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STATES OF TOEPLITZ-CUNTZ ALGEBRAS

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ABSTRACT. We characterize the state space of a Toeplitz-Cuntz algebra $\mathcal{TO}_n$ in terms of positive operator matrices Ω on Fock space which satisfy $\mathrm{sl}\,\Omega \leqslant \Omega$, where $\mathrm{sl}\,\Omega$ is the operator matrix obtained from Ω by taking the trace in the last variable. Essential states correspond to those matrices Ω which are slice-invariant. As an application we show that a pure essential product state of the fixed-point algebra for the action of the gauge group has precisely a circle of pure extensions to $\mathcal{TO}_n$.

KEYWORDS: *Cuntz algebras, Fock space, product states.*

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0. INTRODUCTION

Let $\mathcal{TO}_n$ be the unital C^*-algebra which is universal for collections of n isometries with mutually orthogonal ranges; we call $\mathcal{TO}_n$ a *Toeplitz-Cuntz* algebra. Since their introduction by Cuntz ([8]), these algebras have been profitably used in the study of normal $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on a Hilbert space $\mathcal{H}$. The main idea is as follows. Let $\{v_k \mid 1 \leqslant k \leqslant n\}$ be the distinguished generating isometries in $\mathcal{TO}_n$. Every $*$-representation π of $\mathcal{TO}_n$ on $\mathcal{H}$ on gives rise to an endomorphism α of $\mathcal{B}(\mathcal{H})$ via

$$\alpha(A) = \sum_{k=1}^{n} \pi(v_k) A \pi(v_k)^*, \quad A \in \mathcal{B}(\mathcal{H}),$$

and every endomorphism is of this form for some n and π; see [2], [13], [7].

Arveson has generalized these ideas to the continuous case through the use of product systems. Every representation φ of a continuous product system E on

Hilbert space gives rise to a semigroup $\alpha = \{\alpha_t \mid t > 0\}$ of endomorphisms of $\mathcal{B}(\mathcal{H})$, and φ is said to be *essential* if each α_t is unital; such semigroups are called E_0*-semigroups,* and are the primary objects of study in Arveson's series [2], [3], [4], [5].

One of Arveson's key results is that every product system E has an essential representation. To prove this he associates with E a universal C^*-algebra $C^*(E)$ whose representations are in bijective correspondence with representations of E, characterizes the state space of $C^*(E)$, and then uses this characterization to show that there are always certain states, called *essential* states, whose GNS representations give rise to essential representations of E.

In this paper we develop a discrete version of Arveson's method in which E is a product system over the positive integers $\mathbb{N}$. The algebras which arise as $C^*(E)$ are precisely the Toeplitz-Cuntz algebras: up to isomorphism there is a unique product system E^n over $\mathbb{N}$ for each $n \in \{1, 2, \ldots, \infty\}$, and $C^*(E^n) \cong \mathcal{TO}_n$; see [10]. We write $\mathcal{TO}_\infty$ for the Cuntz algebra $\mathcal{O}_\infty$, a notation which underlies an important advantage of our methods: they apply for both finite and infinite n, so that one does not have to study $\mathcal{O}_\infty$ as a special case. While our methods are motivated by those of Arveson, our exposition avoids any explicit use of product systems: since the C^*-algebra being analyzed is a familiar one, we can use it as a starting point rather than the product system.

Our main result is Theorem 1.9, which characterizes the state space of $\mathcal{TO}_n$ in terms of a class of positive linear functionals on the $*$-algebra $\mathfrak{B}$ of operators on Fock space which have "bounded support"; these functionals are the analogues of Arveson's decreasing locally normal weights ([5]). In Theorem 1.15 we give a reformulation of this result in terms of the so-called *density matrices* associated with these functionals; these are certain infinite operator matrices on full Fock space over an n-dimensional Hilbert space. Roughly speaking, positive linear functionals on $\mathcal{TO}_n$ correspond to positive matrices Ω of trace-class operators with the property that

$$\operatorname{sl}\Omega \leqslant \Omega,$$

where $\operatorname{sl}\Omega$, the *slice* of Ω, is the operator matrix obtained from Ω by "taking the trace in the last variable".

Viewing $\mathcal{TO}_n$ as the universal C^*-algebra of a product system, a state ρ of $\mathcal{TO}_n$ is essential if its associated GNS representation π satisfies

$$\sum \pi(v_k)\pi(v_k)^* = I.$$

In Proposition 2.1 we use the results of Section 1 to give some alternate characterizations of both essentiality and the complementary notion of singularity; perhaps the most useful aspect of this theorem is the characterization of essentiality in terms of invariance under the map β^* defined by

$$\beta^*\rho(v_{i_1}\cdots v_{i_k}v_{j_l}^*\cdots v_{j_1}^*) = \sum_{m=1}^{n} \rho(v_{i_1}\cdots v_{i_k}v_m v_m^* v_{j_l}^*\cdots v_{j_1}^*).$$

When n is finite, essential states of $\mathcal{TO}_n$ are precisely those which factor through the canonical homomorphism of $\mathcal{TO}_n$ onto the Cuntz algebra $\mathcal{O}_n$, and can thus be thought of as states of $\mathcal{O}_n$. This characterization of essentiality can be extended to include the case $n = \infty$, even though $\mathcal{O}_\infty$ is simple. The idea is as follows. When $n = \infty$, Proposition 1.7 characterizes the state space of a certain concrete C^*-algebra $\mathcal{U}$ which contains a copy of $\mathcal{O}_\infty$; $\mathcal{U}$ also contains the compact operators $\mathcal{K}$. We give a canonical procedure for extending states from $\mathcal{O}_\infty$ to $\mathcal{U}$, and show that a state is essential if and only if its canonical extension is zero on $\mathcal{K}$. Section 2 concludes with Theorem 2.2, which gives an alternate approach to the singular-essential decomposition.

A method which has been profitably used to study the state space of $\mathcal{O}_n$ has been to focus first on states of its even-word subalgebra, and then on the problem of extending such states to $\mathcal{O}_n$ ([9], [1], [13], [14], [7], [6]). When n is finite, this even-word subalgebra is a UHF algebra of type n^∞, and is thus somewhat less complicated and better understood than $\mathcal{O}_n$. For example, there is a large supply of states of this algebra readily at hand in the form of product states; indeed, these states have played a key rôle in the study of UHF algebras ([11], [15]). In Section 3 we study the problem of extending a pure periodic product state to $\mathcal{O}_n$. To include the case $n = \infty$ in a unified way we reformulate this problem: we consider instead the even-word subalgebra $\mathcal{F}_n$ of the Toeplitz-Cuntz algebra $\mathcal{TO}_n$, and focus on pure periodic product states of $\mathcal{F}_n$ which are essential. Our main result is Theorem 3.1, which parameterizes the space of all extensions of such a state by probability measures on the circle. Pure extensions correspond to point measures, and we have as an immediate corollary that the space of pure extensions of such a state is precisely a circle.

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1. STATES OF $\mathcal{TO}_n$

From $\mathcal{TO}_n$ to $\mathcal{K}$ and back. Suppose $1 \leqslant i_1, \ldots, i_k \leqslant n$. We call $\mu = (i_1, \ldots, i_k)$ a *multi-index* and define $|\mu| := k$ and $v_\mu := v_{i_1} \cdots v_{i_k}$; of course $v_\emptyset := 1$. The set of all multi-indices will be denoted $\mathcal{W}$, and we define $\mathcal{W}_k := \{\mu \in \mathcal{W} \bigm| |\mu| = k\}$. With this notation, $\mathcal{TO}_n = \overline{\operatorname{span}}\{v_\mu v_\nu^* \mid \mu, \nu \in \mathcal{W}\}$.

When n is finite, the projection $p := 1 - \sum v_i v_i^*$ generates a closed, two-sided ideal $\mathcal{J}_n$ of $\mathcal{TO}_n$ which is isomorphic to the compact operators on an infinite-dimensional, separable Hilbert space; indeed, $\{v_\mu p v_\nu^* \mid \mu, \nu \in \mathcal{W}\}$ is a self-adjoint system of matrix units for $\mathcal{J}_n$ ([8]). Let $q := 1 - p$. Since q is an identity for $C^*(\{v_i q\})$ and $(v_j q)^*(v_i q) = \delta_{ij} q$, the map $v_i \mapsto v_i q$ extends to a $*$-endomorphism β' of $\mathcal{TO}_n$. If we then define $\delta' := \mathrm{id} - \beta'$, one checks easily that $\delta'(v_\mu v_\nu^*) = v_\mu p v_\nu^*$, and consequently $\delta'(\mathcal{TO}_n) \subset \mathcal{J}_n$.

To include the case $n = \infty$ we implement δ' spatially utilizing the Fock representation of $\mathcal{TO}_n$ ([9]). Technically speaking, the representation we are about to define is only unitarily equivalent to the Fock representation; we prefer this version for purely notational reasons. Let

$$\mathcal{E} := \overline{\operatorname{span}}\{v_i \mid 1 \leqslant i \leqslant n\},$$

and more generally, let

$$\mathcal{E}_k := \overline{\operatorname{span}}\{v_\mu \mid \mu \in \mathcal{W}_k\}, \quad k = 0, 1, 2, \ldots,$$

so that $\mathcal{E} = \mathcal{E}_1$. If $f, g \in \mathcal{E}_k$, then $g^* f$ is a scalar multiple of the identity, and the formula $g^* f = \langle f, g \rangle 1$ defines an inner product which makes $\mathcal{E}_k$ a Hilbert space. Notice that the Hilbert space norm on $\mathcal{E}_k$ agrees with the norm $\mathcal{E}_k$ inherits as a subspace of $\mathcal{TO}_n$, and that $\{v_\mu \mid \mu \in \mathcal{W}_k\}$ is an orthonormal basis for $\mathcal{E}_k$. Let

$$F_{\mathcal{E}} := \bigoplus_{k=0}^{\infty} \mathcal{E}_k.$$

By this we mean nothing more than the abstract direct sum of Hilbert spaces; in particular, the inclusion maps $\mathcal{E}_k \hookrightarrow \mathcal{TO}_n$ do *not* factor through the canonical injections $\mathcal{E}_k \hookrightarrow F_{\mathcal{E}}$. We caution the reader that we will think of $\mathcal{E}_k$ in three separate ways: as a subspace of the C^*-algebra $\mathcal{TO}_n$, as a Hilbert space, and as a subspace of $F_{\mathcal{E}}$. This is both a notational advantage and a potential cause of confusion.

