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CORRESPONDENCE OF GROUPOID C^*-ALGEBRAS

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Abstract. Let G_1 and G_2 be topological groupoids. We introduce a notion of correspondence from G_1 to G_2. We show that there exists a correspondence from $C^*_{\rm r}(G_2)$ to $C^*_{\rm r}(G_1)$ if there exists a correspondence from G_1 to G_2. Let f be a homomorphism of G_1 onto G_2. We show that there is a correspondence from G_1 to G_2 if f satisfies certain conditions. Moreover we show that it gives an element of $\mathrm{KK}(C^*_{\rm r}(G_2), C^*_{\rm r}(G_1))$ if f satisfies an additional condition. We study three examples where groupoids are topological spaces, topological groups and transformation groups respectively.

Keywords: *Groupoid, C^*-algebra, correspondence, Hilbert module, Kasparov module,* KK-*group.*

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0. INTRODUCTION

The notion of correspondence was introduced by A. Connes in the theory of von Neumann algebras (cf. [3]). We can define a notion of correspondence between C^*-algebras. In this paper, we introduce a notion of correspondence between groupoids and show that a correspondence between groupoids induces a correspondence between C^*-algebras. If a correspondence between C^*-algebras satisfies an additional condition, then it gives a Kasparov module and an element of the KK-group. We show that if a homomorphism between groupoids satisfies certain conditions, then it gives an element of the KK-group of the associated C^*-algebras.

Let G_1 and G_2 be topological groupoids and let $C^*_{\rm r}(G_1)$ and $C^*_{\rm r}(G_2)$ be their reduced groupoid C^*-algebras respectively. In Section 1, we introduce a notion of correspondence from G_1 to G_2 and show that a correspondence from G_1 to G_2

induces a correspondence from $C^*_{\mathrm{r}}(G_2)$ to $C^*_{\mathrm{r}}(G_1)$ (Theorem 1.4). Let f be a homomorphism of G_1 onto G_2. In general, we cannot construct any homomorphisms between $C^*_{\mathrm{r}}(G_1)$ and $C^*_{\mathrm{r}}(G_2)$. In Section 2, we show that there is a correspondence from G_1 to G_2 if f satisfies certain conditions. It follows from Theorem 1.4 that there is a correspondence from $C^*_{\mathrm{r}}(G_2)$ to $C^*_{\mathrm{r}}(G_1)$. Moreover we show that it gives an element of $\mathrm{KK}(C^*_{\mathrm{r}}(G_2), C^*_{\mathrm{r}}(G_1))$ if f satisfies an additional condition. In Section 3, we study three examples where groupoids are topological spaces, topological groups and transformation groups respectively.

P.S. Muhly, J.N. Renault and D. Williams introduced a notion of equivalence of groupoids in [8]. They showed that if the groupoids are equivalent then the associated C^*-algebras are Morita equivalent ([8], Theorem 2.8). Our definition of correspondence between groupoids is obtained by weakening the conditions in that of equivalence between groupoids introduced by them. The proof of Theorem 1.4 is based on the proof of [8], Theorem 2.8. But we use another trick in the proof of the positivity of the C^*-valued inner product. This trick is useful in later arguments.

Let (V_1, F_1) and (V_2, F_2) be two foliated manifolds and $f : V_1/F_1 \to V_2/F_2$ a K-oriented morphism of quotient spaces. M. Hilsum and G. Skandalis constructed an element $f!$ of $\mathrm{KK}(C^*(V_1, F_1), C^*(V_2, F_2))$ ([5], see also [4]). It is interesting to know the relations between their construction and ours. But it is a problem for further investigation.

1. Correspondence of groupoids

For $i = 1, 2$, let G_i be a second countable locally compact Hausdorff groupoid. We denote by s (resp. r) the source (resp. range) map of G_i. The unit space is denoted by $G_i^{(0)}$. We set $G_{i,x} = s^{-1}(x)$ for $x \in G_i^{(0)}$. We denote by $G_i^{(2)}$ the set of composable pairs. We do not assume that r and s are open, but the existence of the right Haar system implies that these maps are open ([12], I.2.4). Let Z be a second countable locally compact Hausdorff space. We denote by ρ (resp. σ) a continuous map of Z onto $G_1^{(0)}$ (resp. $G_2^{(0)}$). Let $G_1 * Z$ (resp. $Z * G_2$) be the subspace of $G_1 \times Z$ (resp. $Z \times G_2$) consisting of all elements (γ_1, z) (resp. (z, γ_2)) with the property $s(\gamma_1) = \rho(z)$ (resp. $\sigma(z) = r(\gamma_2)$).

Definition 1.1. *A left action of* G_1 on Z is a continuous map $(\gamma, z) \in G_1 * Z \mapsto \gamma \cdot z \in Z$ with the following properties:

(i) $\rho(\gamma \cdot z) = r(\gamma)$ for $(\gamma, z) \in G_1 * Z$;
(ii) $\gamma' \cdot (\gamma \cdot z) = (\gamma'\gamma) \cdot z$ if both sides are defined;
(iii) $\rho(z) \cdot z = z$ for $z \in Z$.

A right action of G_2 on Z is a continuous map $(z, \gamma) \in Z * G_2 \mapsto z \cdot \gamma \in Z$ with properties similar to the above.

We say that the left G_1-space Z is *proper* if the map $(\gamma, z) \in G_1 * Z \mapsto (\gamma \cdot z, z) \in Z \times Z$ is proper, that is, the inverse images of compact sets are compact. The right proper space is defined similarly.

DEFINITION 1.2. Let G_1 and G_2 be a second countable locally compact Hausdorff groupoids and Z a second countable locally compact Hausdorff space. The space Z is *a correspondence from* G_1 to G_2 if it satisfies the following properties:

(i) there exists a left proper action of G_1 on Z such that ρ is an open map;
(ii) there exists a right proper action of G_2 on Z;
(iii) the G_1- and G_2-actions commute;
(iv) the map ρ induces a bijection of Z/G_2 onto $G_1^{(0)}$.

We obtain Definition 1.2 by weakening the conditions in the definition of equivalence between groupoids introduced by Muhly, Renault and Williams ([8], Definition 2.1). Compared with their definition, we do not assume that the G_1- and G_2-actions are free, we do not assume that σ is an open map and, above all, we do not assume that σ induces a bijection of $G_1 \setminus Z$ onto $G_2^{(0)}$. For a subset V of Z, let $[V]_2$ be the saturation of V with respect to the G_2-action, that is, $[V]_2$ is the set of elements $z \cdot \gamma$ of Z with $z \in V$ and $(z, \gamma) \in Z * G_2$. If the condition (iv) of Definition 1.2 is satisfied, then we have $[V]_2 = \rho^{-1}\rho(V)$. Therefore the quotient map $Z \to Z/G_2$ is open if ρ is open. Moreover if the G_2-action is proper, then Z/G_2 is a locally compact Hausdorff space.

Let B be a C^*-algebra. A right Hilbert B-module is a right B-module E with a B-valued inner product such that E is complete with respect to the norm $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$ ([1], 13.1.1). We denote by $\mathcal{L}_B(E)$ the set of bounded adjointable operators on E ([1], 13.2.1) and we denote by $\mathcal{K}_B(E)$ the closure of the linear span of $\{\theta_{\xi,\eta} : \xi, \eta \in E\}$, where $\theta_{\xi,\eta}$ is the element of $\mathcal{L}_B(E)$ defined by $\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta \rangle$ for $\zeta \in E$ ([1], 13.2.3).

