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EXPONENTIAL ORDERING ON BOUNDED SELF-ADJOINT OPERATORS

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Abstract. Reducibility property is proved for bounded self-adjoint operators satisfying the exponential ordering.

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MSC (2000): 47A, 47B.

1. MAIN RESULTS

Let A, B be bounded selfadjoint operators on a Hilbert space $\mathcal{H}$. In [2], the notion of exponential ordering is introduced as the one defined by $\mathrm{e}^A \leqslant \mathrm{e}^B$. In this article, we deal with an infinitesimal version of it. Consider the condition

$$\mathrm{e}^{\kappa A} \leqslant \mathrm{e}^{\kappa B} \quad \text{for some } \kappa > 0,$$

which is equivalent to the following one by Löwner-Heinz' inequality ([3], [4]): there is a positive real κ_0 such that

$$\mathrm{e}^{\kappa A} \leqslant \mathrm{e}^{\kappa B} \quad \text{for all } 0 \leqslant \kappa \leqslant \kappa_0.$$

By the last expression, we see that the condition in fact defines an order relation in the set of bounded selfadjoint operators, which is weaker than the exponential ordering in [2] and will be referred to as *infinitesimal exponential ordering* in what follows.

By power series expansion in the exponential functions, the last condition is further equivalent to

$$B - A + \frac{\kappa}{2}(B^2 - A^2) + \frac{\kappa^2}{3!}(B^3 - A^3) + \cdots \geqslant 0 \quad \text{for sufficiently small } \kappa > 0,$$

which particularly implies the operator inequality $A \leqslant B$: the infinitesimal exponential ordering is finer than the ordinary ordering.

If $B - A$ is invertible, the converse implication is apparently true as remarked in [1].

We here deal with the case when the kernel of $B - A$ is non-trivial and prove the following:

THEOREM. *Let A and B be bounded self-adjoint operators on a Hilbert space. Then the operator inequality $e^{\kappa A} \leqslant e^{\kappa B}$ for some $\kappa > 0$ forces simultaneous decomposability of operators A, B with respect to the orthogonal projection P to the kernel of $B - A$, i.e., $AP = PA$ and $PB = BP$.*

COROLLARY. *If the range of $B - A$ is closed, then the operator inequality $e^{\kappa A} \leqslant e^{\kappa B}$ for some $\kappa > 0$ is equivalent to require $A \leqslant B$, $AP = PA$ and $BP = PB$.*

Proof. The condition is necessary by the theorem. Conversely, if the range of $B - A$ is closed and the condition is satisfied, then the reduced operators $A' = (1 - P)A = A(1 - P)$ and $B' = (1 - P)B = B(1 - P)$ on the closed subspace $(1 - P)\mathcal{H} = (\ker(B - A))^\perp$ satisfy the inequality $A' \leqslant B'$ and $B' - A'$ is invertible because $B' - A'$ is injective with the closed range. Then, as remarked above, we know the inequality $e^{\kappa A'} \leqslant e^{\kappa B'}$ for some (small) $\kappa > 0$. Now the assertion follows if we notice $e^{\kappa B} - e^{\kappa A} = (e^{\kappa B'} - e^{\kappa A'}) \oplus 0$. ∎

When the Hilbert space $\mathcal{H}$ is finite-dimensional, the closedness of the range is automatic and we obtain the following characterization of the infinitesimal exponential ordering. Let A and B be hermitian $n \times n$ matrices. The condition

$$e^{\kappa A} \leqslant e^{\kappa B} \quad \text{for some } \kappa > 0$$

is then equivalent to require $PA = AP$, $PB = BP$ and $A \leqslant B$.

Since a generic operator inequality $A \leqslant B$ (under the assumption that $\ker(B - A) \neq 0$) does not satisfy the reducing property $PA = AP$, $PB = BP$, we have plenty of examples of operator inequality $A \leqslant B$ without satisfying the infinitesimal exponential order relation.

2. PROOF OF THEOREM

We use the notation in the previous section and set $Q = 1-P$, the range projection of $B - A$.