For each integer $k \geqslant 0$, left multiplication by v_i is a linear isometry from $\mathcal{E}_k$ to $\mathcal{E}_{k+1}$, and together these maps induce an isometry $l(v_i)$ on $F_{\mathcal{E}}$. Similarly, right multiplication by v_i induces an isometry $r(v_i)$ on $F_{\mathcal{E}}$. Since $l(v_j)^* l(v_i) = \delta_{ij} I$, the map $v_i \mapsto l(v_i)$ extends to a $*$-representation l of $\mathcal{TO}_n$ on $F_{\mathcal{E}}$; we call this the *Fock representation.* The representation which is more commonly referred to as the Fock representation is unitarily equivalent to l via the unitary $F_{\mathcal{E}} \to \bigoplus_{k=0}^{\infty} \mathcal{E}^{\otimes k}$ determined by

$$v_{i_1} \cdots v_{i_k} \mapsto v_{i_1} \otimes \cdots \otimes v_{i_k}, \quad (i_1, \dots, i_k) \in \mathcal{W}.$$

By [9], l is faithful and irreducible. We will study $\mathcal{TO}_n$ in this representation for the remainder of the paper.

For each pair of vectors $f, g \in F_{\mathcal{E}}$ we will denote by $f \otimes \overline{g}$ the rank-one operator $h \mapsto \langle h, g \rangle f$ on $F_{\mathcal{E}}$. Routine calculations show that when $n < \infty$ we have $l(v_\mu p v_\nu^*) = v_\mu \otimes \overline{v_\nu}$, so the image of the ideal $\mathcal{J}_n$ in the Fock representation is $\mathcal{K}$, the compact operators on $F_{\mathcal{E}}$.

We implement δ' spatially as follows. Define a normal $*$-endomorphism β of $\mathcal{B}(F_{\mathcal{E}})$ by

$$\beta(A) := \sum_{i=1}^{n} r(v_i) A r(v_i)^*, \quad A \in \mathcal{B}(F_{\mathcal{E}}).$$

When n is infinite, the above series converges in the strong operator topology. One easily checks that β implements β' spatially when n is finite; i.e. $\beta(l(v_i)) = l(v_i q)$. Hence $\delta := \mathrm{id} - \beta$ implements δ' spatially when $n < \infty$. Moreover,

$$\delta(l(v_\mu v_\nu^*)) = v_\mu \otimes \overline{v_\nu} \tag{1.1}$$

holds whether or not n is finite, so we always have $\delta \circ l(\mathcal{TO}_n) \subseteq \mathcal{K}$.

OPERATORS OF BOUNDED SUPPORT. Let $P_k := I - \beta^{k+1}(I)$, the orthogonal projection of $F_{\mathcal{E}}$ onto $\bigoplus_{i=0}^{k} \mathcal{E}_i$. Let $\mathfrak{B}_k$ be the von Neumann algebra of all operators $T \in \mathcal{B}(F_{\mathcal{E}})$ satisfying $T = P_k T P_k$, and let

$$\mathfrak{B} := \bigcup_{k=0}^{\infty} \mathfrak{B}_k,$$

the algebra of operators on $F_{\mathcal{E}}$ which have *bounded support.* This algebra is β-invariant; indeed, $\beta^i(P_k) = P_{k+i} - P_{i-1}$ for $i \geqslant 1$, $k \geqslant 0$. Consequently

$$\beta^i(A)\beta^j(B) = 0 \text{ if } A, B \in \mathfrak{B}_k \quad \text{and} \quad |i - j| \geqslant k + 1. \tag{1.2}$$

Since $\beta^k(I) \to 0$ strongly, δ is injective: if $\delta(A) = 0$, then $A = \beta(A)$, and hence $A = \beta^k(A) = \beta^k(A)\beta^k(I) = A\beta^k(I) \to 0$. The following proposition shows that $\mathfrak{B}$ is contained in the range of δ, and establishes a formula for the inverse of δ on $\mathfrak{B}$.

PROPOSITION 1.1. *For each $B \in \mathfrak{B}$, the sum $\sum_{i=0}^{\infty} \beta^i(B)$ converges σ-weakly to a bounded operator $\lambda(B)$ on $F_{\mathcal{E}}$. The map $\lambda : \mathfrak{B} \to \mathcal{B}(F_{\mathcal{E}})$ is injective, $*$-linear, and its restriction to $\mathfrak{B}_k$ is normal, completely positive and has norm $k+1$. Moreover, $\delta|\lambda(\mathfrak{B})$ is the inverse of λ.*

Proof. We follow [5], Theorem 2.2. Fix $k \geqslant 0$. If $A, B \in \mathfrak{B}_k$, then by (1.2)

$$\beta^{i(k+1)}(A)\beta^{j(k+1)}(B) = \delta_{ij}\beta^{j(k+1)}(AB).$$

Consequently $\sum_{j=0}^{\infty} \beta^{j(k+1)}$ is a normal $*$-homomorphism from $\mathfrak{B}_k$ to $\mathcal{B}(F_{\mathcal{E}})$. From this it is easy to see that $\sum_{i=0}^{\infty} \beta^i(B) = \sum_{l=0}^{k} \sum_{j=0}^{\infty} \beta^{j(k+1)+l}(B)$ converges strongly to an operator $\lambda_k(B)$ for each $B \in \mathfrak{B}_k$, and that λ_k is normal, completely positive and $*$-linear. The λ_k's are clearly coherent, and thus define the desired map λ. Since $(P_k - P_{k-1})\lambda(P_k) = (k+1)(P_k - P_{k-1})$, the norm of λ_k is at least $k+1$; since λ_k is the sum of $k+1$ $*$-homomorphisms, we thus have $\|\lambda_k\| = k+1$.

By the normality of β we have $\beta(\lambda(B)) = \sum_{i=1}^{\infty} \beta^i(B)$, and thus $\delta(\lambda(B)) = B$ for each $B \in \mathfrak{B}$. Consequently λ is injective with inverse $\delta|\lambda(\mathfrak{B})$. ∎

PROPOSITION 1.2. *$\lambda(\mathfrak{B})$ and $\lambda(\mathfrak{B} \cap \mathcal{K})$ are irreducible $*$-subalgebras of $\mathcal{B}(F_{\mathcal{E}})$, and $\lambda(\mathfrak{B} \cap \mathcal{K})$ is a dense subalgebra of $l(\mathcal{TO}_n)$.*

Proof. Since λ is $*$-linear, $\lambda(\mathfrak{B})$ and $\lambda(\mathfrak{B} \cap \mathcal{K})$ are self-adjoint subspaces of $\mathcal{B}(F_{\mathcal{E}})$. By (1.1) we have $\lambda(v_\mu \otimes \overline{v_\nu}) = l(v_\mu v_\nu^*)$, and since λ is bounded when restricted to one of the subalgebras $\mathfrak{B}_k$, this shows that $\lambda(\mathfrak{B} \cap \mathcal{K})$ is a dense subspace of $l(\mathcal{TO}_n)$. Suppose $A, B \in \lambda(\mathfrak{B})$; we will show that $AB \in \lambda(\mathfrak{B})$. To begin with, observe that if $C \in \mathfrak{B}_c$ and $D \in \mathfrak{B}_d$, then

$$C\lambda(D) = \sum_{k=0}^{c} C\beta^k(D) \in \mathfrak{B}_{c+d}, \tag{1.3}$$

with a similar equation holding for $\lambda(C)D$. Since $\delta(A), \delta(B) \in \mathfrak{B}$, this implies that both $\delta(A)B = \delta(A)\lambda(\delta(B))$ and $A\delta(B) = \lambda(\delta(A))\delta(B)$ are in $\mathfrak{B}$. Using the identity

$$\delta(AB) = \delta(A)B + A\delta(B) - \delta(A)\delta(B), \tag{1.4}$$

we see that $\delta(AB) \in \mathfrak{B}$. Thus $AB \in \lambda(\mathfrak{B})$, so $\lambda(\mathfrak{B})$ is a $*$-algebra.

If in addition $\delta(A)$ and $\delta(B)$ are compact, then by (1.4), $\delta(AB)$ is compact as well. Thus $AB \in \lambda(\mathfrak{B} \cap \mathcal{K})$, so $\lambda(\mathfrak{B} \cap \mathcal{K})$ is also a $*$-algebra. These algebras are irreducible since $\lambda(\mathfrak{B} \cap \mathcal{K})$ is dense in the irreducible algebra $l(\mathcal{TO}_n)$. ∎

Remark 1.3. If n is finite then $\mathfrak{B} \subset \mathcal{K}$, so that the algebras $\lambda(\mathfrak{B} \cap \mathcal{K})$ and $\lambda(\mathfrak{B})$ coincide. When n is infinite this is not the case. For arbitrary n, (1.3) shows that $\overline{\mathfrak{B}}$ is an ideal in $\overline{\lambda(\mathfrak{B})}$. Moreover, this ideal contains the compacts since $\mathfrak{B}$ is irreducible and has nontrivial intersection with $\mathcal{K}$. When n is finite $\overline{\mathfrak{B}} = \mathcal{K}$.

