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CANONICAL SUBRELATIONS OF ERGODIC EQUIVALENCE RELATIONS-SUBRELATIONS

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Abstract. Given an ergodic measured discrete equivalence relation $\mathcal{R}$ and an ergodic subrelation $\mathcal{S} \subset \mathcal{R}$ of finite index, C. Sutherland showed that they are represented by the cross products $\mathcal{P} \rtimes_\alpha G$ and $\mathcal{P} \rtimes_\alpha H$ of an ergodic subrelation $\mathcal{P} \subset \mathcal{S}$ by a finite group outer action α_G and a subgroup action α_H. This result is strengthened in the sense that the subgroup H may be chosen so that it does not contain any non-trivial normal subgroup of G and that the collection $\{\mathcal{P}, H \subset G, \alpha_G\}$ is invariant for the orbit equivalence of the pair of $\mathcal{R}$ and $\mathcal{S}$. In amenable case of type II_1, a complete invariant for the orbit equivalence of pairs of an ergodic measured discrete equivalence relation and an ergodic subrelation of finite index is obtained.

Keywords: *Orbit equivalence, non-singular transformation, Jones index, measured equivalence relation.*

MSC (2000): Primary 28D99; Secondary 46L37.

1. INTRODUCTION

Let $(X, \mathfrak{B}, m)$ be a Lebesgue space and $\mathcal{R} \subset X \times X$ be a measured discrete equivalence relation. So, each orbit $\mathcal{R}(x) = \{y \in X \mid (y, x) \in \mathcal{R}\}$ is a countable set a.e. x. It is known that every measured discrete equivalence relation $\mathcal{R}$ can be characterized to be a subset

$$\mathcal{R} = \{(gx, x) \mid g \in \mathcal{G},\, x \in X\}$$

where $\mathcal{G}$ is a countable group of non-singular (invertible) transformations on $(X, \mathfrak{B}, m)$ (Feldman and Moore [2]). By m_l, we denote the measure on $\mathcal{R}$ defined by $\mathrm{d}m_l(y, x) = \mathrm{d}m(x)$, $(y, x) \in \mathcal{R}$. A measurable subset $\mathcal{S}$ of the Lebesgue space $(\mathcal{R}, m_l)$ is called a subrelation if it is an equivalence relation set. We say that pairs $\{\mathcal{R}, \mathcal{S}\}$ and $\{\mathcal{R}', \mathcal{S}'\}$ of a measured discrete equivalence relation and a subrelation are orbit equivalence if there exists a measure isomorphism (i.e. a measurable, non-singular and invertible map) φ satisfying

$$\varphi(\mathcal{R}(x)) = \mathcal{R}'(\varphi x) \quad \text{and} \quad \varphi(\mathcal{S}(x)) = \mathcal{S}'(\varphi x) \quad \text{a.e. } x.$$

We will show that given an ergodic measured discrete equivalence relation $\mathcal{R}$ and an ergodic subrelation $\mathcal{S}$ of finite index, there is a system of a subrelation $\mathcal{P} \subset \mathcal{S}$, a finite group G and a subgroup $H \subset G$ and an action $g \in G \mapsto \alpha_g \in N[\mathcal{P}]$ such that

(i) H does not contain any normal subgroup $\neq \{e\}$ of G,
(ii) α_G is outer,
(iii) $\mathcal{R} = \mathcal{P} \rtimes_\alpha G$ and $\mathcal{S} = \mathcal{P} \rtimes_\alpha H$;

where $N[\mathcal{P}]$ denotes the normalizer group of $\mathcal{P}$ (see Section 2). Moreover, the subrelation $\mathcal{P}$, the conjugacy class of the action α_G over $\mathcal{P}$ and the conjugacy class of the pair $\{G, H\}$ of a group and a subgroup satisfying the conditions (i)–(iii) are uniquely determined up to orbit equivalence of the pair $\{\mathcal{R}, \mathcal{S}\}$ (Theorem 4.1). So, we call this system the canonical system of the inclusion $\mathcal{R} \supset \mathcal{S}$. We note that the existence of $\{\mathcal{P}, H \subset G, \alpha_G\}$ satisfying the conditions (ii) and (iii) was shown by Sutherland ([10]).