Then, for sufficiently small $t > 0$, we have the positivity of the operator $(2P+tQ)(\mathrm{e}^{tB} - \mathrm{e}^{tA})(2P+tQ)/t$ and by Taylor expansion of exponential function, the operator inequality

$$\begin{aligned} 0 \leqslant\ & (2P + tQ)(B - A)(2P + tQ) + \frac{t}{2}(2P + tQ)(B^2 - A^2)(2P + tQ) \\ & + \frac{t^2}{6}(2P + tQ)(B^3 - A^3)(2P + tQ) \\ & + \sum_{n\geqslant 4} \frac{t^{n-1}}{n!}(2P + tQ)(B^n - A^n)(2P + tQ). \end{aligned}$$

Since $(B - A)P = 0 = P(B - A)$, we have

$$\begin{aligned} (B^2 - A^2)P &= (B - A)AP = (B - A)QAP, \\ P(B^2 - A^2) &= PA(B - A) = PAQ(B - A), \\ P(B^2 - A^2)P &= 0, \\ P(B^3 - A^3)P &= PA(B - A)AP = PAQ(B - A)QAP, \end{aligned}$$

which is used in the above inequality to get

$$\begin{aligned} 0 \leqslant t^2 & \left(Q(B - A)Q + Q(B - A)QAP + PAQ(B - A)Q + \frac{2}{3}PAQ(B - A)QAP\right) \\ & + \frac{t^3}{2}Q(B^2 - A^2)Q + \frac{t^3}{3}(Q(B^3 - A^3)P + P(B^3 - A^3)Q) \\ & + \frac{t^4}{6}Q(B^3 - A^3)Q + \sum_{n\geqslant 4} \frac{t^{n-1}}{n!}(2P + tQ)(B^n - A^n)(2P + tQ). \end{aligned}$$

Dividing by t^2 and taking the limit $t \to +0$, we obtain the inequality

$$\begin{aligned} 0 &\leqslant Q(B - A)Q + Q(B - A)QAP + PAQ(B - A)Q + \frac{2}{3}PAQ(B - A)QAP \\ &= (Q + PAQ)(B - A)(Q + QAP) - \frac{1}{3}PA(B - A)AP. \end{aligned}$$

Taking into account the identities

$$(Q + QAP)(P - QAP) = 0, \quad AP(P - QAP) = AP,$$

it follows by simultaneous right multiplication by $P-QAP$ and left multiplication by $(P-QAP)^*$ that

$$-\frac{1}{3}PA(B-A)AP \geqslant 0.$$

Since $B-A \geqslant 0$, the above inequality forces the equality $PA(B-A)AP = 0$ and hence

$$(B-A)AP = (B-A)^{1/2}(B-A)^{1/2}AP = 0.$$

Thus the range of AP is contained in the subspace $P\mathcal{H}$ and we have

$$AP = PAP,$$

which yields the commutativity $AP = PA$. Since $BP = AP$ and $PB = PA$, the reducibility of B also follows.

3. EXAMPLES

For a pair of bounded self-adjoint operators (A, B) satisfying $A \leqslant B$, we set

$$\kappa(A,B) = \sup\{\kappa \geqslant 0 : \mathrm{e}^{\kappa A} \leqslant \mathrm{e}^{\kappa B}\},$$

which has the following obvious properties:

$$\begin{cases} \kappa(A+c1, B+c1) = \kappa(A,B) & \text{if } c \text{ is a real number;} \\ \kappa(cA, cB) = \frac{1}{c}\kappa(A,B) & \text{if } c \text{ is a positive real;} \\ \kappa(UAU^*, UBU^*) = \kappa(A,B) & \text{if } U \text{ is a unitary operator.} \end{cases}$$

When A and B are 2×2 hermitian matrices, after the composition of these three operations, the pair (A, B) takes the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \lambda \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \mu \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}$$

with λ, μ reals except for the trivial case that A is a scalar matrix. (Use the angle representation of two projections.)

The condition of majorization $A \leqslant B$ is then equivalent to

$$0 \leqslant \lambda \sin^2\theta + \mu\cos^2\theta \leqslant \lambda\mu,$$

which particularly implies $\lambda \geqslant 0$, $\mu \geqslant 0$.