Decreasing positive linear functionals

Definition 1.4. Suppose ω is a linear functional on $\mathfrak{B}$. We say that ω is *decreasing* if $\omega \circ \delta$ is positive on $\mathfrak{B}$; that is, if

$$\omega(B^*B) - \omega(\beta(B^*B)) \geqslant 0, \quad B \in \mathfrak{B}.$$

Proposition 1.5. *Suppose ω is a linear functional on $\mathfrak{B}$. Then $\omega \circ \delta$ is positive on $\lambda(\mathfrak{B})$ iff ω is positive and decreasing, in which case $\omega \circ \delta$ extends uniquely to a positive linear functional $\Delta\omega$ of norm $\omega(P_0)$ on the C^*-algebra $\overline{\lambda(\mathfrak{B})}$.*

Proof. Suppose $\omega \circ \delta$ is positive on $\lambda(\mathfrak{B})$. Then ω is decreasing since $\mathfrak{B} \subseteq \lambda(\mathfrak{B})$, and ω is positive since $\omega = \omega \circ \delta \circ \lambda$ and λ is positive.

Conversely, suppose ω is positive and decreasing. Then $(B_1, B_2) \mapsto \omega(B_2^* B_1)$ defines a positive semi-definite sesquilinear form on $\mathfrak{B}$, so there is a Hilbert space $\mathcal{H}_\omega$ and a linear map $\Omega : \mathfrak{B} \to \mathcal{H}_\omega$ such that $\Omega(\mathfrak{B})$ is dense in $\mathcal{H}_\omega$ and $\langle \Omega(B_1), \Omega(B_2) \rangle = \omega(B_2^* B_1)$ for every $B_1, B_2 \in \mathfrak{B}$. Since ω is decreasing, $\Omega(B) \mapsto \Omega(\beta(B))$ extends uniquely to a linear contraction T on $\mathcal{H}_\omega$. Define $\tau : \mathbb{Z} \to \mathcal{B}(\mathcal{H}_\omega)$ by

$$\tau(k) := \begin{cases} T^k & \text{if } k \geqslant 0 \\ T^{*(-k)} & \text{if } k < 0. \end{cases} \tag{1.5}$$

Since T is a contraction, τ is positive definite.

Suppose now that $B \in \mathfrak{B}$. By (1.4),

$$\begin{aligned} \delta(\lambda(B)^*\lambda(B)) &= \delta(\lambda(B^*))\lambda(B) + \lambda(B^*)\delta(\lambda(B)) - \delta(\lambda(B^*))\delta(\lambda(B)) \\ &= B^*\lambda(B) + \lambda(B^*)B - B^*B = \sum_{i=0}^{\infty} B^*\beta^i(B) + \sum_{j=1}^{\infty} \beta^j(B^*)B, \end{aligned}$$

where by (1.3) each of the sums is only finitely non-zero. Since

$$\langle \tau(k)\Omega(B), \Omega(B) \rangle = \begin{cases} \omega(B^*\beta^k(B)) & \text{if } k \geqslant 0 \\ \omega(\beta^{-k}(B^*)B) & \text{if } k < 0, \end{cases}$$

we thus have

$$\omega \circ \delta(\lambda(B)^*\lambda(B)) = \sum_{k=-\infty}^{\infty} \langle \tau(k)\Omega(B), \Omega(B) \rangle \geqslant 0,$$

so $\omega \circ \delta$ is positive on $\lambda(\mathfrak{B})$.

Since $\lambda(P_0) = \sum_{n=0}^{\infty} \beta^n(I - \beta(I)) = I$, the algebra $\lambda(\mathfrak{B})$ unital. Hence to show that $\omega \circ \delta$ is bounded on $\lambda(\mathfrak{B})$ it suffices to establish the relation

$$\limsup_{k\to\infty} |\omega \circ \delta(A^k)|^{\frac{1}{k}} \leqslant \|A\|, \quad A \in \lambda(\mathfrak{B}). \tag{1.6}$$

For this, suppose $A \in \lambda(\mathfrak{B})$ and k is a positive integer. Then $\delta(A) \in \mathfrak{B}_a$ for some positive integer a, and repeated applications of (1.4) and (1.3) give that $\delta(A^k) \in \mathfrak{B}_{ka}$. Now P_{ka} is the unit in $\mathfrak{B}_{ka}$, so

$$|\omega \circ \delta(A^k)| \leqslant \omega(P_{ka})\|\delta(A^k)\|.$$

Since ω is decreasing,

$$\omega(P_{ka}) = \sum_{i=0}^{ka} \omega(\beta^i(P_0)) \leqslant \sum_{i=0}^{ka} \omega(P_0) = (ka+1)\omega(P_0).$$

These last two inequalities together with the fact that $\|\delta\| \leqslant 2$ give

$$|\omega \circ \delta(A^k)| \leqslant 2(ka+1)\omega(P_0)\|A\|^k,$$

from which (1.6) follows immediately. Finally,

$$\|\Delta\omega\| = \omega \circ \delta(I) = \omega \circ \delta(\lambda(P_0)) = \omega(P_0). \quad ∎$$

DEFINITION 1.6. Denote by $\mathcal{P}_\beta$ the cone of decreasing positive linear functionals on $\mathfrak{B}$, partial-ordered by the relation

$$\omega_1 \leqslant \omega_2 \quad \text{iff} \quad \omega_2 - \omega_1 \in \mathcal{P}_\beta.$$

PROPOSITION 1.7. *The map $\omega \mapsto \Delta\omega$ is an affine order isomorphism of $\mathcal{P}_\beta$ onto the positive part of the dual of $\overline{\lambda(\mathfrak{B})}$.*

Proof. It is clear from the construction that $\omega \mapsto \Delta\omega$ is affine. To see that it is surjective, suppose ρ is a positive linear functional on $\overline{\lambda(\mathfrak{B})}$. Let $\omega = \rho\circ\lambda$. Then $\omega \circ \delta$ agrees with ρ on $\lambda(\mathfrak{B})$ and hence is positive. By Proposition 1.5 this implies that ω is positive and decreasing, and clearly $\Delta\omega = \rho$. Since $\omega(B) = \Delta\omega(\lambda(B))$ for each $B \in \mathfrak{B}$, the map $\omega \mapsto \Delta\omega$ is injective. Lastly, observe that $\omega_1 \leqslant \omega_2$ iff $\omega_2 - \omega_1 \in \mathcal{P}_\beta$, which by Proposition 1.5 is equivalent to the condition that $(\omega_2 - \omega_1) \circ \delta$ be positive on $\lambda(\mathfrak{B})$. This in turn is obviously equivalent to the condition that $\Delta\omega_1 \leqslant \Delta\omega_2$. ∎

When n is finite recall that $\overline{\lambda(\mathfrak{B})} = \overline{\lambda(\mathfrak{B} \cap \mathcal{K})} = l(\mathcal{TO}_n)$, so we have an affine isomorphism

$$\omega \mapsto \Delta\omega \circ l \tag{1.7}$$

of $\mathcal{P}_\beta$ onto the state space of $\mathcal{TO}_n$. When n is infinite the algebra $\overline{\lambda(\mathfrak{B})}$ properly contains $l(\mathcal{O}_\infty)$, and consequently (1.7) gives an affine map of $\mathcal{P}_\beta$ onto the state space of $\mathcal{O}_\infty$ which is not injective. The following definition is the key to identifying a subcone of $\mathcal{P}_\beta$ on which this map is an isomorphism.

DEFINITION 1.8. A linear functional ω on $\mathfrak{B}$ is *locally normal* if its restriction to each of the von Neumann subalgebras $\mathfrak{B}_k$ is normal. We denote by $\mathcal{W}_\beta$ the subcone of $\mathcal{P}_\beta$ consisting of all decreasing positive linear functionals on $\mathfrak{B}$ which are locally normal.

We now prove our main theorem, which improves on Proposition 1.7 in the sense that it characterizes the state space not just of $\mathcal{TO}_n$ for n finite, but also of $\mathcal{O}_\infty$.

THEOREM 1.9. *Suppose $\{v_1, \ldots, v_n\}$ are the distinguished generating isometries of the Toeplitz-Cuntz algebra $\mathcal{TO}_n$; we include the case $n = \infty$ by writing $\mathcal{TO}_\infty$ for the Cuntz algebra $\mathcal{O}_\infty$. Let $\mathcal{E} \subseteq \mathcal{TO}_n$ be the closed linear span of $\{v_1, \ldots, v_n\}$, let $F_\mathcal{E}$ be full Fock space over $\mathcal{E}$, let $\mathfrak{B}$ be the algebra of operators on $F_\mathcal{E}$ which have bounded support, and let $\mathcal{W}_\beta$ be the partially-ordered cone of decreasing locally normal positive linear functionals on $\mathfrak{B}$. For each $\omega \in \mathcal{W}_\beta$ there is a unique positive linear functional ρ on $\mathcal{TO}_n$ which satisfies*

$$\rho(v_\mu v_\nu^*) = \omega(v_\mu \otimes \overline{v_\nu}), \quad \mu, \nu \in \mathcal{W}. \tag{1.8}$$

Moreover, the map $\omega \mapsto \rho$ is an affine order isomorphism of $\mathcal{W}_\beta$ onto the positive part of the dual of $\mathcal{TO}_n$.

Proof. Suppose $\omega \in \mathcal{W}_\beta$, and let $\rho := \Delta\omega \circ l$, where l is the Fock representation of $\mathcal{TO}_n$ and $\Delta\omega$ is as in Proposition 1.5. Then ρ satisfies (1.8), and ρ is uniquely determined by (1.8) since elements of the form $v_\mu v_\nu^*$ have dense linear span in $\mathcal{TO}_n$. If n is finite then every linear functional on $\mathfrak{B}$ is automatically locally normal, so that $\mathcal{W}_\beta$ is all of $\mathcal{P}_\beta$. Since $\overline{\lambda(\mathfrak{B})} = l(\mathcal{TO}_n)$, the theorem thus reduces to Proposition 1.7.