The uniqueness of the canonical system will lead us to classifying the pairs of an amenable ergodic measured discrete equivalence relation and an ergodic subrelation of finite index. As a matter of fact, a generalization of Dye's theorem is obtained (Theorem 4.2). Namely, the conjugacy class of the pair $\{G, H\}$ of a finite group and a subgroup appearing in the canonical system is a complete invariant for the orbit equivalence in case of amenable relations of type II_1. About type III case, the classification will be discussed elsewhere ([7]).

An idea to prove the main theorem (Theorem 4.1) is to develop a discrete decomposition theorem for an index cocycle. Namely, as shown in [3], the pair $\{\mathcal{R}, \mathcal{S}\}$ provides an index cocycle. It is known that if a cocycle is a Radon-Nikodym derivative then a type III relation is decomposed into a type II_∞ relation and a $\mathbb{Z}$-action satisfying a scaling-down property through a cross product. So, the problem is what an analogue of the discrete decomposition of the pair $\mathcal{R} \supset \mathcal{S}$ for an index cocycle is.

For this, we will introduce the index ratio set (Definition 2.4). This is the pair of a finite group and a subgroup, whose conjugacy class is invariant for the orbit equivalence. Then, roughly speaking, the subrelation $\mathcal{P}$ and the action α_G

will be obtained in such a way that the type II_∞ relation and a $\mathbb{Z}$-action are obtained in the discrete decomposition of a type III relation using the Krieger's ratio set. The computation of an index ratio set will be described in the example of a measured discrete equivalence relation and an ergodic subrelation arising from a labeled graph (Section 5).

2. INDEX RATIO SET

Let $\mathcal{R} \supset \mathcal{S}$ be an ergodic measured discrete equivalence relation and an ergodic subrelation on $(X, \mathfrak{B}, m)$ (see [2]). We let

$$\begin{aligned}
[\mathcal{R}] = \{\psi \mid \psi & \text{ a mesurable, invertible, non-singular transformation such that} \\
& \psi x \in \mathcal{R}(x) \text{ a.e. } x\}, \\
[\mathcal{R}]_* = \{\psi \mid \psi & \text{ an invertible, non-singular map from a measurable subset} \\
& \mathrm{Dom}(\psi) \text{ onto a measurable subset } \mathrm{Im}(\psi) \text{ such that} \\
& \psi x \in \mathcal{R}(x) \text{ a.e. } x \in \mathrm{Dom}\,\psi\}, \text{ and} \\
N[\mathcal{R}] = \{\psi \mid \psi & \text{ a measurable, invertible, non-singular transformation such} \\
& \text{that } \psi(\mathcal{R}(x)) = \mathcal{R}(\psi(x)) \text{ a.e. } x\}.
\end{aligned}$$

We note that both $[\mathcal{R}]$ and $N[\mathcal{R}]$ are groups.

It is known from [3] that the function $x \in X \mapsto \#\{\mathcal{S}(y) \mid (y, x) \in \mathcal{R}\}$ is measurable and is a constant $\leqslant \infty$ a.e. x. By $[\mathcal{R} : \mathcal{S}]$, we denote this constant and call it the index of $\mathcal{S}$. The Jones index ([8]) of the Krieger factor and the subfactor constructed from the pair $\mathcal{R}$ and $\mathcal{S}$ is equal to $[\mathcal{R} : \mathcal{S}]$.

