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STRONG FELLER SEMIGROUPS ON C^*-ALGEBRAS

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Abstract. On a given C^*-algebra, the existence of a strong Feller semigroup is equivalent to the existence of an approximation of the identity by weakly compact completely positive endomorphisms. As a corollary, there are strong Feller semigroups on any nuclear C^*-algebra, but one can find also strong Feller semigroups on non nuclear C^*-algebras.

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1. INTRODUCTION

A non commutative strong Feller kernel is a completely positive endomorphism Φ of a C^*-algebra $\mathcal{A}$, the canonical extension of which maps the enveloping von Neumann algebra into the algebra of multipliers [i.e., with obvious notations, $\Phi^{**}(\mathcal{A}^{**}) \subset M(\mathcal{A})$]. At a very abstract level, to be strongly Feller is something like a "regularizing kernel" property.

Strong Feller semigroups (i.e., one parameter semigroups of non commutative strong Feller kernels) provide a geometrical structure on the underlying C^*-algebra, and their generators can be viewed as a kind of non commutative elliptic operators. This was the point of view in [7], where the strong Feller property was the crucial one allowing to solve the Dirichlet problem in C^*-algebras.

A natural question is: on which C^*-algebras does there exist a strong Feller semigroup?

A more general question is: on which C^*-algebras does there exist an approximation of the identity by strong Feller kernels?

Our main result is that the two are equivalent: on a given separable C^*-algebra $\mathcal{A}$, there exists a strong Feller semigroup if and only if there exists an approximation of the identity by strong Feller kernels, and also if and only if there exists an approximation of the identity by weakly compact completely positive endomorphisms of $\mathcal{A}$. This is a weak nuclearity requirement which can be interpreted in two ways.

The first one is that we meet a "strong Feller" or "weakly compact approximation property", analogous to other weakenings of nuclearity such as E. Kirchberg's exactness ([5]) or U. Haagerup's completely bounded approximation property (cf. [2], [1]), and which can raise intrinsic interest.

Another way to read this result is that, even in the nuclear case, it leads to an approximation of the identity of $\mathcal{A}$ by completely positive endomorphisms which mutually commute and which are infinitely divisible. Infinite divisibility (i.e., the existence of a semigroup and its associated generator) provides at least a starting point for a non commutative geometry on the C^*-algebra. And conversely, if one requires infinite divisibility, the strong Feller property is the best one is allowed to expect.

Anyway, the world of C^*-algebras splits in two parts: those which have and those which have not a strong Feller semigroup. To the first category clearly belong all nuclear C^*-algebras.

But there exist strong Feller semigroups on C^*-algebras which are not nuclear: for instance the canonical semigroup on the reduced C^*-algebra of a free group with finitely or countably many generators. In a companion paper, more focused on harmonic analysis, it will be shown that there exist strong Feller semigroups on the reduced C^*-algebras of groups which are usually considered as "weakly amenable", such as $\mathrm{SL}(2,\mathbb{Z})$, $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$ or $\mathrm{SL}(2,\mathbb{Q}_p)$.

This paper is organized as follows: next section (Section 2) is devoted to technical preliminaries, mainly on how to deduce a semigroup from an approximation of the identity. Section 3 deals with the connection between weakly compact and strong Feller completely positive endomorphisms. Section 4 is the proof of the main result: C^*-algebras on which there exists a strong Feller semigroup are those on which there exists a weakly compact completely positive approximation of the identity. Section 5 provides examples, comments, and a hint to the fact that the approximation properties we meet behave well with respect to semi-split extensions of C^*-algebras.

Last section is devoted to a more constructive approach of the existence properties of Section 4. It provides canonical ways to deduce a strong Feller semigroup from a semigroup which is strongly Feller only in the resolvent sense.

What we show is that, if Δ is the generator of the latter, then $\mathrm{Log}\,(\mathrm{Id}+\Delta)$ and Δ^β ($\beta\in]0,1[$) are generators of strong Feller semigroups.

This has heuristic implications. For instance, if l is the canonical length function on a free group G, then not only l but also $\mathrm{Log}\,(1+l)$, $\mathrm{Log}\,(1+\mathrm{Log}\,(1+l))$, etc. generate a strong Feller semigroup on the reduced C^*-algebra of G, which means that one cannot expect any control on the growth speed of the generator of a strong Feller semigroup.

2. TECHNICAL PRELIMINARIES AND NOTATIONS

LEMMA 2.1. *There exists a universal constant K such that, if $\mathcal{E}$ is a Banach space and T a linear endomorphism of $\mathcal{E}$ with $\|T\|\leqslant 1$, then one has*

$$\|(T-\mathrm{Id})\exp[t(T-\mathrm{Id})]\|\leqslant K\cdot t^{-1/2}$$

*for any t in $\mathbb{R}_+^*$ (*Id *is the identity automorphism of $\mathcal{E}$).*

Proof. Let $E(t)$ be the integer part of t: one has

$$(\mathrm{Id}-T)\exp(tT)=\sum_{k=0}^{E(t)}\left(1-\frac{k}{t}\right)\frac{t^kT^k}{k!}-\sum_{k=E(t)+1}^{\infty}\left(\frac{k}{t}-1\right)\frac{t^kT^k}{k!}.$$

With $T=\mathrm{Id}$ in the above formula, one obtains

$$\sum_{k=0}^{E(t)}\left(1-\frac{k}{t}\right)\frac{t^k}{k!}=\sum_{k=E(t)+1}^{\infty}\left(\frac{k}{t}-1\right)\frac{t^k}{k!}$$

and for any T with $\|T\|\leqslant 1$,

$$\begin{aligned}\|(\mathrm{Id}-T)\exp(tT)\|&\leqslant\sum_{k=0}^{E(t)}\left(1-\frac{k}{t}\right)\frac{t^k}{k!}+\sum_{k=E(t)+1}^{\infty}\left(\frac{k}{t}-1\right)\frac{t^k}{k!}\\&=2\cdot\sum_{k=0}^{E(t)}\left(1-\frac{k}{t}\right)\frac{t^k}{k!}=2\cdot\frac{t^{E(t)}}{E(t)!}.\end{aligned}$$

We have then

$$\|(T-\mathrm{Id})\exp[t(T-\mathrm{Id})]\|\leqslant 2\cdot\mathrm{e}^{-t}\cdot\frac{t^{E(t)}}{E(t)!}$$

which, by Stirling formula, provides the result.

[By Stirling formula, there exists $t_0 > 1$ such that, for $t \geqslant t_0$:

$$E(t)! \geqslant \sqrt{2\pi E(t)}\mathrm{e}^{-E(t)-1}E(t)^{E(t)}$$

hence

$$\|(\mathrm{Id}-T)\exp[t(T-\mathrm{Id})]\| \leqslant \frac{2e}{\sqrt{2\pi E(t)}}\left(\frac{E(t)+1}{E(t)}\right)^{E(t)} \leqslant \frac{2\mathrm{e}^2}{\sqrt{2\pi E(t)}} \leqslant \frac{2\mathrm{e}^2}{\sqrt{2\pi(t-1)}}$$

which allows to choose K such that $\frac{2\mathrm{e}^2}{\sqrt{2\pi(t-1)}} \leqslant K\cdot t^{-1/2}$ for $t \geqslant t_0$ and $2\cdot\exp(2t) \leqslant K \cdot t^{-1/2}$ for $t \leqslant t_0$.] ∎

2.2. Terminology.

2.2.1. All along this paper, the word "semigroup" stands for a one parameter pointwise norm continuous semigroup of completely positive contractions of a C^*-algebra.

2.2.2. Let us define the generator Δ of such a semigroup $\{\Phi_t\}_{t\in\mathbb{R}_+}$ by the formula

$$\Delta(a) = \lim_{t\searrow 0}\frac{1}{t}(a-\Phi_t(a)),$$

its domain being the dense linear space of those a in $\mathcal{A}$ for which the limit exists.

