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ON $\mathbb{Z}/2\mathbb{Z}$-GRADED KK-THEORY AND ITS RELATION WITH THE GRADED Ext-FUNCTOR

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ABSTRACT. This paper studies the relation between KK-theory and the Ext-functor of Kasparov for $\mathbb{Z}_2$-graded C^*-algebras. We use an approach similar to the picture of J. Cuntz in the ungraded case. We show that the graded Ext-functor coincides with $\mathbb{Z}_2$-equivariant KK-theory up to a shift in dimension and that the graded KK-functor can be expressed in terms of $\mathbb{Z}_2$-equivariant KK-theory. We derive a (double) exact sequence relating both theories.

KEYWORDS: *$\mathbb{Z}_2$-graded C^*-algebras, superquasimorphism, Cuntz picture, universal algebras, double exact sequence.*

MSC (2000): 46L80.

INTRODUCTION

This paper is an investigation of the relation between the graded Ext-functor of Kasparov and $\mathbb{Z}_2$-graded KK-theory. Throughout the paper we assume that all algebras and homomorphisms are $\mathbb{Z}_2$-graded and we study the KK- and Ext-functors for graded C^*-algebras as defined by Kasparov (see [13]). Although it is clear from his paper that both functors are closely connected, it is only in the trivially graded case that this relationship is given a precise and definitive form, the general case has remained unsettled in [13]. The importance of this question is reflected by the fact that the natural map from extensions to the odd KK-group is used in [17] to prove the existence of long exact sequences for the graded KK-functor. The main result of this paper is the construction of a double exact sequence relating both theories. Since this notion is probably unfamiliar to most readers we briefly explain what it means. There is a natural map from Ext_* to

KK_{*+1} and also from KK_{*+1} to Ext_*. These maps are not actually inverse to each other, but their composition can be computed from certain natural involutions of the groups involved. In addition, both maps are part of two six-term exact sequences involving the same six groups but having opposite sense of direction. They are linked by the condition that starting from any group in this sequence mapping back and forth in both directions is multiplication by 2. Hence the kernels and cokernels may be computed from natural involutions of the groups (at least when tensored with $\mathbb{Z}[\frac{1}{2}]$). To achieve this result we use techniques which are due mainly to J. Cuntz. In particular, we obtain an algebraic description of the graded KK-functor (and of the Ext-functor) similar to the picture of Cuntz in the ungraded case. These results are the core of the author Ph.D. thesis.

The paper is divided into four sections. In the first section we develop the language of superquasimorphisms (as analogs of quasihomomorphisms of Cuntz). We give a description of $\mathrm{KK}(A, B)$ in terms of superquasimorphisms and show that it agrees with Kasparovs original definition (Proposition 1.5). We describe the KK-product in this picture and prove its associativity (Theorems 1.7 and 1.10). The fact that our product coincides with Kasparov's is not proved. This ultimately follows from the characterization of the KK-product in [11], but can probably also be shown directly.

In the second part we introduce the universal C^*-algebras χA to obtain another description of the KK-functor as homotopy classes of homomorphisms from χA to the compact operators of the universal graded Hilbert B-module (Theorem 2.1) using the results of Section 1, and also the universal algebras qA of Cuntz and εA of R. Zekri to get a similar description for the Ext-functor. We study their relation and obtain the natural maps $\alpha : \mathrm{KK}_* \to \mathrm{Ext}_{*+1}$ and $\beta : \mathrm{Ext}_{*+1} \to \mathrm{KK}_*$ coming from maps between the universal algebras. A fundamental construction of Cuntz in its $\mathbb{Z}_2$-equivariant form (due to J. Weidner and R. Zekri) shows that $\mathrm{KK}_*(A, B) \simeq \mathrm{Ext}_{*+1}(\chi A, B)$. It also shows that graded Ext-theory coincides up to a shift in dimension with $\mathbb{Z}_2$-equivariant KK-theory. These results are in fact the major technical step leading to our main theorem (Theorem 3.6), which is proved in Section 3, after some additional technicalities.

The last section contains some miscellanous results, particular cases and examples. The rather simple but nonetheless important observation of (4.3) and (4.4) provides a slight generalization of the Green-Julg theorem in the particular case of $\mathbb{Z}_2$. This is used together with $\mathbb{Z}_2$-equivariant Bott Periodicity to obtain another double exact sequence relating the $\mathbb{Z}_2$-equivariant KK-functor with ungraded KK (Theorem 4.2). Both double exact sequences may be identified if either argument is trivially graded. We show that the definition of the elementary graded K-functor due to van Daele coincides with Kasparovs definition (of $KK(\mathbb{C}, A)$). We look at possible applications of the double exact sequence, define graded E-theory and compute the graded K-groups of some graded Cuntz algebras $\mathcal{O}_{n,m}$ (Proposition 4.8).

1. Superquasimorphisms and KK

Definition 1.1. Let A, B be $\mathbb{Z}_2$-graded C^*-algebras. A *superquasimorphism* from A to B is a triple (ϕ, G, μ), where ϕ is a graded homomorphism from A to a $\mathbb{Z}_2$-graded algebra D with $J \triangleleft D$ a graded (invariant) ideal and $G \in D$ of degree 1, so that

$$(G - G^*)\phi(x) \in J, \quad (1 - G^2)\phi(x) \in J, \quad [\phi(x), G] = \phi(x)G - G\phi(\overline{x}) \in J$$

for every $x \in A$ and $\mu : J \to B$ is a graded homomorphism. We write symbolically

$$A \xrightarrow{\phi} D \overset{G}{\triangleright} J \xrightarrow{\mu} B$$

or shortly $\Phi : A \longrightarrow\!\!\triangleright\, B$.

A *map* of two superquasimorphisms Φ_1, Φ_2 is a commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{\phi_1} & D_1 & \overset{G_1}{\triangleright} & J_1 & \xrightarrow{\mu_1} & B \\
\| & & \downarrow & & \downarrow & & \| \\
A & \xrightarrow{\phi_2} & D_2 & \overset{G_2}{\triangleright} & J_2 & \xrightarrow{\mu_2} & B.
\end{array}$$

$D_1 \to D_2$ maps G_1 to G_2. We will say that Φ_1 *is contained in* Φ_2 if the vertical homomorphisms are injective and that Φ_2 *is a quotient of* Φ_1 if they are surjective.

If $f : A' \to A$, $g : B \to B'$ are homomorphisms and $\Phi : A \longrightarrow\!\!\triangleright\, B$ is a superquasimorphism, then the composition $g \circ \Phi \circ f$ gives a superquasimorphism from A' to B'.

Let $q_t : B[0,1] \to B$ be evaluation at time t. A superquasimorphism Π from A to $B[0,1]$ will be called a *homotopy* from Φ to Ψ, if $\Phi = q_0 \circ \Pi$ and $\Psi = q_1 \circ \Pi$. Φ and Ψ are *equivalent* if there is a chain $\{\Phi_i,\ i = 0, \ldots, n\}$ consisting of finitely many superquasimorphisms Φ_i, such that $\Phi_0 = \Phi$, $\Phi_n = \Psi$ and for each i, Φ_i and Φ_{i+1} are connected either by a map or by a homotopy. For example, we may consider a chain of maps $\Phi_0 \to \Phi_1 \leftarrow \cdots \to \Phi_n$. If two superquasimorphisms can be connected by such a chain of maps we write $\Phi \simeq \Psi$ and $\Phi \sim \Psi$ if they are equivalent.

A superquasimorphism is *special* if D is generated by $\phi(A)$ and G; it is *normal* if $G = G^*$ and G^2 is a projection such that $(1 - G^2)\phi(A) = 0$.

Let $\mathcal{K} = \mathcal{K}(H)$ be the algebra of compact operators on a separable Hilbert space and $\widehat{\mathcal{K}} = \widehat{M_2}(\mathcal{K})$ with the standard "off-diagonal" grading on $\widehat{M_2}$, i.e.

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Choose a standard isomorphism $\widehat{\mathcal{K}} \otimes \widehat{\mathcal{K}} = \widehat{\mathcal{K}}$, such that the identity $\mathrm{id}_{\widehat{\mathcal{K}}}$ is homotopic to the embedding j_0 mapping $x \in \widehat{\mathcal{K}}$ to $e \otimes x$ with e a minimal projection of degree 0 in the left upper corner of $\widehat{M_2}(\mathcal{K})$.

LEMMA 1.2. (i) *Every superquasimorphism contains a special superquasimorphism. If $\Phi \sim \Psi$ for two special superquasimorphisms then the connecting chain of mappings and homotopies can be realized by special superquasimorphisms.*

(ii) *Every superquasimorphism from A to $\widehat{\mathcal{K}} \otimes B$ is equivalent to a normal one. An equivalence of two normal superquasimorphisms can be realized by a chain of normal superquasimorphisms.*

Proof. (i) The first statement is immediate, the second follows by specializing all maps and homotopies simultaneously.

(ii) Let $G_t = (\frac{2-t}{2})G + \frac{t}{2}G^*$. The homotopy $\Phi_t : A \xrightarrow{\phi} D \overset{G_t}{\triangleright} J \xrightarrow{\mu} B$ connects (ϕ, G, μ) and (ϕ, G_1, μ) with G_1 selfadjoint. If $(\phi, G, \mu) : A \longrightarrow\!\!\triangleright\, \widehat{\mathcal{K}} \otimes B$ is given with $G = G^*$, then this is equivalent to

$$\left(\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \sin(\frac{\pi}{2}G) & \cos(\frac{\pi}{2}G) \\ \cos(\frac{\pi}{2}G) & -\sin(\frac{\pi}{2}G) \end{pmatrix}, \mathrm{id} \otimes \mu \right) : A \longrightarrow\!\!\triangleright\, \widehat{M_2}(\widehat{\mathcal{K}} \otimes B) \simeq \widehat{\mathcal{K}} \otimes B$$

which is normal. The procedure of simultaneous normalization applied to a chain of mappings and homotopies gives the second statement. ∎

REMARK 1.3. It follows from Lemma 1.2 that, when considering equivalence classes, we can restrict ourselves to special and (independently) to normal superquasimorphisms.

Denote the set of equivalence classes of superquasimorphisms from A to $\widehat{\mathcal{K}} \otimes B$ by $\mathrm{KK}(A, B)$. On this set there is a (commutative) addition, defined by

$$[\Phi_1] + [\Phi_2] = [\Phi_1 \oplus \Phi_2]$$

$$\Phi_1 \oplus \Phi_2 : \; A \longrightarrow\!\!\triangleright\, M_2(\widehat{\mathcal{K}} \otimes B) \simeq \widehat{\mathcal{K}} \otimes B.$$

The result will be independent up to an equivalence from the chosen isomorphism $M_2(\widehat{\mathcal{K}}) \simeq \widehat{\mathcal{K}}$ if M_2 (left upper corner $(\widehat{M_2}(\mathcal{K}))$) is identified with the left upper corner of $\widehat{M_2}(\mathcal{K})$.

Proposition 1.4. KK(A, B) *is an abelian group.*

Proof. Let (ϕ, G) be a normal superquasimorphism (μ is assumed injective and dropped from the notation). The inverse of $[(\phi, G)]$ is $[(G\overline{\phi}G, -G)]$. ∎

We want to show that our definition agrees with the one of Kasparov in [13] in case A is separable and B is σ-unital. Assume we are given a countably generated graded Kasparov A-B-bimodule E. By the Stabilization Theorem (cf. [12]), E can be embedded as a direct summand into the universal graded Hilbert-B-module

$$\widehat{H}_B = H_B \perp H_B^{\mathrm{op}}.$$

H_B is the module of all square summable B-sequences (with respect to $\|\cdot\|$ in B) with natural grading obtained from the grading of B. H_B^{op} is H_B with opposite grading, i.e. $H_B^{\mathrm{op}(0)} \simeq H_B^{(1)}$ and vice versa. We thus obtain a superquasimorphism

$$\Phi = (\phi, G) : A \longrightarrow \mathcal{L}(E) \triangleright \mathcal{K}(E) \hookrightarrow \mathcal{K}(\widehat{H}_B) \simeq \widehat{\mathcal{K}} \otimes B.$$

The last isomorphism is given with respect to a standard basis of $\widehat{H}_B$. Obviously a homotopy of Kasparov modules leads to equivalent superquasimorphisms (the embedding $\mathcal{K}(E) \hookrightarrow \mathcal{K}(\widehat{H}_B)$ causes no problem since any two orthocomplemented embeddings obtained from the Stabilization Theorem are unitarily equivalent). This gives a well-defined map $[E] \to [\Phi]$.

On the other hand, let

$$\Phi : A \xrightarrow{\phi} D \overset{G}{\triangleright} J \xrightarrow{\mu} \widehat{\mathcal{K}} \otimes B$$

be a given superquasimorphism. Define $E = \overline{\operatorname{span} \mu(J)(\widehat{H}_B)}$. It is easily checked that

$$\mathcal{K}(E) \simeq \overline{\mu(J)(\widehat{\mathcal{K}} \otimes B)\mu(J)}$$

so that $(E, \widetilde{\phi}, \widetilde{G})$ defines an A-B-bimodule, where $\widetilde{\phi}$ is the composition

$$A \xrightarrow{\phi} D \longrightarrow \mathcal{L}(E) \simeq \mathcal{M}(\mathcal{K}(E))$$

and $\widetilde{G}$ is the image of G in $\mathcal{L}(E)$. If (ϕ', G', μ') is contained in (ϕ, G, μ) then $E' \subset E$ and E and E' are homotopic. On the other hand, if (ϕ', G', μ') is a quotient of (ϕ, G, μ), then $E = E'$, thus superquasimorphisms that are connected by maps give rise to homotopic Kasparov modules. Applying the above construction to homotopies we conclude that the map $[\Phi] \to [E]$ is well defined (although the module arising from this construction is not in general countably generated, it will be as soon as we restrict ourselves to the class of special superquasimorphisms).