It remains only to show that when n is infinite,

$$\omega \mapsto \Delta\omega \circ l \tag{1.9}$$

maps $\mathcal{W}_\beta$ bijectively onto the positive part of the dual of $\mathcal{O}_\infty$. To begin with, suppose $\omega_1, \omega_2 \in \mathcal{W}_\beta$ are such that $\Delta\omega_1 \circ l = \Delta\omega_2 \circ l$. Then $\Delta\omega_1$ and $\Delta\omega_2$ agree on $l(\mathcal{O}_\infty) \supseteq \lambda(\mathfrak{B} \cap \mathcal{K})$, so for each $K \in \mathfrak{B} \cap \mathcal{K}$ we have

$$\omega_1(K) = \Delta\omega_1(\lambda(K)) = \Delta\omega_2(\lambda(K)) = \omega_2(K).$$

Since ω_1 and ω_2 are locally normal, this implies that $\omega_1 = \omega_2$. Thus (1.9) is injective.

To show surjectivity, suppose ρ is a positive linear functional on $\mathcal{O}_\infty$. Define ω_0 on $\mathfrak{B} \cap \mathcal{K}$ by $\omega_0 := \rho \circ l^{-1} \circ \lambda$, and for each k let T_k be the unique positive trace-class operator in $\mathfrak{B}_k$ such that $\omega_0(K) = \operatorname{tr}(T_k K)$ for each $K \in \mathfrak{B}_k \cap \mathcal{K}$. The formula

$$\omega(B) := \operatorname{tr}(T_k B), \quad B \in \mathfrak{B}_k, \tag{1.10}$$

gives the unique extension of ω_0 to a positive linear functional ω on $\mathfrak{B}$ which is locally normal. Once we establish that ω is decreasing, surjectivity of (1.9) follows immediately: for each $K \in \mathfrak{B} \cap \mathcal{K}$,

$$\Delta\omega(\lambda(K)) = \omega \circ \delta(\lambda(K)) = \omega_0(K) = \rho \circ l^{-1}(\lambda(K)),$$

so that $\rho = \Delta\omega \circ l$.

We will show that $\omega \circ \delta$ is positive on $\lambda(\mathfrak{B})$; by Proposition 1.5 this implies that ω is decreasing. For each $B \in \mathfrak{B}$ define a function $\varphi_B : \mathbb{Z} \to \mathbb{C}$ by

$$\varphi_B(k) := \begin{cases} \omega(B^*\beta^k(B)) & \text{if } k \geqslant 0 \\ \omega(\beta^{-k}(B^*)B) & \text{if } k < 0. \end{cases}$$

As in the proof of Proposition 1.5,

$$\omega \circ \delta(\lambda(B)^*\lambda(B)) = \sum_{k=-\infty}^{\infty} \varphi_B(k), \quad B \in \mathfrak{B},$$

so it suffices to show that each φ_B is positive definite. We will do this by showing that φ_K is positive definite for each $K \in \mathfrak{B} \cap \mathcal{K}$, and that any φ_B can be obtained as a pointwise limit of such functions.

Let $E_k := \beta^k(I) - \beta^{k+1}(I)$, the orthogonal projection of $F_\mathcal{E}$ onto $\mathcal{E}_k$. For each $z \in \mathbb{T}$, let $U_z := \sum_{k=0}^{\infty} z^k E_k$. Each of the unitaries U_z is a multiplier of $\mathfrak{B} \cap \mathcal{K}$, and since $\beta(U_z) = \overline{z} U_z \beta(I)$ we have

$$\varphi_K(k) = z^k \varphi_{U_z K}(k), \quad K \in \mathfrak{B} \cap \mathcal{K},\ k \in \mathbb{Z}.$$

Let $\widehat{\varphi}_K$ denote the Fourier transform of φ_K. For each $K \in \mathfrak{B} \cap \mathcal{K}$ and $z \in \mathbb{T}$,

$$\widehat{\varphi}_K(z) = \sum_{k=-\infty}^{\infty} \varphi_K(k) z^k = \sum_{k=-\infty}^{\infty} \varphi_{U_{\bar{z}}K}(k) = \rho(\lambda(U_{\bar{z}}K)^* \lambda(U_{\bar{z}}K)) \geqslant 0,$$

so that $\widehat{\varphi}_K$ is positive. By Herglotz's Theorem, this implies that φ_K is positive definite.

Lastly, suppose $B \in \mathfrak{B}$, say $B \in \mathfrak{B}_m$. Let (K_α) be a bounded net in $\mathfrak{B}_m \cap \mathcal{K}$ which converges to B in the strong operator topology on $\mathfrak{B}_m$. Then $K_\alpha^* \to B^*$ in the σ-weak topology, and hence $\beta^k(K_\alpha^*) \to \beta^k(B^*)$ σ-weakly for any $k \geqslant 0$. Since this latter net is bounded in norm, $\beta^k(K_\alpha^*) K_\alpha \to \beta^k(B^*) B$ weakly, hence σ-weakly. From this it is apparent that $\varphi_{K_\alpha}(k) \mapsto \varphi_B(k)$ for each $k \in \mathbb{Z}$. Thus φ_B is positive definite. ∎

We conclude this section by giving a reformulation of Theorem 1.9 in terms of density matrices. Suppose ω is a locally normal linear functional on $\mathfrak{B}$. Then for each positive integer k there is a unique trace-class operator T_k in $\mathfrak{B}_k$ such that

$$\omega(B) = \mathrm{tr}(T_k B), \quad B \in \mathfrak{B}_k.$$

These density operators are coherent in the sense that $T_k = P_k T_{k+1} P_k$ for each k. Define $\Omega_{ij} : \mathcal{E}_j \to \mathcal{E}_i$ by

$$\Omega_{ij} := E_i T_k E_j,$$

where E_i is the orthogonal projection of $F_{\mathcal{E}}$ onto $\mathcal{E}_i$, and k is any integer greater than both i and j; we think of T_k as having operator matrix $(\Omega_{ij})_{i,j=0}^k$. The *density matrix* of ω is the infinite operator matrix $\Omega := (\Omega_{ij})$. If $x \in \mathcal{E}_j$ and $y \in \mathcal{E}_i$, we may write $\langle \Omega x, y \rangle$ rather than $\langle \Omega_{ij} x, y \rangle$.

Definition 1.10. Suppose $\Omega = (\Omega_{ij})$ is an infinite operator matrix; i.e., $\Omega_{ij} \in \mathcal{B}(\mathcal{E}_j, \mathcal{E}_i)$ for every pair i, j of nonnegative integers. For each k let T_k be the operator in $\mathfrak{B}_k$ determined by

$$\langle T_k x, y \rangle = \langle \Omega x, y \rangle \quad x \in \mathcal{E}_j,\, y \in \mathcal{E}_i,\, 0 \leqslant i, j \leqslant k.$$

We say that Ω is *positive* if each T_k is positive, and *locally trace-class* if each T_k is trace-class.

It is evident that an infinite operator matrix Ω is the density matrix of a locally normal linear functional if and only if it is locally trace-class, and that a density matrix Ω is positive if and only if its associated linear functional ω is positive.

We now characterize those density matrices which correspond to linear functionals which are decreasing. For this, we first need a lemma.

LEMMA 1.11. *Suppose T is a trace-class operator on a separable Hilbert space $\mathcal{H}$, and $U_1, U_2, U_3, \ldots$ are isometries on $\mathcal{H}$ with mutually orthogonal ranges. Then*

$$\sum_{k=1}^{\infty} U_k^* T U_k \tag{1.11}$$

converges in trace-class norm (and hence in operator norm as well).

Proof. First suppose $T \geqslant 0$. Let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}$. If $l > m \geqslant 1$, then

$$\begin{aligned}\operatorname{tr}\Big(\sum_{k=1}^{l} U_k^* T U_k - \sum_{k=1}^{m} U_k^* T U_k\Big) &= \sum_{i=1}^{\infty} \sum_{k=m+1}^{l} \langle U_k^* T U_k e_i, e_i\rangle \\ &= \sum_{k=m+1}^{l} \sum_{i=1}^{\infty} \langle T U_k e_i, U_k e_i\rangle.\end{aligned}$$

But

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \langle T U_k e_i, U_k e_i\rangle \leqslant \operatorname{tr} T < \infty,$$

so the sequence $\Big(\sum_{k=1}^{m} U_k^* T U_k\Big)_{m=1}^{\infty}$ is Cauchy in the trace-class norm. Since the algebra of trace-class operators is complete in this norm, the infinite sum (1.11) converges as claimed. Since every trace-class operator can be written as a linear combination of four positive trace-class operators, (1.11) converges in trace-class norm for every trace-class operator T. ∎

DEFINITION 1.12. Suppose T is a trace-class operator on $F_{\mathcal{E}}$. The *slice* of T is the operator

$$\operatorname{sl} T := \sum_{k=1}^{n} r(v_k)^* T r(v_k),$$

where $r(v_k)$ is right creation by v_k on $F_{\mathcal{E}}$. Note that when $n = \infty$ the sum converges to a trace-class operator (Lemma 1.11).

If $\Omega = (\Omega_{ij})$ is a locally trace-class operator matrix, we denote by $\operatorname{sl}\Omega$ the locally trace-class operator matrix $(\operatorname{sl}\Omega_{i+1,j+1})$.

REMARK 1.13. Suppose $S \in \mathcal{B}(\mathcal{E}_j, \mathcal{E}_i)$ and $A \in \mathcal{B}(\mathcal{E})$ are trace-class operators such that $T(xy) = (Sx)(Ay)$ for $x \in \mathcal{E}_j$ and $y \in \mathcal{E}$. (The unitary $x \otimes y \mapsto xy$

transforms T into the tensor product $S \otimes A$.) Then

$$\begin{aligned}(\operatorname{sl} T)x &= \sum_{k=1}^{n} r(v_k)^* T r(v_k) x = \sum_{k=1}^{n} r(v_k)^* T(x v_k) \\ &= \sum_{k=1}^{n} r(v_k)^*(Sx)(A v_k) = \sum_{k=1}^{n} \langle A v_k, v_k \rangle Sx = \operatorname{tr}(A) Sx,\end{aligned}$$

so slicing has the effect of taking the trace in the last variable.