With this convention, once can write $\Phi_t = \mathrm{e}^{-t\Delta}$.

2.2.3. The resolvent maps will be completely positive contractions $R_\lambda = \lambda\int_0^\infty \mathrm{e}^{-t\lambda}\Phi_t\mathrm{d}t = \lambda(\lambda\cdot\mathrm{Id}+\Delta)^{-1}(\lambda\in\mathbb{R}_+)$; each R_λ will be written shortly $\frac{\lambda}{\lambda+\Delta}$. We shall make a constant use of the obvious equality

$$\mathrm{Id} - \frac{\lambda}{\lambda+\Delta} = \frac{\Delta}{\lambda+\Delta},$$

which implies that $\frac{\Delta}{\lambda+\Delta}$ is a bounded endomorphism of $\mathcal{A}$, with norm less than 2. [Notice that [3] or [10] call 'resolvent' the maps $(\lambda\cdot\mathrm{Id}+\Delta)^{-1}$.]

2.2.4. It is more or less obvious that, if Ψ is a completely positive endomorphism of a C^*-algebra $\mathcal{A}$ with $\|\Psi\| \leqslant \alpha$, then $\{\exp[t(\Psi-\alpha\cdot\mathrm{Id})]\}_{t\in\mathbb{R}_+}$ is a semigroup of completely positive contractions of $\mathcal{A}$.

In particular, $\frac{\Delta}{\lambda+\Delta}$ is the generator of such a semigoup, as well as any finite linear combination $\sum_k \alpha_k\frac{\Delta}{\lambda+\Delta}$, with every α_k positive.

Similarly, if $\{\Psi_1,\ldots,\Psi_n\}$ is a finite set of completely positive contractions of $\mathcal{A}$, then $n\cdot\mathrm{Id}-\Psi_1+\cdots+\Psi_n$ generates a semigroup $\{\exp(t[\Psi_1+\cdots+\Psi_n-n\cdot\mathrm{Id}])\}_{t\in\mathbb{R}_+}$ of completely positive contractions of $\mathcal{A}$.

LEMMA 2.3. *Let $\mathcal{A}$ be a C^*-algebra and $\{\Psi_n\}_{n\in\mathbb{N}}$ a sequence of completely positive contractions of $\mathcal{A}$. Suppose that:*

(i) *the $\{\Psi_n\}$ commute, i.e., $\Psi_m \circ \Psi_n = \Psi_n \circ \Psi_m$, $\forall\, n, m \in \mathbb{N}$;*

(ii) *the subspace $\mathfrak{A} = \Big\{a \in \mathcal{A} \,\Big|\, \sum_n \|a - \Psi_n(a)\| < +\infty\Big\}$ is dense in $\mathcal{A}$.*

Then, there exists a semigroup $\{\Phi_t\}_{t\in\mathbb{R}_+}$ of completely positive contractions of $\mathcal{A}$ such that, for any t in $\mathbb{R}_+$, the sequence of endomorphisms $\{\exp[t(\Psi_1 + \cdots + \Psi_n - n\cdot \mathrm{Id})]\}_{n\in\mathbb{N}}$ converges pointwise to Φ_t.

Moreover, for any $t > 0$, one has $\lim\limits_{n\to\infty} \|(\mathrm{Id} - \frac{1}{n}(\Psi_1 + \cdots + \Psi_n)) \circ \Phi_t\| = 0$ in the Banach algebra of bounded endomorphisms of $\mathcal{A}$.

Proof. Set $\Phi_t^n = \exp[t(\Psi_1 + \cdots + \Psi_n - n\cdot \mathrm{Id})]$ for t in $\mathbb{R}_+$ and n in $\mathbb{N}$. For n and m in $\mathbb{N}$, $m > n$, set $\delta_{n,m} = (m-n)\mathrm{Id} - (\Psi_{n+1} + \cdots + \Psi_m)$ and write $\Phi_t^m = \Phi_t^n \circ \exp[-t\delta_{n,m}]$. For a in $\mathfrak{A}$ one has

$$\|\Phi_t^m - \Phi_t^n(a)\| = \|\Phi_t^n(\exp[-t\delta_{n,m}](a) - a)\| \leqslant \|(\exp[-t\delta_{n,m}](a) - a)\| \leqslant t\|\delta_{n,m}(a)\|.$$

By definition of $\mathfrak{A}$, $\|\delta_{n,m}(a)\|$ tends to 0 when $n \to \infty$, uniformly in m, so that the sequence $\{\Phi_t^n(a)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for any a in $\mathfrak{A}$. By norm density of $\mathfrak{A}$, this property extends to all a in $\mathcal{A}$. We have proved the existence of the pointwise norm limit semigroup $\{\Phi_t\}$.

Its pointwise norm continuity comes from the inequality

$$\|\Phi_t(a) - a\| = \lim_n \|\Phi_t^n(a) - a\| \leqslant t \cdot \lim_n \|(\Psi_1 + \cdots + \Psi_n - n\cdot \mathrm{Id})(a)\|, \quad \forall\, a \in \mathfrak{A}.$$

For the last property, notice that one can write $\Phi_t = \Phi_t^n \circ \lim\limits_m \exp[-t\delta_{n,m}]$, which implies

$$\begin{aligned} &\Big\|\Big(\mathrm{Id} - \frac{1}{n}(\Psi_1 + \cdots + \Psi_n)\Big) \circ \Phi_t\Big\| \\ &\quad\leqslant \Big\|\Big(\mathrm{Id} - \frac{1}{n}(\Psi_1 + \cdots + \Psi_n)\Big) \circ \exp[t(\Psi_1 + \cdots + \Psi_n - n\cdot \mathrm{Id})]\Big\| \\ &\quad\leqslant K \cdot (nt)^{-1/2} \end{aligned}$$

by Lemma 2.1. ∎

Next lemma is a bit less obvious and more technical:

LEMMA 2.4. *Let $\mathcal{A}$ be a C^*-algebra and $\{\Psi_n\}_{n\in\mathbb{N}}$ a sequence of completely positive contractions of $\mathcal{A}$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a dense sequence in $\mathcal{A}$.*

Suppose that:

$$\|(\mathrm{Id} - \Psi_n)(x)\| \leqslant 2^{-n}\|x\|$$

for any $n \geqslant 1$, and any x in the (finite dimensional) subspace of $\mathcal{A}$ generated by the $\{\psi_j^k(a_l) \mid 0 \leqslant j,\, l < n;\, k \leqslant n^2\}$.

Then the sequence $\{\{\exp[t(\Psi_1 + \cdots + \Psi_n - n \cdot \mathrm{Id})]\}_{t\in\mathbb{R}_+}\}_{n\in\mathbb{N}}$ of semigroups on $\mathcal{A}$ converges in the resolvent sense towards a semigroup $\{\Phi_t\}_{t\in\mathbb{R}_+}$ of completely positive contractions of $\mathcal{A}$.

More precisely, if Δ is the generator of $\{\Phi_t\}_{t\in\mathbb{R}_+}$, then one has

$$\lim_{n\to\infty}[\mathrm{Id} + \alpha(n \cdot \mathrm{Id} - (\Psi_1 + \cdots + \Psi_n))]^{-1}(a) = [\mathrm{Id} + \alpha\Delta]^{-1}(a)$$

for any α in $\mathbb{R}_+^*$ and any a in $\mathcal{A}$ (where $[\mathrm{Id} + \alpha\Delta]^{-1} = \frac{1}{\alpha}\int_0^\infty \mathrm{e}^{-t/\alpha}\Phi_t\,\mathrm{d}t$ is the resolvent map $R_{1/\alpha}$).