If $N = [\mathcal{R} : \mathcal{S}] < \infty$ then one can get the set of transformations φ_i in $[\mathcal{R}]$, $i = 0, 1, \ldots, N-1$, such that $\varphi_0 = \mathrm{id}$, and $\mathcal{R}(x) = \bigcup_{i=0}^{N-1} \mathcal{S}(\varphi_i x)$. These φ_i are called choice functions ([3]). If $(x, y) \in \mathcal{R}$ and $0 \leqslant i \leqslant N-1$, then an integer j is uniquely determined by $(\varphi_i y, \varphi_j x) \in \mathcal{S}$. Thus, we have the permutation $\sigma(x, y) \in \Sigma_N$ defined by $\sigma(x, y)(i) = j$. Here the Σ_N means the set of all permutations on the N objects. Obviously, $\sigma : (x, y) \in \mathcal{R} \mapsto \sigma(x, y) \in \Sigma_N$ is a homomorphism and is called the index cocycle of the pair $\mathcal{R}$ and $\mathcal{S}$ ([3]). We let $[\mathcal{R} : \mathcal{S}] = N < \infty$ and set

$$\begin{aligned}
\mathbf{r}_0(\mathcal{S}) = \{\theta \in \Sigma_N \mid & \text{ there exists for any measurable subset } E \text{ of positive} \\
& \text{measure a partial transformation } \varphi \in [\mathcal{S}]_* \text{ such that} \\
& \mathrm{Dom}(\varphi), \mathrm{Im}(\varphi) \subset E, \text{ and } \sigma(\varphi x, x) = \theta, \forall x \in \mathrm{Dom}(\varphi)\}.
\end{aligned}$$

Thus $\mathbf{r}_0(\mathcal{S})$ is a subgroup of Σ_N. By $\mathrm{Ker}(\sigma)$, we denote the subrelation $\{(x, y) \in \mathcal{R} \mid \sigma(x, y) = e\} \subset \mathcal{S}$.

LEMMA 2.1. $\#\{\mathrm{Ker}(\sigma)\text{-ergodic components}\} \leqslant N!$.

Proof. Consider the subrelation $\mathcal{Q}$ of $\mathcal{S}$ defined by

$$\mathcal{Q} = \{(x,y) \in \mathcal{S} \mid \sigma(x,y) \in \mathbf{r}_0(\mathcal{S})\}.$$

Let us choose any finite partition $\{A_\lambda \mid \lambda \in \Lambda\}$ of X consisting of $\mathcal{Q}$-invariant measurable subsets of positive measure. If $(x,y),(x,z) \in \mathcal{S}$ and if $x \in A_\gamma, y \in A_\lambda, z \in A_\mu$ and if $\lambda \neq \mu$, then since A_λ and A_μ are disjoint $\mathcal{Q}$-invariant sets, $\sigma(z,y) \neq e$. Hence $\sigma(y,x) \neq \sigma(z,x)$. This implies $\#(\Lambda) \leqslant \#(\Sigma_N) = N!$. We will show that both of the partitions of X by the $\mathcal{Q}$-ergodic components and by the $\mathrm{Ker}(\sigma)$-ergodic components respectively coincide with each other. For this let $\{A_\lambda \mid \lambda \in \Lambda\}$ be the finite partition consisting of all $\mathcal{Q}$-ergodic components.

Since $\mathrm{Ker}(\sigma) \subset \mathcal{Q}$, every $\mathcal{Q}$-invariant set is $\mathrm{Ker}(\sigma)$-invariant. We want to show that each A_λ is a $\mathrm{Ker}(\sigma)$-ergodic component. For this we let $\lambda \in \Lambda$ and E and F be measurable subsets of A_λ of positive measure. Since the restriction $\mathcal{Q}|_{A_\lambda}$ of $\mathcal{Q}$ to the set A_λ is ergodic, we obtain a $\varphi \in [\mathcal{Q}]_*$, and a $\theta \in \mathbf{r}_0(\mathcal{S})$ satisfying

$$\mathrm{Dom}(\varphi) \subset E, \quad \mathrm{Im}(\varphi) \subset F \quad \text{and} \quad \sigma(\varphi x, x) = \theta, \quad \forall\, x \in \mathrm{Dom}(\varphi).$$