Proposition 1.5. *The two maps constructed above are inverse to each other.*

Proof. The successive application

$$[E] \longrightarrow [\Phi] \longrightarrow [E]$$

takes E to itself. In contrast, starting with a special superquasimorphism Φ, mapping back and forth will not be the identity. One has natural maps

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, equal] & D \arrow[r, phantom, "\triangleright"] \arrow[d] & J \arrow[r, "\mu"] \arrow[d] & \widehat{\mathcal{K}} \otimes B \\
A \arrow[r, "\widetilde{\phi}"] & \mathcal{L}(E) \arrow[r, phantom, "\triangleright"] & \mathcal{K}(E) \arrow[r, hook] & \widehat{\mathcal{K}} \otimes B\ .
\end{tikzcd}$$

The last square when completed with $\widehat{\mathcal{K}} \otimes B = \widehat{\mathcal{K}} \otimes B$ need not be commutative, since the original module $E = \overline{\operatorname{span} \mu(J)(\widehat{H}_B)}$ need not be orthocomplemented in general. It is always commutative up to a homotopy, because the embedding $E \subset \widehat{H}_B$ is homotopic to an orthocomplemented embedding $E \perp \widehat{H}_B \simeq \widehat{H}_B$ by the Stabilization Theorem, and any two of these are unitarily equivalent. This completes the proof of the equivalence. ∎

Construction 1.6. Let A be separable and

$$\Phi : A \xrightarrow{\phi} D_1 \overset{F}{\triangleright} J_1 \xrightarrow{\nu} B, \quad \Psi : B \xrightarrow{\widehat{\psi}} \widehat{D}_2 \overset{\widehat{G}}{\triangleright} \widehat{J}_2 \xrightarrow{\widehat{\mu}} C$$

be superquasimorphisms with D_1 separable (for instance if Φ is special). Let

$$J_1 \xrightarrow{\psi} D_2 \overset{G}{\triangleright} J_2 \xrightarrow{\mu} C$$

be contained in $(\nu \circ \Psi)$ so that D_2 is separable.

Put $R = \overline{\psi(J_1) D_2 \psi(J_1)} \subset D_2$ and $J = R \cap J_2$. The image of G in $D_2/J_2 \supset R/J$ lies in the idealizer of the latter algebra and hence defines an element of $\mathcal{M}(R/J)$ which is denoted G'. By [15], 3.12.10, the map $R \to R/J$ extends to a surjective homomorphism $\mathcal{M}(R) \to \mathcal{M}(R/J)$. Let I be the kernel of this map and $\widetilde{G}$ any preimage of G' of degree 1. Let Δ be the separable subalgebra of $\mathcal{M}(R)$ generated by $\widetilde{\psi}(D_1)$, R, $\widetilde{G}$ and 1, where $\widetilde{\psi}$ is the obvious extension of ψ. Put $L = \Delta \cap I$, then $R \cdot L = R \cap L = J$. By KTT (shortly for Kasparov's Technical Theorem) (Theorem 4 in Chapter 3 of [13]) choose elements $M, N \in \mathcal{M}(R)_0$ with $0 \leqslant M \leqslant 1$, $N = 1 - M$ and

$$M \cdot R \subset J, \quad N \cdot L \subset J, \quad [M, \Delta] \subset J.$$

So $\widetilde{F} = \widetilde{\psi}(F)$ satisfies $[\widetilde{F}, \widetilde{G}] \equiv 0 \operatorname{mod} L$. Put

$$H = \sqrt{M}\widetilde{F} + \sqrt{N}\widetilde{G}, \quad \rho(x) = \widetilde{\psi} \circ \phi(x), \quad x \in A.$$

Then

$$(H - H^*)\rho(x) \in J, \quad (1 - H^2)\rho(x) \in J, \quad [\rho(x), H] \in J$$

for all $x \in A$, so (ρ, H, μ) gives a superquasimorphism from A to C. This is called the product of Φ and Ψ and written $\Phi \# \Psi$.

Theorem 1.7. *The construction above gives a bilinear product*

$$\mathrm{KK}(A, B) \times \mathrm{KK}(B, C) \longrightarrow \mathrm{KK}(A, C).$$

Proof. Bilinearity is clear.

To prove that it is well defined note first that the equivalence class of the product does not depend on the choice of $\widetilde{G}$ and M. Let G_0, M_0 and G_1, M_1 be two different choices for $\widetilde{G}, M$ as above. Let Δ' be the separable subalgebra of $\mathcal{M}(R)$ generated by $\widetilde{\psi}(D_1)$, R, G_0, G_1, M_0, M_1, 1 and $L' = \Delta' \cap I$. By KTT there exist $M', N' \in \mathcal{M}(R)_0$, $0 \leqslant M' \leqslant 1$, $N' = 1 - M'$ with

$$M' \cdot R \subset J, \quad N' \cdot L' \subset J, \quad [M', \Delta'] \subset J,$$

and these conditions remain valid if one replaces Δ', L' by Δ_i, L_i $(i = 0, 1)$ respectively. By convexity of the set of elements M that fulfil the KTT conditions for a given set of algebras, it follows that we can continuously deform M_0 to M' in our formulas. Then connect G_0 to G_1 by $G_t = (1-t)G_0 + tG_1$ and afterwards deform M' back to M_1.

It will now be shown that the product does not depend on the choice of representatives. For instance, if Φ' is a quotient of Φ then nothing has to be changed in the construction above so that we get the same result. If Ψ' is a quotient of Ψ we have surjective homomorphisms $R \to R'$, $\mathcal{M}(R) \to \mathcal{M}(R')$, so that the resulting superquasimorphism will be a quotient of the original one (up to homotopy).

Assume that $\Psi' = (\psi', G', \mu')$ is contained in Ψ. The algebra $R' = \overline{\psi'(J_1) D_2' \psi'(J_1)}$ is a subalgebra of R such that the hereditary subalgebra it generates is the whole of R. Hence there is an embedding $\mathcal{M}(R') \hookrightarrow \mathcal{M}(R)$ mapping the subalgebras Δ', L', R' of $\mathcal{M}(R')$ into the corresponding algebras Δ, L, R and in particular mapping the elements G', M' to elements of the same kind (note that M fulfils the KTT conditions for R, L, Δ as well as R', L', Δ'). So the product $\Phi \# \Psi'$ is contained (up to homotopy) in $\Phi \# \Psi$.

The hardest part is the following. Assume that Φ' is contained in Φ. The product with Φ' is constructed in the multipliers of $R' = \overline{\psi(J_1')D_2\psi(J_1')}$. Put

$$\widehat{R} = \begin{pmatrix} R' & \overline{R'R} \\ \overline{RR'} & R \end{pmatrix}, \quad \widehat{J} = \widehat{R} \cap M_2(J).$$

We can assume that both products are constructed in $\mathcal{M}(\widehat{R})$, which contains a copy of each $\mathcal{M}(R')$ and $\mathcal{M}(R)$ in the upper left respective lower right corner. In the following, all the structures refering to the first product carry a "$\,'\,$". In both cases the algebras $\widetilde{\psi}'(D_1')$ and $\widetilde{\psi}(D_1')$ may be replaced by the diagonal

$$\widehat{\psi}(x) = \begin{pmatrix} \widetilde{\psi}'(x) & 0 \\ 0 & \widetilde{\psi}(x) \end{pmatrix}, \quad x \in D_1'.$$

Also we replace $\widetilde{G}'$ and $\widetilde{G}$ respectively by $\widehat{G} = \widetilde{G}' \oplus \widetilde{G}$. Consider the subalgebra $\overline{R}$ of $\widehat{R}[0,1]$ which is obtained by rotating R' from the upper left to the lower right corner, i.e. $\overline{R}$ consists of elements

$$\overline{y}(t) = \begin{pmatrix} y \cdot \cos^2(\frac{\pi}{2}t) & y \cdot \cos(\frac{\pi}{2}t) \cdot \sin(\frac{\pi}{2}t) \\ y \cdot \cos(\frac{\pi}{2}t) \cdot \sin(\frac{\pi}{2}t) & y \cdot \sin^2(\frac{\pi}{2}t) \end{pmatrix}, \quad y \in R',\, t \in [0,1].$$

Let $\overline{\Delta}$ be the subalgebra of $\mathcal{M}(\widehat{R}[0,1])$ generated by $\overline{R}$, the constant sections $\widehat{\psi}(x)$, $x \in D_1'$, the constant function $\widehat{G}$ and 1. Let $\overline{J} = \overline{R} \cap \widehat{J}[0,1]$, $\overline{L} = \overline{\Delta} \cap \overline{I}$ with $\overline{I}$ the kernel of $\mathcal{M}(\widehat{R}[0,1]) \to \mathcal{M}(\widehat{R}/\widehat{J}[0,1])$. Choose $\overline{M}, \overline{N}$ satisfying the KTT conditions for $\overline{\Delta}$, $\overline{R}$, $\overline{L}$, $\overline{J}$ and construct a superquasimorphism from A to $C[0,1]$ as above. Evaluation at $t = 0$ is essentially the primed product and evaluation at $t = 1$ gives a product constructed in $\mathcal{M}(R)$ with elements M_1, N_1 satisfying KTT for Δ_1, $R'+J$, L_1, J (Δ_1 is generated by $\widetilde{\psi}(D_1')$, $R'+J$, $\widetilde{G}$ and 1 and $L_1 = \Delta_1 \cap L$). This shows that both products are equivalent.

Finally, if $\overline{\Phi} : A \longrightarrow\!\!\triangleright\, B[0,1]$ or $\overline{\Psi} : B \longrightarrow\!\!\triangleright\, C[0,1]$ are homotopies of superquasimorphisms, then Construction 1.6 yields a superquasimorphism from A to $C[0,1]$ connecting the products of the evaluated representatives at $t = 0$, $t = 1$ respectively up to natural mappings. This completes the proof. ∎

REMARKS 1.8. (i) One can simplify the verification that the product is well defined if one restricts the class of admissible representatives in Construction 1.6 to, say special superquasimorphisms or some other suitable class. The advantage of the more general construction lies in its greater flexibility.

(ii) If one of the representatives above is actually a homomorphism (i.e. $\phi(A) \subset J_1$ or $\psi(B) \subset J_2$), the result of the construction is the same as ordinary composition by homomorphisms. Assume first that Φ is a homomorphism. The image of A in $\mathcal{M}(R)$ actually lies in R. Put

$$\widehat{R} = \begin{pmatrix} R & \overline{RD_2} \\ \overline{D_2R} & D_2 \end{pmatrix} \subset M_2(D_2).$$

Let $\widehat{H} = H \oplus G \in \mathcal{M}(\widehat{R}))$ and rotate $\big(\left(\begin{smallmatrix} \rho & 0 \\ 0 & 0 \end{smallmatrix}\right), \widehat{H}, \mathrm{id} \otimes \mu\big)$ to $\big(\left(\begin{smallmatrix} 0 & 0 \\ 0 & \rho \end{smallmatrix}\right), \widehat{H}, \mathrm{id} \otimes \mu\big)$. If $\psi : J_1 \to C$ is a homomorphism (so that $R = J$) replace M, N by $M' = 1$, $N' = 0$ and the claim follows.

Lemma 1.9. *Let A be separable and $\Phi = (\phi, F) : A \to D_1 \triangleright J_1$, $\Psi = (\psi, G, \mu) : J_1 \to D_2 \triangleright J_2 \to C$ superquasimorphisms with D_1, D_2 separable. There is a representative $\Psi' = (\psi', G', \mu')$ in the equivalence class of Ψ which is the restriction of an superquasimorphism from $E_1 = J_1 + \phi(A)$ to C, such that the product of Φ and Ψ' is given by $(\psi' \circ \phi, G', \mu')$.*

Proof. Form the product of $E_1 \xrightarrow{\mathrm{id}} D_1 \overset{F}{\triangleright} J_1$ with Ψ. Its restriction to J_1 is the product of $J_1 \to D_1 \triangleright J_1$ with Ψ, hence equivalent to Ψ and the other part of the statement is then clear by construction. ∎

Theorem 1.10. *The product* $\mathrm{KK}(A, B) \times \mathrm{KK}(B, C) \to \mathrm{KK}(A, C)$ *is associative.*

Proof. We assume that A, B are separable, B, C, D stable (for simplicity). For $\mathbf{x} \in \mathrm{KK}(A, B)$, $\mathbf{y} \in \mathrm{KK}(B, C)$, $\mathbf{z} \in \mathrm{KK}(A, C)$ choose representatives

$$A \xrightarrow{\phi} D_1 \overset{F}{\triangleright} J_1 \xrightarrow{\nu} B, \quad B \xrightarrow{\psi'} D_2' \overset{G'}{\triangleright} J_2' \xrightarrow{\mu'} C, \quad C \xrightarrow{\omega'} D_3' \overset{H'}{\triangleright} J_3' \xrightarrow{\pi'} D.$$

The assertion is clear if one of the factors is a homomorphism. In the general case, we can without changing the result, replace the product (in either order) by the product of

$$A \xrightarrow{\phi} D_1 \overset{F}{\triangleright} J_1, \quad J_1 \xrightarrow{\psi} D_2 \overset{G}{\triangleright} J_2, \quad J_2 \xrightarrow{\omega} D_3 \overset{H}{\triangleright} J_3 \xrightarrow{\pi} D$$

with D_1, D_2, D_3 separable. Furthermore, one can arrange that ψ and ω extend to homomorphisms $D_1 \to D_2$, $D_2 \to D_3$ respectively, that the product of (ϕ, F) and (ψ, G) is $(\psi \circ \phi, G)$ and that (ω, H, π) is the restriction of a superquasimorphism from $E_2 = J_2 + \psi(E_1)$ with $E_1 = J_1 + \phi(A)$, which is the product of $E_2 \to D_2 \triangleright J_2$ with (ω, H, π). This follows from Lemma 1.9 and associativity in the special case when one of the factors is a homomorphism.

Then $(\mathbf{x}\cdot\mathbf{y})\cdot\mathbf{z}$ is the class of $(\omega\circ\psi\circ\phi, H, \pi)$ and $\mathbf{x}\cdot(\mathbf{y}\cdot\mathbf{z})$ is the product of (ϕ, F) and $(\omega\circ\psi, H, \pi)$ which is something like $(\omega\circ\psi\circ\phi, K, \pi)$ with $K = \sqrt{M}\widetilde{F} + \sqrt{N}H$. On the other hand, H is something like

$$
\begin{aligned}
&H = \sqrt{M_2}\big(\sqrt{M_1}\widetilde{F} + \sqrt{N_1}\widetilde{G}\big) + \sqrt{N_2}\widetilde{H},\\
&N_1 \cdot [\widetilde{F}, \widetilde{G}] \subset J_2, \quad M_1 \cdot J_1 \subset J_2,\\
&N_2 \cdot [\widetilde{F}, \widetilde{H}] \subset J_3,\ N_2 \cdot [\widetilde{G}, \widetilde{H}] \subset J_3, \quad M_2 \cdot J_2 \subset J_3.
\end{aligned}
$$

Thus $[H, K] \geqslant 0 \bmod J_3$ and the claim follows from Lemma 11 of [16]. ∎

2. UNIVERSAL ALGEBRAS AND THE Ext-FUNCTOR

From the presentation of the KK-groups as homotopy classes of superquasimorphisms one obtains quite naturally a description using universal algebras — an analogue of Cuntz description for trivially graded algebras.

