollary now follows from Theorem 3.8, since the polynomials are dense in $\mathcal{A}(\mathcal{P})$. ∎

In case the model for T provided by Theorem 5.2 consists only of the multiplication operator part, i.e. in case $P(C_T)^s(\mathbf{1}_{\mathcal{H}})$ converges strongly to 0 for $s\to\infty$, we can strengthen this result. Let $A: H^2_{\mathcal{H}}(\rho_P^m)\to B^2_{\mathcal{H}}(\mathcal{P},\mu)$ be the isomorphism intertwining M_z^* on $H^2_{\mathcal{H}}(\rho_P^m)$ and M^* on $B^2_{\mathcal{H}}(\mathcal{P},\mu)$ mentioned in the beginning of this paragraph. Then

$$H^\infty(\mathcal{P})\to\mathcal{L}(\mathcal{H}),\quad f\mapsto V^*A^{-1}M_{\check{f}}{}^*AV, \tag{5.7}$$

where $V:\mathcal{H}\to H^2_{\mathcal{H}}(\rho_P^m)$ is the isometry constructed in Theorem 3.8, $\check{f}$ is the holomorphic map $z\mapsto\overline{f(\overline{z})}$ on $\mathcal{P}$ and $M_{\check{f}}$ is the bounded operator of multiplication with $\check{f}$ on $B^2_{\mathcal{H}}(\mathcal{P},\mu)$, defines a continuous algebra homomorphism with norm less or equal to $\|A\|\,\|A^{-1}\|$, mapping the coordinate functions to the components of T. Thus (5.7) gives a continuous $H^\infty(\mathcal{P})$-functional calculus for T.

In a forthcoming paper ([9]), the developed standard model for (P,m)-positive multioperators T will be applied to give necessary conditions for the existence of non-trivial joint invariant subspaces of T.

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