By definition of $\mathbf{r}_0(\mathcal{S})$, there exists a $\psi \in [\mathcal{S}]_*$ satisfying

$$\mathrm{Dom}(\psi), \mathrm{Im}(\psi) \subset \mathrm{Im}(\varphi) \quad \text{and} \quad \sigma(\psi x, x) = \theta^{-1}, \quad \forall\, x \in \mathrm{Dom}(\varphi).$$

Then,

$$\sigma(\psi \cdot \varphi x, x) = \sigma(\psi \cdot \varphi x, \varphi x)\sigma(\varphi x, x) = \theta^{-1}\theta = e, \quad \forall\, x \in \varphi^{-1}(\mathrm{Dom}(\psi)).$$

Hence $\psi \cdot \varphi \in [\mathrm{Ker}(\sigma)]_*$, $\mathrm{Dom}(\psi \cdot \varphi) \subset E$, and $\mathrm{Im}(\psi \cdot \varphi) \subset F$. On the other hand, since $\mathrm{Ker}(\sigma) \subset \mathcal{Q}$, the set A_λ is a $\mathrm{Ker}(\sigma)$-invariant. Therefore A_λ is a $\mathrm{Ker}(\sigma)$-ergodic component. ∎

Throughout the rest of this section, we let $\{A_\lambda \mid \lambda \in \Lambda\}$ be the partition of X consisting of all $\mathrm{Ker}(\sigma)$-ergodic components. Let $\lambda \in \Lambda$ and set

$$\begin{aligned}\mathbf{r}_\lambda(\mathcal{R}) = \{\theta \in \Sigma_N \mid\ & \text{there exists for any measurable subset } A \subset A_\lambda \text{ of positive} \\ & \text{measure a } \varphi \in [\mathcal{R}]_* \text{ such that } \mathrm{Dom}(\varphi), \mathrm{Im}(\varphi) \subset A \text{ and} \\ & \sigma(\varphi x, x) = \theta, \forall\, x \in \mathrm{Dom}(\varphi)\},\\ \mathbf{r}_\lambda(\mathcal{S}) = \{\theta \in \Sigma_N \mid\ & \text{there exists for any measurable subset } A \subset A_\lambda \text{ of positive} \\ & \text{measure a } \varphi \in [\mathcal{S}]_* \text{ such that } \mathrm{Dom}(\varphi), \mathrm{Im}\,\varphi \subset A, \text{ and} \\ & \sigma(\varphi x, x) = \theta, \forall\, x \in \mathrm{Dom}(\varphi)\}.\end{aligned}$$

Then both $\mathbf{r}_\lambda(\mathcal{R})$ and $\mathbf{r}_\lambda(\mathcal{S})$ are the subgroups of Σ_N.

Lemma 2.2. *Let* $\lambda \in \Lambda$. *Then*

(i) $\sigma(y,x) \in \mathbf{r}_\lambda(\mathcal{S})$, *a.e.* $(y,x) \in \mathcal{S}$ *with* $x, y \in A_\lambda$, *and*

(ii) $\sigma(y,x) \in \mathbf{r}_\lambda(\mathcal{R})$, *a.e.* $(y,x) \in \mathcal{R}$ *with* $x, y \in A_\lambda$.