Let A be a $\mathbb{Z}_2$-graded C^*-algebra. Consider the $\mathbb{Z}_2$-graded C^*-algebra $\mathcal{X}A = C^*(A, F)$ which is generated freely by A together with a symmetry F of odd degree. There exist no (algebraic) relations besides those of A and the single relation $F = F^* = F^{-1}$, in other words if A^+ is the algebra A with unit adjoined then $\mathcal{X}A$ is the free product of A^+ with the Clifford algebra C_1 amalgamated over $\mathbb{C}$. Giving a homomorphism $\mathcal{X}A \to D$ is the same as giving a homomorphism $\phi : A \to D$ and a selfadjoint partial isometry $G \in D$ of degree 1 such that $(1 - G^2)\phi(A) = 0$.

Let XA be the ideal generated by A and let χA be the ideal that is generated by all (graded) commutators of the form $[x, F] = xF - F\overline{x}$, $x \in A$. One has the universal normal superquasimorphism

$$A \longrightarrow \mathcal{X}A \overset{F}{\triangleright} \chi A.$$

For this, let $\Phi : A \xrightarrow{\phi} D \overset{G}{\triangleright} J \to B$ be an arbitrary normal superquasimorphism. We may complete the following diagram as indicated by the dashed arrow

$$
\begin{array}{ccccccc}
A & \longrightarrow & \mathcal{X}A & \overset{F}{\triangleright} & \chi A & \overset{\varphi}{\dashrightarrow} & B\\
\| & & \downarrow & & \downarrow & & \|\\
A & \xrightarrow{\phi} & D & \overset{G}{\triangleright} & J & \longrightarrow & B.
\end{array}
$$

Every normal superquasimorphism Φ from A to B thus defines a homomorphism $\chi A \xrightarrow{\varphi} B$. It is easy to see that $\varphi = \varphi'$ in case $\Phi \simeq \Phi'$ and that homotopies are preserved.

Conversely, every $\varphi : \chi A \to B$ immediately defines a superquasimorphism from A to B such that homotopies are preserved. For any graded algebras A, B let $[A, B]$ denote the set of homotopy classes of (graded) homomorphisms from A to B. We have shown

Theorem 2.1. $\mathrm{KK}(A, B) \simeq [\chi A, \widehat{\mathcal{K}} \otimes B]$. ∎

Let $\mathcal{X}^2 A = \mathcal{X}\mathcal{X}A = C^*(A, F, G)$ and $\overline{\rho}_0 : \mathcal{X}^2 A \to \mathcal{X}A$ be the homomorphism sending F and G to F and ρ_0 the restriction to $\chi^2 A$.

Proposition 2.2. (i) *There is a homomorphism* $\psi_A : \chi A \to \widehat{M_2}(\chi^2 A)$ *so that* $\rho_0 \circ \psi_A : \chi A \to \widehat{M_2}(\chi A)$ *is homotopic to*

$$j_0 = \begin{pmatrix} \mathrm{id}_{\chi A} & 0 \\ 0 & 0 \end{pmatrix}.$$

(ii) *The Kasparov product of* $[\varphi_1] \in [\chi A, \widehat{\mathcal{K}} \otimes B]$ *and* $[\varphi_2] \in [\chi B, \widehat{\mathcal{K}} \otimes C]$ *is* $[(\mathrm{id}_{\widehat{\mathcal{K}}} \otimes \varphi_2) \circ \pi \circ (\mathrm{id}_{\widehat{M}_2} \otimes \chi(\varphi_1)) \circ \psi_A]$ *(π is the surjection $\chi(\widehat{\mathcal{K}} \otimes B) \to \widehat{\mathcal{K}} \otimes \chi B$).*

Proof. (i) The homomorphism ψ_A is the (normalized) product of the superquasimorphisms $A \to \mathcal{X}A \triangleright \chi A$ and $\chi A \to \mathcal{X}\chi A \triangleright \chi^2 A$. Let $R = \overline{\chi A(\mathcal{X}\chi A)\chi A}$. Then ρ_0 maps R surjectively onto χA so that we can extend it to a map $\mathcal{M}(R) \to \mathcal{M}(\chi A)$. Let M', N' be the images of $M, N \in \mathcal{M}(R)$ as in the construction of the product above. Since R is mapped to χA, we may continuously deform M' to 1 to obtain the desired homotopy.

(ii) Is obvious. ∎

Recall the construction of the Cuntz algebra $qA \triangleleft QA$ (see [6]). Its grading is the natural one induced by the grading of A. Let τ be the natural $\mathbb{Z}_2$-action on QA exchanging the two copies of A and κ the product of τ with the grading automorphism on A, i.e. $\kappa : \iota(x) \leftrightarrow \overline{\iota}(\overline{x})$ if $\iota, \overline{\iota} : A \to A * A$ are the natural embeddings. The map κ preserves the grading and maps qA to itself. Then one has the identities

$$XA = QA \rtimes_\kappa \mathbb{Z}_2^{\mathrm{od}} \quad \chi A = qA \rtimes_\kappa \mathbb{Z}_2^{\mathrm{od}}. \tag{2.1}$$

By the notation $\mathbb{Z}_2^{\mathrm{od}}$ we mean that the symmetry F' implementing κ is taken of degree 1 instead of degree 0, which is another possibility. Explicitly, the identity (2.1) is given by the formulas

$$XA \ni \left\{ \begin{array}{ccc} x & \leftrightarrow & \iota(x) \\ F\overline{x}F & \leftrightarrow & \overline{\iota}(x) \\ F & \leftrightarrow & F' \end{array} \right\} \in QA \rtimes_\kappa \mathbb{Z}_2^{\mathrm{od}}.$$

Assume that A is separable and B σ-unital. From Theorem 2.1 one gets the wellknown result that $\mathrm{KK}(A,B)$ is given as homotopy classes of homomorphisms $[qA, \mathcal{K}\otimes B]$ in the case when the grading on both A and B is trivial. A homomorphism $\chi A \to \widehat{\mathcal{K}} \otimes B$ may be extended (up to homotopy) to a map $(\phi, G) : \mathcal{X}A \to \mathcal{M}(\widehat{\mathcal{K}} \otimes B)$. Moreover, G may be taken to be a symmetry, we can even assume that $G = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \widehat{M_2}(\mathcal{M}(\mathcal{K}\otimes B))$ (ditto for homotopies). Now $\phi : A \to \widehat{M_2}(\mathcal{M}(\mathcal{K}\otimes B))$ is of the form

$$\begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix} \quad \text{and} \quad \phi_0 - \phi_1 \in \mathcal{K}\otimes B.$$

The pair (ϕ_0, ϕ_1) defines a quasihomomorphism from A to $\widehat{\mathcal{K}} \otimes B$ in the sense of [7] or, in other words, a homomorphism from qA to $\widehat{\mathcal{K}} \otimes B$. Obviously this construction is bijective. If in addition the algebras A and B are equipped with an action of the compact group G that commutes with the $\mathbb{Z}_2$-action, Theorem 2.1 generalizes to the equivariant setting by the following: χA is now a $G \times \mathbb{Z}_2$-invariant ideal in $C^*(A, F)$, with F taken G-invariant of degree 1. We have that $\mathcal{K}\otimes B \simeq \mathcal{K}\otimes\mathcal{K}(L_2(G\times\mathbb{Z}_2))\otimes B$ carries the natural $G\times\mathbb{Z}_2$-action which is the product of the trivial action on the first factor, the action induced by the left regular representation on the second and the action on B. Generalized in this way, we can write Theorem 2.1 as $\mathrm{KK}_G(A,B) \simeq [\chi A, \mathcal{K}\otimes B]_{G\times\mathbb{Z}_2}$, the right hand side meaning homotopy classes of $G\times\mathbb{Z}_2$-equivariant homomorphisms. In case the $\mathbb{Z}_2$-grading on A and B is trivial, we get just as above that $\mathrm{KK}_G(A,B) \simeq [qA, \mathcal{K}\otimes B]_G$, in particular $\mathrm{KK}_{\mathbb{Z}_2}(A,B)$ equals homotopy classes of $\mathbb{Z}_2$-graded homomorphisms $[qA, \widehat{\mathcal{K}}\otimes B]$.

Another crossed product of qA by an action of $\mathbb{Z}_2$ is the Zekri algebra εA (again the grading is induced from A) and it is important in connection with Kasparov's graded Ext-functor. There is a natural homomorphism

$$\mathrm{Ext}(A,B) \longrightarrow [\varepsilon A, \widehat{\mathcal{K}}\otimes B]$$

which is an isomorphism when A is separable and B σ-unital. Surjectivity follows easily from the Stabilization Theorem while injectivity stems from the congruence of the two equivalence relations "compact perturbation" and "homotopy" in $\mathbb{Z}_2$-equivariant KK-Theory which, as is shown below, coincides up to a dimension shift with the Ext-functor as above, cf. [1], [13].

Definition 2.3. $\mathrm{Ex}(A,B)$ is the set $[qA, \widehat{\mathcal{K}}\otimes B] = \mathrm{KK}_{\mathbb{Z}_2}(A,B)$ and $\mathrm{Ex}_i(A,B) = \mathrm{Ex}(\mathrm{S}^i A, B)$, $\mathrm{S}A$ denoting the suspension of A, i.e. $\mathrm{S}A = C_0(\mathbb{R}, A)$.

Recall the following fundamental construction of Cuntz (Theorem 1.6 of [6]) of which a $\mathbb{Z}_2$-equivariant version has been provided by J. Weidner and R. Zekri.

Theorem 2.4. (Cuntz) *Let A be separable, $\mathbb{Z}_2$-graded. There is a homomorphism $\varphi_A : qA \longrightarrow M_2(q^2A)$, so that*

$$\pi_0 \circ \varphi_A \sim \begin{pmatrix} \mathrm{id}_{qA} & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \varphi_A \circ \pi_0 \sim \begin{pmatrix} \mathrm{id}_{q^2A} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\pi_0 : qA \to A$ is the canonical homomorphism sending $\bar{\iota}A$ to 0. ∎

Theorem 2.5. (Weidner/Zekri) *Let A be separable, $\mathbb{Z}_2$-graded. Then:*
$\widehat{\mathcal{K}} \otimes \varepsilon A$ is homotopy equivalent with $\widehat{\mathcal{K}} \otimes q\varepsilon A$ and also with $\widehat{\mathcal{K}} \otimes \varepsilon qA$;
$\widehat{\mathcal{K}} \otimes \chi A$ is homotopy equivalent with $\widehat{\mathcal{K}} \otimes q\chi A$ and also with $\widehat{\mathcal{K}} \otimes \chi qA$.

Proof. The proof of the first statement can be taken literally from [18]. Only minor and obvious changes are necessary to get the second statement. ∎

Remarks 2.6. (i) The Cuntz construction gives an associative product on Ex that coincides with Kasparov product. A and B are Ex-equivalent if and only if qA and qB are stably homotopy equivalent. This follows from the fact that $\pi_{0*} \in \mathrm{Ex}(qA, A)$ is an equivalence. Hence we may present the group $\mathrm{Ex}(A, B)$ as homotopy classes $[\widehat{\mathcal{K}} \otimes qA, \widehat{\mathcal{K}} \otimes qB]$, the product being defined as composition of homomorphisms. The corresponding statement for $\mathrm{KK}(A, B)$ and $\chi A, \chi B$ is not true (see [11]).

(ii) From the existence of long exact sequences for the Ex-functor one gets that $\varepsilon A \sim_{\mathrm{Ex}} \mathrm{S}A$ since the "universal semisplit extension" of A by εA is contractible (see [19]). From the equivariant Cuntz construction one then obtains

$$\mathrm{Ex}_1(A, B) \simeq [q\mathrm{S}A, \widehat{\mathcal{K}} \otimes B] \simeq [q\varepsilon A, \widehat{\mathcal{K}} \otimes B] \simeq [\varepsilon A, \widehat{\mathcal{K}} \otimes B],$$

so there is a natural isomorphism $\mathrm{Ext}(A, B) \xrightarrow{\sim} \mathrm{Ex}_1(A, B)$ in case B is σ-unital.

(iii) $\mathrm{KK}_*(A, B) \simeq \mathrm{Ex}_*(\chi A, B)$. This does not mean that KK is just a special case of Ex, since the product is different. However, it can be derived from Ex-product in a certain sense (see [11]).

Let us take a closer look at the relation between KK and Ex. Consider the following homomorphisms.

Let $\alpha_0 : \chi A \to \widehat{M_2}(qA)$ be defined by restriction from the formulas

$$\alpha_0(x) = \begin{pmatrix} \iota(x) & 0 \\ 0 & \bar{\iota}(\overline{x}) \end{pmatrix} \quad \text{for } x \in A \quad \text{and} \quad \alpha_0(F) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

Let $\beta_0 : qA \to \chi A = qA \rtimes_\kappa \mathbb{Z}_2^{\mathrm{od}}$ be the natural embedding (with the $\mathbb{Z}_2$-action κ as defined above).

The homomorphisms α_0, β_0 induce functorial mappings

$$\alpha : \mathrm{Ex}_*(A,B) \longrightarrow \mathrm{KK}_*(A,B)$$
$$\beta : \mathrm{KK}_*(A,B) \longrightarrow \mathrm{Ex}_*(A,B).$$

One can also consider the map $\alpha_1 : \chi A \widehat{\otimes} C_1 \to \widehat{M_2}(\varepsilon A)$ given by

$$\alpha_1(x\widehat{\otimes}1) = \begin{pmatrix} x & 0 \\ 0 & \overline{x} \end{pmatrix} \quad \text{for } x \in A$$

and

$$\alpha_1(F\widehat{\otimes}1) = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}, \quad \alpha_1(1\widehat{\otimes}T) = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix},$$

T being the generator of C_1 and $\beta_1 : \varepsilon A \to \chi A \widehat{\otimes} C_1$ with

$$\beta_1(x) = x\widehat{\otimes}1 \quad \text{for } x \in A \quad \text{and} \quad \beta_1(V) = \mathrm{i}(F\widehat{\otimes}T). \tag{2.3}$$

Indeed, these maps can be used equally well to define α and β (see below). It is not hard to show that α commutes with products using its definition (see [11] for a proof of this result).