Proof. Set $\theta_n = \frac{1}{n}(\Psi_1+\cdots+\Psi_n)$ and $\Delta_n = n\cdot\mathrm{Id}-(\Psi_1+\cdots+\Psi_n) = n(\mathrm{Id}-\theta_n)$. Compute, for $n, p > 0$ and $l \leqslant n$:

$$(\mathrm{Id}-\Psi_{n+p})[\mathrm{Id}+\alpha\Delta_n]^{-1}(a_l) = (\mathrm{Id}-\Psi_{n+p})\frac{1}{1+n\alpha}\sum_k \left(\frac{n\alpha}{1+n\alpha}\right)^k \theta_n^k(a_l) = A+B$$

with

$$\begin{aligned} A &= (\mathrm{Id} - \Psi_{n+p})\frac{1}{1+n\alpha}\sum_{k=0}^{(n+p)^2}\left(\frac{n\alpha}{1+n\alpha}\right)^k \theta_n^k(a_l) \\ &= \frac{1}{1+n\alpha}\sum_{k=0}^{(n+p)^2}\left(\frac{n\alpha}{1+n\alpha}\right)^k (\mathrm{Id} - \Psi_{n+p})\theta_n^k(a_l) \end{aligned}$$

and

$$B = (\mathrm{Id} - \Psi_{n+p})\frac{1}{1+n\alpha}\sum_{k=(n+p)^2+1}\left(\frac{n\alpha}{1+n\alpha}\right)^k \theta_n^k(a_l).$$

The assumptions imply $\|A\| \leqslant 2^{-n-p}\|a_l\|$. Moreover, one has

$$\|B\| \leqslant \frac{2}{1+n\alpha}\|a_l\| \sum_{k=(n+p)^2+1}\left(\frac{n\alpha}{1+n\alpha}\right)^k \leqslant 2\left(\frac{n\alpha}{1+n\alpha}\right)^{(n+p)^2}\|a_l\|.$$

Fix $\alpha_0 > 0$ and α in $]0, \alpha_0]$. It is easy to check that, for n large enough, one has $\left(\frac{n\alpha}{1+n\alpha}\right)^n \leqslant \mathrm{e}^{-1/2\alpha_0}$ and

$$\left(\frac{n\alpha}{1+n\alpha}\right)^{(n+p)^2} \leqslant \left(\frac{n\alpha}{1+n\alpha}\right)^{n(n+p)} \leqslant \mathrm{e}^{-(n+p)/2\alpha_0};$$

hence the inequality

$$\|(\mathrm{Id} - \Psi_{n+p})[\mathrm{Id} + \alpha\Delta_n]^{-1}(a_l)\| \leqslant 2\|a_l\|(2^{-n-p} + \mathrm{e}^{-(n+p)/2\alpha_0}).$$

For fixed l, for n large enough and $m > n$, one has

$$\begin{aligned}\|(\mathrm{Id}+\alpha\Delta_m)[\mathrm{Id}+\alpha\Delta_n]^{-1}(a_l)-a_l\| &= \alpha\Big\|\sum_{p=1}^{m-n}(\mathrm{Id}-\Psi_{n+p})[\mathrm{Id}+\alpha\Delta_n]^{-1}(a_l)\Big\| \\ &\leqslant C(2^{-n}+\mathrm{e}^{-n/2\alpha_0})\end{aligned}$$

and, as $[\mathrm{Id}+\alpha\Delta_m]^{-1}$ is a contraction,

$$\|[\mathrm{Id}+\alpha\Delta_m]^{-1}(a_l)-[\mathrm{Id}+\alpha\Delta_n]^{-1}(a_l)\| \leqslant C(2^{-n}+\mathrm{e}^{-n/2\alpha_0}),$$

where the constant C depends only on α and $\|a_l\|$.

The sequence $n \to [\mathrm{Id}+\alpha\Delta_n]^{-1}(a_l)$ is a Cauchy sequence for any l in $\mathbb{N}$; by density, for any a in $\mathcal{A}$, the sequence $n \to [\mathrm{Id}+\alpha\Delta_n]^{-1}(a)$ has the Cauchy property and $\rho_\alpha(a) = \lim_{n\to\infty}[\mathrm{Id}+\alpha\Delta_n]^{-1}(a)$ exists.

For fixed $a = a_l$, the convergence above is uniform for α in $]0,\alpha_0]$, so that $\lim_{\alpha\to 0}\rho_\alpha(a) = a$ for $a = a_l$, then for all a in $\mathcal{A}$.

Moreover, the $\{\rho_\alpha\}$ satisfy the same resolvent equation

$$\alpha\rho_\alpha - \beta\rho_\beta = (\alpha-\beta)\rho_\alpha\rho_\beta, \quad \forall\,\alpha,\beta \text{ in } \mathbb{R}_+^*$$

than the $\{[\mathrm{Id}+\alpha\Delta_n]^{-1}\}$, which in particular implies that all the ρ_α have the same kernel and the same range $\mathfrak{D}$. Making α tending to zero, one sees that the ρ_α are one to one and that $\mathfrak{D}$ is dense, so that $\Delta = \frac{1}{\alpha}(\rho_\alpha^{-1} - \mathrm{Id})$ is a densely defined closed operator, which clearly does not depend on α. Every $\rho_\alpha = [\mathrm{Id}+\alpha\Delta]^{-1}$ is a contraction, which by Hille-Yoshida theorem implies that Δ is the generator of a continuous semigroup $\{\Phi_t\}$ of contractions of $\mathcal{A}$ (cf. [3], Corollary 12.3 at page 263 and [10], IX.7, Corollary), with $\Phi_t = \lim_{n\to\infty}(\rho_{t/n})^n$ completely positive. ∎

3. STRONG FELLER KERNELS AND WEAKLY COMPACT ENDOMORPHISMS

3.1. Let $\mathcal{A}$ be a C^*-algebra. Its enveloping von Neumann algebra will be denoted $\mathcal{A}^{**}$; its multipliers algebra is

$$M(\mathcal{A}) = \{x \in \mathcal{A}^{**} \mid xa \in \mathcal{A} \text{ and } ax \in \mathcal{A},\ \forall\, a \in \mathcal{A}\}.$$

$M(\mathcal{A})$ contains $\mathcal{A}$ as a two-sided ideal. If $\mathcal{A}$ has a unit, then $M(\mathcal{A}) = \mathcal{A}$. (For the multipliers algebra and its topology, cf. [6].)

If Ψ is a completely positive endomorphism of $\mathcal{A}$, then by double duality Ψ extends to a completely positive normal endomorphism Ψ^{**} of $\mathcal{A}^{**}$ with the same norm.

3.2. DEFINITIONS.

1. An endomorphism of a C^*-algebra $\mathcal{A}$ is *weakly compact* if the image of the unit ball $\mathcal{A}_1$ of $\mathcal{A}$ is a σ-$(\mathcal{A}, \mathcal{A}^*)$ relatively compact subset of $\mathcal{A}$.

2. A completely positive endomorphism Ψ of $\mathcal{A}$ will be said to have *the strong Feller property*, and will be called a *strong Feller completely positive endomorphism* of $\mathcal{A}$, if its canonical extension maps the enveloping von Neumann algebra into the multipliers algebra, i.e.

$$\Psi^{**}[\mathcal{A}^{**}] \subset M(\mathcal{A}).$$

[Heuristically, it turns "Borel functions" into "continuous functions".]

3. A *strong Feller approximation of the identity* on a C^*- algebra $\mathcal{A}$ is a net $\{\Psi_\alpha\}$ of strong Feller completely positive endomorphisms of $\mathcal{A}$ such that

$$\lim_\alpha \|\Psi_\alpha(a) - a\| = 0, \quad \forall\, a \in \mathcal{A}.$$

4. A *strong Feller semigroup* on $\mathcal{A}$ is a semigroup $\{\Phi_t\}_{t\in\mathbb{R}_+}$ of completely positive endomorphisms of $\mathcal{A}$ such that Φ_t has the strong Feller property for every t in $\mathbb{R}_+^*$.