DEFINITION 1.3. Let A and B be C^*-algebras. The pair (E, ϕ) is *a correspondence from* A *to* B if it satisfies the following properties:

(i) E is a right Hilbert B-module;
(ii) ϕ is a $*$-homomorphism of A into $\mathcal{L}_B(E)$.

If ϕ is a map of A into $\mathcal{K}_B(E)$, then $(E, \phi, 0)$ is a Kasparov module for trivially graded C^*-algebras (A, B) ([1], 17.1.1) and gives an element $[E]$ of $\mathrm{KK}(A, B)$.

Note that a $*$-homomorphism between C^*-algebras induces a correspondence of C^*-algebras and it gives a Kasparov module ([1], 17.1.2).

For $i = 1, 2$, let λ^i be a right Haar system of G_i. Let $C_{\mathrm{c}}(G_i)$ be the $*$-algebra of continuous functions with compact supports, where the product and the involution are defined as follows:

$$(ab)(\gamma) = \int_{G_i} a(\gamma\gamma'^{-1})b(\gamma')\,\mathrm{d}\lambda^i_{s(\gamma)}(\gamma'),$$
$$a^*(\gamma) = \overline{a(\gamma^{-1})}$$

for $a, b \in C_{\mathrm{c}}(G_i)$ and $\gamma \in G_i$. For $x \in G_i^{(0)}$, we set $H_{i,x} = L^2(G_{i,x}, \lambda^i_x)$. We define a representation $\pi_{i,x}$ of $C_{\mathrm{c}}(G_i)$ on $H_{i,x}$ by

$$(\pi_{i,x}(a)\zeta)(\gamma) = \int_{G_i} a(\gamma\gamma'^{-1})\zeta(\gamma')\,\mathrm{d}\lambda^i_x(\gamma')$$

for $a \in C_{\mathrm{c}}(G_i)$, $\zeta \in H_{i,x}$ and $\gamma \in G_{i,x}$. We define the reduced norm $\|a\|$ by

$$\|a\| = \sup_{x \in G_i^{(0)}} \|\pi_{i,x}(a)\|.$$

The reduced groupoid C^*-algebra $C^*_{\mathrm{r}}(G_i)$ is the completion of $C_{\mathrm{c}}(G_i)$ by the reduced norm (cf. [2]).

THEOREM 1.4. *Let (G_i, λ^i) be a second countable locally compact Hausdorff groupoid with a right Haar system λ^i for $i = 1, 2$ and Z a correspondence from G_1 to G_2. Then there exists a correspondence from $C^*_{\mathrm{r}}(G_2)$ to $C^*_{\mathrm{r}}(G_1)$.*

Proof. We set $\widetilde{A} = C_{\mathrm{c}}(G_1)$, $\widetilde{B} = C_{\mathrm{c}}(G_2)$ and $\widetilde{E} = C_{\mathrm{c}}(Z)$. For $a \in \widetilde{A}$ and $\xi \in \widetilde{E}$, we define a function ξa on Z by

$$(\xi a)(z) = \int_{G_1} \xi(\gamma \cdot z)a(\gamma)\,\mathrm{d}\lambda^1_{\rho(z)}(\gamma) \qquad (z \in Z).$$

As in [8], we can show that $\xi a \in \widetilde{E}$.

For $\xi, \eta \in \widetilde{E}$, $\gamma \in G_1$ and $z \in Z$ with $r(\gamma) = \rho(z)$, we set

$$\langle \xi, \eta \rangle(\gamma) = \int_{G_2} \overline{\xi(z \cdot \gamma'^{-1})}\eta(\gamma^{-1} \cdot z \cdot \gamma'^{-1})\,\mathrm{d}\lambda^2_{\sigma(z)}(\gamma').$$

The above integral exists since the G_2-action is proper. It follows from the condition (iv) of Definition 1.2 that $\langle \xi, \eta \rangle(\gamma)$ is independent of the choice of z. As in

[8], we can show that $\langle \xi, \eta \rangle \in \widetilde{A}$. Let M be the closed subset of $G_1 \times Z$ consisting of elements (γ, z) with the property $r(\gamma) = \rho(z)$. We denote by $S(\gamma, z)$ the integral which defines $\langle \xi, \eta \rangle(\gamma)$. Then S is a continuous function on M. We fix $(\gamma_0, z_0) \in M$. For $\varepsilon > 0$, there exist a neighborhood V of γ_0 and a neighborhood U of z_0 such that $|S(\gamma_0, z_0) - S(\gamma, z)| < \varepsilon$ for every $(\gamma, z) \in M \cap (V \times U)$. Since ρ is open, $\rho(U)$ is a neighborhood of $\rho(z_0)$. Since r is continuous, there exists a neighborhood W of γ_0 such that $r(W) \subset \rho(U)$. Then, for every $\gamma \in W \cap V$, there exists $z \in U$ such that $r(\gamma) = \rho(z)$, that is, $(\gamma, z) \in M \cap (V \times U)$. Since we have $\langle \xi, \eta \rangle(\gamma) = S(\gamma, z)$ for $(\gamma, z) \in M$, this implies the continuity of $\langle \xi, \eta \rangle$. The function $\langle \xi, \eta \rangle$ has compact support since the G_1-action is proper.

Next we show that $\langle \xi, \xi \rangle \geqslant 0$ for every $\xi \in \widetilde{E}$. Since we do not have a $\widetilde{B}$-valued inner product, our proof is different from [8] and [14]. For $x \in G_1^{(0)}$, let X_x be the subset of $G_1 \times G_1 \times Z$ consisting of elements (γ, γ', z) with the property $s(\gamma) = s(\gamma') = \rho(z) = x$. For $\xi \in \widetilde{E}$, we define a function ψ_ξ on X_x by

$$\psi_\xi(\gamma, \gamma', z) = \int_{G_2} \overline{\xi(\gamma \cdot z \cdot \gamma_2^{-1})}\xi(\gamma' \cdot z \cdot \gamma_2^{-1}) \, \mathrm{d}\lambda^2_{\sigma(z)}(\gamma_2).$$

Then ψ_ξ is continuous on X_x since the G_2-action is proper. It follows from the condition (iv) of Definition 1.2 that we have $\psi_\xi(\gamma, \gamma', z) = \psi_\xi(\gamma, \gamma', z')$ for (γ, γ', z), $(\gamma, \gamma', z') \in X_x$. We fix an element $z_0 \in Z$ with $\rho(z_0) = x$. For $\gamma, \gamma' \in G_{1,x}$ and $z \in Z$ with $r(\gamma) = \rho(z)$, we have

$$\langle \xi, \xi \rangle(\gamma \gamma'^{-1}) = \psi_\xi(\gamma, \gamma', \gamma^{-1} \cdot z) = \psi_\xi(\gamma, \gamma', z_0).$$

Then we have, for every $\zeta \in C_{\mathrm{c}}(G_{1,x})$,

$$\begin{aligned}(\pi_{1,x}(\langle \xi, \xi \rangle)\zeta | \zeta) &= \int_{G_1}\int_{G_1} \psi_\xi(\gamma, \gamma', z_0)\zeta(\gamma')\overline{\zeta(\gamma)} \, \mathrm{d}\lambda^1_x(\gamma') \, \mathrm{d}\lambda^1_x(\gamma) \\ &= \int_{G_2} \left| \int_{G_1} \xi(\gamma \cdot z_0 \cdot \gamma_2^{-1})\zeta(\gamma) \, \mathrm{d}\lambda^1_x(\gamma) \right|^2 \mathrm{d}\lambda^2_{\sigma(z_0)}(\gamma_2) \geqslant 0.\end{aligned}$$

Since $C_{\mathrm{c}}(G_{1,x})$ is dense in $H_{1,x}$, we have $\pi_{1,x}(\langle \xi, \xi \rangle) \geqslant 0$ for every $x \in G_1^{(0)}$. Since the field of representations $\{\pi_{1,x} : x \in G_1^{(0)}\}$ is faithful, we have $\langle \xi, \xi \rangle \geqslant 0$.