The $\mathbb{Z}_2$-grading provides $\mathrm{KK}(A,B)$ with more structure than that of an abelian group. Every $\mathbf{x} \in \mathrm{KK}(A,B)$ may be composed with the grading automorphisms of A and B. These represent canonical elements of order 2 in $\mathrm{KK}(A,A)$ (resp. $\mathrm{KK}(B,B)$). In this way the abelian group $\mathrm{KK}(A,B)$ inherits a bigraded structure or a structure of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-module. We denote the corresponding elements in $\mathrm{KK}(A,A)$ by $*$ (or $*_A$ if it is convenient to specify the particular algebra) and write $*\mathbf{x}$, $\mathbf{x}*$ to denote the left (resp. right) conjugate of $\mathbf{x}$.

Of particular interest is the diagonal conjugation $\mathbf{x}^* = *\mathbf{x}*$. We will call it *the involution*. The involution commutes with the product in the sense that $\mathbf{x}^*\mathbf{y}^* = (\mathbf{xy})^*$ and with the natural map $\tau_D : \mathrm{KK}(A,B) \to \mathrm{KK}(A\widehat{\otimes}D, B\widehat{\otimes}D)$ for any D. Hence it commutes with the general form of the product.

If $\mathbf{x} \in \mathrm{KK}(A,B)$ and $\mathbf{y} \in \mathrm{KK}(B,A)$ are such that $\mathbf{xy} = 1_A$, $\mathbf{yx} = 1_B$ and $\mathbf{x}^* = \mathbf{x}$ then $\mathrm{KK}(A,D)$ ($\mathrm{KK}(D,A)$) and $\mathrm{KK}(B,D)$ (resp. $\mathrm{KK}(D,B)$) are naturally isomorphic as bigraded groups for all D. We will say that A and B are *strongly* KK-equivalent in this case.

EXAMPLE 2.7. Let C_1 be the complex Clifford algebra with one odd generator and $S = C_0(\mathbb{R})$. One has $*_{C_1} = -\mathbf{1}_{C_1}$ in $\mathrm{KK}(C_1,C_1)$ but $*_S = \mathbf{1}_S$ in $\mathrm{KK}(S,S)$. It follows that $\mathbf{x}^* = *\mathbf{x}* = -\mathbf{x}$ for every $\mathbf{x} \in \mathrm{KK}(S,C_1)$ so that S and C_1 are not strongly KK-equivalent although they are KK-equivalent in the usual sense.

Returning to the relation of KK and Ex, the significance of the canonical involution is as follows. The composition $\alpha \circ \beta$ is a map of $\mathrm{KK}(A,B)$ to itself which sends an element $\mathbf{x}$ to $\mathbf{x}+\mathbf{x}^*$. Similarly, the composition $\beta\circ\alpha$ is determined by a natural involution of the group $\mathrm{Ex}(A,B)$. Recall that this group is a module over $\mathrm{Ex}(\mathbb{C},\mathbb{C})$ which is $\mathbb{Z}[\xi]$ with $\xi^2=1$, the module structure being implemented by Kasparov product. Now, there are two ways of taking this product, depending on whether one uses the ordinary tensor product (as is customary in equivariant KK-theory) or the graded tensor product. Recall that for any graded D there are natural maps

$$\sigma_D : \mathrm{Ex}(A,B) \longrightarrow \mathrm{Ex}(A\otimes D, B\otimes D)$$

and

$$\tau_D : \mathrm{Ex}(A,B) \longrightarrow \mathrm{Ex}(A\widehat{\otimes} D, B\widehat{\otimes} D)$$

commuting with products. Using these, one has two pairings

$$\begin{aligned}\mathrm{Ex}(\mathbb{C},\mathbb{C})\otimes \mathrm{Ex}(A,B) &\longrightarrow \mathrm{Ex}(A,B)\\ \mathrm{Ex}(\mathbb{C},\mathbb{C})\widehat{\otimes}\mathrm{Ex}(A,B) &\longrightarrow \mathrm{Ex}(A,B).\end{aligned}$$

Be careful: even if some of the algebras are trivially graded so that the graded and ordinary tensor product coincide, the natural transformations τ_D and σ_D may be different. For example the map

$$\tau_{C_1} : \mathrm{Ex}(\mathbb{C},\mathbb{C}) \longrightarrow \mathrm{Ex}(C_1, C_1)$$

is an isomorphism (since $\tau_{C_1}^2$ is), but

$$\sigma_{C_1} : \mathrm{Ex}(\mathbb{C},\mathbb{C}) \longrightarrow \mathrm{Ex}(C_1, C_1)$$

is not! (If D is trivially graded then $\tau_D=\sigma_D$.) One gets two natural involutions on $\mathrm{Ex}(A,B)$ (taking the product with the element ξ) that will be denoted $\flat$ and $\sharp$ respectively. In general, they are different. The involution $\flat$ is implemented by conjugation of an Ex-cycle with a unitary of odd degree while $\sharp$ is obtained by composing $\flat$ with the grading automorphism (on the left or on the right). Both involutions are preserved under Ex-products in the sense that $\mathbf{x}^\flat\cdot\mathbf{y}=(\mathbf{x}\cdot\mathbf{y})^\flat=\mathbf{x}\cdot\mathbf{y}^\flat$ and the same with $\sharp$. Also, by the way they were defined, $\flat$ commutes with the map σ_D and $\sharp$ with τ_D for any D. Given any element $\mathbf{x}\in\mathrm{Ex}(A,B)$ the composite $(\beta\circ\alpha)(\mathbf{x})$ is $\mathbf{x}-\mathbf{x}^\sharp$.

The map β does not commute with the products.

3. THE MAIN THEOREM

A main result of this paper is the construction of a double exact sequence relating the KK- and Ex-functor. In the proof it will be useful to have an explicit construction for the stable homotopy equivalence $\varphi_A : \chi A \to M_2(q\chi A)$ with inverse π_0 which is compatible with the homomorphism ψ_A in Proposition 1.5 in the sense that its construction is built on Kasparov's Technical Theorem rather than Pedersen's theorem on the lifting of derivations (as in the proof of Weidner and Zekri's Theorem following Cuntz). Also rather than dealing with the universal algebras q^2A, $q\varepsilon A$, $\chi\varepsilon A$, $\chi^2 A$ etc. we will prefer to work with the ideals generated by the former algebras in Q^2A, QEA, XEA, $X^2A, \ldots$ respectively.

Definition 3.1. We denote by I_{q^2A}, $I_{q\varepsilon A}$, $I_{\chi\varepsilon A}$, $I_{\chi^2 A}$ etc. the ideals generated by q^2A, $q\varepsilon A$, $\chi\varepsilon A$, $\chi^2 A$ etc., respective in Q^2A, QEA, XEA, X^2A etc., respective.

It is convenient to have the following:

Lemma 3.2. *Let τ_1, τ_2 denote the canonical $\mathbb{Z}_2$-actions on the inner and outer Q of $Q^2A = Q(QA)$ respectively. The embedding $q^2A \subset I_{q^2A}$ is a stable equivariant homotopy equivalence with respect to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action (τ_1, τ_2).*

Proof. The proof is quite similar to the corresponding result in [18]. ∎

Remark 3.3. Lemma 3.2 implies that $q\varepsilon A, \chi\varepsilon A, \chi^2 A$ etc. (in fact any combination we wish to choose) are stably homotopy equivalent to $I_{q\varepsilon A}, I_{\chi\varepsilon A}, I_{\chi^2 A}$ etc. respective on dualizing the above statement in the proper way. We do not bother to check here that, for instance $q^2A \rtimes_{\tau_1} \mathbb{Z}_2$ is equivalent with $q(qA \rtimes_{\tau_1} \mathbb{Z}_2) = q\varepsilon A$ which is rather straightforward.

Construction 3.4. Let $I = I_{q\varepsilon A}$ be as above. One has a natural map $QEA \to \mathcal{M}(I)$ and below we will not distinguish between elements in QEA and their images in $\mathcal{M}(I)$. Putting $R = I + Q\varepsilon A$, $L = I + qEA$ one has $R \cdot L = R \cap L = I$. For this, it suffices to check that $\iota(x)(\iota(y) - \bar\iota(y)) \in I$ for $x \in \varepsilon A$, $y \in EA$. But $\iota(x)(\iota(y) - \bar\iota(y)) \equiv \iota(xy) - \bar\iota(xy) \equiv 0 \mod I$.

By KTT, we choose elements $M, N \in \mathcal{M}(I)_0$ with $M + N = 1$, $M \cdot R \subset I$, $N \cdot L \subset I$, $[QEA, M] \subset I$ and put

$$S = \begin{pmatrix} \sqrt[4]{M}\,\iota(V)\,\sqrt[4]{M} & \sqrt{N} \\ \sqrt{N} & -\sqrt[4]{M}\,\bar\iota(V)\,\sqrt[4]{M} \end{pmatrix}$$

$$W = \begin{pmatrix} \sin(\frac{\pi}{2}S) & \cos(\frac{\pi}{2}S) \\ \cos(\frac{\pi}{2}S) & -\sin(\frac{\pi}{2}S) \end{pmatrix}.$$

Define $\widetilde{\varphi}_A : EA \to M_4(\mathcal{M}(I))$ by

$$\widetilde{\varphi}_A(x) = \begin{pmatrix} \iota(x) & 0 & 0 & 0 \\ 0 & \bar{\iota}(x) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x \in A,\ \widetilde{\varphi}_A(V) = W.$$

It follows that $\widetilde{\varphi}_A(\varepsilon A) \subset M_4(I_{q\varepsilon A})$.

Much in the same way, we define $\widetilde{\psi}_A : \chi A \to \widehat{M}_2(I_{\chi^2 A})$. For this, let now $I = I_{\chi^2 A}$, $R = X\chi A + I$, $L = \chi\mathcal{X}A + I$ so that $R \cdot L = R \cap L = I$. With corresponding elements M, N put $S = \sqrt[4]{M}F\sqrt[4]{M} + \sqrt[4]{N}G\sqrt[4]{N}$,

$$H = \begin{pmatrix} \sin(\frac{\pi}{2}S) & \cos(\frac{\pi}{2}S) \\ \cos(\frac{\pi}{2}S) & -\sin(\frac{\pi}{2}S) \end{pmatrix}$$

and

$$\widetilde{\psi}_A(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad x \in A,\ \widetilde{\psi}_A(F) = H.$$

Lemma 3.5. (i) *$\widetilde{\varphi}_A$ is a stable homotopy equivalence with inverse π_0.*

(ii) *The stable homotopy class of $\widetilde{\psi}_A$ corresponds to that of ψ_A under $\chi^2 A \hookrightarrow I_{\chi^2 A}$.*

Proof. (i) $\pi_0 : I_{q\varepsilon A} \to \varepsilon A$ is surjective and extends to $\overline{\pi}_0 : \mathcal{M}(I) \to \mathcal{M}(\varepsilon A)$. This map sends L onto EA so that $\overline{\pi}_0(N)\cdot EA \subset \varepsilon A$. In the composition $\pi_0 \circ \widetilde{\varphi}_A$, we can therefore deform $\overline{\pi}_0(N)$ continuously to 0 which gives the homotopy $\pi_0 \circ \widetilde{\varphi}_A \sim j_0$. Now since π_0 is a homotopy equivalence and left inverse to $\widetilde{\varphi}_A$, the result follows.

(ii) Let $I = I_{\chi^2 A}, R, L$ as in the second part of Construction 3.4. Let $D = \overline{\chi A(\mathcal{X}\chi A)\chi A}$, $J = D \cap \chi^2 A$, $R' = \overline{DRD}$, $J' = R' \cap I$. Also let

$$\widehat{R} = \begin{pmatrix} R' & \overline{R'R} \\ \overline{RR'} & R \end{pmatrix}, \quad \widehat{J} = \widehat{R} \cap M_2(I).$$

It follows that $D/J \simeq R/J \simeq R'/J' \simeq \chi A \widehat{\otimes} C_1$. There is a map $\mathcal{M}(D) \to \mathcal{M}(R')$ so that the superquasimorphism

$$A \longrightarrow \widehat{M}_2(\mathcal{M}(D)) \triangleright \widehat{M}_2(J) \xrightarrow{j} \widehat{M}_2(I)$$

that represents $j \circ \psi_A$ can be replaced by

$$A \longrightarrow \widehat{M}_2(\mathcal{M}(\widehat{R})) \triangleright \widehat{M}_2(\widehat{J}) \longrightarrow \widehat{M}_4(I)$$

on mapping $\mathcal{M}(D)$ into the left upper corner of $\mathcal{M}(\widehat{R})$. To prove the result we use a rotation argument as in the proof of Theorem 1.7. ∎

Let $j = \mathrm{id}\widehat{\otimes}1$ be the embedding of A into $A\widehat{\otimes}C_1$ and $j' = j\widehat{\otimes}\mathrm{id} : A\widehat{\otimes}C_1 \to \widehat{M}_2(A)$.

THEOREM 3.6. ($\sharp$-sequence) *Let A be separable. The following diagram represents two exact sequences with opposite sense of direction*

$$\begin{tikzcd}
\mathrm{Ex}_0(A,B) \arrow[r, shift left, "\alpha"] & \mathrm{KK}_0(A,B) \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\beta'"] & \mathrm{Ex}_1(A\widehat{\otimes}C_1,B) \arrow[l, shift left, "\alpha'"] \arrow[d, shift left, "j^*"] \\
\mathrm{Ex}_0(A\widehat{\otimes}C_1,B) \arrow[u, shift left, "j^*"] \arrow[r, shift left, "\beta'"] & \mathrm{KK}_1(A,B) \arrow[l, shift left, "\alpha'"] \arrow[r, shift left, "\alpha"] & \mathrm{Ex}_1(A,B). \arrow[l, shift left, "\beta"] \arrow[u, shift left, "j'^*"]
\end{tikzcd} \tag{3.1}$$

Here $\alpha : \mathrm{Ex}_1(A,B) \to \mathrm{KK}_1(A,B)$ is induced by $\alpha_1 : \chi A\widehat{\otimes}C_1 \to \widehat{M}_2(\varepsilon A)$ and $\beta : \mathrm{KK}_1(A,B) \to \mathrm{Ex}_1(A,B)$ is induced by $\beta_1 : \varepsilon A \to \chi A\widehat{\otimes}C_1$.