LEMMA 3.3. *Let Φ be a completely positive endomorphism of $\mathcal{A}$. Then the following properties are equivalent:*

(i) $\Phi^{**}(\mathcal{A}^{**}) \subset \mathcal{A}$;

(ii) Φ *is weakly compact;*

(iii) Φ^{**} *is continuous from $\mathcal{A}^{**}$ equipped with the Mackey-Arens topology into $\mathcal{A}$ with the norm topology;*

(iv) Φ *is continuous from the unit ball of $\mathcal{A}$ equipped with the restriction of the σ-strong topology of $\mathcal{A}^{**}$, into $\mathcal{A}$ with the norm topology.*

Proof. (i) $\Rightarrow$ (ii) The unit ball of $\mathcal{A}^{**}$ is σ-$(\mathcal{A}^{**}, \mathcal{A}^*)$ compact.

(ii) $\Rightarrow$ (i) By Kaplansky density theorem, if Φ is weakly compact then Φ^{**} maps the unit ball of $\mathcal{A}^{**}$ into the σ-weak closure of $\Phi(\mathcal{A}_1)$ in $\mathcal{A}^{**}$, which is still contained in $\mathcal{A}$.

(i) $\Rightarrow$ (iii) If (i) is satisfied, then the image by the transposed map Φ^* of the unit ball of $\mathcal{A}^*$ is σ-$(\mathcal{A}^*, \mathcal{A}^{**})$ compact; hence (iii) by definition of the Mackey-Arens topology (cf. [9], Chapter III, Section 5, p. 153).

(iii) $\Rightarrow$ (iv) is Theorem 5.7 in [9], Chapter III.

(iv) $\Rightarrow$ (i) Use again Kaplansky density theorem. ∎

REMARK 3.4. Suppose that there exists a strong Feller approximation of the identity on a separable C^*-algebra $\mathcal{A}$; then there exists a sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ of completely positive weakly compact contractions of $\mathcal{A}$ which converges pointwise to the identity.

Proof. Start with a sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ of completely positive strong Feller endomorphisms of $\mathcal{A}$ which converges pointwise to the identity. If $\{b_m\}$ is a positive increasing approximate unit for $\mathcal{A}$, then by the previous lemma, $\left\{\frac{b_m\Psi_m(b_n\cdot b_n)b_m}{\|b_m\Psi_m(b_n^2)b_m\|}\right\}_{n,m\in\mathbb{N}}$ is a set of weakly compact completely positive contractions of $\mathcal{A}$ from which it is not difficult to extract a sequence which converges norm pointwise to the identity:

Choose $\mathcal{E}$ a finite subset in $\mathcal{A}$ and $\varepsilon > 0$, then $n \in \mathbb{N}$ such that $\|b_n\|^2 \geqslant 1-\varepsilon$ and $\|b_nab_n - a\| \leqslant \varepsilon\ \forall a \in \mathcal{E}$; notice that, when m tends to infinity, one has for any a in $\mathcal{E}$.

$$\lim_m \|\Psi_m(b_nab_n) - b_naB_n\| = \lim_m \|b_m(b_nab_n)b_m - b_nab_n\| = 0$$

hence

$$\limsup_m \|b_m\Psi_m(b_nab_n)b_m - a\| \leqslant \varepsilon$$

and, as $\lim_m \|b_m\Psi_m(b_n^2)b_m\| = \|b_n^2\| \geqslant 1-\varepsilon$,

$$\limsup_m \left\|\frac{b_m\Psi_m(b_nab_n)b_m}{\|b_m\Psi_m(b_n^2)b_m\|} - a\right\| \leqslant \frac{2\varepsilon}{1-\varepsilon}, \quad \forall a \in \mathcal{E}. \qquad \blacksquare$$

4. WEAKLY COMPACT APPROXIMATION OF THE IDENTITY AND STRONG FELLER SEMIGROUPS

PROPOSITION 4.1. *Let $\mathcal{A}$ be a separable C^*-algebra. Then the following properties are equivalent:*

(i) *there exists an approximation of the identity of $\mathcal{A}$ by weakly compact completely positive endomorphisms of $\mathcal{A}$;*

(ii) *there exists on $\mathcal{A}$ a strong Feller approximation of the identity;*

(iii) *there exists on $\mathcal{A}$ a strong Feller semigroup of completely positive contractions.*

The proposition will be an immediate application of the following lemma:

LEMMA 4.2. *Let $\mathcal{A}$ be a separable C^*-algebra. Then the following properties are equivalent:*

(i) *there exists on $\mathcal{A}$ a strong Feller approximation of the identity;*

(i′) *there exists an approximation of the identity of $\mathcal{A}$ by weakly compact completely positive endomorphisms of $\mathcal{A}$;*

(ii) *there exists a commuting sequence of strong Feller completely positive contractions of $\mathcal{A}$ which converges to the identity; commuting means: $\Psi_n \circ \Psi_m = \Psi_m \circ \Psi_n, \forall\, n, m \in \mathbb{N}$;*

(iii) *there exists a semigroup $\{\Phi_t\}_{t\in\mathbb{R}_+}$ of completely positive contractions of $\mathcal{A}$, of which the resolvent maps $R_\lambda = \lambda \int\limits_0^\infty \mathrm{e}^{-\lambda t}\Phi_t \mathrm{d}t$ have the strong Feller property, for all λ in $\mathbb{R}_+^*$;*

(iv) *there exists a strong Feller semigroup of completely positive contractions of $\mathcal{A}$.*

Proof. (i) $\Leftrightarrow$ (i′) by Lemma 3.3 and Remark 3.4.

Implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious.

We shall prove (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv). We start with the second (an easier) one.

(ii) $\Rightarrow$ (iv) By replacing if required the (commuting) sequence of strong Feller completely positive contractions of $\mathcal{A}$ approximating the identity of $\mathcal{A}$ by a subsequence, one can suppose that there exists a sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ of strong Feller completely positive contractions of $\mathcal{A}$ such that

1. the $\{\Psi_n\}$ commute: $\Psi_m \circ \Psi_n = \Psi_n \circ \Psi_m$, $\forall\, n, m \in \mathbb{N}$;
2. the subspace $\mathfrak{A} = \Big\{a \in \mathcal{A} \,\Big|\, \sum\limits_n \|a - \Psi_n(a)\| < \infty\Big\}$ is dense in $\mathcal{A}$.

By Lemma 2.3, there exists a semigroup $\{\Phi_t\}_{t\in\mathbb{R}_+}$ of completely positive contractions of $\mathcal{A}$ such that, for any t in $\mathbb{R}_+$, the sequence of endomorphisms $\{\exp[t(\Psi_1 + \cdots + \Psi_n - n \cdot \mathrm{Id})]\}_{n\in\mathbb{N}}$ converge pointwise to Φ_t.

Moreover, the same lemma implies $\Phi_t = \operatorname{norm}\lim\limits_{n\to\infty} \frac{1}{n}(\Psi_1 + \cdots + \Psi_n) \circ \Phi_t$ for any $t > 0$, so Φ_t has the strong Feller property, as a norm limit in the Banach spaces $\mathfrak{L}(\mathcal{A})$ and $\mathfrak{L}(\mathcal{A}^{**})$ of the strong Feller endomorphisms $\frac{1}{n}(\Psi_1 + \cdots + \Psi_n) \circ \Phi_t$, hence the semigroup $\{\Phi_t\}_{t\in\mathbb{R}_+}$ is a strong Feller semigroup.

(i) $\Leftrightarrow$ (iii) The proof is divided into several steps.

Step 1. By assumption (i) and Remark 3.4, there exists a sequence $\{\Psi_n\}$ of weakly compact completely positive contractions approximating the identity of $\mathcal{A}$, such that $\Psi_n^{**}(\mathcal{A}^{**}) \subset \mathcal{A}$ for any n.