For $b \in \widetilde{B}$ and $\xi \in \widetilde{E}$, we define a function $b\xi$ on Z by

$$(b\xi)(z) = \int_{G_2} b(\gamma^{-1})\xi(z \cdot \gamma^{-1}) \, \mathrm{d}\lambda^2_{\sigma(z)}(\gamma) \qquad (z \in Z).$$

As in [8], we can show that $b\xi \in \widetilde{E}$.

We will show that $\langle b\xi, b\xi\rangle \leqslant \|b\|^2\langle\xi,\xi\rangle$ for $b \in \widetilde{B}$ and $\xi \in \widetilde{E}$, where $\|b\|$ is the norm of $C^*_{\mathrm{r}}(G_2)$. In [8], they showed this inequality using the results [8], Proposition 2.10 and [13], Proposition 4.2. Here we show the inequality directly. For $x \in G_1^{(0)}$, let X_x and $\psi_{b\xi}$ be as above. We fix an element $z_0 \in Z$ with $\rho(z_0) = x$. It follows from the condition (iv) of Definition 1.2 that we have

$$\langle b\xi, b\xi\rangle(\gamma\gamma'^{-1}) = \psi_{b\xi}(\gamma,\gamma',z_0)$$

for $\gamma,\gamma' \in G_{1,x}$. For $\zeta \in C_{\mathrm{c}}(G_{1,x})$, we define a function $\widetilde{\xi}$ on $G_{2,\sigma(z_0)}$ by

$$\widetilde{\xi}(\gamma_2) = \int_{G_1} \xi(\gamma\cdot z_0\cdot\gamma_2^{-1})\zeta(\gamma)\,\mathrm{d}\lambda^1_x(\gamma) \qquad (\gamma_2 \in G_{2,\sigma(z_0)}).$$

Since the G_2-action is proper, we have $\widetilde{\xi} \in C_{\mathrm{c}}(G_{2,\sigma(z_0)})$. Since we have

$$\int_{G_1} (b\xi)(\gamma\cdot z_0\cdot\gamma_2^{-1})\zeta(\gamma)\,\mathrm{d}\lambda^1_x(\gamma) = (\pi_{2,\sigma(z_0)}(b)\widetilde{\xi})(\gamma_2),$$

it follows that

$$\begin{aligned}(\pi_{1,x}(\langle b\xi, b\xi\rangle)\zeta|\zeta) &= \int_{G_1}\int_{G_1} \psi_{b\xi}(\gamma,\gamma',z_0)\zeta(\gamma')\overline{\zeta(\gamma)}\,\mathrm{d}\lambda^1_x(\gamma')\,\mathrm{d}\lambda^1_x(\gamma)\\ &= \int_{G_2} |(\pi_{2,\sigma(z_0)}(b)\widetilde{\xi})(\gamma_2)|^2\,\mathrm{d}\lambda^2_{\sigma(z_0)}(\gamma_2)\\ &= \|\pi_{2,\sigma(z_0)}(b)\widetilde{\xi}\|^2 \leqslant \|b\|^2\|\widetilde{\xi}\|^2.\end{aligned}$$

By a similar calculation we have $(\pi_{1,x}(\langle\xi,\xi\rangle)\zeta|\zeta) = \|\widetilde{\xi}\|^2$. Since $C_{\mathrm{c}}(G_{1,x})$ is dense in $H_{1,x}$, we have

$$\pi_{1,x}(\langle b\xi, b\xi\rangle) \leqslant \|b\|^2\pi_{1,x}(\langle\xi,\xi\rangle)$$

for every $x \in G_1^{(0)}$. Therefore we have $\langle b\xi, b\xi\rangle \leqslant \|b\|^2\langle\xi,\xi\rangle$.

We denote by E the completion of $\widetilde{E}$ with respect to the norm $\|\xi\| = \|\langle\xi,\xi\rangle\|^{1/2}$. Then E is a right Hilbert $C^*_{\mathrm{r}}(G_1)$-module and a left $C^*_{\mathrm{r}}(G_2)$-module. We define a $*$-homomorphism ϕ of $C^*_{\mathrm{r}}(G_2)$ to $\mathcal{L}_{C^*_{\mathrm{r}}(G_1)}(E)$ by $\phi(b)\xi = b\xi$ for $b \in C^*_{\mathrm{r}}(G_2)$ and $\xi \in E$. Then (E,ϕ) is a correspondence from $C^*_{\mathrm{r}}(G_2)$ to $C^*_{\mathrm{r}}(G_1)$. ∎

2. HOMOMORPHISMS OF GROUPOIDS

Let G_1 and G_2 be as in Section 1 and let f be a continuous homomorphism of G_1 onto G_2. We denote by $f^{(0)}$ the restriction of f to $G_1^{(0)}$, which is a map onto $G_2^{(0)}$. Let H be the kernel of f, that is, the set of all $\gamma \in G_1$ such that $f(\gamma) \in G_2^{(0)}$. Then H is a closed subgroupoid of G_1 and we have $H^{(0)} = G_1^{(0)}$. We have a natural right action of H on G_1. Since H is closed, this action is proper. We define a map $(r,s)_H$ of H into $H^{(0)} \times H^{(0)}$ by $(r,s)_H(\gamma) = (r_H(\gamma), s_H(\gamma))$ for $\gamma \in H$, where r_H and s_H are the range map and the source map of H respectively. In this section we will prove the following theorems.

THEOREM 2.1. *Let G_1 and G_2 be two second countable locally compact Hausdorff groupoids, let f be a continuous homomorphism of G_1 onto G_2 and let H be the kernel of f. Suppose that the following conditions are satisfied:*

(C1) *the quotient map $q : G_1 \to G_1/H$ is an open map;*

(C2) *the map $r : G_1 \to G_1^{(0)}$ is an open map;*

(C3) *the map $(r,s)_H : H \to H^{(0)} \times H^{(0)}$ is a proper map;*

(C4) *$f(G_{1,x}) = G_{2,f(x)}$ for all $x \in G_1^{(0)}$;*

(C5) *f is an open map;*

(C6) *$f^{(0)}$ is locally one-to-one.*

Then G_1/H is a correspondence from G_1 to G_2.

THEOREM 2.2. *Let (G_i, λ^i) be a second countable locally compact Hausdorff groupoid with a right Haar system λ^i for $i = 1,2$ and let f be a continuous homomorphism of G_1 onto G_2. Suppose that the conditions* (C1), (C3)–(C6) *and the following condition are satisfied:*

(C7) *$f^{(0)}$ is a proper map.*

Then there exists a correspondence (E, ϕ) from $C_{\mathrm{r}}^*(G_2)$ to $C_{\mathrm{r}}^*(G_1)$ such that the range of ϕ is contained in $\mathcal{K}_{C_{\mathrm{r}}^*(G_1)}(E)$. Therefore $(E, \phi, 0)$ is a Kasparov module for $(C_{\mathrm{r}}^*(G_2), C_{\mathrm{r}}^*(G_1))$ and it gives an element of $\mathrm{KK}(C_{\mathrm{r}}^*(G_2), C_{\mathrm{r}}^*(G_1))$.