Lemma 1.14. *Suppose ω is a locally-normal linear functional on $\mathfrak{B}$ with density matrix Ω. Then $\omega \circ \beta$ has density matrix* $\operatorname{sl}\Omega$.

Proof. Let T_k be the trace-class operator in $\mathfrak{B}_k$ such that $\omega(B) = \operatorname{tr}(T_k B)$ for $B \in \mathfrak{B}_k$, so that T_k has operator matrix $(\Omega_{ij})_{i,j=0}^{k}$. Suppose $B \in \mathfrak{B}_k$. Then $\beta(B) \in \mathfrak{B}_{k+1}$, and thus

$$\begin{aligned}\omega \circ \beta(B) = \operatorname{tr}(T_{k+1}\beta(B)) &= \sum_{i=1}^{\infty} \operatorname{tr}(T_{k+1} r(v_i) B r(v_i)^*) \\ &= \sum_{i=1}^{\infty} \operatorname{tr}(r(v_i)^* T_{k+1} r(v_i) B) = \operatorname{tr}((\operatorname{sl} T_{k+1}) B).\end{aligned}$$

Thus $\omega \circ \beta$ has density matrix $\operatorname{sl}\Omega$. ∎

We now give our reformulation of Theorem 1.9.

Theorem 1.15 *Suppose $\{v_1, \ldots, v_n\}$ are the distinguished generating isometries of the Toeplitz-Cuntz algebra $\mathcal{TO}_n$; we include the case $n = \infty$ by writing $\mathcal{TO}_\infty$ for the Cuntz algebra $\mathcal{O}_\infty$. Let $\mathcal{E} \subseteq \mathcal{TO}_n$ be the closed linear span of $\{v_1, \ldots, v_n\}$, and let $F_\mathcal{E}$ be full Fock space over $\mathcal{E}$. Suppose Ω is a positive locally trace-class operator matrix on $F_\mathcal{E}$ which satisfies $\operatorname{sl}\Omega \leqslant \Omega$. Then there is a unique positive linear functional ρ on $\mathcal{TO}_n$ which satisfies*

$$\rho(v_\mu v_\nu^*) = \langle \Omega v_\mu, v_\nu \rangle, \quad \mu, \nu \in \mathcal{W}.$$

Moreover, the map $\Omega \mapsto \rho$ is an affine order isomorphism of such operator matrices onto the positive part of the dual of $\mathcal{TO}_n$.

Proof. The equation $\omega(v_\mu \otimes \overline{v_\nu}) = \langle \Omega v_\mu, v_\nu \rangle$ establishes an affine order isomorphism $\omega \mapsto \Omega$ between $\mathcal{W}_\beta$ and positive locally trace-class operator matrices on $F_\mathcal{E}$ which satisfy $\operatorname{sl}\Omega \leqslant \Omega$, so the theorem follows immediately from Theorem 1.9. ∎

2. SINGULAR AND ESSENTIAL STATES

A state ρ of $\mathcal{TO}_n$ with GNS representation $\pi : \mathcal{TO}_n \to \mathcal{B}(\mathcal{H})$ is said to be *essential* if $\sum \pi(v_i v_i^*)$ is the identity operator on $\mathcal{H}$, and *singular* if $\sum_{\mu \in \mathcal{W}_k} \pi(v_\mu v_\mu^*)$ decreases strongly to zero in k. When n is finite $\mathcal{TO}_n$ has a unique ideal $\mathcal{J}_n$, and it is not hard to show that essential states of $\mathcal{TO}_n$ are precisely those which are singular with respect to this ideal; similarly, singular states are $\mathcal{J}_n$-essential. As a result, every state of $\mathcal{TO}_n$ has a unique decomposition into essential and singular components, a result which was generalized to the case $n = \infty$ in [13].

We can view the singular/essential decomposition of a state ρ of $\mathcal{O}_\infty$ as the decomposition with respect to an ideal as follows. Let ω be the unique decreasing locally normal positive linear functional on $\mathfrak{B}$ such that $\rho = \Delta\omega \circ l$, as in Theorem 1.9. The C^*-algebra $\overline{\lambda(\mathfrak{B})}$ contains the ideal $\mathcal{K}$ of compact operators on $F_\mathcal{E}$ (Remark 1.3), so we can decompose the functional $\Delta\omega$ of Proposition 1.7 with respect to this ideal. Restricting to $\overline{\lambda(\mathfrak{B} \cap \mathcal{K})} = l(\mathcal{O}_\infty)$ and pulling back to $\mathcal{O}_\infty$ gives the singular/essential decomposition of ρ; this will follow from Proposition 2.1 (1d) and (2d).

Again allowing n to be either finite or infinite, we follow [13] and define a positive linear functional $\alpha^*\rho$ by

$$\alpha^*\rho(x) := \sum_{i=1}^{n} \rho(v_i x v_i^*), \quad x \in \mathcal{TO}_n.$$

In [13], Corollary 2.9, Laca characterized singular and essential states of $\mathcal{TO}_n$ using the monotonically nonincreasing sequence $(\|\alpha^{*k}\rho\|)_{k=1}^{\infty}$: ρ is essential iff this sequence is constant and singular iff it converges to zero.

Let ω be such that $\rho = \Delta\omega \circ l$, as in Theorem 1.9. Then $\omega \circ \beta$ is a locally normal positive linear functional on $\mathfrak{B}$ which is decreasing since $(\omega \circ \beta) \circ \delta = (\omega \circ \delta) \circ \beta$ is positive on $\mathfrak{B}$. Define

$$\beta^*\rho := \Delta(\omega \circ \beta) \circ l.$$

If $\mu, \nu \in \mathcal{W}$, then

$$\begin{aligned} \beta^*\rho(v_\mu v_\nu^*) &= \omega \circ \beta(v_\mu \otimes \overline{v_\nu}) = \omega\Big(\sum_{i=1}^{n} r(v_i)(v_\mu \otimes \overline{v_\nu}) r(v_i)^*\Big) \\ &= \sum_{i=1}^{n} \omega(v_\mu v_i \otimes \overline{v_\nu v_i}) = \sum_{i=1}^{n} \rho(v_\mu v_i v_i^* v_\nu^*). \end{aligned} \tag{2.1}$$

In particular $\|\beta^{*k}\rho\| = \beta^{*k}\rho(1) = \sum_{\mu\in\mathcal{W}_k} \rho(v_\mu v_\mu^*) = \|\alpha^{*k}\rho\|$, so Laca's characterization can be stated in terms of β^*. What is not immediately apparent is that essentiality is equivalent to β^*-invariance.

We remind the reader of the notation $E_k := \beta^k(I) - \beta^{k+1}(I)$, the orthogonal projection of $F_{\mathcal{E}}$ onto $\mathcal{E}_k$.

PROPOSITION 2.1. *Suppose ρ is a positive linear functional on $\mathcal{TO}_n$ and ω is the unique decreasing locally normal positive linear functional on $\mathfrak{B}$ such that $\rho = \Delta\omega \circ l$. Let Ω be the density matrix of ω. Statements* (1a)–(1f) *below are equivalent, as are statements* (2a)–(2d)*:*

(1a) *ρ is essential;*	(2a) *ρ is singular;*
(1b) *$\omega(E_k)$ is constant in k;*	(2b) $\lim \omega(E_k) = 0$*;*
(1c) *$\omega \circ \delta(B) = 0$ for each $B \in \mathfrak{B}$;*	(2c) *ρ is normal in the Fock representation;*
(1d) *$\Delta\omega$ is $\mathcal{K}$-singular;*	(2d) *$\Delta\omega$ is $\mathcal{K}$-essential.*
(1e) $\operatorname{sl}\Omega = \Omega$*;*	
(1f) $\rho = \beta^*\rho$*;*	

Proof. Since $\{v_\mu \mid \mu \in \mathcal{W}_k\}$ is an orthonormal basis for $\mathcal{E}_k$ and ω is locally normal,

$$\omega(E_k) = \sum_{\mu\in\mathcal{W}_k} \omega(v_\mu \otimes \overline{v_\mu}) = \sum_{\mu\in\mathcal{W}_k} \rho(v_\mu v_\mu^*) = \|\alpha^{*k}\rho\|.$$

Thus (1a) $\Leftrightarrow$ (1b) and (2a) $\Leftrightarrow$ (2b) follow from [13], Corollary 2.9.

(1b) $\Leftrightarrow$ (1c) Since $\delta(E_k) = E_k - E_{k+1}$, (1c) $\Rightarrow$ (1b) is immediate. For the converse, simply observe that $\omega \circ \delta|\mathfrak{B}$ is a positive linear functional whose restriction to the C^*-algebra $\mathfrak{B}_k$ has norm $\omega \circ \delta(P_k) = \omega \circ \delta\Big(\sum_{i=0}^{k} E_i\Big) = 0$.

(1c) $\Leftrightarrow$ (1d) Since $\mathfrak{B} \subseteq \lambda(\mathfrak{B})$ and $\Delta\omega = \omega \circ \delta$ on $\lambda(\mathfrak{B})$, (1c) implies that $\Delta\omega$ vanishes on $\mathfrak{B}$, and hence on $\overline{\mathfrak{B}}$. Since $\mathcal{K} \subseteq \overline{\mathfrak{B}}$, this gives (1d). Conversely, if $\Delta\omega(K) = 0$ for each $K \in \mathcal{K}$, then $\omega \circ \delta(K) = 0$ for each $K \in \mathfrak{B} \cap \mathcal{K}$. Since δ is σ-weakly continuous and maps $\mathfrak{B}_k$ into $\mathfrak{B}_{k+1}$, (1c) follows from local normality of ω.