Fix a dense countable subset $\{a_l\}_{l\in\mathbb{N}}$ of $\mathcal{A}$ and replace the sequence $\{\Psi_n\}$ by a subsequence, in such a way that the assumptions of Lemma 2.4 are satisfied, i.e.

$\|(\mathrm{Id}-\Psi_n)(x)\| \leqslant 2^{-n}\|x\|$ for any $n \geqslant 1$, and any x in the subspace of $\mathcal{A}$ generated by the $\{\psi_j^k(a_l) \mid 0 \leqslant j,\, l < n;\, k \leqslant n^2\}$.

Then, by Lemma 2.4, there exists a semigroup $\{\Phi_t\}_{t\in\mathbb{R}_+}$ with generator Δ such that

$$\lim_{n\to\infty}[\mathrm{Id} - \alpha(\Psi_1 + \cdots + \Psi_n - n \cdot \mathrm{Id})]^{-1}(a) = [\mathrm{Id} + \alpha\Delta]^{-1}(a) = R_{1/\alpha}(a)$$

for any α in $\mathbb{R}_+^*$ and any a in $\mathcal{A}$.

Step 2. Moreover, let $1_{\mathcal{A}}$ be the unit in A^{**} (which is also the unit in the multiplier algebra $M(\mathcal{A})$). Set $[\mathrm{Id} - \alpha(\Psi_1 + \cdots + \Psi_n - n \cdot \mathrm{Id})]^{-1}(1_{\mathcal{A}}) = y_n$ (from now on, we shall not distinguish between a completely positive endomorphism of $\mathcal{A}$ and its canonical extension to $\mathcal{A}^{**}$). One has

$$y_n = \frac{1}{n+1}1_{\mathcal{A}} + \frac{\alpha}{n+1}(\Psi_1 + \cdots + \Psi_n)(y_n) \in M(\mathcal{A}).$$

As $\Psi_j(1_{\mathcal{A}^{**}})$ belongs to $\mathcal{A}$, one could have assumed also, in Step 1 above,

$$\|(\mathrm{Id} - \Psi_n)\Psi_j^k(1_{\mathcal{A}^{**}})\| \leqslant 2^{-n}, \quad \forall\, j = 1, \ldots, n-1;\, \forall\, k = 1, \ldots, n^2.$$

One can then easily prove, just as in the proof of Lemma 2.4, that the sequence $\{[\mathrm{Id} - \alpha(\Psi_1 + \cdots + \Psi_n - n\cdot\mathrm{Id})]^{-1}(1_{\mathcal{A}^{**}})\}$ is norm-converging to $[\mathrm{Id} + \alpha\Delta]^{-1}(1_{\mathcal{A}^{**}})$, which implies that $[\mathrm{Id} + \alpha\Delta]^{-1}(1_{\mathcal{A}^{**}})$ is a multiplier of $\mathcal{A}$.

Notice that, for a completely positive contraction Ψ of $\mathcal{A}$, one has $\Psi^{**}[M(\mathcal{A})] \subset M(\mathcal{A})$ if and only if $\Psi^{**}(1_{\mathcal{A}^{**}}) \in M(\mathcal{A})$. What we have proved in fact is $[\mathrm{Id} + \alpha\Delta]^{-1}[M(\mathcal{A})] \subset M(\mathcal{A})$ (see Remark 4.3.2 below).

Step 3. Set now, for $n < m$ in $\mathbb{N}^*$:

$$\begin{aligned}
\theta_n &= \frac{1}{n}(\Psi_1 + \cdots + \Psi_n);\\
\Delta_n &= n \cdot \mathrm{Id} - (\Psi_1 + \cdots + \Psi_n);\\
\delta_{nm} &= \Delta_m - \Delta_n = (m-n)\cdot \mathrm{Id} - (\Psi_{n+1} + \cdots + \Psi_m).
\end{aligned}$$

For x in $\mathcal{A}^{**}$, define $y_{nm} = [\mathrm{Id} + \alpha\Delta_n]^{-1}[\mathrm{Id} + \frac{\alpha}{\alpha n+1}\delta_{nm}]^{-1}(x)$ and compute, for α in $\mathbb{R}_+^*$:

$$\begin{aligned}
x &= \left[\mathrm{Id} + \frac{\alpha}{\alpha n+1}\delta_{nm}\right][\mathrm{Id} + \alpha\Delta_n]y_{nm}\\
&= \left[\mathrm{Id} + \frac{\alpha}{\alpha n+1}\delta_{nm}\right][(\alpha n+1)\mathrm{Id} - \alpha(\Psi_1 + \cdots + \Psi_n)]y_{mn}\\
&= \left[(\alpha n+1)\mathrm{Id} - \alpha\delta_{nm} - \alpha(\Psi_1 + \cdots + \Psi_n) - \frac{\alpha^2 n}{\alpha n+1}\delta_{nm}\theta_n\right] y_{mn}\\
&= \left[\mathrm{Id} + \alpha\Delta_m - \frac{\alpha^2 n}{\alpha n+1}\delta_{nm}\theta_n\right] y_{mn};
\end{aligned}$$

hence

$$\begin{aligned}
&[\mathrm{Id}+\alpha\Delta_m]^{-1}x \\
&\quad = y_{mn} - [\mathrm{Id}+\alpha\Delta_m]^{-1}\left[\frac{\alpha^2 n}{\alpha n+1}\delta_{nm}\theta_n\right] y_{mn} \\
&\quad = [\mathrm{Id}+\alpha\Delta_n]^{-1}\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_{nm}\right]^{-1} x \\
&\qquad - [\mathrm{Id}+\alpha\Delta_m]^{-1}\left[\frac{\alpha^2 n}{\alpha n+1}\delta_{nm}\theta_n\right][\mathrm{Id}+\alpha\Delta_n]^{-1}\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_{nm}\right]^{-1} x
\end{aligned}$$

and

$$\begin{aligned}
&[\mathrm{Id}+\alpha\Delta_m]^{-1} - [\mathrm{Id}+\alpha\Delta_n]^{-1}\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_{nm}\right]^{-1} \\
&\quad = [\mathrm{Id}+\alpha\Delta_m]^{-1}\left[\frac{\alpha^2 n}{\alpha n+1}\delta_{nm}\theta_n\right][\mathrm{Id}+\alpha\Delta_n]^{-1}\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_{nm}\right]^{-1} \\
&\quad = \frac{\alpha^2 n}{\alpha n+1}[\mathrm{Id}+\alpha\Delta_m]^{-1}[\Delta_m-\Delta_n][\mathrm{Id}+\alpha\Delta_n]^{-1}\theta_n\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_{nm}\right]^{-1} \\
&\quad = \frac{\alpha n}{\alpha n+1}\left([\mathrm{Id}+\alpha\Delta_n]^{-1} - [\mathrm{Id}+\alpha\Delta_m]^{-1}\right)\theta_n\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_{nm}\right]^{-1}.
\end{aligned}$$

Step 4. Apply Lemma 2.4 to the sequence $\{\Psi_n, \Psi_{n+1}, \ldots\}$, thus there exists a semigroup with generator δ_n such that the sequence $m \to \left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_{nm}\right]^{-1}$ of completely positive endomorphisms of $\mathcal{A}$ converge pointwise to the completely positive contraction $\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_n\right]^{-1}$, and the formula above becomes, when m goes to infinity:

$$\begin{aligned}
&[\mathrm{Id}+\alpha\Delta]^{-1} - [\mathrm{Id}+\alpha\Delta_n]^{-1}\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_n\right]^{-1} \\
&\quad = \frac{\alpha n}{\alpha n+1}\left([\mathrm{Id}+\alpha\Delta]^{-1} - [\mathrm{Id}+\alpha\Delta_n]^{-1}\right)\theta_n\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_n\right]^{-1}.
\end{aligned}$$

By Step 2 above,

$$\frac{\alpha n}{\alpha n+1}\left([\mathrm{Id}+\alpha\Delta]^{-1} - [\mathrm{Id}+\alpha\Delta_n]^{-1}\right)\theta_n\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_n\right]^{-1}$$

is a difference of two strong Feller completely positive endomorphisms of $\mathcal{A}$.