P.S. Muhly and D.P. Williams introduced the notion of a proper groupoid for a principal groupoid in [9]. If H is principal, then it is a proper groupoid if and only if $(r,s)_H$ is proper ([9], Lemma 2.1). Here we do not assume that H is principal and we do not use the term "proper groupoid".

First, we will prove Theorem 2.1. We assume that the conditions (C1) and (C2) are satisfied. Set $Z = G_1/H$. We define a map ρ of Z onto $G_1^{(0)}$ by $\rho(q(\gamma)) = r(\gamma)$ for $\gamma \in G_1$ and a map σ of Z onto $G_2^{(0)}$ by $\sigma(q(\gamma)) = s(f(\gamma))$ for $\gamma \in G_1$. These mappings are well-defined. The maps ρ and σ are continuous and ρ is an open map by (C2). Let $G_1 * Z$ and $Z * G_2$ be the sets defined in Section 1. Define a left action of G_1 on Z by $\gamma \cdot q(\gamma') = q(\gamma\gamma')$ for $(\gamma, q(\gamma')) \in G_1 * Z$. This is well-defined and the action is continuous by (C1). Then we have the following proposition.

PROPOSITION 2.3. *Suppose that the condition* (C3) *is satisfied. Then the left action of* G_1 *on* Z*, defined above, is proper.*

Proof. Let α_1 be the map of $G_1 * Z$ into $Z \times Z$ defined by $\alpha_1(\gamma, z) = (\gamma \cdot z, z)$. Let K_1 and K_2 be compact subsets of Z. We will show that $\alpha_1^{-1}(K_1 \times K_2)$ is compact. It follows from the condition (C1) that, for $i = 1, 2$, there exists a compact subset K'_i of G_1 such that $q(K'_i) = K_i$. Let X be the closed subset of $G_1 \times H \times G_1$ consisting of elements $(\gamma_1, \gamma_2, \gamma_3)$ with the properties $s(\gamma_1) = r_H(\gamma_2)$ and $s_H(\gamma_2) = r(\gamma_3)$. We define a map ψ of X into G_1 by $\psi(\gamma_1, \gamma_2, \gamma_3) = \gamma_1\gamma_2\gamma_3$. Let (γ, z) be an element of $G_1 * Z$ such that $\alpha_1(\gamma, z) \in K_1 \times K_2$. Then there exist elements γ_1 of K'_1 and γ_2 of K'_2 such that $q(\gamma_1) = \gamma \cdot z$ and $q(\gamma_2) = z$. Since we have $q(\gamma_1) = \gamma \cdot z = q(\gamma\gamma_2)$, there exists an element γ_0 of H such that $\gamma_1\gamma_0 = \gamma\gamma_2$. Then $(\gamma_1, \gamma_0, \gamma_2^{-1})$ is an element of $(K'_1 \times H \times K_2'^{-1}) \cap X$ and we have $\psi(\gamma_1, \gamma_0, \gamma_2^{-1}) = \gamma$. We set $C_i = s(K'_i)$ for $i = 1, 2$, which is a compact subset of $G_1^{(0)} = H^{(0)}$. Then γ_0 is an element of $(r, s)_H^{-1}(C_1 \times C_2)$. We set $C = (r, s)_H^{-1}(C_1 \times C_2)$, which is a compact set by (C3). We set $K' = (K'_1 \times C \times K_2'^{-1}) \cap X$, which is also a compact set. Therefore $\psi(K')$ is a compact set and (γ, z) is an element of $\psi(K') \times K_2$. Thus we have proved that $\alpha_1^{-1}(K_1 \times K_2)$ is a subset of $\psi(K') \times K_2$. This completes the proof of the proposition. ∎

We will define the right action of G_2 on Z. To do this, we assume that the condition (C4) is satisfied. Let (z, γ_2) be an element of $Z * G_2$ and γ_1 an element of G_1 such that $z = q(\gamma_1)$. Since we have $s(f(\gamma_1)) = r(\gamma_2)$, there exists the product $f(\gamma_1)\gamma_2$. By (C4), there exists $\gamma \in G_1$ such that $f(\gamma) = f(\gamma_1)\gamma_2$ and $r(\gamma) = \rho(z)$. We define $z \cdot \gamma_2$ to be $q(\gamma)$. We will show that this is well-defined. Let γ'_1 be an element of G_1 such that $z = q(\gamma'_1)$ and γ' an element of G_1 such that $f(\gamma') = f(\gamma'_1)\gamma_2$ and $r(\gamma') = \rho(z)$. There exists an element $\gamma'' \in H$ such that $\gamma_1 = \gamma'_1\gamma''$ and we have $f(\gamma_1) = f(\gamma'_1)$. Thus we have $f(\gamma') = f(\gamma)$. On the other hand, there exists an element $\gamma_0 = \gamma^{-1}\gamma'$. Since f is a homomorphism, $f(\gamma_0)$ is an element of $G_2^{(0)}$ and γ_0 is an element of H. This implies that $q(\gamma) = q(\gamma')$. Therefore $z \cdot \gamma_2$ is well-defined.

PROPOSITION 2.4. *Suppose that the conditions* (C5) *and* (C6) *are satisfied. Then the map* $(z, \gamma) \in Z * G_2 \to z \cdot \gamma \in Z$ *is a proper action of* G_2 *on* Z.

Proof. We will show that the above map is continuous. We fix an element (z, γ_2) of $Z * G_2$. Let V be an open neighborhood of $z \cdot \gamma_2$ in Z. By (C6), we may assume that $f^{(0)}$ is one-to-one on $\rho(V)$. Let γ_1 be an element of G_1 such that $z = q(\gamma_1)$ and γ an element of G_1 such that $f(\gamma) = f(\gamma_1)\gamma_2$ and $r(\gamma) = \rho(z)$. Then we have $z \cdot \gamma_2 = q(\gamma)$. By (C5), $f(q^{-1}(V))$ is an open neighborhood of $f(\gamma)$ in G_2. We denote by β_2 the map of $G_2^{(2)}$ into G_2 defined by $\beta_2(\gamma', \gamma'') = \gamma'\gamma''$. Since β_2 is continuous, there exist an open neighborhood W' of $f(\gamma_1)$ and an open neighborhood U of γ_2 such that β_2 maps $(W' \times U) \cap G_2^{(2)}$ into $f(q^{-1}(V))$. We denote by W'' the intersection of $f^{-1}(W')$ and $r^{-1}(r(q^{-1}(V)))$, which is an open neighborhood of γ_1 by (C2). Set $W = q(W'')$, which is an open neighborhood of z by (C1). Let (z', γ_2') be an element of the intersection of $W \times U$ and $Z * G_2$ and γ_1' an element of W'' such that $z' = q(\gamma_1')$. Since $(f(\gamma_1'), \gamma_2')$ is an element of the intersection of $W' \times U$ and $G_2^{(2)}$, $f(\gamma_1')\gamma_2'$ belongs to $f(q^{-1}(V))$. Therefore there exists an element γ' of $q^{-1}(V)$ such that $f(\gamma') = f(\gamma_1')\gamma_2'$. We have $f^{(0)}(r(\gamma')) = f^{(0)}(r(\gamma_1'))$ and $r(\gamma')$ and $r(\gamma_1')$ belong to $\rho(V)$. By (C6), we have $r(\gamma') = r(\gamma_1') = \rho(z')$. Thus we have $z' \cdot \gamma_2' = q(\gamma') \in V$. This implies that the map is continuous.