(1c) $\Leftrightarrow$ (1e) By Lemma 1.14, $\omega = \omega \circ \beta$ iff $\operatorname{sl}\Omega = \Omega$.

(1c) $\Leftrightarrow$ (1f) By Proposition 1.7, $\omega = \omega \circ \beta$ iff $\rho = \beta^*\rho$.

(2a) $\Rightarrow$ (2c) This follows from [13], Theorem 2.11.

(2c) $\Rightarrow$ (2b) Suppose $\rho = \varphi \circ l$ for some $\varphi \in \mathcal{B}(F_{\mathcal{E}})_*$. Then φ and $\Delta\omega$ agree on $l(\mathcal{TO}_n)$, so $\varphi \circ \lambda$ and ω agree on $\mathfrak{B} \cap \mathcal{K}$. Fix k, and let $\{P_\alpha\}$ be a net of finite rank projections which increases to E_k. By the normality of φ and local normality of λ,

$$\omega(E_k) = \lim \omega(P_\alpha) = \lim \varphi \circ \lambda(P_\alpha) = \varphi \circ \lambda(E_k) = \varphi(\beta^k(I)),$$

which decreases to zero.

(2c) ⇔ (2d) Suppose again that $\rho = \varphi \circ l$ for some $\varphi \in \mathcal{B}(F_{\mathcal{E}})_*$. Fix $B \in \mathfrak{B}$, say $B \in \mathfrak{B}_m$, and let $\{K_\alpha\}$ be a net of compact operators in $\mathfrak{B}_m$ which converges σ-weakly to B. Then

$$\Delta\omega(\lambda(B)) = \omega(B) = \lim \omega(K_\alpha) = \lim \varphi \circ \lambda(K_\alpha) = \varphi(\lambda(B)),$$

so that φ extends $\Delta\omega$ as well. The converse is immediate. ∎

THEOREM 2.2. *Suppose ρ is a positive linear functional on $\mathcal{TO}_n$, and let ω be the unique decreasing locally normal positive linear functional on $\mathfrak{B}$ such that $\rho = \Delta\omega \circ l$. Then $\omega \circ \delta|\mathfrak{B}$ extends uniquely to a normal positive linear functional φ on $\mathcal{B}(F_{\mathcal{E}})$, and the singular part of ρ is $\varphi \circ l$.*

Proof. Suppose ω is a decreasing locally normal positive linear functional on $\mathfrak{B}$. Then $\omega \circ \delta|\mathfrak{B}$ is a locally normal positive linear functional, and $\omega \circ \delta(P_k) = \omega(E_0) - \omega(E_{k+1}) \leqslant \omega(E_0)$ for every $k \geqslant 0$. It follows that $\omega \circ \delta$ is bounded on $\mathfrak{B}$: if $B \in \mathfrak{B}_k$, then $|\omega \circ \delta(B)| \leqslant \omega \circ \delta(P_k)\|B\| \leqslant \omega(E_0)\|B\|$. Thus $\omega \circ \delta$ extends uniquely to a positive linear functional ψ on $\overline{\mathfrak{B}}$. By Remark 1.3, $\mathcal{K} \subset \overline{\mathfrak{B}}$, so there is a unique $\varphi \in \mathcal{B}(F_{\mathcal{E}})_*$ which coincides with ψ on $\mathcal{K}$. But then $\varphi(K) = \omega \circ \delta(K)$ for every $K \in \mathfrak{B} \cap \mathcal{K}$, which by local normality implies that φ extends $\omega \circ \delta|\mathfrak{B}$.

Let $\rho_s := \varphi \circ l$. By Proposition 2.1 (2c), ρ_s is singular. Let ω_s be the unique decreasing locally normal positive linear functional on $\mathfrak{B}$ such that $\rho_s = \Delta\omega_s \circ l$, and let ω_e be the locally normal linear functional $\omega - \omega_s$. By Theorem 1.9 and Proposition 2.1 (1c), the proof will be complete once we establish that ω_e is positive and $\omega_e \circ \delta|\mathfrak{B} = 0$.

For every $K \in \mathfrak{B} \cap \mathcal{K}$ we have $\omega_e(K) = \omega(K) - \omega_s(K) = \omega(K) - \varphi(\lambda(K))$, so by local normality we have $\omega_e = \omega - \varphi \circ \lambda$. Consequently $\omega_e \circ \delta|\mathfrak{B} = 0$.

To show that ω_e is positive, we fix a positive integer m and show that the bounded linear functional $\omega_e|\mathfrak{B}_m$ achieves its norm at P_m, the identity element of the C^*-algebra $\mathfrak{B}_m$.

For every $k \geqslant 0$ we have $\omega_e = \omega_e \circ \beta^k$, so

$$\begin{aligned}\|\omega_e|\mathfrak{B}_m\| &= \|\omega_e \circ \beta^k|\mathfrak{B}_m\| \leqslant \|\omega \circ \beta^k|\mathfrak{B}_m\| + \|\omega_s \circ \beta^k|\mathfrak{B}_m\| \\ &= \omega \circ \beta^k(P_m) + \omega_s \circ \beta^k(P_m) = \sum_{i=0}^{m}(\omega(E_{k+i}) + \omega_s(E_{k+i})) \\ &\leqslant (m+1)(\omega(E_k) + \omega_s(E_k)),\end{aligned}$$

since $\omega(E_k)$ and $\omega_s(E_k)$ are monotonically nonincreasing in k. By Proposition 2.1 (2b), $\lim_{k\to\infty} \omega_s(E_k) = 0$, so $\|\omega_e|\mathfrak{B}_m\| \leqslant (m+1) \lim_{k\to\infty} \omega(E_k)$. On the other hand,

$$\omega_e(P_m) = \omega_e \circ \beta^k(P_m) = \sum_{i=0}^{m}(\omega(E_{k+i}) - \omega_s(E_{k+i})) \geqslant (m+1)(\omega(E_{k+m}) - \omega_s(E_k)),$$

so $\omega_e(P_m) \geqslant (m+1) \lim_{k\to\infty} \omega(E_k)$. Thus ω_e is positive. ∎

3. EXTENDING PRODUCT STATES

For each $\lambda \in \mathbb{T}$ the isometries $\{\lambda v_i \mid 1 \leqslant i \leqslant n\}$ satisfy the relations $(\lambda v_j)^*(\lambda v_i) = \delta_{ij}1$, and hence there is a $*$-endomorphism γ_λ of $\mathcal{TO}_n$ such that $\gamma_\lambda(v_i) = \lambda v_i$. Each γ_λ is actually an automorphism since $\gamma_\lambda \circ \gamma_{\lambda^{-1}}$ is the identity on $\mathcal{TO}_n$; in fact γ is a continuous automorphic action of $\mathbb{T}$ on $\mathcal{TO}_n$, called the *gauge* action. Denote by $\mathcal{F}_n$ the fixed-point algebra of this action, and let Φ denote the canonical conditional expectation of $\mathcal{TO}_n$ onto $\mathcal{F}_n$; that is,

$$\Phi(x) := \int_{\mathbb{T}} \gamma_\lambda(x) \mathrm{d}m(\lambda), \quad x \in \mathcal{TO}_n,$$

where m is normalized Haar measure. In terms of generating monomials,

$$\mathcal{F}_n = \overline{\mathrm{span}}\{v_\mu v_\nu^* \mid |\mu| = |\nu|\} \quad \text{and} \quad \Phi(v_\mu v_\nu^*) = \begin{cases} v_\mu v_\nu^* & \text{if } |\mu| = |\nu| \\ 0 & \text{otherwise.} \end{cases}$$

We are interested in *product states* of $\mathcal{F}_n$. To explain what we mean by this, let $\widetilde{\mathcal{K}} := \mathcal{K}(\mathcal{E}) \times \mathbb{C}$, endowed with the structure of a unital $*$-algebra via $(A,\lambda)(B,\mu) = (AB + \lambda B + \mu A, \lambda\mu)$ and $(A,\lambda)^* = (A^*, \overline{\lambda})$. When $\mathcal{E}$ is infinite-dimensional, $\widetilde{\mathcal{K}}$ is $*$-isomorphic to the concrete C^*-algebra $\mathcal{K}(\mathcal{E}) + \mathbb{C}I$, but when $\dim \mathcal{E} < \infty$ this is not the case. Nevertheless, it is not difficult to show that there is a unique C^*-norm on $\widetilde{\mathcal{K}}$. The map

$$v_{i_1} \cdots v_{i_m} v_{j_m}^* \cdots v_{j_1}^* \mapsto (v_{i_1} \otimes \overline{v_{j_1}}, 0) \otimes \cdots \otimes (v_{i_m} \otimes \overline{v_{j_m}}, 0) \otimes (0,1) \otimes (0,1) \otimes \cdots$$

embeds $\mathcal{F}_n$ in the infinite tensor product $\widetilde{\mathcal{K}}^{\otimes\infty}$. If $(\rho_k)_{k=1}^\infty$ is a sequence of states of $\mathcal{K}(\mathcal{E})$, so that $\widetilde{\rho}_k(K,\lambda) = \rho_k(K) + \lambda$ defines a sequence of states of $\widetilde{\mathcal{K}}$, we call the restriction of the product state $\bigotimes_{k=1}^{\infty} \widetilde{\rho}_k$ to $\mathcal{F}_n$ a *product state* of $\mathcal{F}_n$.

Now suppose $(e_k)_{k=1}^{\infty}$ is a sequence of unit vectors in $\mathcal{E}$. For each k, let ρ_k denote the vector state of $\mathcal{K}(\mathcal{E})$ corresponding to e_k, and let ρ denote the corresponding product state of $\mathcal{F}_n$. It is evident that ρ is pure and determined by

$$\rho(v_{i_1}\cdots v_{i_m}v_{j_m}^*\cdots v_{j_1}^*) = \langle v_{i_1}, e_1\rangle\cdots\langle v_{i_m}, e_m\rangle\langle e_m, v_{j_m}\rangle\cdots\langle e_1, v_{j_1}\rangle. \tag{3.1}$$

The remainder of this paper is devoted to classifying all extensions to $\mathcal{TO}_n$ of such a state.