One checks

$$\|(\mathrm{Id}-\theta_n)[\mathrm{Id}+\alpha\Delta_n]^{-1}\| = \frac{1}{\alpha n}\|\alpha\Delta_n[\mathrm{Id}+\alpha\Delta_n]^{-1}\| \leqslant \frac{2}{\alpha n},$$

which implies

$$\left\|(\mathrm{Id}-\theta_n)[\mathrm{Id}+\alpha\Delta_n]^{-1}\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_n\right]^{-1}\right\| \leqslant \frac{2}{\alpha n},$$

so that

$$\begin{aligned}\Big\|[\mathrm{Id}+\alpha\Delta]^{-1} - \theta_n[\mathrm{Id}+\alpha\Delta_n]^{-1}\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_n\right]^{-1} \\ - \frac{\alpha n}{\alpha n+1}\left([\mathrm{Id}+\alpha\Delta]^{-1}-[\mathrm{Id}+\alpha\Delta_n]^{-1}\right)\theta_n\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_n\right]^{-1}\Big\| \leqslant \frac{2}{\alpha n}.\end{aligned}$$

It follows that

$$\begin{aligned}\theta_n[\mathrm{Id}+\alpha\Delta_n]^{-1}\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_n\right]^{-1} \\ - \frac{\alpha n}{\alpha n+1}\left([\mathrm{Id}+\alpha\Delta]^{-1}-[\mathrm{Id}+\alpha\Delta_n]^{-1}\right)\theta_n\left[\mathrm{Id}+\frac{\alpha}{\alpha n+1}\delta_n\right]^{-1}\end{aligned}$$

is a bounded linear endomorphism of $\mathcal{A}$, the canonical extension of which maps $\mathcal{A}^{**}$ into $M(\mathcal{A})$. As n goes to infinity, we get the same property for $[\mathrm{Id}+\alpha\Delta]^{-1}$. The proof is complete. ∎

4.3. Remarks.

4.3.1. Notice that, for a given semigroup, all the resolvent maps have the strong Feller property if and only if one resolvent map has the strong Feller property. (For any pair λ, μ in $\mathbb{R}_+^*$, $\frac{\lambda+\Delta}{\mu+\Delta}$ is bounded endomorphism of $\mathcal{A}$; see 2.2.2 above.)

4.3.2. Let Φ be a 2-positive endomorphism of $\mathcal{A}$. Its canonical extension Φ^{**} maps $M(\mathcal{A})$ into $M(\mathcal{A})$ if and only if $\Phi^{**}(1_{\mathcal{A}^{**}})$ is a multiplier of $\mathcal{A}$.

Proof. The 'only if' part is trivial. Suppose $\Phi^{**}(1_{\mathcal{A}^{**}}) \in M(\mathcal{A})$ and let $\{b_\alpha\}$ be a positive increasing approximate unit for $\mathcal{A}$; fix a in $\mathcal{A}$, then $\{a^*\Phi^{**}(1_{\mathcal{A}^{**}} - b_\alpha)a\}$ is a decreasing net in $\mathcal{A}$ which converges weakly, hence in norm to 0. Replacing $\{b_\alpha\}$ by $\{b_\alpha^{-1/2}\}$ we get that the net $\{a^*\Phi^{**}[(1_{\mathcal{A}^{**}} - b_\alpha)^2]a\}$ is norm convergent to 0, as well as the net $\{a^*\Phi^{**}(1_{\mathcal{A}^{**}} - b_\alpha)x(1_{\mathcal{A}^{**}} - b_\alpha)]a\}$ for any x in $M(\mathcal{A})$; this proves that $a^*\Phi^{**}(x)a$ is an element of $\mathcal{A}$, as the norm limit of the net

$$a^*\Phi(b_\alpha x + x b_\alpha + b_\alpha x b_\alpha)a.$$

5. STRONG FELLER APPROXIMATION PROPERTY AND NUCLEARITY

As an obvious corollary of Proposition 4.1, we get:

PROPOSITION 5.1. *There exists a strong Feller semigroup on any separable nuclear C^*-algebra.*

The converse is not true: the following example shows that there exist strong Feller semigroups on separable non-nuclear C^*-algebras.

PROPOSITION 5.2. *There exists a strong Feller semigroup on the reduced C^*-algebra of a free group with finitely many generators.*

Proof. Let $G = \mathbb{F}_n$ be the free group with n generators, and l be the "length of the word" function. It is known (cf. [2], Lemma 1.2) that l is a conditionally negative type function. As a consequence, there exists a semigroup $\{\Phi_t\}_{t\in\mathbb{R}_+}$ of completely positive unit preserving endomorphisms of the reduced C^*-algebra $\mathcal{A} = C^*_{\mathrm{red}}(G)$ characterized by

$$\Phi_t[\lambda(f)] = \lambda(\mathrm{e}^{-tl} f) \quad \text{for } f \text{ in } \mathbb{C}(G),\ t \text{ in } \mathbb{R}_+.$$

Lemma 1.4 of [2] implies that, for f a function of G with finite support,

$$\|\Phi_t[\lambda(f)]\| = \|\lambda(\mathrm{e}^{-tl} f)\| \leqslant \sum_n (n+1)\mathrm{e}^{-tn} \|f \cdot \chi_n\|_2$$

where χ_n is the characteristic function of the subset of G of elements with length n, and $\|\cdot\|_2$ is the l^2-norm.

As $\|f\|_2^2 = \sum\limits_n \|f \cdot \chi_n\|_2^2$, we get the inequality:

$$\|\Phi_t[\lambda(f)]\|^2 \leqslant C_t \cdot \|f\|_2^2 \quad \text{with} \quad C_t = \Big[\sum_n (n+1)^2 \mathrm{e}^{-2tn}\Big]^{1/2}$$

which implies assumption (iv) of Lemma 3.3 above. ∎

LEMMA 5.3. *Let $\mathcal{A}$ be a separable C^*-algebra, and $\mathcal{B}$ an hereditary sub-C^*-algebra. Suppose that there exists a strong Feller semigroup on $\mathcal{A}$. Then there exists a strong Feller semigroup on $\mathcal{B}$.*

Proof. Applying Lemma 4.2, to a weakly compact completely positive approximation $\{\Psi_n\}_{n\in\mathbb{N}}$ of the identity of $\mathcal{A}$ and an approximate unit $\{\beta_n\}_{n\in\mathbb{N}}$ for $\mathcal{B}$, one can associate a weakly compact completely positive approximation $\{\mathcal{B} \ni b \to \beta_n^* \Psi_n(b) \beta_n\}_{n\in\mathbb{N}}$ of the identity of $\mathcal{B}$.

LEMMA 5.4. *Let I be a closed ideal of the separable C^*-algebra $\mathcal{A}$, and suppose that there exists a completely positive lifting for the canonical projection from $\mathcal{A}$ onto $\mathcal{A}/I$.*

Then there exists a strong Feller semigroup on $\mathcal{A}$ if and only if there exists a strong Feller semigroup on both C^*-algebras I and $\mathcal{A}/I$.

Proof. Let π be the canonical projection from $\mathcal{A}$ onto $\mathcal{A}/I$, and $\wedge$ a completely positive map from $\mathcal{A}/I$ into $\mathcal{A}$ such that $\pi \circ \wedge = \mathrm{Id}_{\mathcal{A}/I}$.