Let α_2 be the map of $Z * G_2$ into $Z \times Z$ defined by $\alpha_2(z, \gamma) = (z, z \cdot \gamma)$. We will show that α_2 is proper. We define a map Φ of Z onto G_2 by $\Phi(q(\gamma)) = f(\gamma)$. Since we have $\Phi^{-1}(V) = q(f^{-1}(V))$ for every subset V of G_2, Φ is continuous by (C1). Set $Z^x = \rho^{-1}(x)$ for $x \in G_1^{(0)}$. Then Φ is one-to-one on Z^x. Let Y be the closed subset of $Z \times Z$ consisting of elements (z_1, z_2) with the property $\rho(z_1) = \rho(z_2)$. For $(z_1, z_2) \in Y$, the product $\Phi(z_1)^{-1}\Phi(z_2)$ is defined and $(z_1, \Phi(z_1)^{-1}\Phi(z_2))$ is an element of $Z * G_2$. Define a map Ψ of Y into $Z * G_2$ by $\Psi(z_1, z_2) = (z_1, \Phi(z_1)^{-1}\Phi(z_2))$. Since Φ is continuous, Ψ is also continuous. Since we have $\Phi(z \cdot \gamma) = \Phi(z)\gamma$, $\Psi \circ \alpha_2$ is the identity map. For $(z_1, z_2) \in Y$, set $\gamma = \Phi(z_1)^{-1}\Phi(z_2)$. Then we have $\Phi(z_1 \cdot \gamma) = \Phi(z_2)$. Since Φ is one-to-one on $Z^{\rho(z_1)}$, we have $z_1 \cdot \gamma = z_2$. Therefore $\alpha_2 \circ \Psi$ is the identity map. Thus we have proved that $\alpha_2^{-1} = \Psi$. Hence α_2 is a homeomorphism and it is a proper map. ∎

Proof of Theorem 2.1. It is clear that the G_1- and the G_2-actions commute. Let $z = q(\gamma)$ and $z' = q(\gamma')$ be elements of Z such that $\rho(z) = \rho(z')$. Then we have $z' = z \cdot \gamma_2$ for $\gamma_2 = f(\gamma^{-1}\gamma')$. This implies that ρ induces a bijection of Z/G_2 onto $G_1^{(0)}$. By virtue of Propositions 2.3 and 2.4, Z is a correspondence from G_1 to G_2. ∎

Since α_2 is one-to-one from the proof of Proposition 2.4, the action of G_2 on Z is free, that is, $z \cdot \gamma_2 = z$ implies that $\gamma_2 = \sigma(z)$. Therefore $\rho : Z \to G_1^{(0)}$ is a principal fibration with structure groupoid G_2 (cf. [5]).

Second, we will prove Theorem 2.2. Since G_i has a Haar system, the range and the source maps are open. In particular, the condition (C2) is satisfied. Therefore G_1/H is a correspondence from G_1 to G_2 by Theorem 2.1 and there exists a correspondence from $C_r^*(G_2)$ to $C_r^*(G_1)$ by Theorem 1.4. Let (E, ϕ) be the correspondence constructed in the proof of Theorem 1.4 from $Z = G_1/H$. Let q_1 be the quotient map of Z onto $G_1 \setminus Z$. Since q and s are open, q_1 is open. Then $G_1 \setminus Z$ is a locally compact Hausdorff space since the G_1-action is proper. We define a continuous map $\widetilde{\sigma}$ of $G_1 \setminus Z$ onto $G_2^{(0)}$ by $\widetilde{\sigma}(q_1(z)) = \sigma(z)$. Let Ω be the subspace of $(G_1 \setminus Z) \times G_2$ consisting of all elements (w, γ) with the property $\widetilde{\sigma}(w) = r(\gamma)$.

Lemma 2.5. *Suppose that the condition* (C7) *is satisfied. Then $\widetilde{\sigma}$ is a proper map.*

Proof. Let R_H be the image of $(r,s)_H$. Then R_H is an equivalence relation of $G_1^{(0)}$. We denote by q_H the quotient map of $G_1^{(0)}$ onto $G_1^{(0)}/R_H$. We define a map φ of $G_1 \setminus Z$ onto $G_1^{(0)}/R_H$ by $\varphi(q_1 \circ q(\gamma)) = q_H \circ s(\gamma)$. Then φ is a homeomorphism. We define a continuous map $\widetilde{f}^{(0)}$ of $G_1^{(0)}/R_H$ onto $G_2^{(0)}$ by $\widetilde{f}^{(0)}(q_H(u)) = f^{(0)}(u)$. Then $\widetilde{f}^{(0)}$ is a proper map by (C7). Therefore $\widetilde{\sigma}$ is proper since $\widetilde{f}^{(0)} \circ \varphi = \widetilde{\sigma}$. ∎

We set $\widetilde{B} = C_{\mathrm{c}}(G_2)$ and $\widetilde{E} = C_{\mathrm{c}}(Z)$ as in the proof of Theorem 1.4. For $\xi, \eta \in \widetilde{E}$, define a function $\omega(\xi,\eta)$ on Ω by

$$\omega(\xi,\eta)(q_1(z),\gamma_2) = \int_{G_1} \xi(\gamma_1 \cdot z)\overline{\eta(\gamma_1 \cdot z \cdot \gamma_2)}\, \mathrm{d}\lambda^1_{\rho(z)}(\gamma_1)$$

for $(z,\gamma_2) \in Z * G_2$. Then $\omega(\xi,\eta)$ is an element of $C_{\mathrm{c}}(\Omega)$ since the G_1- and the G_2-actions are proper.

Lemma 2.6. *For $\xi, \eta, \zeta \in \widetilde{E}$ and $b \in \widetilde{B}$, the following equations hold:*

$$(\theta_{\xi,\eta}\zeta)(z) = \int_{G_2} \omega(\xi,\eta)(q_1(z),\gamma_2^{-1})\zeta(z \cdot \gamma_2^{-1})\, \mathrm{d}\lambda^2_{\sigma(z)}(\gamma_2),$$

$$\omega(b\xi,\eta)(q_1(z),\gamma_2) = \int_{G_2} b(\gamma_2\gamma^{-1})\omega(\xi,\eta)(q_1(z \cdot (\gamma_2\gamma^{-1})),\gamma)\, \mathrm{d}\lambda^2_{s(\gamma_2)}(\gamma).$$

Proof. Let $X_{\rho(z)}$ be the set defined as in the proof of Theorem 1.4. We define a continuous function $\psi_{\eta,\zeta}$ on $X_{\rho(z)}$ by

$$\psi_{\eta,\zeta}(\gamma,\gamma',z') = \int_{G_2} \overline{\eta(\gamma \cdot z' \cdot \gamma_2^{-1})}\zeta(\gamma' \cdot z' \cdot \gamma_2^{-1})\, \mathrm{d}\lambda^2_{\sigma(z')}(\gamma_2).$$