Now the following is easy to check:

PROPOSITION. *Assume that* $\cos\theta\sin\theta \neq 0$. *Then, for* $\lambda \geqslant 0$, $\mu \geqslant 0$, *we have*

$$\begin{cases} \kappa(A,B) = +\infty & \text{if and only if } \lambda \geqslant 1 \text{ and } \mu \geqslant 1; \\ 0 < \kappa(A,B) < +\infty & \text{if and only if } (\lambda-1)(\mu-1) < 0,\ \lambda\sin^2\theta + \mu\cos^2\theta < \lambda\mu; \\ \kappa(A,B) = 0 & \text{if and only if } (\lambda-1)(\mu-1) < 0,\ \lambda\sin^2\theta + \mu\cos^2\theta = \lambda\mu. \end{cases}$$

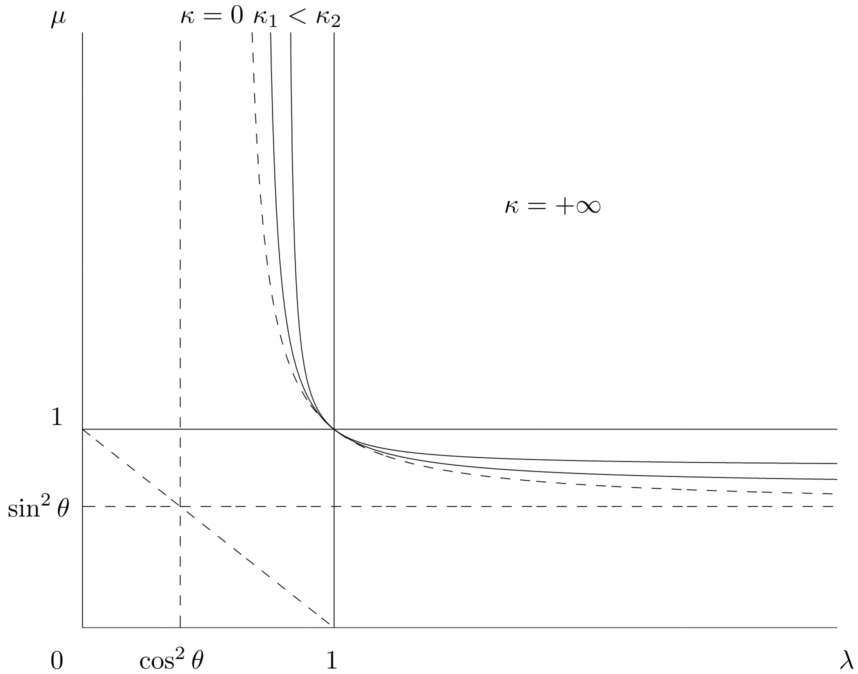

Figure 1.

For example, choose $\sin\theta = \cos\theta = 1/\sqrt{2}$ and

$$\lambda_n^{-1} = \frac{1}{2} - \frac{1}{n}, \quad \mu_n^{-1} = \frac{3}{2} - \frac{1}{n}$$

for $n \geqslant 3$. Then

$$B_n = \frac{2n}{(n-2)(3n-2)} \begin{pmatrix} 2n-2 & n \\ n & 2n-2 \end{pmatrix}$$

majorates A with the limit

$$B = \lim_{n\to\infty} B_n = \frac{2}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and these satisfy $\kappa(A, B_n) > 0$, $\kappa(A, B) = 0$.

Now we are ready to construct an example of bounded self-adjoint operators $A' \leqslant B'$ with no infinitesimal exponential order relation and having the trivial kernel for the difference $B' - A'$. Let $A' \leqslant B'$ be defined on the Hilbert space $\bigoplus_{n \geqslant 3} \mathbb{C}^2$ by

$$A' = \bigoplus_{n \geqslant 3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B' = \bigoplus_{n \geqslant 3} B_n.$$

Then clearly $\ker(B' - A') = \{0\}$. If $\kappa = \kappa(A', B') = \inf\{\kappa(A, B_n) : n \geqslant 3\}$ is strictly positive, $e^{\kappa A} \leqslant e^{\kappa B_n}$ for any $n \geqslant 3$ and therefore, by taking the limit $n \to \infty$, $e^{\kappa A} \leqslant e^{\kappa B}$, which is impossible because $\kappa(A, B) = 0$.

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REFERENCES

1. M. Fujii, J.F. Jiang, E. Kamei, K. Tanahashi, A characterization of chaotic order and a problem, *J. Inequalities Applications*, to appear.
2. F. Hansen, Selfadjoint means and operator monotone functions, *Math. Ann.* **256** (1981), 29–35.
3. E. Heinz, Beiträge zur Störungstheorie der Spektralzerlegung, *Math. Ann.* **123**(1951), 415–438.
4. K. Löwner, Über monotone Matrixfunktionen, *Math. Z.* **38**(1934), 177–216.

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