All mappings occurring in this double sequence commute with the long exact sequences of the KK- *and* Ex-*groups.*

Proof. Consider the short exact sequence

$$0 \longrightarrow \chi A \longrightarrow XA \longrightarrow A\widehat{\otimes}C_1 \longrightarrow 0. \tag{3.2}$$

Define $s : A\widehat{\otimes}C_1 \to XA$ by $s(x\widehat{\otimes}1) = (1/2)(x + F\overline{x}F)$, $s(x\widehat{\otimes}T) = (1/2)(xF + F\overline{x})$. Then s is a cross section to $XA \to A\widehat{\otimes}C_1$ and completely positive, since it is a contraction of the homomorphism $\sigma : A\widehat{\otimes}C_1 \to \widehat{M}_2(XA)$,

$$\sigma(x\widehat{\otimes}1) = \frac{1}{2}\begin{pmatrix} x + F\overline{x}F & xF - F\overline{x} \\ Fx - \overline{x}F & \overline{x} + FxF \end{pmatrix}, \quad \sigma(1\widehat{\otimes}T) = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}$$

with T the generator of C_1. We may apply the long exact sequence of Ex-theory to obtain

$$\begin{tikzcd}
\mathrm{Ex}_0(XA,B) \arrow[r] & \mathrm{Ex}_0(\chi A,B) \arrow[r] & \mathrm{Ex}_1(A\widehat{\otimes}C_1,B) \arrow[d] \\
\mathrm{Ex}_0(A\widehat{\otimes}C_1,B) \arrow[u] & \mathrm{Ex}_1(\chi A,B) \arrow[l] & \mathrm{Ex}_1(XA,B). \arrow[l]
\end{tikzcd}$$

$\mathrm{Ex}_*(\chi A,B) \simeq \mathrm{KK}_*(A,B)$ by the equivariant version of the Cuntz construction and XA is stably homotopic equivalent to A. Check that $\rho : XA \to \widehat{M}_2(A)$, $\rho(x) = \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $x \in A$, $\rho(F) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is a stable homotopic inverse to the embedding $A \hookrightarrow XA$. The composition $\chi A \hookrightarrow XA \xrightarrow{\rho} \widehat{M}_2(A)$ is equal to $\chi A \xrightarrow{\alpha_0} \widehat{M}_2(qA) \xrightarrow{\pi_0} \widehat{M}_2(A)$. This implies that the map $\mathrm{Ex}_*(A,B) \simeq \mathrm{Ex}_*(XA,B) \to \mathrm{KK}_*(A,B)$ is just α.

Now, $A \hookrightarrow XA \to A\widehat{\otimes}C_1$ sends x to $x\widehat{\otimes}1$, so that $\mathrm{Ex}_*(A\widehat{\otimes}C_1, B) \to \mathrm{Ex}_*(A, B)$ is j^*.

The connecting map $\mathrm{Ex}_*(\chi A, B) \to \mathrm{Ex}_{*+1}(A\widehat{\otimes}C_1, B)$ is the Ex-product with the image of the sequence (3.2) in $\mathrm{Ex}_1(A\widehat{\otimes}C_1, \chi A) \simeq \mathrm{Ex}_1(A, \chi A\widehat{\otimes}C_1)$. Using the homomorphism σ, it is easily checked that this element is just the class of the homomorphism $\beta_1 : \varepsilon A \to \chi A\widehat{\otimes}C_1$. The Ex-product corresponds to ordinary composition of homomorphisms in this case.

Identifying $\mathrm{KK}_*(A, B)$ and $\mathrm{KK}_*(\mathrm{S}A\widehat{\otimes}C_1, B)$ one obtains an exact sequence with arrows reversed. We claim that the map $\mathrm{KK}_*(A, B) \to \mathrm{Ex}_*(A, B)$ is equal to β.

First we replace A by $\mathrm{S}A = A(0, 1)$. We regard an element of $\mathrm{Ex}_0(\mathrm{S}A, B)$ as the homotopy class of a homomorphism $q\mathrm{S}A \to \widehat{\mathcal{K}}\otimes B$ or equivalently of $q\varepsilon A \to \widehat{\mathcal{K}}\otimes B$. Two such homomorphisms correspond under the surjection $\pi : \varepsilon A \to M_2(\mathrm{S}A)$, induced by

$$\overline{\pi}(x) = \begin{pmatrix} \cos^2(\frac{\pi}{2}t)x & \sin(\frac{\pi}{2}t)\cos(\frac{\pi}{2}t)x \\ \sin(\frac{\pi}{2}t)\cos(\frac{\pi}{2}t)x & \sin^2(\frac{\pi}{2}t)x \end{pmatrix}, \quad \overline{\pi}(P) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Big(\text{resp.} \quad \overline{\pi}(V) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = 1 - 2P\Big).$$

This follows upon tensoring the extension

$$0 \longrightarrow S \longrightarrow \mathbb{C}[0, 1) \longrightarrow \mathbb{C} \longrightarrow 0$$

by A and codifying it as homomorphism $\varepsilon A \to M_2(\mathrm{S}A)$. We must then check the commutativity (up to stable homotopy) of the following diagram

$$\begin{tikzcd}
M_2(q\varepsilon A) \arrow[r, "\beta_0"] & M_2(\chi\varepsilon A) \arrow[r, "j_0"] & \widehat{M}_4(\chi\varepsilon A) \arrow[r, "\chi(\pi)"] & \widehat{M}_8(\chi\mathrm{S}A) \arrow[d, equal] \\
\varepsilon A \arrow[u, "\varphi_A"] \arrow[r, "\beta_1"] & \chi A\widehat{\otimes}C_1 \arrow[r, "\psi_A"] & \widehat{M}_2(\chi^2 A\widehat{\otimes}C_1) \arrow[r, "\chi(\gamma)"] & \widehat{M}_8(\chi\mathrm{S}A) .
\end{tikzcd}$$

Here the composition $\chi(\gamma)\circ\psi_A$ is the KK-product with a generator of $\mathrm{KK}(C_1, S)$. One checks that γ is equal to $\pi\circ\alpha_1$. Replacing $q\varepsilon A$, $\chi\varepsilon A$, $\chi^2 A$ by $I_{q\varepsilon A}$, $I_{\chi\varepsilon A}$, $I_{\chi^2 A}$ respectively as suggested by Lemma 3.2 we have reduced the question to the commutativity of

$$\begin{tikzcd}
M_4(I_{q\varepsilon A}) \arrow[r, "\widetilde{\beta}_0"] & M_4(I_{\chi\varepsilon A}) \arrow[r, "\mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & F \end{smallmatrix}\right)", "\sim"'] & \widehat{M}_4(I_{\chi\varepsilon A}) \\
\varepsilon A \arrow[u, "\widetilde{\varphi}_A"] \arrow[r, "\beta_1"] & \chi A\widehat{\otimes}C_1 \arrow[r, "\widetilde{\psi}_A"] & \widehat{M}_2(I_{\chi^2 A}\widehat{\otimes}C_1). \arrow[u, "\chi(\alpha_1)"']
\end{tikzcd}$$

Here $\widetilde{\beta}_0$ is the natural extension of the embedding β_0 to $I_{q\varepsilon A} \hookrightarrow I_{\chi\varepsilon A} \simeq I_{q\varepsilon A} \rtimes_\kappa \mathbb{Z}_2^{\mathrm{od}}$. This map extends to a map on the multipliers $\mathcal{M}(I_{q\varepsilon A}) \hookrightarrow \mathcal{M}(I_{\chi\varepsilon A}) \simeq \mathcal{M}(I_{q\varepsilon A}) \rtimes_\kappa \mathbb{Z}_2^{\mathrm{od}}$ and the images of the elements M, N in the construction of $\widetilde{\varphi}_A$ then satisfy the KTT conditions also for the algebras $R' = R \rtimes_\kappa \mathbb{Z}_2^{\mathrm{od}}$, $L' = L \rtimes_\kappa \mathbb{Z}_2^{\mathrm{od}}$ etc. in $\mathcal{M}(I_{\chi\varepsilon A})$.

Explicitely, $\widetilde{\beta}_0 \circ \widetilde{\varphi}_A$ is given by the formulas

$$x \mapsto \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & F\overline{x}F & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x \in A,$$

$$V \mapsto H = \begin{pmatrix} \sin(\frac{\pi}{2}S) & \cos(\frac{\pi}{2}S) \\ \cos(\frac{\pi}{2}S) & -\sin(\frac{\pi}{2}S) \end{pmatrix}, \quad S = \begin{pmatrix} \sqrt[4]{M}V\sqrt[4]{M} & \sqrt{N} \\ \sqrt{N} & -F\sqrt[4]{M}V\sqrt[4]{M}F \end{pmatrix}$$

with $M \cdot X\varepsilon A \subset I_{\chi\varepsilon A}$, $N \cdot \chi EA \subset I_{\chi\varepsilon A}$, $[M, XEA] \subset I_{\chi\varepsilon A}$.

The homomorphism $I_{\chi^2 A}\widehat{\otimes}C_1 \xrightarrow{\chi(\alpha_1)} \widehat{M}_2(I_{\chi\varepsilon A})$ is not necessarily extendible to multipliers. By an argument we have already used twice (cf. the proof of Theorem 1.7) we find an equivalent homomorphism in the homotopy class of $\chi(\alpha_1) \circ \widetilde{\psi}_A$ which does admit an extension to $XA\widehat{\otimes}C_1 \to \mathcal{M}(I_{\chi\varepsilon A})$ by rotating in the appropriate sense from upper left to lower right corner in the algebra

$$\begin{pmatrix} \overline{\mathrm{Im}(\chi(\alpha_1))\widehat{M}_2(I_{\chi\varepsilon A})\mathrm{Im}(\chi(\alpha_1))} & \overline{\mathrm{Im}(\chi(\alpha_1))\widehat{M}_2(I_{\chi\varepsilon A})} \\ \overline{\widehat{M}_2(I_{\chi\varepsilon A})\mathrm{Im}(\chi(\alpha_1))} & \widehat{M}_2(I_{\chi\varepsilon A}) \end{pmatrix}.$$

The formulas for the images of $XA\widehat{\otimes}C_1$ essentially stay the same. Finally, we can choose the images of M, N in $\mathcal{M}(I_{\chi\varepsilon A})$ in such a way that they satisfy the KTT conditions for the same algebras as in the composition $\widetilde{\beta}_0 \circ \widetilde{\varphi}_A$. (Indeed, we can assume that they are exactly the same.)

Now, the composition $\chi(\alpha_1) \circ \widetilde{\psi}_A \circ \beta_1$ will be equivalent to $\lambda : \varepsilon A \to \widehat{M}_4(I_{\chi\varepsilon A})$ with

$$\lambda(x) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \overline{x} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x \in A$$

$$\lambda(V) = \mathrm{i}H'T, \qquad T = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix},$$

$$H' = \begin{pmatrix} \sin(\frac{\pi}{2}S') & \cos(\frac{\pi}{2}S') \\ \cos(\frac{\pi}{2}S') & -\sin(\frac{\pi}{2}S') \end{pmatrix}, \quad S' = \begin{pmatrix} \sqrt[4]{N}F\sqrt[4]{N} & \sqrt[4]{M}V\sqrt[4]{M} \\ \sqrt[4]{M}V\sqrt[4]{M} & -\sqrt[4]{N}F\sqrt[4]{N} \end{pmatrix}.$$

Consider $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $U^* = -U$. Then $S'U$ is a "compact perturbation" of

$$S = \begin{pmatrix} \sqrt[4]{M}V\sqrt[4]{M} & -\sqrt{N}F \\ -F\sqrt{N} & -\sqrt[4]{M}V\sqrt[4]{M} \end{pmatrix}$$

in the sense that $(S'U - S)\begin{pmatrix} x & 0 \\ 0 & \overline{x} \end{pmatrix} \in \widehat{M_2}(I_{\chi\varepsilon A})$, $x \in A$. One has $US' = -S'U$, so that $H'\begin{pmatrix} U & 0 \\ 0 & -U \end{pmatrix} = \mathrm{i}H'T$ is a compact perturbation of

$$H = \begin{pmatrix} \sin(\frac{\pi}{2}S) & \cos(\frac{\pi}{2}S) \\ \cos(\frac{\pi}{2}S) & -\sin(\frac{\pi}{2}S) \end{pmatrix}.$$

From this we conclude that $\lambda(V)$ may be continuously deformed to H. Also we can replace S by $\mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)S$, but this just gives the same formula as in the composition $\widetilde{\beta}_0 \circ \widetilde{\varphi}_A$ under the identification

$$\mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix} : M_4(I_{\chi\varepsilon A}) \xrightarrow{\sim} \widehat{M_4}(I_{\chi\varepsilon A}).$$

We have shown that β_0 and β_1 induce the same map $\mathrm{KK}_*(A, B) \xrightarrow{\beta} \mathrm{Ex}_*(A, B)$. Similarly, the map $\mathrm{Ex}_1(A, B) \xrightarrow{\alpha} \mathrm{KK}_1(A, B)$ is described by α_1. If this is true for $B = \mathrm{S}A$, it is true for all B since both α and β are induced by homomorphisms of universal algebras in the first variable which are independent of B. On the other hand, the case of arbitrary A is obtained from $A = \mathbb{C}$ by tensoring with A (graded tensor product, see Remark 3.7), in which case the statement is easy to see.

It remains to show that α, β and j^* commute with long exact sequences. This is trivial for those parts of the sequences coming from homomorphisms between the algebras involved. The connecting map δ is given as the product with the image of

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

in $\mathrm{Ex}_1(B, J)$ in the case of Ex_* and as product with the image of this element under α in $\mathrm{KK}_1(B, J)$ in case of KK. Since α commutes with products, the result follows in this case.