For $\gamma_1 \in G_{1,\rho(z)}$ and $z_0 \in Z$ with $r(\gamma_1) = \rho(z)$, we have, as in the proof of Theorem 1.4,

$$\langle \eta,\zeta\rangle(\gamma_1) = \psi_{\eta,\zeta}(\gamma_1,\rho(z),\gamma_1^{-1} \cdot z_0) = \psi_{\eta,\zeta}(\gamma_1,\rho(z),z).$$

Since we have

$$(\theta_{\xi,\eta}\zeta)(z) = \int_{G_1} \xi(\gamma_1 \cdot z)\langle \eta, \zeta\rangle(\gamma_1)\, \mathrm{d}\lambda^1_{\rho(z)}(\gamma_1),$$

the first equation follows. The second equation follows from a direct computation. ∎

Proposition 2.7. *Suppose that the condition* (C7) *is satisfied. Let b be an element of $\widetilde{B}$ and let C be a compact subset of G_2 such that the interior of C contains the support of b. For every $\varepsilon > 0$, there exist positive elements ξ_i and η_i of $\widetilde{E}$ $(i = 1, \ldots, n)$ such that*

$$\Big| \sum_{i=1}^{n} \omega(b\xi_i, \eta_i)(w, \gamma_2) - b(\gamma_2) \Big| < \varepsilon$$

for all $(w, \gamma_2) \in \Omega$ and $\sum_{i=1}^{n} \omega(b\xi_i, \eta_i)(w, \gamma_2) = 0$ if $\gamma_2 \notin C$.

Proof. Set $K = s(C)$. Let U be a relatively compact open subset of G_2 such that $K \subset U$. Since $\widetilde{\sigma}$ is proper by Lemma 2.5, $\widetilde{\sigma}^{-1}(K)$ and $\widetilde{\sigma}^{-1}(r(\overline{U}))$ are compact. Since q_1 and r are open, there exist a relatively compact open set U_0 in Z and a compact subset K_0 of U_0 such that $q_1(U_0) = \widetilde{\sigma}^{-1}(r(U))$ and $q_1(K_0) = \widetilde{\sigma}^{-1}(K)$. We denote by $\widetilde{K}$ the intersection of $K_0 \times K$ and $Z * G_2$ and by $\widetilde{U}$ the intersection of $U_0 \times U$ and $Z * G_2$. Let α_2 be the homeomorphism of $Z * G_2$ onto Y defined in the proof of Proposition 2.4. Since $\alpha_2(\widetilde{K})$ is a closed subset of the diagonal of $Z \times Z$, there exist non-negative elements $\xi_1, \ldots, \xi_n$ of $C_{\mathrm{c}}(Z)$ such that if we define an element φ of $C_{\mathrm{c}}(Y)$ by $\varphi(z_1, z_2) = \sum_{i=1}^{n} \xi_i(z_1)\xi_i(z_2)$, then the support of φ is contained in $\alpha_2(\widetilde{U})$ and φ is positive on $\alpha_2(\widetilde{K})$. Set $\kappa_0 = \sum_{i=1}^{n} \omega(\xi_i, \xi_i)$. Then we have $\kappa_0(w, \widetilde{\sigma}(w)) > 0$ if $\widetilde{\sigma}(w) \in K$ and $\kappa_0(w, \gamma) = 0$ if $\gamma \notin U$. Define a continuous function h on Z by

$$h(z) = \int_{G_2} \kappa_0(q_1(z \cdot \gamma^{-1}), \gamma)\, \mathrm{d}\lambda^2_{\sigma(z)}(\gamma).$$

Then there exists a continuous function $\widetilde{h}$ on $G_1 \setminus Z$ such that $h = \widetilde{h} \circ q_1$. Note that $\widetilde{h}$ is positive on $\widetilde{\sigma}^{-1}(K)$ and zero outside $\widetilde{\sigma}^{-1}(s(U))$. Let $\widetilde{k}$ be a non-negative continuous function on $G_1 \setminus Z$ such that $\widetilde{k} = \widetilde{h}^{-1}$ on $\widetilde{\sigma}^{-1}(K)$. Define an element η_i of $C_{\mathrm{c}}(Z)$ by $\eta_i(z) = \widetilde{k}(q_1(z))\xi_i(z)$. We set $\kappa = \sum_{i=1}^{n} \omega(\xi_i, \eta_i)$. Since $\kappa(q_1(z), \gamma) = \widetilde{k}(q_1(z \cdot \gamma))\kappa_0(q_1(z), \gamma)$, $\int \kappa(q_1(z \cdot \gamma^{-1}), \gamma)\, \mathrm{d}\lambda^2_{\sigma(z)}(\gamma)$ is one if $\sigma(z) \in K$ and zero outside $s(U)$.

Set $\kappa_b = \sum_{i=1}^{n} \omega(b\xi_i, \eta_i)$. Choose U so small that it has the following property: $|b(\gamma_2\gamma^{-1}) - b(\gamma_2)| < \varepsilon$ for $\gamma_2 \in C$ and $\gamma \in U$ with $s(\gamma_2) = s(\gamma)$. It follows from Lemma 2.6 that we have, for $(z, \gamma_2) \in Z * G_2$ and $\gamma_2 \in C$,

$$|\kappa_b(q_1(z), \gamma_2) - b(\gamma_2)| \leqslant \int_{G_2} |b(\gamma_2\gamma^{-1}) - b(\gamma_2)| \kappa(q_1(z \cdot (\gamma_2\gamma^{-1})), \gamma) \, \mathrm{d}\lambda^2_{s(\gamma_2)}(\gamma) < \varepsilon.$$

Denote by D the support of b and by V the interior of C. Let β_2 be the map of $G_2^{(2)}$ onto G_2 defined by $\beta_2(\gamma', \gamma'') = \gamma'\gamma''$ and Q the set defined by $Q = \beta_2((D \times U) \cap G_2^{(2)})$. Then $\kappa_b(q_1(z), \gamma_2) \neq 0$ implies that $\gamma_2 \in Q$. We choose U so small that $Q \subset V$. Then we have $\kappa_b(w, \gamma_2) = 0$ if $\gamma_2 \notin C$. ∎

Proof of Theorem 2.2. Let b and C be as in Proposition 2.7. For $\varepsilon > 0$, let ξ_i and η_i be elements which satisfy the condition of Proposition 2.7. For $\zeta \in \widetilde{E}$, we set

$$g = \sum_{i=1}^{n} \theta_{b\xi_i, \eta_i}\zeta - b\zeta.$$

For $x \in G_1^{(0)}$ and $\delta \in C_c(G_{1,x})$, we will calculate $(\pi_{1,x}(\langle g, g\rangle)\delta|\delta)$. Fix an element $z_0 \in Z$ with $\rho(z_0) = x$. As in the proof of Theorem 1.4, we have

$$(\pi_{1,x}(\langle g, g\rangle)\delta|\delta) = \int_{G_2} \left| \int_{G_1} g(\gamma \cdot z_0 \cdot \gamma_2^{-1})\delta(\gamma) \, \mathrm{d}\lambda^1_x(\gamma) \right|^2 \mathrm{d}\lambda^2_{\sigma(z_0)}(\gamma_2).$$