Let $\{\Psi_n\}_{n\in\mathbb{N}}$ be a weakly compact completely positive approximation of the identity of $\mathcal{A}$. Then Lemma 5.3 provides the existence of a strong Feller semigroup on I, and $\{\pi \circ \Psi_n \circ \wedge\}_{n\in\mathbb{N}}$ is obviously a weakly compact approximation of the identity of $\mathcal{A}/I$.

Conversely, let $\{\Phi_n\}_{n\in\mathbb{N}}$ a weakly compact approximation of the identity of the quotient algebra $\mathcal{A}/I$, $\{\theta_n\}_{n\in\mathbb{N}}$ a weakly compact approximation of the identity of I, and $\{\beta_n\}_{n\in\mathbb{N}}$ an increasing approximate unit for I which is quasi-central in $\mathcal{A}$.

It is not difficult then to extract from the family

$$\{\mathcal{A} \ni a \to \theta_k(\beta_l^{1/2} a \beta_l^{1/2}) + (1-\beta_l)^{1/2} \wedge (\Phi_m(\pi(a)))(1-\beta_l)^{1/2}\}_{k,l,m\in\mathbb{N}}$$

of weakly compact completely positive endomorphisms of $\mathcal{A}$, a sequence which converges norm pointwise to the identity of $\mathcal{A}$. ∎

5.5. REMARKS AND QUESTIONS.

5.5.1. Proposition 5.2 shows that a strong Feller completely positive endomorphism of a C^*-algebra is not always a pointwise norm limit of completely positive endomorphisms with finite rank. Is it a pointwise norm limit of completely bounded endomorphisms with finite rank?

5.5.2. Let I be a closed ideal of a C^*-algebra $\mathcal{A}$ on which there exists a weakly compact completely positive approximation of the identity. Does there always exist a completely positive lifting for the canonical projection from $\mathcal{A}$ onto $\mathcal{A}/I$? (Cf. [7] for sufficient conditions, which are based on a good relationship between the ideal I and a strong Feller semigroup $\{\Phi_t\}$ on $\mathcal{A}$.)

6. Generation of strong Feller semigroups

In this section, we start with a semigroup with generator Δ, $\{\Phi_t = \mathrm{e}^{-t\Delta}\}_{t\in\mathbb{R}_+}$, on a separable C^*-algebra, and we show that $\mathrm{Log}\,(\mathrm{Id}+\Delta)$, as well as Δ^β for any β in $]0,\,1[$, are again generators of a semigroup, which has the strong Feller property as soon as the resolvent maps of Φ have the strong Feller property.

For β in $]0,\,1[$, let us define the constant

$$K_\beta = \left[\int_0^\infty \frac{1}{t+1} t^{-\beta}\,\mathrm{d}t\right]^{-1} = \frac{\sin(\beta\pi)}{\pi}.$$

Lemma 6.1. *For β in $]0,1[$, the formula (cf.* [6]*, or* [4], V.3.11, 3.50*)*

$$(\mathrm{Id}+\Delta)^{-\beta} = K_\beta \int_0^\infty \frac{1}{t+1+\Delta} t^{-\beta}\,\mathrm{d}t$$

defines a completely positive contraction of $\mathcal{A}$, which has the strong Feller property whenever the resolvent maps of Φ have the strong Feller property.

Proof. One can write

$$\int_0^\infty \frac{1}{t+1+\Delta} t^{-\beta}\,\mathrm{d}t = \int_0^\infty\int_0^\infty t^{-\beta}\mathrm{e}^{-s(t+1)}\Phi_s\,\mathrm{d}s\mathrm{d}t,$$

which defines a completely positive endomorphism of $\mathcal{A}$ with norm less than

$$\int_0^\infty\int_0^\infty t^{-\beta}\mathrm{e}^{-s(t+1)}\,\mathrm{d}s\mathrm{d}t = K_\beta^{-1};$$

hence one infers the complete positivity and contractivity of $(\mathrm{Id}+\Delta)^{-\beta}$.

The strong Feller property comes from the fact that we have a norm absolutely convergent integral of strong Feller kernels. ∎

6.2. The strong feller semigroup generated by $\mathrm{Log}(\mathrm{Id}+\Delta)$. The usual properties of functional integrals insure that, for β and γ in $\mathbb{R}_+^*$ such that $\beta+\gamma<1$, one has $(\mathrm{Id}+\Delta)^{-\beta}\circ(\mathrm{Id}+\Delta)^{\gamma} = (\mathrm{Id}+\Delta)^{-\beta+\gamma}$. So, we can define a semigroup $\{(\mathrm{Id}+\Delta)^{-\beta}\}_{\beta\in\mathbb{R}_+}$ by extending the previous family through the formula

$$\forall\beta\in\mathbb{R}_+\quad (\mathrm{Id}+\Delta)^{-\beta} = [(\mathrm{Id}+\Delta)^{-\beta/n}]^n,\quad \forall\,n\in\mathbb{N},\, n>\beta.$$

Define Log (Id + Δ) as the generator of this semigroup, which is a strong Feller semigroup as soon as the resolvent of Φ are strong Feller endomorphisms of $\mathcal{A}$. Notice that the converse implication is true: if $(\mathrm{Id}+\Delta)^{-1/2}$ has the strong Feller property, the same is true for its square $(\mathrm{Id}+\Delta)^{-1}$, and for all the resolvent maps of Φ by Remark 4.3.

6.3. Tentative definition of $\boldsymbol{\Delta^\beta}$. For β in $]0, 1[$, let us define the operator $\underline{\Delta}^\beta$ by the formula

$$\underline{\Delta}^\beta = K_{1-\beta}\int_0^\infty \frac{\Delta}{t+\Delta}t^{\beta-1}\,\mathrm{d}t.$$

(Cf. [4], V.3.50.)

Notice that, for a in $\mathcal{A}$, $\|\frac{\Delta}{t+\Delta}(a)\|$ is smaller than $2\|a\|$ so there is no problem of convergence at the neighbourhood of 0, and the domain of $\underline{\Delta}^\beta$ is the space of a in $\mathcal{A}$ for which $\lim\limits_{T\to\infty}\int_0^T \frac{\Delta}{t+\Delta}(a)t^{\beta-1}\mathrm{d}t$ exists in $\mathcal{A}$.

Notice also that, for a in the domain of Δ, one has $\|\frac{\Delta}{t+\Delta}(a)\| \leqslant \frac{1}{t}\|\Delta a\|$, so that the integral is absolutely convergent:

$$K_{1-\beta}\int_0^\infty \Big\|\frac{\Delta}{t+\Delta}(a)\Big\|t^{\beta-1}\mathrm{d}t < +\infty.$$

In particular, one has $\mathrm{dom}(\Delta)\subset\mathrm{dom}(\underline{\Delta}^\beta)$, which proves that Δ^β has a dense domain.

Proposition 6.4. *The operator* $\underline{\Delta}^\beta$ *is closable and its closure* Δ^β *is the generator of a semigroup.*

If the resolvent maps of Φ *have the strong Feller property, then* $\{\mathrm{e}^{-t\Delta^\beta}\}$ *is a strong Feller semigroup.*

Proof. For ε and T in $\mathbb{R}$, $0<\varepsilon<T$, define $\Delta^\beta_{\varepsilon,T} = K_{1-\beta}\int_\varepsilon^T \frac{\Delta}{t+\Delta}t^{\beta-1}\,\mathrm{d}t$. This is a bounded linear endomorphism of $\mathcal{A}$, which can be written as $\alpha_{\varepsilon,T}\mathrm{Id}-\Psi_{\varepsilon,T}$, where $\alpha_{\varepsilon,T}=\beta^{-1}K_{1-\beta}[T^\beta-\varepsilon^\beta]$ belongs to $\mathbb{R}^*_+$ and $\Psi_{\varepsilon,T}=K_{1-\beta}\int_\varepsilon^T\frac{1}{t+\Delta}t^\beta\,\mathrm{d}t$ is a completely positive endomorphism of $\mathcal{A}$, with norm less than $\alpha_{\varepsilon,T}$. It follows that $\{\Phi_t^{(\beta),\varepsilon,T}=\mathrm{e}^{-t\Delta^\beta_{\varepsilon,T}}\}$ is (cf. 2.2.3) a semigroup of completely positive contractions of $\mathcal{A}$.