It is easy to see that it is also true for $j_* = j^* \circ \tau_{C_1}$, which follows for instance from the associativity of the product. For example, in case of the long exact sequence in the first variable of Ex_*, one has the commutative diagram

$$\begin{tikzcd} \mathrm{Ex}_*(B,D) \arrow[r] & \mathrm{Ex}_*(B,D\widehat{\otimes}C_1) \\ \mathrm{Ex}_{*+1}(J,D) \arrow[r] \arrow[u, "\delta"] & \mathrm{Ex}_{*+1}(J,D\widehat{\otimes}C_1). \arrow[u, "\delta"'] \end{tikzcd}$$

For β, consider the following diagram in the contravariant case (the covariant case is clear by associativity)

$$\begin{tikzcd} \varepsilon B \arrow[r] & q\varepsilon B \arrow[r] \arrow[d] & qJ \arrow[d] \\ & \chi\varepsilon B \arrow[r] & \chi J \end{tikzcd}$$

which clearly commutes. ∎

Remark 3.7. The sucessive application $\alpha' \circ \beta'$ in (3.1) is $\alpha \circ \beta$ applied to the group $\mathrm{KK}_*(\mathrm{S}A\widehat{\otimes}C_1, B) \simeq \mathrm{KK}_*(A,B)$. Recall that this identification changes the sign of the involution so that $\alpha' \circ \beta' = \mathrm{id} - *$. Also the embedding $A \xrightarrow{1\widehat{\otimes}\mathrm{id}} \widehat{M_2}\widehat{\otimes}A$ is $\mathrm{id} + \sharp$ on Ex_*. It follows that starting at any point of the double sequence (3.1), mapping back and forth in both directions, is multiplication by 2.

Consider the $\mathbb{Z}_2$-graded C^*-algebra $\widehat{S}$ of continuous functions on the real line with decomposition into even and odd functions. Evaluation of a function at time $t = 0$ gives a graded homomorphism

$$\widehat{S} \xrightarrow{e^*} \mathbb{C}.$$

Proposition 3.8. $\mathrm{KK}(A,B) \simeq \mathrm{Ex}(\widehat{S}\widehat{\otimes}A, B)$ *and the map* α *is just* e^*.

Proof. Consider the surjection $p : \chi A \to \chi\mathbb{C}\widehat{\otimes}A$, sending $x \in A$ to $1_{\mathbb{C}}\widehat{\otimes}x$ and F to $F\widehat{\otimes}1$. Then p induces an isomorphism

$$\mathrm{KK}_*(A,B) \simeq \mathrm{Ex}_*(\chi A, B) \xrightarrow{\sim} \mathrm{Ex}_*(\chi\mathbb{C}\widehat{\otimes}A, B)$$

by the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \chi A \arrow[r] \arrow[d, "p"] & XA \arrow[r] \arrow[d, "p"] & A\widehat{\otimes}C_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \chi\mathbb{C}\widehat{\otimes}A \arrow[r] & X\mathbb{C}\widehat{\otimes}A \arrow[r] & A\widehat{\otimes}C_1 \arrow[r] & 0 \end{tikzcd}$$

together with the Five Lemma and the fact that XA and $X\mathbb{C}\widehat{\otimes}A$ are stably homotopic equivalent under p. The commutativity of the connecting homomorphism with p follows at once from its definition. Recall that it is simply the product with the extension under consideration.

Now $\chi\mathbb{C}$ is isomorphic as a graded algebra with $\widehat{M_2}(\widehat{S})$. Identifying $\widehat{S}$ with the algebra of continuous functions on the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ with decomposition into even and odd functions, this isomorphism can be realized as the map

$$1_{\mathbb{C}} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad F \mapsto \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix}. \tag{3.3}$$

Surjectivity is obvious and injectivity follows from Proposition 6.1 in [7], where it is proved that this map is injective when restricted to $q\,\mathbb{C}$. But $\chi\,\mathbb{C}$ is just $q\,\mathbb{C}\rtimes_\tau\mathbb{Z}_2^{\mathrm{od}}$, so the result follows. Using this identification, evaluation corresponds to the map $\rho_{\mathbb{C}} : \chi\mathbb{C} \to \widehat{M_2}(\mathbb{C})$ constructed above. This completes the proof. ∎

Remark 3.9. We could also have derived Theorem 3.6 using the short exact sequence

$$0 \longrightarrow \mathrm{S}A\widehat{\otimes}C_1 \xrightarrow{i} \widehat{S}\widehat{\otimes}A \xrightarrow{e} A \longrightarrow 0 \tag{3.4}$$

instead of (3.2). We leave the proof to the reader. The map β is however less easy to handle in this picture than by universal algebras because it involves Bott Periodicity.

4. Additional results

4.1. More on double exact sequences. In the following, $A^\vee$ will denote the C^*-algebra A but with trivial $\mathbb{Z}_2$-action. Recall that $\mathrm{Ex}_0(A,B)$ is just $\mathrm{KK}_{\mathbb{Z}_2}(A,B)$. If $A=\mathbb{C}$ one has a natural isomorphism

$$\mathrm{K}_{\mathbb{Z}_2}(B) \simeq \mathrm{K}(B\rtimes\mathbb{Z}_2^\vee). \tag{4.1}$$

We will frequently omit the reference to the action in the crossed product. If nothing else is said, it is tacitly assumed that the crossed product is taken with respect to the grading automorphism, i.e. $B\rtimes\mathbb{Z}_2^\vee$ is equal to $(B\widehat{\otimes}C_1)^\vee$. The double sequence (3.1) then reduces to (compare [10])

$$\begin{tikzcd} \mathrm{K}_0(A\rtimes\mathbb{Z}_2^\vee) \arrow[r, shift left] \arrow[d, shift left] & \mathrm{K}_0(A) \arrow[l, shift left] \arrow[r, shift left] & \mathrm{K}_1(A^\vee) \arrow[l, shift left] \arrow[d, shift left] \\ \mathrm{K}_0(A^\vee) \arrow[u, shift left] \arrow[r, shift left] & \mathrm{K}_1(A) \arrow[l, shift left] \arrow[r, shift left] & \mathrm{K}_1(A\rtimes\mathbb{Z}_2^\vee) \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd} \tag{4.2}$$

The result (4.1) can be generalized to the case of the bifunctor $\mathrm{Ex}(A, B)$ if either A or B is trivially graded, as one sees by the following purely formal argument. Assume first that A is trivially graded. Writing

$$\widehat{\mathcal{K}} \otimes B \simeq (\mathcal{K} \otimes B \widehat{\otimes} C_1) \widehat{\otimes} C_1 = (\mathcal{K} \otimes B \rtimes \mathbb{Z}_2^{\mathrm{od}}) \widehat{\otimes} C_1$$

one obtains the isomorphism

$$[qA, \widehat{\mathcal{K}} \otimes B] \xrightarrow{\sim} [qA, \mathcal{K} \otimes B \rtimes \mathbb{Z}_2^{\vee}] \tag{4.3}$$

by the correspondence

$$\phi(x) = \phi_0(x) + \phi_1(x) T' \quad \leftrightarrow \quad \widetilde{\phi}(x) = \phi_0(x) + \phi_1(x)$$

with T' an odd generator of C_1 and ${T'}^2 = -1$. If B is trivially graded one gets in the same way

$$[qA, \widehat{\mathcal{K}} \otimes B] \simeq [q(A \rtimes \mathbb{Z}_2^{\mathrm{od}}), \mathcal{K} \otimes B \widehat{\otimes} C_1] \xrightarrow{\sim} [q(A \rtimes \mathbb{Z}_2)^{\vee}, \mathcal{K} \otimes B] \tag{4.4}$$

on identifying

$$\phi(x) = \phi_0(x) + \phi_1(x) T \quad \leftrightarrow \quad \widetilde{\phi}(x) = \phi_0(x) + \phi_1(x)$$

with $T \in C_1$ selfadjoint, i.e. $T^2 = 1$.

Let $\widehat{\mathrm{S}}A = \widehat{S} \otimes A$ (ordinary tensor product). We call $\widehat{\mathrm{S}}A$ *the graded suspension* of A. Kasparov's Periodicity Theorem (Theorem 7 of Part 5 of [13]) gives full Bott Periodicity for the Ex-functor also with respect to graded suspensions:

$$\mathrm{Ex}(A, B) \simeq \mathrm{Ex}(\widehat{\mathrm{S}}^2 A, B) \simeq \mathrm{Ex}(A, \widehat{\mathrm{S}}^2 B) \simeq \mathrm{Ex}(\widehat{\mathrm{S}}A, \widehat{\mathrm{S}}B). \tag{4.5}$$

In particular, the map

$$\sigma_{\widehat{\mathrm{S}}} : \mathrm{Ex}(A, B) \longrightarrow \mathrm{Ex}(\widehat{\mathrm{S}}A, \widehat{\mathrm{S}}B)$$

is an isomorphism for all A, B. By the natural transformation α, Ex-equivalence implies KK-equivalence, so the first two isomorphisms in (4.5) carry over to the KK-groups, but the last one does not, since σ_D is not defined for KK-theory. (It is still possible, that $\widehat{S}$ preserves strong KK-equivalence (i.e. in case the equivalences are invariant under the involution on KK). This hypothesis is supported by the sequences (4.8) and (4.9) below.)

The next result shows that the graded suspension provides a link between the theories of A and $A^{\vee}$. There is an exact sequence

$$0 \longrightarrow \mathrm{S}A \otimes C_1 \longrightarrow \widehat{\mathrm{S}}A \longrightarrow A \longrightarrow 0 \tag{4.6}$$

and $\mathrm{S}A \otimes C_1 \simeq \mathrm{S}A^{\vee} \otimes C_1$. Also, one has $\mathrm{Ex}_*(\mathrm{S}A^{\vee} \otimes C_1, B) \simeq \mathrm{Ex}_*(\mathrm{S}A^{\vee}, B \widehat{\otimes} C_1) \simeq \mathrm{KK}_{*+1}(A^{\vee}, B^{\vee})$ by (4.3). Let $\natural$ be the involution on $\mathrm{KK}_*(A^{\vee}, B^{\vee})$ given by composing an element in this group with the grading automorphisms of A and B (viewed as elements in $\mathrm{KK}(A^{\vee}, A^{\vee})$ and $\mathrm{KK}(B^{\vee}, B^{\vee})$).

THEOREM 4.2. ($\flat$-sequence) *The following sequences with opposite sense of direction are exact*

$$\begin{array}{ccccc}
\mathrm{Ex}_0(A,B) & \underset{i}{\overset{r}{\rightleftarrows}} & \mathrm{KK}_0(A^\vee,B^\vee) & \underset{r'}{\overset{i'}{\rightleftarrows}} & \mathrm{Ex}_1(A,\widehat{\mathrm{S}}B) \\
e_* \uparrow\downarrow e'_* & & & & e'_* \uparrow\downarrow e_* \\
\mathrm{Ex}_0(A,\widehat{\mathrm{S}}B) & \underset{i'}{\overset{r'}{\rightleftarrows}} & \mathrm{KK}_1(A^\vee,B^\vee) & \underset{r}{\overset{i}{\rightleftarrows}} & \mathrm{Ex}_1(A,B).
\end{array} \tag{4.7}$$

Here r is the restriction homomorphism $\mathrm{KK}_{\mathbb{Z}_2} \to \mathrm{KK}_{\{1\}}$. The composition $e \circ e'$ (and $e' \circ e$) is $1-\flat$, $i \circ r$ (and $i' \circ r'$) is $1+\flat$. Also $r \circ i = 1-\natural$ and $r' \circ i' = 1+\natural$.

Proof. The clockwise sequence is the long exact Ex-sequence in the second argument applied to (4.6) making identifications according to (4.5) and the sequence in the opposite sense is obtained replacing B by $\widehat{\mathrm{S}}B$ and using equivariant Bott periodicity $\widehat{S}^2B \sim_{\mathrm{Ex}} B$ (or equivalently applying the long exact sequence to the first variable). The connecting map r is the product with the image of (4.6) in $\mathrm{Ex}_1(B, \mathrm{S}B \otimes C_1)$. Write $\widehat{S} = C_0(-\frac{\pi}{2}, \frac{\pi}{2})$ with grading $f(t) \mapsto f(-t)$. Then the sequence (4.6) for $A = \mathbb{C}$ corresponds to the homomorphism

$$\varepsilon\mathbb{C} \longrightarrow M_2(S \otimes C_1) \subset M_2(\widehat{S})$$

$$1_{\mathbb{C}} \mapsto \begin{pmatrix} \cos^2 t & \cos t\,|\sin t| \\ \cos t\,|\sin t| & \sin^2 t \end{pmatrix}, \quad V \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is equal to $\mathrm{id} \otimes 1 : \varepsilon\mathbb{C} \to \varepsilon\mathbb{C} \otimes C_1$.

So r maps the homomorphism $\phi : qA \to \widehat{\mathcal{K}} \otimes B$ to $\phi \otimes 1 : qA \to \widehat{\mathcal{K}} \otimes B \otimes C_1 \simeq \mathcal{K} \otimes B^\vee \otimes C_1$ and this corresponds to $\phi^\vee = \phi : qA^\vee \to \mathcal{K} \otimes B^\vee$ under the isomorphism $\mathrm{Ex}(A, B^\vee \otimes C_1) \simeq \mathrm{KK}(A^\vee, B^\vee)$ as in (4.5).

$e_* \circ e'_*$ is the product of the canonical Bott element $\beta_2 \in \mathrm{Ex}(\mathbb{C}, \widehat{S}^2)$ followed by the evaluation map $\widehat{S}^2 \xrightarrow{e^2} \mathbb{C}$. To compute it, it is convenient to adopt the formal setup of [13]. We simultanously allow a $\mathbb{Z}_2$-action and another (formal) $\mathbb{Z}_2$-grading on the algebras and view $\mathrm{KK}_{\mathbb{Z}_2}(A,B)$ as a theory on bigraded algebras. Let $C_1^\wedge$ be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra generated by one selfadjoint unitary of bidegree (1,1) and $\widehat{M_2^\wedge} = C_2^\wedge = C_1^\wedge \widehat{\otimes} C_1^\wedge$ where the graded tensor product is with respect to the second grading only. The algebra $\widehat{S}$ is taken to be trivially graded in the second factor of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The Bott element β_2 of Kasparov is really constructed in $\mathrm{KK}_{\mathbb{Z}_2}(\mathbb{C}, \widehat{M_2^\wedge}(\widehat{S}^2))$. After evaluating, only the homomorphism part of the Bott element will carry some nontrivial information and this gives the class of the projection $1_{\widehat{M_2^\wedge}}$ in $\widehat{\mathcal{K}} \otimes \mathcal{K}^\wedge$. The suffix $\wedge$ refers to the second grading while $\widehat{\ }$ refers to the first. Now $1_{\widehat{M_2^\wedge}}$ is the sum of a minimal projection of bidegree (0,0) in the left upper corner of $\widehat{M}_2^{\wedge}$ and the projection obtained from this by conjugating with a unitary of bidegree (1,1). Conjugation with a unitary of bidegree (0,1) corresponds to taking the inverse and conjugating with a unitary of bidegree (1,0) is the map $\flat$. Hence we obtain the result for $e_* \circ e'_*$. For $e'_* \circ e_* = \sigma_{\widehat{S}}(e_* \circ e'_*)$ the same argument is valid.