By Lemma 2.6, we have

$$g(\gamma \cdot z_0 \cdot \gamma_2^{-1}) = \int_{G_2} (\kappa_b(q_1(z_0 \cdot \gamma_2^{-1}), {\gamma'_2}^{-1}) - b({\gamma'_2}^{-1}))\zeta(\gamma \cdot z_0 \cdot (\gamma'_2\gamma_2)^{-1}) \, \mathrm{d}\lambda^2_{r(\gamma_2)}(\gamma'_2).$$

Define an element $\widetilde{\zeta}$ of $C_c(G_{2,\sigma(z_0)})$ by

$$\widetilde{\zeta}(\gamma_2) = \int_{G_1} \zeta(\gamma \cdot z_0 \cdot \gamma_2^{-1})\delta(\gamma) \, \mathrm{d}\lambda^1_x(\gamma).$$

By Proposition 2.7, we have

$$(\pi_{1,x}(\langle g, g\rangle)\delta|\delta) \leqslant \varepsilon^2 \int_{G_2} \left(\int_{G_2} \chi_C({\gamma'_2}^{-1}) |\widetilde{\zeta}(\gamma'_2\gamma_2)| \, \mathrm{d}\lambda^2_{r(\gamma_2)}(\gamma'_2) \right)^2 \mathrm{d}\lambda^2_{\sigma(z_0)}(\gamma_2),$$

where χ_C is the characteristic function of C. There exists a constant M such that $\int \chi_C(\gamma)\,\mathrm{d}\lambda^2_x(\gamma)$ and $\int \chi_C(\gamma^{-1})\,\mathrm{d}\lambda^2_x(\gamma)$ are smaller than M for every $x \in G_2^{(0)}$. Then we have

$$\begin{aligned}(\pi_{1,x}(\langle g,g\rangle)\delta|\delta) &\leqslant \varepsilon^2 M \int\limits_{G_2}\int\limits_{G_2} \chi_C(\gamma_2\gamma_3^{-1})|\widetilde{\zeta}(\gamma_3)|^2\,\mathrm{d}\lambda^2_{\sigma(z_0)}(\gamma_3)\,\mathrm{d}\lambda^2_{\sigma(z_0)}(\gamma_2)\\ &= \varepsilon^2 M \int\limits_{G_2}\Big(|\widetilde{\zeta}(\gamma_3)|^2 \int\limits_{G_2} \chi_C(\gamma_4)\,\mathrm{d}\lambda^2_{r(\gamma_3)}(\gamma_4)\Big)\,\mathrm{d}\lambda^2_{\sigma(z_0)}(\gamma_3)\\ &\leqslant \varepsilon^2 M^2 \|\widetilde{\zeta}\|^2.\end{aligned}$$

Since $\|\widetilde{\zeta}\|^2 = (\pi_{1,x}(\langle\zeta,\zeta\rangle)\delta|\delta)$, we have $\langle g,g\rangle \leqslant \varepsilon^2 M^2\langle\zeta,\zeta\rangle$. Therefore we have $\Big\|\sum\limits_{i=1}^{n}\theta_{b\xi_i,\eta_i} - \phi(b)\Big\| \leqslant \varepsilon M$. This implies that $\phi(b)$ is an element of $\mathcal{K}_{C^*_{\mathrm{r}}(G_1)}(E)$. ∎

3. EXAMPLES

Let G_i $(i = 1,2)$, f and H be as in Theorem 2.1. Suppose that they satisfy the conditions (C1)–(C6). Set $Z = G_1/H$. Denote by λ_i a right Haar system of G_i. It follows from Theorems 1.4 and 2.1 that we have a correspondence from $C^*_{\mathrm{r}}(G_2)$ to $C^*_{\mathrm{r}}(G_1)$. Denote by (E,ϕ) the correspondence constructed in the proof of Theorem 1.4. If the condition (C7) is satisfied, then $(E,\phi,0)$ is a Kasparov module and gives an element of $\mathrm{KK}(C^*_{\mathrm{r}}(G_2), C^*_{\mathrm{r}}(G_1))$ by Theorem 2.2. In this section, we will study three examples where groupoids G_i are topological spaces, topological groups and transformation groups, respectively.

(i) *Topological spaces.* Let X_i be a topological space and suppose that $G_i = X_i$. Then f is a continuous map of X_1 onto X_2 and $C^*_{\mathrm{r}}(G_i)$ is the C^*-algebra $C_0(X_i)$ of continuous functions on X_i vanishing at infinity. Note that $f = f^{(0)}$, $H = X_1$ and $X_1/H = X_1$. We have $E = C_0(X_1)$ and it is naturally a right Hilbert $C_0(X_1)$-module. Then ϕ is a $*$-homomorphism of $C_0(X_2)$ into the multiplier algebra $M(C_0(X_1))$ of $C_0(X_1)$ defined by $\phi(b) = b(f(x))$ for $b \in C_0(X_2)$ and $x \in X_1$. If (C7) is satisfied, then f is a proper map and ϕ maps $C_0(X_2)$ into $C_0(X_1)$.

(ii) *Topological groups.* Let Γ_i be a topological group and suppose that $G_i = \Gamma_i$. Then f is a homomorphism of Γ_1 onto Γ_2 and H is the kernel of f. By (C5), f is an open map. Therefore Γ_1/H is isomorphic to Γ_2 as topological groups. We identify Γ_1/H and Γ_2. Then f is the quotient map of Γ_1 onto Γ_1/H. Since $G_i^{(0)} = \{e\}$, $f^{(0)}$ is a trivial map and (C7) is always satisfied. Note also that

H is a compact group by (C3). Let ν_i be a right Haar measure on Γ_i. Set $\lambda_i = \nu_i$. Let Δ_i be the modular function of Γ_i and let ν_0 be a Haar measure of H. We may suppose that ν_i and ν_0 satisfy the following equation:

$$\int_{\Gamma_2}\int_{H} a(gh)\Delta_2(\dot{g})\,\mathrm{d}\nu_0(h)\,\mathrm{d}\nu_2(\dot{g}) = \int_{\Gamma_1} a(g)\Delta_1(g)\,\mathrm{d}\nu_1(g)$$

for $a \in C_{\mathrm{c}}(\Gamma_1)$.

Set $\pi_i = \pi_{i,e}$, where $\pi_{i,e}$ is the representation of $C^*_{\mathrm{r}}(\Gamma_i)$ on $H_{i,e} = L^2(\Gamma_i, \nu_i)$. We define an anti$*$-automorphism $a \mapsto \check{a}$ of $C^*_{\mathrm{r}}(\Gamma_i)$ by $\check{a}(g) = a(g^{-1})$ for $a \in C_{\mathrm{c}}(\Gamma_i)$. Note that $Z = \Gamma_1/H = \Gamma_2$. Since H is compact, $\xi \circ f$ is an element of $C_{\mathrm{c}}(\Gamma_1)$ for every $\xi \in C_{\mathrm{c}}(Z)$. By a calculation as in the proof of Theorem 1.4, we have

$$\|\pi_1((\xi \circ f)^\vee)\zeta\|^2 = \nu_0(H)(\pi_1(\langle \xi, \xi\rangle)\zeta|\zeta)$$

for $\xi \in C_{\mathrm{c}}(Z)$ and $\zeta \in C_{\mathrm{c}}(\Gamma_1)$. This implies that $\|\xi \circ f\|_{C^*_{\mathrm{r}}(\Gamma_1)} = \nu_0(H)^{1/2}\|\xi\|_E$ for $\xi \in C_{\mathrm{c}}(Z)$. Therefore there exists a unique linear map $f_* : E \to C^*_{\mathrm{r}}(\Gamma_1)$ such that $f_*(\xi) = \xi \circ f$ for $\xi \in C_{\mathrm{c}}(Z)$. Then we have $\|f_*(\xi)\| = \nu_0(H)^{1/2}\|\xi\|_E$. We have, for $\xi, \eta \in E$, $a \in C^*_{\mathrm{r}}(\Gamma_1)$ and $b \in C_{\mathrm{c}}(\Gamma_2)$,

$$\begin{aligned} f_*(\xi a) &= \check{a}\, f_*(\xi) \\ \langle \xi, \eta\rangle &= \nu_0(H)^{-1}(f_*(\eta)f_*(\xi)^*)^\vee \\ f_*(b\xi) &= \nu_0(H)^{-1}f_*(\xi)f_*(b)^\vee. \end{aligned}$$

The last equation does not hold for every $b \in C^*_{\mathrm{r}}(\Gamma_2)$ since we cannot define $f_*(b)$ if b does not belong to $C_{\mathrm{c}}(\Gamma_2)$.