For fixed $t\geqslant 0$ and $0<\varepsilon'<\varepsilon<T$, we have

$$\begin{aligned}\|\Phi_t^{(\beta),\varepsilon,T}-\Phi_t^{(\beta),\varepsilon',T}\| &= \|\Phi_t^{(\beta),\varepsilon,T}\circ(\mathrm{Id}-\Phi_t^{(\beta),\varepsilon',\varepsilon})\|\\ &\leqslant \|\mathrm{Id}-\Phi_t^{(\beta),\varepsilon',\varepsilon})\| \leqslant t\|\Delta^\beta_{\varepsilon,\varepsilon'}\| \leqslant 2tK_{1-\beta}\varepsilon^\beta/\beta\end{aligned}$$

which tends to 0 with ε, uniformly in ε' and T.

This implies the esistence of the semigroup $\{\Phi_t^{(\beta),T} = \lim_{\varepsilon\to 0} \Phi_t^{(\beta),\varepsilon,T}\}$, where the limit is a norm limit in the Banach algebra of endomorphisms of $\mathcal{A}$.

For $T < T'$ and a in the domain of $\underline{\Delta}^\beta$, compute

$$\begin{aligned}\|[\Phi_t^{(\beta),T'} - \Phi_t^{(\beta),T}](a)\| &= \|\Phi_t^{(\beta),T} \circ (\mathrm{Id} - \Phi_t^{(\beta),T,T'})(a)\| \\ &\leqslant \|(\mathrm{Id} - \Phi_t^{(\beta),T,T'})(a)\| \leqslant t\|\Delta^\beta_{T,T'}(a)\|\end{aligned}$$

with tends to 0 as T tends to infinity, uniformly in T'.

For any a in the domain of $\underline{\Delta}^\beta$,and by density for any a in $\mathcal{A}$, $\Phi_t^{(\beta)}(a) = \lim_{T\to\infty} \Phi_t^{(\beta),T}(a)$ exists, and we get a one parameter semigroup $\{\Phi_t^{(\beta)}\}_{t\geqslant 0}$ of completely positive contractions of $\mathcal{A}$.

Fix again a in the domain of $\underline{\Delta}^\beta$; one has $\underline{\Delta}^\beta a = \lim_{T\to\infty}\lim_{\varepsilon\to 0} \Delta^\beta_{\varepsilon,T} a$ and can compute

$$\begin{aligned}a - \Phi_t^{(\beta)}(a) &= \lim_{T\to\infty}\lim_{\varepsilon\to 0} a - \Phi_t^{(\beta),\varepsilon,T}(a) = \lim_{T\to\infty}\lim_{\varepsilon\to 0} \int_0^t \Phi_s^{(\beta),\varepsilon,T}(\Delta^\beta_{\varepsilon,T} a) \\ &= \lim_{T\to\infty}\lim_{\varepsilon\to 0} \int_0^t \Phi_s^{(\beta),\varepsilon,T}(\Delta^\beta a) = \int_0^t \Phi_s^{(\beta)}(\Delta^\beta a)\end{aligned}$$

by Lebesgue dominated convergence theorem.

From which we deduce that the generator of the semigroup $\Phi^{(\beta)}$ is an extension of $\underline{\Delta}^\beta$, which implies firstly that it has a dense domain, so that $\Phi^{(\beta)}$ is pointwise norm continuous, and secondly that $\underline{\Delta}^\beta$ is closable.

Let us denote Δ^β the generator of the semigroup $\Phi^{(\beta)}$; the resolvent map $(\mathrm{Id} + \Delta^\beta)^{-1}$ is a completely positive contraction of $\mathcal{A}$ which commutes with all the resolvent maps $\frac{\lambda}{\lambda+\Delta}$ of the semigroup Φ.

Let a belong to the domain of Δ^β and define $b = a + \Delta^\beta a$, so that $a = (\mathrm{Id} + \Delta^\beta)^{-1}(b)$. We have then $\frac{\lambda}{\lambda+\Delta}(a) = (\mathrm{Id} + \Delta^\beta)^{-1}\frac{\lambda}{\lambda+\Delta}(b)$, which means

$$\frac{\lambda}{\lambda+\Delta}(b) = \frac{\lambda}{\lambda+\Delta}(a) + \Delta^\beta \frac{\lambda}{\lambda+\Delta}(a).$$

From $\lim_{\lambda\to\infty} \frac{\lambda}{\lambda+\Delta}(a) = a$ and $\lim_{\lambda\to\infty} \frac{\lambda}{\lambda+\Delta}(b) = b$, we get $\lim_{\lambda\to\infty} \Delta^\beta\frac{\lambda}{\lambda+\Delta}(a) = \Delta^\beta a$. As every $\frac{\lambda}{\lambda+\Delta}(a)$ belongs to the domain of Δ, we deduce that the domain of Δ is a core for Δ^β, and so is the domain of $\underline{\Delta}^\beta$ which is larger; we have proved that Δ^β is the closure of $\underline{\Delta}^\beta$.

We suppose now that all the resolvent maps of the semigroup Φ have the strong Feller property, and we come back to the notations of the beginning of this proof: $\Delta^{\beta}_{\varepsilon,T} = \alpha_{\varepsilon,T}\mathrm{Id} - \Psi_{\varepsilon,T}$, where $\Psi_{\varepsilon,T} = K_{1-\beta}\int_{\varepsilon}^{t} \frac{1}{t+\Delta} t^{\beta}\,\mathrm{d}t$ is a completely positive strong Feller endomorphism of $\mathcal{A}$, as a convergent integral of strong Feller endomorphisms.

Notice that, as above, one can build a semigroup $\{\Phi_t^{(\beta),T,+\infty} = \lim_{T'\to\infty} \Phi_t^{(\beta),T,T'}\}$ (pointwise norm limit) and that we have

$$\begin{aligned}
&\|(\mathrm{Id} - (\alpha_{\varepsilon,T})^{-1}\Psi_{\varepsilon,T}) \circ \Phi_t^{(\beta)}\| \\
&\quad = \|(\mathrm{Id} - (\alpha_{\varepsilon,T})^{-1}\Psi_{\varepsilon,T}) \circ \Phi_t^{(\beta),\varepsilon,T} \circ \Phi_t^{(\beta),\varepsilon} \circ \Phi_t^{(\beta),T,+\infty}\| \\
&\quad \leqslant \|(\mathrm{Id} - (\alpha_{\varepsilon,T})^{-1}\Psi_{\varepsilon,T}) \circ \Phi_t^{(\beta),\varepsilon,T}\| \\
&\quad = \|(\mathrm{Id} - (\alpha_{\varepsilon,T})^{-1}\Psi_{\varepsilon,T}) \circ \mathrm{e}^{t[\alpha_{\varepsilon,T}\mathrm{Id} - \Psi_{\varepsilon,T}]}\| \leqslant K(t\alpha_{\varepsilon,T})^{-1/2}
\end{aligned}$$

by Lemma 2.1.

As $\lim_{T\to\infty}\lim_{\varepsilon\to 0} \alpha_{\varepsilon,T} = +\infty$, we have, for all $t > 0$

$$\Phi_t^{(\beta)} = \lim_{T\to\infty}\lim_{\varepsilon\to 0}(\alpha_{\varepsilon,T})^{-1}\Psi_{\varepsilon,T} \circ \Phi_t^{(\beta)}$$

(norm limit in the Banach algebra of bounded endomorphism of $\mathcal{A}$), which implies the strong Feller property for the semigroup $\Phi^{(\beta)}$. ∎

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