Next, we consider the maps $i \circ r$ and $i' \circ r'$. We can restrict ourselves to the case $A = \widehat{S}$, $B = \widehat{S}$. Also $i' \circ r'$ is just $\sigma_{\widehat{S}}(i \circ r)$ so that it suffices to consider $i \circ r$. This is the class of the homomorphism

$$j : \widehat{S} \xrightarrow{\mathrm{id} \otimes 1} \widehat{S} \otimes C_1 \simeq S \otimes C_1 \hookrightarrow \widehat{S}$$

sending $f = f(t)$ to $\widetilde{f} = f(2t + \frac{\pi}{2} \bmod 1), t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ in $\mathrm{Ex}(\widehat{S}, \widehat{S}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Now $r(\mathbf{1}_{\widehat{S}}) = \mathbf{1}_S$ is a generator of $\mathrm{KK}(S, S)$, so that $j_* = i \circ r(\mathbf{1}_{\widehat{S}})$ is a generator of the kernel of e'_* which is $(1 + \flat)(\mathrm{Ext}_*(\widehat{S}, \widehat{S}))$. Hence $j_* = \pm(1 + \flat)(\mathbf{1}_{\widehat{S}})$. If the sign is negative, then necessarily $r(j_*) = -2 \cdot \mathbf{1}_S$. But the restricted homomorphism $j^\vee : S \to S$ viewed as quasihomomorphism $(j^\vee, 0)$ is homotopic to $(\mathrm{id}_S, \overline{\mathrm{id}_S})$ with $\overline{\mathrm{id}_S}$ sending $f(t)$ to $f(1-t)$, $t \in [0, 1]$. So $r(j_*) = \mathbf{1}_S - (-\mathbf{1}_S) = 2 \cdot \mathbf{1}_S$ and consequently $j_* = (1 + \flat)(\mathbf{1}_{\widehat{S}})$.

Consider the map i'. It sends the homomorphism

$$S \otimes qA^\vee \xrightarrow{\mathrm{id}_S \otimes \phi} S \otimes \mathcal{K} \otimes B^\vee$$

to

$$S \otimes qA \xrightarrow{\mathrm{id}_S \otimes \widetilde{\phi}} S \otimes \widehat{\mathcal{K}} \otimes B \otimes C_1 \hookrightarrow \widehat{\mathcal{K}} \otimes \widehat{S}B$$

with $\widetilde{\phi}(x) = \phi(x_0)_0 + \phi(x_0)_1 T + \phi(x_1)_0 T + \phi(x_1)_1$, T a generator of C_1. Restricting this by r' gives $\mathrm{id}_S \otimes \phi + \mathrm{id}_S \otimes (\overline{\mathrm{id}_B} \circ \phi \circ \overline{\mathrm{id}_A})$. Thus $r' \circ i' = 1 + \natural$.

The map $r \circ i$ is $r' \circ i'$ applied to $\mathrm{KK}_*(A^\vee, S^2 B^\vee)$ where the $\mathbb{Z}_2$-action on S^2 (when viewed in the graded category) is complex conjugation. Restricting, this equals to taking the inverse so that $r \circ i$ is equal to $1 - \natural$. ∎

Applying the long exact sequences of KK instead of Ex, one obtains the double exact sequences

$$\begin{tikzcd}
\mathrm{KK}_0(A,B) \arrow[r, shift left, "r_1"] \arrow[d, shift left, "e^*"] & \mathrm{KK}_1(A^\vee,B) \arrow[l, shift left, "i_1"] \arrow[r, shift left, "i'_1"] & \mathrm{KK}_1(\widehat{\mathrm{S}}A,B) \arrow[l, shift left, "r'_1"] \arrow[d, shift left, "e'^*"] \\
\mathrm{KK}_0(\widehat{\mathrm{S}}A,B) \arrow[u, shift left, "e'^*"] \arrow[r, shift left, "r'_1"] & \mathrm{KK}_0(A^\vee,B) \arrow[l, shift left, "i'_1"] \arrow[r, shift left, "i_1"] & \mathrm{KK}_1(A,B) \arrow[l, shift left, "r_1"] \arrow[u, shift left, "e^*"]
\end{tikzcd} \tag{4.8}$$

$$
\begin{tikzcd}
\mathrm{KK}_0(A,B) \arrow[r, shift left, "r_2"] \arrow[d, shift left, "e'_*"] & \mathrm{KK}_1(A,B^\vee) \arrow[l, shift left, "i_2"] \arrow[r, shift left, "i'_2"] & \mathrm{KK}_1(A,\widehat{S}B) \arrow[l, shift left, "r'_2"] \arrow[d, shift left, "e_*"] \\
\mathrm{KK}_0(A,\widehat{S}B) \arrow[u, shift left, "e_*"] \arrow[r, shift left, "r'_2"] & \mathrm{KK}_0(A,B^\vee) \arrow[l, shift left, "i'_2"] \arrow[r, shift left, "i_2"] & \mathrm{KK}_1(A,B). \arrow[l, shift left, "r_2"] \arrow[u, shift left, "e'_*"]
\end{tikzcd}
\tag{4.9}
$$

Let $\natural_1$ ($\natural_2$) be the involution on $\mathrm{KK}_*(A^\vee,B)$ ($\mathrm{KK}_*(A,B^\vee)$) given by composing with the grading automorphism of A (B), and $*_l$, $*_r$ denote left and right conjugation on $\mathrm{KK}(A,B)$.

Corollary 4.3. *In the sequence* (4.8) *the maps $i'_1\circ r'_1$ and $i_1\circ r_1$ are $1-*_l$, $e^*\circ e^{'*}$ and $e^{'*}\circ e^*$ is $1+*_l$; $r_1\circ i_1$ is $1-\natural_1$ and $r'_1\circ i'_1$ is $1+\natural_1$. The same for* (4.9) *with $*_l$ replaced by $*_r$ and $\natural_1$ replaced by $\natural_2$.*

Proof. First, $i_1\circ r_1$ is the product with the image of the element $\mathbf{1}_A+\mathbf{1}_A^\flat\in \mathrm{Ex}(A,A)$ in $\mathrm{KK}(A,A)$ which is the element $\mathbf{1}_A-*_A$. Then $e^*\circ e^{'*}$ is the product with $\mathbf{1}_A+*_A$. The same for $i'_1\circ r'_1$ and $e^{'*}\circ e^*$. The map $r'_1\circ i'_1$ is the product with the element corresponding to the composition $\mathrm{S}A\otimes C_1\hookrightarrow \widehat{\mathrm{S}}A\hookrightarrow \widehat{\mathrm{S}}A\otimes C_1\simeq \mathrm{S}A\otimes C_1$ in $\mathrm{KK}(\mathrm{S}A\otimes C_1,\mathrm{S}A\otimes C_1)=\mathrm{KK}(A^\vee,A^\vee)$. This element is equal to $\mathbf{1}_{A^\vee}-\mathbf{1}_{A^\vee}^{\natural_1}$. The composition $r_1\circ i_1$ is the same as $r'_1\circ i'_1$ applied to $\widehat{\mathrm{S}}A\otimes C_1\simeq \mathrm{S}A^\vee\otimes C_1$. This gives $1+\natural_1$. The argument for the second sequence is completely analogous. ∎

Remark 4.4. The sequence (4.7) may be thought of as a generalization of (4.2) though not of (3.1). If either A or B are graded, trivially the double sequences (3.1) and (4.7) coincide. In particular, as far as ordinary K-theory and K-homology are concerned, these are equally well expressed in terms of the graded suspension: $\mathrm{K}_*(A)\simeq \mathrm{K}_*^{\mathbb{Z}_2}(\widehat{\mathrm{S}}A)$. This result can be generalized to the following. If A and B are arbitrary graded algebras and D is trivially graded then $\mathrm{Ex}(D,A\widehat{\otimes}B)\simeq \mathrm{Ex}(D,A\otimes B)$ and the same in the other variable. By (4.3) and (4.4), these groups can be computed using the crossed product by the grading action, which is $(A\widehat{\otimes}B\widehat{\otimes}C_1)^\vee$ in the first case and $((A\otimes B)\widehat{\otimes}C_1)^\vee$ in the second. But, as trivially graded algebras, these are isomorphic as a simple calculation shows.

4.5. Elementary K-theory. Let us show that $\mathrm{KK}(\mathbb{C},A)$ coincides with the elementary K-functor for $\mathbb{Z}_2$-graded Banach algebras considered in [9]. Recall its definition. For a complex unital graded C^*-algebra one consideres equivalence classes of elements of odd degree (which can be taken selfadjoint) whose square is 1 in $\widehat{M}_2(M_n(A))$ for some n. The equivalence relation is twofold: homotopy of such elements in $\widehat{M}_2(M_n(A))$ and secondly taking the direct limit in $\widehat{M}_2(M_\infty(A)^+)$ with respect to the embedding $[x] \mapsto [x \oplus k_1]$ for $x \in \widehat{M}_2(M_n(A))$, $k_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \widehat{M}_2(A)$. The set of equivalence classes forms an abelian group with the obvious notion for the sum of two elements. This group is the odd K-group of the graded algebra A, denoted $\mathrm{K}_1(A)$. If A is not unital define $\mathrm{K}_1(A)$ as the K-group of the algebra A with a unit adjoined (complex case).

Let $\widetilde{Q}\mathbb{C}^+ = C^*(F, G)$ be the universal C^*-algebra generated by two symmetries of odd degree. Recall that $\widetilde{Q}\mathbb{C}^+$ can be represented as a crossed product of a trivially graded algebra isomorphic to $C(\mathbb{T})$ by $\mathbb{Z}_2$, such that the grading automorphism is equal to the dual action on $C(\mathbb{T})\rtimes\mathbb{Z}_2$. Let $\widetilde{q}\mathbb{C}$ be the invariant ideal generated by the difference $F - G$. This is just the same as $C_0(\mathbb{R})\rtimes_\alpha\mathbb{Z}_2^{\mathrm{od}}$, $\alpha(f)(t) = f(-t)$ and this is isomorphic as a graded algebra with $\widehat{q}\mathbb{C}$ which is $q\mathbb{C}$ with the $\mathbb{Z}_2$-action τ exchanging the two copies of $\mathbb{C}$ (see [7], Proposition 6.3). Regard $\widehat{q}\mathbb{C}$ as a subalgebra of $\chi\mathbb{C}$ in the following way. The projection $\iota(1_\mathbb{C})$ corresponds to $\frac{1}{2}[1_\mathbb{C} + F\,1_\mathbb{C}\,F + \mathrm{i}(F\,1_\mathbb{C} - 1_\mathbb{C}\,F)]$ and $\bar{\iota}(1_\mathbb{C})$ corresponds to $\frac{1}{2}[1_\mathbb{C} + F\,1_\mathbb{C}\,F - \mathrm{i}(F\,1_\mathbb{C} - 1_\mathbb{C}\,F)]$. Then $\chi\mathbb{C} = \widehat{q}\mathbb{C}\rtimes_\tau\mathbb{Z}_2^{\mathrm{od}} = \widehat{q}\mathbb{C}\widehat{\otimes}C_1$, so that $[\widehat{q}\mathbb{C}, \widehat{\mathcal{K}} \otimes A]$ is isomorphic with $\mathrm{KK}_1(\mathbb{C}, A) \simeq [\chi\mathbb{C}, \widehat{\mathcal{K}} \otimes A\widehat{\otimes}C_1]$.

Let $x_n \in \widehat{M}_2(M_n(A))$ be a representative defining an element in $\mathrm{K}_1(A)$ of [9] and x its image in $\widehat{M}_2(M_\infty(A)^+)$. Also let $k \in \widehat{M}_2(M_\infty(A)^+)$ be the direct limit of $k_n = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes 1 \in \widehat{M}_2 \otimes M_n(A)$. Then x and k are symmetries of odd degree such that $x - k \in \widehat{M}_2(M_n(A)) \subset \widehat{M}_2(\mathcal{K} \otimes A)$. This gives the map $\mathrm{K}_1(A) \to \mathrm{KK}_1(\mathbb{C}, A)$. On the other hand, one can extend a homomorphism $\varphi : \widetilde{q}\mathbb{C} \to \widehat{\mathcal{K}} \otimes A$ (up to homotopy) to a unital homomorphism $\widetilde{Q}\mathbb{C}^+ \xrightarrow{\varphi^+} \mathcal{M}(\widehat{\mathcal{K}} \otimes A)$ giving a pair of odd symmetries (f, g) whose difference is compact. Replacing (f, g) by $(f \oplus -f, g \oplus -f)$ in $\widehat{M}_2(\mathcal{M}(\widehat{\mathcal{K}} \otimes A))$, the latter is homotopic to a pair of the form $(k \oplus -k, \widetilde{x})$. Choosing a (fixed) isomorphism of $\widehat{M}_2(\mathcal{M}(\widehat{\mathcal{K}} \otimes A))$ with $\mathcal{M}(\widehat{\mathcal{K}} \otimes A)$ such that $(k \oplus -k)$ corresponds to k one gets a symmetry x with $x - k \in \widehat{\mathcal{K}} \otimes A$. Also the elementary K-functor commutes with inductive limits as one easily sees. Thus x will be homotopic to the image of some $x_n \in \widehat{M}_2(M_\infty(A)^+)$ with $x_n - k_n \in \widehat{M}_2(M_n(A))$. This gives the inverse to the map constructed above, since the same arguments are valid for homotopies.

4.6. AF-ALGEBRAS AND CLASSIFICATION OF SYMMETRIES. Let us turn to some applications of the former results. Consider the following case. Let A be a UHF-algebra (or more generally AF-algebra), γ a K-trivial symmetry (i.e. γ can be strongly approximated by inner automorphisms) and $B = A\rtimes_\gamma\mathbb{Z}_2$. Let A^γ be the corresponding graded algebra.

The groups $\mathrm{K}_0(B), \mathrm{K}_1(B)$ can be used to classify the (stable conjugacy class of the) symmetry γ. (Recall that $\mathrm{K}_0(B)$ is a scaled ordered group.) If we assume γ to be locally (finitely) representable (whence $\mathrm{K}_1(B) = 0$) this classification is also complete. It is an open question whether the classification is complete (for AF-algebras) in general and what conditions have to be imposed on $\mathrm{K}_0(B), \mathrm{K}_1(B)$ (cf. [2]).