(iii) *Transformation groups.* Let Γ_i be a topological group and X_i a right Γ_i-space. Define $G_i = X_i \times \Gamma_i$. The groupoid structure of G_i is defined as follows: $r(x,g) = x$, $s(x,g) = xg$ and $(x,g)(xg,g') = (x,gg')$, where we identify $G_i^{(0)}$ with X_i. Moreover, suppose that there exist a map $f^{(0)}$ of X_1 onto X_2 and a homomorphism φ of Γ_1 onto Γ_2 such that $f(x,g) = (f^{(0)}(x), \varphi(g))$ and $f^{(0)}(xg) = f^{(0)}(x)\varphi(g)$. By (C5), $f^{(0)}$ and φ are open maps. Let Ξ be the kernel of φ. We identify Γ_1/Ξ with Γ_2. Then φ is the quotient map. We have $H = X_1 \times \Xi$ and $Z = X_1 \times \Gamma_2$. The condition (C3) is satisfied if and only if the Ξ-action is proper. The map $\rho : Z \to X_1$ is defined by $\rho(x, g_2) = x$, and the G_1-action on Z is defined by $(xg_1^{-1}, g_1)\cdot(x, g_2) = (xg_1^{-1}, g_1\cdot g_2)$ for $(xg_1^{-1}, g_1) \in G_1$ and $(x, g_2) \in Z$. The map $\sigma : Z \to X_2$ is defined by $\sigma(x, g_2) = f^{(0)}(x)g_2$, and the G_2-action on Z is defined by $(x, g_2)\cdot(f^{(0)}(x)g_2, g_3) = (x, g_2g_3)$ for $(x, g_2) \in Z$ and $(f^{(0)}(x)g_2, g_3) \in G_2$.

Let ν_i be a right Haar measure on Γ_i and Δ_i the modular function of Γ_i. Set $\widetilde{\nu}_i = \Delta_i \nu_i$; this is a left Haar measure on Γ_i. In this example, we may choose ν_1 and ν_2 independently. The right Haar system λ^i is given by the formula:

$$\int_{G_i} a(\gamma)\,\mathrm{d}\lambda^i_x(\gamma) = \int_{\Gamma_i} a(xg^{-1}, g)\,\mathrm{d}\nu_i(g)$$

for $a \in C_{\mathrm{c}}(G_i)$. Let μ_i be a positive Radon measure on X_i such that the support of μ_i is X_i. Define a measure m_i on G_i by

$$m_i = \int_{X_i} \lambda^i_x \,\mathrm{d}\mu_i(x).$$

Then we have

$$L^2(G_i, m_i) = \int_{X_i}^{\oplus} L^2(G_{i,x}, \lambda^i_x)\,\mathrm{d}\mu_i(x).$$

Define a faithful representation π_i of $C^*_{\mathrm{r}}(G_i)$ on $L^2(G_i, m_i)$ by

$$\pi_i = \int_{X_i}^{\oplus} \pi_{i,x}\,\mathrm{d}\mu_i(x).$$

Denote by $\|\cdot\|_{L^2(Z)}$ the norm of $L^2(Z, \mu_1 \times \widetilde{\nu}_2)$. Note that we use here the left Haar measure $\widetilde{\nu}_2$. It follows from the proof of Theorem 1.4 that $C_{\mathrm{c}}(Z)$ is a right $C_{\mathrm{c}}(G_1)$-module. Set $\pi(\xi)\zeta = \xi\zeta$ for $\xi \in C_{\mathrm{c}}(Z)$ and $\zeta \in C_{\mathrm{c}}(G_1)$, that is,

$$(\pi(\xi)\zeta)(x, g_2) = \int_{\Gamma_1} \xi(xg^{-1}, g\cdot g_2)\zeta(xg^{-1}, g)\,\mathrm{d}\nu_1(g)$$

for $(x, g_2) \in Z$. By a calculation as in the proof of Theorem 1.4, we have

$$\|\pi(\xi)\zeta\|^2_{L^2(Z)} = (\pi_1(\langle \xi, \xi\rangle)\zeta|\zeta).$$

Therefore we can extend $\pi(\xi)$ to a bounded operator of $L^2(G_1, m_1)$ to $L^2(Z, \mu_1 \times \widetilde{\nu}_2)$, which we denote again by $\pi(\xi)$. Since we have $\|\pi(\xi)\| = \|\xi\|_E$ for $\xi \in C_{\mathrm{c}}(Z)$, we can extend π to an isometry of E to $\mathcal{L}(L^2(G_1, m_1), L^2(Z, \mu_1 \times \widetilde{\nu}_1))$, which we denote again by π. Then we have, for $\xi, \eta \in E$ and $a \in C^*_{\mathrm{r}}(G_1)$,

$$\pi_1(\langle \xi, \eta\rangle) = \pi(\xi)^*\pi(\eta)$$
$$\pi(\xi a) = \pi(\xi)\pi_1(a).$$

Moreover we suppose that $f^{(0)}$ is proper. Then $f_*^{(0)}(\mu_1)$ is a positive Radon measure on X_2 and we may assume that $\mu_2 = f_*^{(0)}(\mu_1)$. Define an isometry U of $L^2(G_2, m_2)$ to $L^2(Z, \mu_1 \times \widetilde{\nu}_2)$ by $(U\zeta)(x, g_2) = \zeta(f^{(0)}(x)g_2, g_2^{-1})$. Let $\mu_1 = \int_{X_2} \mu_y \, \mathrm{d}\mu_2(y)$ be the decomposition of μ_1 by $f^{(0)}$. Note that μ_y is a positive Borel measure on X_1 such that μ_y is supported by $(f^{(0)})^{-1}(y)$. Then we have

$$(U^*\zeta)(yg_2^{-1}, g_2) = \int_{X_1} \zeta(x, g_2^{-1}) \, \mathrm{d}\mu_y(x).$$

Set $P = UU^*$. Then we have, for $b \in C_{\mathrm{r}}^*(G_2)$ and $\xi \in E$,

$$U\pi_2(b)U^*\pi(\xi) = P\pi(b\xi).$$

REMARK. In the study of foliations, homomorphisms between holonomy groupoids appears in many cases. For example, see [6], [7], [10], [11], [15]. It is interesting to apply our results to these homomorphisms. But interesting holonomy groupoids are sometimes non-Hausdorff. Therefore it is necessary to extend our results to non-Hausdorff groupoids.

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