One can always extend ρ by precomposing with Φ; the resulting extension $\rho\circ\Phi$ is called the *gauge-invariant* extension. The most extreme situation is when this extension is pure, in which case it is the unique state extending ρ. By [14], Theorem 4.3, this occurs precisely when the sequence (e_k) is *aperiodic* in the sense that the series

$$\sum_{i=1}^{\infty}(1-|\langle e_i, e_{i+p}\rangle|) \tag{3.2}$$

diverges for each positive integer p. In all other cases we say that (e_k) is *periodic*, and call the smallest positive integer p for which the series in (3.2) converges the *period* of (e_k).

Suppose then that (e_k) has finite period p. Notice from (3.1) that if we multiply each of the vectors e_k by a complex number of modulus one, we obtain a sequence which gives rise to the same product state ρ. Consequently, we are free to rephase so that $\langle e_i, e_{i+p}\rangle$ is always real and nonnegative.

THEOREM 3.1. *Suppose $\{v_1,\dots,v_n\}$ are the distinguished generating isometries of the Toeplitz-Cuntz algebra $\mathcal{TO}_n$; we include the case $n=\infty$ by writing $\mathcal{TO}_\infty$ for the Cuntz algebra $\mathcal{O}_\infty$. Let $\mathcal{E}\subseteq\mathcal{TO}_n$ be the closed linear span of $\{v_1,\dots,v_n\}$, and suppose (e_k) is a sequence of unit vectors in $\mathcal{E}$ which is periodic with finite period $p\geqslant 1$, and for which $\langle e_i, e_{i+p}\rangle$ is always nonnegative. Let ρ be the corresponding pure product state of $\mathcal{F}_n$ determined by* (3.1). *There is an affine isomorphism $\sigma\mapsto\rho_\sigma$ from $P(\mathbb{T})$, the space of Borel probability measures on the circle $\mathbb{T}$, to the space of all states of $\mathcal{TO}_n$ which extend ρ, given by*

$$\rho_\sigma(v_{i_1}\cdots v_{i_k}v_{j_l}^*\cdots v_{j_1}^*) = \lambda_{k,l}\langle v_{i_1}, e_1\rangle\cdots\langle v_{i_k}, e_k\rangle\langle e_l, v_{j_l}\rangle\cdots\langle e_1, v_{j_1}\rangle, \tag{3.3}$$

where

$$\lambda_{k,l} = \begin{cases}\widehat{\sigma}\left(\frac{k-l}{p}\right)\prod_{i=1}^{\infty}\langle e_{l+i}, e_{k+i}\rangle & \text{if } k-l\in p\mathbb{Z}\\ 0 & \text{otherwise,}\end{cases} \tag{3.4}$$

and $\widehat{\sigma}$ is the Fourier transform of σ.

Proof. For each $k \geqslant 0$ let $\mathbf{e}_k := e_1 \cdots e_k \in \mathcal{TO}_n$, where of course $\mathbf{e}_0 := 1$. Note that $\mathbf{e}_k \in \mathcal{E}_k = \overline{\mathrm{span}}\{v_\mu \mid \mu \in \mathcal{W}_k\}$. Equation (3.1) can now be written more tersely as

$$\rho(v_\mu v_\nu^*) = \langle v_\mu, \mathbf{e}_{|\mu|}\rangle\langle \mathbf{e}_{|\nu|}, v_\nu\rangle, \quad \mu, \nu \in \mathcal{W}, \ |\mu| = |\nu|,$$

which in turn extends by linearity and continuity to

$$\rho(xy^*) = \langle x, \mathbf{e}_k\rangle\langle \mathbf{e}_k, y\rangle, \quad x, y \in \mathcal{E}_k.$$

Similarly, (3.3) can be written as

$$\rho_\sigma(xy^*) = \lambda_{k,l}\langle x, \mathbf{e}_k\rangle\langle \mathbf{e}_l, y\rangle, \quad x \in \mathcal{E}_k, \ y \in \mathcal{E}_l. \tag{3.5}$$

Suppose now that $\widetilde{\rho}$ is a state of $\mathcal{TO}_n$ which extends ρ. By Schwarz inequality,

$$|\widetilde{\rho}(xy^*)| \leqslant \rho(xx^*)^{\frac{1}{2}}\rho(yy^*)^{\frac{1}{2}} = |\langle x, \mathbf{e}_k\rangle\langle y, \mathbf{e}_l\rangle|, \quad x \in \mathcal{E}_k, \ y \in \mathcal{E}_l. \tag{3.6}$$

Let $x_1 := \langle x, \mathbf{e}_k\rangle\mathbf{e}_k$, $x_2 := x - x_1$, $y_1 := \langle y, \mathbf{e}_l\rangle\mathbf{e}_l$ and $y_2 := y - y_1$. Then

$$\widetilde{\rho}(xy^*) = \widetilde{\rho}(x_1y_1^*) + \widetilde{\rho}(x_1y_2^*) + \widetilde{\rho}(x_2y_1^*) + \widetilde{\rho}(x_2y_2^*) = \widetilde{\rho}(x_1y_1^*);$$

that is,

$$\widetilde{\rho}(xy^*) = \widetilde{\rho}(\mathbf{e}_k\mathbf{e}_l^*)\langle x, \mathbf{e}_k\rangle\langle \mathbf{e}_l, y\rangle, \quad x \in \mathcal{E}_k, \ y \in \mathcal{E}_l. \tag{3.7}$$

For each $k, l \geqslant 0$ define

$$\lambda_{k,l} := \widetilde{\rho}(\mathbf{e}_k\mathbf{e}_l^*).$$

Comparing (3.5) and (3.7), it is apparent that we must exhibit a Borel probability measure σ such that (3.4) is satisfied.

For each positive integer k

$$\sum_{\mu \in \mathcal{W}_k} \widetilde{\rho}(v_\mu v_\mu^*) = \sum_{\mu \in \mathcal{W}_k} |\langle v_\mu, \mathbf{e}_k\rangle|^2 = \|\mathbf{e}_k\|^2 = 1,$$

so by [13], Corollary 2.9, $\widetilde{\rho}$ is essential. By Proposition 2.1 (1f), this implies that $\widetilde{\rho} = \beta^*\widetilde{\rho}$. In particular, for each $k, l \geqslant 0$

$$\begin{aligned}\lambda_{k,l} = \widetilde{\rho}(\mathbf{e}_k\mathbf{e}_l^*) = \beta^*\widetilde{\rho}(\mathbf{e}_k\mathbf{e}_l^*) &= \sum_{i=1}^n \widetilde{\rho}(\mathbf{e}_k v_i v_i^* \mathbf{e}_l^*) \\ &= \sum_{i=1}^n \widetilde{\rho}(\mathbf{e}_{k+1}\mathbf{e}_{l+1}^*)\langle \mathbf{e}_k v_i, \mathbf{e}_{k+1}\rangle\langle \mathbf{e}_{l+1}, \mathbf{e}_l v_i\rangle \qquad \text{(by (3.7))} \\ &= \sum_{i=1}^n \lambda_{k+1,l+1}\langle v_i, e_{k+1}\rangle\langle e_{l+1}, v_i\rangle = \lambda_{k+1,l+1}\langle e_{l+1}, e_{k+1}\rangle,\end{aligned}$$

and by induction

$$\lambda_{k,l} = \lambda_{k+j,l+j} \prod_{i=1}^{j} \langle e_{l+i}, e_{k+i} \rangle \quad \forall\, j. \tag{3.8}$$

Suppose now that p divides $|k-l|$. Since we have phased the sequence (e_i) so that $\langle e_i, e_{i+p} \rangle$ is always nonnegative, the assumption that (e_i) has period p means that $\sum |1 - \langle e_i, e_{i+p} \rangle| < \infty$. By [12], Proposition 1.2, it follows that $\sum |1 - \langle e_i, e_{i+m} \rangle| < \infty$ whenever p divides m. In particular we have

$$\sum_{i=1}^{\infty} |1 - \langle e_{l+i}, e_{k+i} \rangle| < \infty,$$

which implies that there is a positive integer i_0 such that

$$\lim_{j\to\infty} \prod_{i=i_0}^{i_0+j} \langle e_{l+i}, e_{k+i} \rangle$$

exists and is nonzero. Together with (3.8), this shows that $\lim_{m\to\infty} \lambda_{k+m,l+m}$ exists; indeed

$$\lim_{m\to\infty} \lambda_{k+m,l+m} = \lambda_{k+i_0-1,l+i_0-1} \Big(\prod_{i=i_0}^{\infty} \langle e_{l+i}, e_{k+i} \rangle \Big)^{-1}.$$

Since this limit depends only on the quantity $k-l$, we can define a function $\tau : \mathbb{Z} \to \mathbb{C}$ by $\tau_{a-b} := \lim_{m\to\infty} \lambda_{ap+m,bp+m}$ for $a, b \geqslant 0$.

We claim that τ is positive definite; that is, we claim that for any finite collection $z_0, \ldots, z_m$ of complex numbers, the sum $\sum_{a,b=0}^{m} z_a \overline{z}_b \tau_{a-b}$ is real and non-negative. To see this, define a sequence (w_i) in $\mathcal{TO}_n$ by $w_i := \sum_{a=0}^{m} z_a \mathbf{e}_{(a+i)p}$. Then

$$\begin{aligned} \sum_{a,b=0}^{m} z_a \overline{z}_b \tau_{a-b} &= \lim_{i\to\infty} \sum_{a,b=0}^{m} \lambda_{(a+i)p,(b+i)p} z_a \overline{z}_b \\ &= \lim_{i\to\infty} \sum_{a,b=0}^{m} \widetilde{\rho}(\mathbf{e}_{(a+i)p} \mathbf{e}_{(b+i)p}^*) z_a \overline{z}_b = \lim_{i\to\infty} \widetilde{\rho}(w_i w_i^*) \geqslant 0, \end{aligned}$$

as claimed.