As an example of specific interest, assume that A is the CAR-algebra. $D = \mathrm{K}_0(A)$ is 2-divisible without 2-torsion. The double sequence (4.2) takes the form

$$\begin{tikzcd}
\mathrm{K}_0(B) \arrow[r, shift left, "p_2"] \arrow[d, shift left, "i_2"] & \mathrm{K}_0(A^\gamma) \arrow[l, shift left, "i_2"] \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[d, shift left] \\
D \arrow[u, shift left, "p_1"] \arrow[r, shift left, "0"] & \mathrm{K}_1(A^\gamma) \arrow[l, shift left, "0"] \arrow[r, shift left, "\sim"] & \mathrm{K}_1(B). \arrow[l, shift left, "\sim"] \arrow[u, shift left]
\end{tikzcd}$$

The mapping $D \xrightarrow{i_1} \mathrm{K}_0(B) \xrightarrow{p_1} D$ is $1 + \gamma_* = 2$. Since D is 2-divisible without 2-torsion it follows that $\mathrm{K}_1(B) \to \mathrm{K}_1(A^\gamma)$ and $\mathrm{K}_1(A^\gamma) \to \mathrm{K}_1(B)$ are isomorphisms. On the other hand, the composite (in both directions) is multiplication by 2, hence $\mathrm{K}_1(B) = \mathrm{K}_1(A^\gamma)$ is 2-divisible without 2-torsion (see [2], Proposition 2). But one can see more as follows from the following proposition.

Proposition 4.7. *Let A be a (trivially graded) C^*-algebra such that* $\mathrm{K}_0(A)$, $\mathrm{K}_1(A)$ *are* 2*-divisible without* 2*-torsion. Let γ be a symmetry (automorphism of order* 2*) of A with A^γ the corresponding graded algebra and $B = A \rtimes_\gamma \mathbb{Z}_2$. Then* $\mathrm{K}_*(B)$ *and* $\mathrm{K}_*(A^\gamma)$ *are* 2*-divisible without* 2*-torsion.*

The same reasoning holds of course for K*-homology groups and more general* KK*-groups.*

Proof. Consider the double sequence (4.2). Since the K-groups of A are 2-divisible without 2-torsion the sequence splits at $\mathrm{K}_0(A)$ and $\mathrm{K}_1(A)$. Thus it is sufficient to consider double exact sequences which are of the following form

$$\begin{tikzcd}
0 \arrow[r, shift left] & D_1 \arrow[l, shift left] \arrow[r, shift left, "i_1"] & A_1 \arrow[l, shift left, "p_1"] \arrow[r, shift left, "\sigma"] & A_2 \arrow[l, shift left, "\tau"] \arrow[r, shift left, "p_2"] & D_2 \arrow[l, shift left, "i_2"] \arrow[r, shift left] & 0 \arrow[l, shift left]
\end{tikzcd}$$

with D_1, D_2 being 2-divisible without 2-torsion. The result now follows from a diagram chase. ∎

Examples of algebras satisfying the hypothesis of Proposition 3.8 are the CAR-algebra and other UHF- or AF-algebras with 2-divisible K_0-group, also the even Cuntz algebras $\mathcal{O}_n$ with $n = 2m$. More generally, since these algebras are in the UCT-class, any tensor product of one of these algebras with an arbitrary C^*-algebra B will have 2-divisible K-theory without 2-torsion by the Künneth Theorem for tensor products. The UCT also shows that the condition 2-divisible without 2-torsion then holds for $\mathrm{KK}(A, B)$ and $\mathrm{KK}(B, A)$ for any B (since $\mathrm{KK}(A, A)$ has this property).

PROPOSITION 4.8. *Let γ be a* K*-trivial symmetry of the C^*-algebra A (i.e. $\gamma^* = \mathbf{1}_A \in \mathrm{KK}(A, A)$) and assume that $\mathrm{K}_0(A)$ has no 2-torsion and $\mathrm{K}_1(A)$ is 2-divisible. Then with $\mathrm{K}_*(B), \mathrm{K}_*(A^\gamma)$ as above one has that $\mathrm{K}_0(B), \mathrm{K}_0(A^\gamma)$ have no 2-torsion and $\mathrm{K}_1(B), \mathrm{K}_1(A^\gamma)$ are 2-divisible.*

Proof. Diagram chase. ∎

In particular, the crossed product $A \rtimes_\gamma \mathbb{Z}_2$ of a C^*-algebra by a symmetry γ can be UHF only if $\mathrm{K}_1(A)$ is 2-divisible (for example this does not hold for the Bunce-Deddens algebras, cf. [14]). The example of Blackadar then shows that $\mathrm{K}_1(A)$ need not be trivial.

Of course, these are very particular examples. However, the double sequence (4.2) may be used in general as a means of classification for symmetries of a given algebra A (whether it is AF or not).

4.9. $\mathbb{Z}_2$-GRADED E-THEORY. The double exact sequence (3.1) can also be obtained for the E-theory of Connes and Higson ([3]) (and other semisplit theories factoring over KK as the bivariant cyclic homology/cohomology theory). Let

$$\mathrm{E}_{\mathbb{Z}_2}(A, B) = [[\widehat{\mathcal{K}} \otimes \mathrm{S}A, \widehat{\mathcal{K}} \otimes \mathrm{S}B]]$$

denote $\mathbb{Z}_2$-equivariant E-theory, the right hand side meaning homotopy classes of $\mathbb{Z}_2$-equivariant asymptotic morphisms. For graded A and B define

$$\mathrm{E}(A, B) := [[\widehat{S} \widehat{\otimes} A, \widehat{\mathcal{K}} \otimes B]].$$

One checks that this semigroup is in fact a group, the inverse of an element is given (as for KK) by composing with the grading automorphism and conjugating with a unitary of odd degree. Moreover, this theory has the same relation with $\mathbb{Z}_2$-equivariant E-theory as KK has with Ex. There is a natural map from $\mathrm{KK}(A, B)$ to $\mathrm{E}(A, B)$ commuting with products and a double exact sequence as in (3.1) for

E and $\mathrm{E}_{\mathbb{Z}_2}$. The proof is straightforward. Reversing the argument above we could also define

$$\mathrm{E}(A,B) := [[\chi A, \widehat{\mathcal{K}} \otimes B]]. \tag{4.10}$$

The natural transformation $\mathrm{KK}_*(A,B) \to \mathrm{E}_*(A,B)$ becomes very conceptual substituting asymptotic morphisms for homomorphisms.

Finally, define an asymptotic superquasimorphism from A to B by the diagram

$$A \overset{\phi_t}{\rightsquigarrow} D \overset{F_t}{\triangleright} J \overset{\mu_t}{\rightsquigarrow} B.$$

Then $(\phi_t)_{t\in[1,\infty)}$ is an equivariant asymptotic morphism from A to D, $(F_t)_{t\in[1,\infty)}$ is a family of elements of odd degree in D (both strictly continuous with respect to J) such that

$$(F_t - F_t^*)\phi_t(A) \subset J,$$

$$(1 - F_t^2)\phi_t(A) \subset J,$$

$$[\phi_t(A), F_t]_+ \subset J,$$

and these are norm continuous. Moreover $(\mu_t)_{t\in[1,\infty)}$ is an equivariant asymptotic morphism from J to B. Define $\mathrm{E}(A,B)$ as equivalence classes of asymptotic superquasimorphisms from A to $\widehat{\mathcal{K}} \otimes B$, the notion of equivalence of two such application being totally analogous to the equivalence of superquasimorphisms. All three definitions are equivalent. Note that we may apply a normalizing procedure to asymptotic superquasimorphisms in the same way as to ordinary superquasimorphisms. Hence we can assume $F_t = F_t^*$, $(1 - F_t^2)\phi_t(A) = 0$. A normal asymptotic superquasimorphism from A to $\widehat{\mathcal{K}} \otimes B$ defines an asymptotic morphism from χA to $\widehat{\mathcal{K}} \otimes B$ in the following way. Mapping A to $\phi_t(A)$ and F to F_t gives a (strictly continuous with respect to J) asymptotic morphism from $\mathcal{X}A$ to D which restricted to χA is norm continuous and maps χA into J. Composing with $J \rightsquigarrow \widehat{\mathcal{K}} \otimes B$, we get the desired asymptotic morphism $\chi A \rightsquigarrow \widehat{\mathcal{K}} \otimes B$. The same arguments as in the proof of Theorem 2.1 show that this map respects the equivalence relations and is bijective on the quotients (by the equivalence relations).

4.10. K-THEORY OF GRADED CUNTZ ALGEBRAS. As an example let us apply the double exact sequence to compute the K-groups of graded Cuntz algebras (compare [4]). Let $\mathcal{O}_{n,m} = C^*(S_1,\dots,S_n,T_1,\dots,T_m)$ be the simple C^*-algebra generated by n even and m odd isometries subject to the relations

$$\sum_{1\leqslant i\leqslant n} S_iS_i^* + T_jT_j^* = 1, \quad S_i^*S_i = T_j^*T_j = 1.$$

PROPOSITION 4.11.

$$\begin{aligned}
\mathrm{K}_0(\mathcal{O}_{n,m}) &\simeq \mathbb{Z}_{|n-m-1|} && n-1 \neq m\\
&\simeq \mathbb{Z} && n-1 = m;\\
\mathrm{K}_1(\mathcal{O}_{n,m}) &= 0 && n-1 \neq m\\
&\simeq \mathbb{Z} && n-1 = m.
\end{aligned}$$

Proof. $\mathrm{Ex}_*(\mathcal{O}_{n,m}) \simeq \mathrm{K}_*(\mathcal{O}_{n,m} \rtimes \mathbb{Z}_2)$. The crossed product $\mathcal{O}_{n,m} \rtimes \mathbb{Z}_2$ is seen to be isomorphic to a Cuntz-Krieger algebra generated by $2(n+m)$ partial isometries $S_{i1} = S_i(\frac{1-F}{2})$, $S_{i2} = S_i(\frac{1+F}{2})$, $T_{j1} = T_j(\frac{1-F}{2})$, $T_{j2} = T_j(\frac{1+F}{2})$, F a generator of $\mathbb{Z}_2$, subject to the relations

$$\sum_i S_{i1}S_{i1}^* + \sum_j T_{j1}T_{j1}^* = P, \quad \sum_i S_{i2}S_{i2}^* + \sum_j T_{j2}T_{j2}^* = Q,$$

$$S_{i1}^*S_{i1} = T_{j2}^*T_{j2} = P, \quad S_{i2}^*S_{i2} = T_{j1}^*T_{j1} = Q$$

for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$. Let $\widetilde{A}$ be the matrix corresponding to these relations. The K-groups of this algebra can be computed as in [5]:

$$\mathrm{Ex}_0(\mathcal{O}_{n,m}) \simeq \operatorname{coker} A, \quad A = \begin{pmatrix} n-1 & m \\ m & n-1 \end{pmatrix} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2, \text{ in particular}$$

$$\begin{aligned}
\mathrm{Ex}_0(\mathcal{O}_{n,m}) &\text{is finite of order } |(n-1)^2 - m^2| && (n-1)^2 \neq m^2;\\
&\simeq \mathbb{Z} \oplus \mathbb{Z} && n=1,\, m=0;\\
&\simeq \mathbb{Z} && n=0,\, m=1;\\
&\simeq \mathbb{Z} \oplus \mathbb{Z}_m && n-1 = m > 0;\\
\mathrm{Ex}_1(\mathcal{O}_{n,m}) &= 0; && (n-1)^2 \neq m^2;\\
&\simeq \mathbb{Z} \oplus \mathbb{Z} && n=1,\, m=0;\\
&\simeq \mathbb{Z} && n=0,\, m=1;\\
&\simeq \mathbb{Z} && n-1 = m > 0.
\end{aligned}$$

An inspection shows that the involution $\flat$ on the group $\mathrm{Ex}_0(\mathcal{O}_{n,m})$ is given by exchanging the two generators in this presentation. Assume that $(n-1)^2 \neq m^2$. Then $\mathrm{K}_1(\mathcal{O}_{n+m-1}) = 0$ so that the map $\mathrm{Ex}_0(\mathcal{O}_{n,m}) \to \mathrm{K}_0(\mathcal{O}_{n,m})$ is surjective, so the latter group is finite cyclic of order $|n-1-m|$. It also follows from the double sequence (4.2) that $\mathrm{K}_0(\mathcal{O}_{n+m-1}) \to \mathrm{Ex}_0(\mathcal{O}_{n,m})$ is injective so that $\mathrm{K}_1(\mathcal{O}_{n,m}) = 0$.

The case $m = 0$ is clear, so let $n-1 = m > 0$. The composition $\mathrm{K}_0(\mathcal{O}_{n+m-1}) \to \mathrm{Ex}_0(\mathcal{O}_{n,m}) \to \mathrm{K}_0(\mathcal{O}_{n+m-1})$ is multiplication by 2 (check that the corresponding involution on $\mathrm{K}_0(\mathcal{O}_{n+m-1})$ is trivial), so the kernel of the first map is $\mathbb{Z}_2$ and its image is $\mathbb{Z}_m$. It follows that $\mathrm{K}_0(\mathcal{O}_{n,m}) \simeq \mathbb{Z}$ and $\mathrm{K}_1(\mathcal{O}_{n,m}) \simeq \mathbb{Z}$. The last identity follows from the fact that composition $\mathrm{Ex}_1(\mathcal{O}_{n,m}) \to \mathrm{K}_1(\mathcal{O}_{n,m}) \to \mathrm{Ex}_1(\mathcal{O}_{n,m})$ is multiplication by 2 and the second map is surjective.

Finally assume $n = 0$, $m = 1$. The composition $\mathrm{K}_0(\mathcal{O}_1) \to \mathrm{Ex}_0(\mathcal{O}_{0,1}) \to \mathrm{K}_0(\mathcal{O}_1)$ is multiplication by 2, so the double sequence (4.2) leaves only two possibilities — either $\mathrm{K}_0(\mathcal{O}_{0,1}) = 0$ and $\mathrm{K}_1(\mathcal{O}_{0,1}) \simeq \mathbb{Z}_2$ or $\mathrm{K}_0(\mathcal{O}_{0,1}) \simeq \mathbb{Z}_2$ and $\mathrm{K}_1(\mathcal{O}_{0,1}) = 0$. The result follows from the exact sequence

$$0 \longrightarrow \mathrm{S}C_1 \longrightarrow \mathcal{O}_{0,1} \longrightarrow C_1 \longrightarrow 0. \quad \blacksquare$$

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