By Herglotz's Theorem there is Borel probability measure σ on $\mathbb{T}$ such that $\tau = \widehat{\sigma}$. We claim that (3.4) is satisfied. The case $k - l \in p\mathbb{Z}$ follows immediately from (3.8) by letting $j \to \infty$. If p does not divide $k-l$, then by [12], Proposition 1.2, the series $\sum_{i=1}^{\infty}(1-|\langle e_{l+i}, e_{k+i}\rangle|)$ diverges, so that the infinite product $\prod_{i=1}^{\infty}|\langle e_{l+i}, e_{k+i}\rangle|$ diverges as well; in particular,

$$\lim_{j\to\infty}\prod_{i=1}^{j}\langle e_{l+i}, e_{k+i}\rangle = 0.$$

Since by (3.6) we have $|\lambda_{k+j,l+j}| \leqslant 1$ for each j, it follows from (3.8) that $\lambda_{k,l} = 0$. This completes the proof that every extension of ρ is of the form ρ_σ.

Conversely, suppose σ is a Borel probability measure on $\mathbb{T}$. Define coefficients $\lambda_{k,l}$ as in (3.4), and, resuming the notation and terminology of Section 1, define a locally normal linear functional ω on $\mathfrak{B}$ by

$$\omega(B) := \sum_{k,l=0}^{m}\lambda_{k,l}\langle B\mathbf{e}_l, \mathbf{e}_k\rangle, \quad B \in \mathfrak{B}_m. \tag{3.9}$$

We claim that ω is positive and decreasing, and that the functional $\Delta\omega \circ l$ of Proposition 1.5 is the desired state ρ_σ satisfying (3.5).

For $c = 0, 1, \ldots, p-1$, let $\mathcal{H}_c$ be the Hilbert space inductive limit of the isometric inclusions $\mathcal{E}_m \hookrightarrow \mathcal{E}_{m+1}$ determined by

$$x_1 \cdots x_m \mapsto x_1 \cdots x_m e_{m+c+1}, \quad x_i \in \mathcal{E}.$$

Modulo the isomorphisms $x_1 \cdots x_m \in \mathcal{E}_m \mapsto x_1 \otimes \cdots \otimes x_m \in \mathcal{E}^{\otimes m}$, $\mathcal{H}_c$ is just the infinite tensor product $\mathcal{E}^{\otimes\infty}$ with canonical unit vector $e_{c+1} \otimes e_{c+2} \otimes e_{c+3} \otimes \cdots$ introduced in [16]. Consequently [12], Proposition 1.1 applies: if (f_i) is a sequence of unit vectors in $\mathcal{E}$ such that $\sum_{i=1}^{\infty}|1 - \langle e_{c+i}, f_i\rangle| < \infty$, then $f_1 f_2 f_3 \cdots$ is a unit vector in $\mathcal{H}_c$. In particular, for each $a \geqslant 0$ we can define a vector $f_{c,a} \in \mathcal{H}_c$ by

$$f_{c,a} := e_{ap+c+1}e_{ap+c+2}e_{ap+c+3}\cdots.$$

By (3.4), $\lambda_{ap+c,bp+c} = \widehat{\sigma}(a-b)\langle f_{c,b}, f_{c,a}\rangle$.

Suppose now that B is an operator of bounded support on $F_{\mathcal{E}}$. Choose M

so that $B \in \mathfrak{B}_{Mp+p-1}$. Then

$$\begin{aligned}
\omega(B^*B) &= \sum_{k,l=0}^{Mp+p-1} \lambda_{k,l} \langle B\mathbf{e}_l, B\mathbf{e}_k\rangle = \sum_{c=0}^{p-1} \sum_{a,b=0}^{M} \lambda_{ap+c,bp+c} \langle B\mathbf{e}_{bp+c}, B\mathbf{e}_{ap+c}\rangle \\
&= \sum_{c=0}^{p-1} \sum_{a,b=0}^{M} \widehat{\sigma}(a-b) \langle f_{c,b}, f_{c,a}\rangle \langle B\mathbf{e}_{bp+c}, B\mathbf{e}_{ap+c}\rangle \\
&= \sum_{c=0}^{p-1} \int_{\mathbb{T}} \sum_{a,b=0}^{M} \gamma^{b-a} \langle f_{c,b} \otimes B\mathbf{e}_{bp+c}, f_{c,a} \otimes B\mathbf{e}_{ap+c}\rangle \,\mathrm{d}\sigma(\gamma) \\
&= \sum_{c=0}^{p-1} \int_{\mathbb{T}} \Big\langle \sum_{b=0}^{M} \gamma^b f_{c,b} \otimes B\mathbf{e}_{bp+c}, \sum_{a=0}^{M} \gamma^a f_{c,a} \otimes B\mathbf{e}_{ap+c} \Big\rangle \mathrm{d}\sigma(\gamma) \\
&= \sum_{c=0}^{p-1} \int_{\mathbb{T}} \Big\| \sum_{a=0}^{M} \gamma^a f_{c,a} \otimes B\mathbf{e}_{ap+c} \Big\|^2 \mathrm{d}\sigma(\gamma) \geqslant 0,
\end{aligned}$$

so ω is positive.

To see that ω is decreasing, suppose $B \in \mathfrak{B}_m$. Then $\beta(B) \in \mathfrak{B}_{m+1}$, so

$$\begin{aligned}
\omega \circ \beta(B) &= \sum_{k,l=0}^{m+1} \lambda_{k,l} \langle \beta(B)\mathbf{e}_l, \mathbf{e}_k\rangle = \sum_{k,l=0}^{m+1} \lambda_{k,l} \sum_{i=1}^{n} \langle r(v_i) B r(v_i)^* \mathbf{e}_l, \mathbf{e}_k\rangle \\
&= \sum_{k,l=1}^{m+1} \lambda_{k,l} \sum_{i=1}^{n} \langle e_l, v_i\rangle \langle v_i, e_k\rangle \langle B\mathbf{e}_{l-1}, \mathbf{e}_{k-1}\rangle \\
&= \sum_{k,l=1}^{m+1} \lambda_{k,l} \langle e_l, e_k\rangle \langle B\mathbf{e}_{l-1}, \mathbf{e}_{k-1}\rangle \\
&= \sum_{k,l=0}^{m} \lambda_{k+1,l+1} \langle e_{l+1}, e_{k+1}\rangle \langle B\mathbf{e}_l, \mathbf{e}_k\rangle = \omega(B)
\end{aligned}$$

since from (3.4) it is evident that $\lambda_{k,l} = \lambda_{k+1,l+1} \langle e_{l+1}, e_{k+1}\rangle$ for every k, l.

Let $\rho_\sigma = \Delta\omega \circ l$. If $x \in \mathcal{E}_k$ and $y \in \mathcal{E}_l$, then

$$\rho_\sigma(xy^*) = \omega(x \otimes \overline{y}) = \lambda_{k,l} \langle (x \otimes \overline{y})\mathbf{e}_l, \mathbf{e}_k\rangle = \lambda_{k,l} \langle x, \mathbf{e}_k\rangle \langle \mathbf{e}_l, y\rangle,$$

so ρ_σ satisfies (3.5) as claimed.

Since the Fourier transform is linear, it is evident from the defining formulas (3.3) and (3.4) that the map $\sigma \mapsto \rho_\sigma$ is affine. It remains only to show that $\sigma \mapsto \rho_\sigma$ is injective. For this, suppose $a \in \mathbb{Z}$. Since we have phased the sequence (e_i) so that $\langle e_i, e_{i+p}\rangle$ is always nonnegative, by [12], Proposition 1.2, the assumption that the series $\sum|1 - \langle e_i, e_{i+p}\rangle|$ converges implies that $\sum|1 - \langle e_i, e_{i+ap}\rangle|$ also converges. Consequently, the infinite product $\prod\langle e_i, e_{i+ap}\rangle$ converges; that is, there is a positive integer i_a such that $\prod\limits_{i=i_a}^{\infty} \langle e_i, e_{i+ap}\rangle$ exists and is nonzero. Since

$$\rho_\sigma(\mathbf{e}_{i_a+ap-1}\mathbf{e}^*_{i_a-1}) = \widehat{\sigma}(a) \prod_{i=i_a}^{\infty} \langle e_i, e_{i+ap}\rangle$$

and the Fourier transform is injective, this shows that $\sigma \mapsto \rho_\sigma$ is injective. ∎

COROLLARY 3.2. *Suppose ρ is a pure essential product state of $\mathcal{F}_n$ with finite period p. Then the gauge group acts p-to-1 transitively on the extensions of ρ to pure states of $\mathcal{TO}_n$. In particular, ρ has precisely a circle of extensions to pure states of $\mathcal{TO}_n$.*

Proof. Suppose $\widetilde{\rho}$ is an extension of ρ to a pure state of $\mathcal{TO}_n$. Then there is a Borel probability measure σ on $\mathbb{T}$ such that $\widetilde{\rho}$ is the extension ρ_σ of Theorem 3.1. Moreover, since $\widetilde{\rho}$ is pure, there is a $z \in \mathbb{T}$ such that σ is the point measure at z. If $\lambda \in \mathbb{T}$, then the pure state $\widetilde{\rho} \circ \gamma_\lambda$ is equal to ρ_φ, where φ is the point measure at $\lambda^p z$. Thus the gauge group acts p-to-1 transitively on the extensions of ρ to pure states of $\mathcal{TO}_n$. ∎

REMARK 3.3. We conjecture that Corollary 3.2 holds more generally for non-product states.

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