Proof. (i) Let $\varphi \in [\mathcal{S}]_*$ be such that $\mathrm{Dom}(\varphi)$, $\mathrm{Im}(\varphi) \subset A_\lambda$ and $\sigma(\varphi x, x) =$ a constant $= \theta$, $\forall\, x \in \mathrm{Dom}(\varphi)$. We show that $\theta \in \mathbf{r}_\lambda(\mathcal{S})$. Since the restriction $\mathrm{Ker}(\sigma)|_{A_\lambda}$ is ergodic, there exist for every set $E \subset A_\lambda$ of positive measure, partial transformations $\psi_i \in [\mathcal{P}]_*$, $i = 1, 2$ satisfying that $\mathrm{Dom}(\psi_1) \subset E$, $\mathrm{Im}\,\psi_1 \subset \mathrm{Dom}(\varphi)$ and $\mathrm{Dom}(\psi_2) \subset \mathrm{Im}(\varphi \cdot \psi_1)$, $\mathrm{Im}(\psi_2) \subset E$. So, we see that $\mathrm{Dom}(\psi_2 \cdot \varphi \cdot \psi_1), \mathrm{Im}(\psi_2 \cdot \varphi \cdot \psi_1) \subset E$ and that

$$\begin{aligned}\sigma(\psi_2 \cdot \varphi \cdot \psi_1 x, x) =& \sigma(\psi_2 \cdot \varphi \cdot \psi_1 x, \varphi \cdot \psi_1 x)\sigma(\varphi \cdot \psi_1 x, \psi_1 x)\sigma(\psi_1 x, x)\\ =& e \cdot \theta \cdot e = \theta,\end{aligned}$$

$\forall\, x \in \mathrm{Dom}(\psi_2 \cdot \varphi \cdot \psi_1) \subset E$. Since $\psi_2 \cdot \varphi \cdot \psi_1 \in [\mathcal{S}]_*$, we have $\theta \in \mathbf{r}_\lambda(\mathcal{S})$. The proof of (ii) is similarly done. So, we omit it. ∎

Lemma 2.3. *There exist permutations* $\theta_{\lambda,\mu} \in \Sigma_N$, $\lambda, \mu \in \Lambda$ *satisfying the following conditions:*

(i) *For a.e.* $(y,x) \in \mathcal{R}$ *with* $y \in A_\mu$ *and* $x \in A_\lambda$, $(y,x) \in \mathcal{S}$ *if and only if* $\sigma(y,x) \in \mathrm{r}_\mu(\mathcal{S}) \cdot \theta_{\mu,\lambda}$;

(ii) $\theta_{\lambda,\mu} \cdot \theta_{\mu,\gamma} = \theta_{\lambda,\gamma}$, $\theta_{\lambda,\lambda} = e$;

(iii) $\theta_{\lambda,\mu} \cdot \mathbf{r}_\mu(\mathcal{S}) \cdot \theta_{\lambda,\mu}^{-1} = \mathbf{r}_\lambda(\mathcal{S})$, $\theta_{\lambda,\mu} \cdot \mathbf{r}_\mu(\mathcal{R}) \cdot \theta_{\lambda,\mu}^{-1} = \mathbf{r}_\lambda(\mathcal{R})$.

Proof. We choose and fix a λ_0 in Λ. Let $\lambda \in \Lambda$, then since $\mathcal{S}$ is ergodic there exists a partial transformation $\varphi \in [\mathcal{S}]_*$ such that $\mathrm{Dom}(\varphi) \subset A_{\lambda_0}$, $\mathrm{Im}(\varphi) \subset A_\lambda$, $\sigma(\varphi x, x) =$ a constant, $\forall\, x \in \mathrm{Dom}(\varphi)$, and such that $\varphi x = x$, $\forall\, x \in \mathrm{Dom}(\varphi)$, if $\lambda = \lambda_0$. By φ_λ and $\theta_{\lambda,\lambda_0}$ we denote such a partial transformation φ and the corresponding constant in Σ_N. If φ' is another choice of a partial transformation in $[\mathcal{S}]_*$ having the corresponding constant θ', then since $\mathrm{Ker}(\sigma)|_{A_{\lambda_0}}$ is ergodic, we obtain a partial transformation $\psi \in [\mathrm{Ker}(\sigma)]_*$ such that $\mathrm{Dom}(\psi) \subset \mathrm{Dom}(\varphi_\lambda)$, $\mathrm{Im}(\psi) \subset \mathrm{Dom}(\varphi')$. Then,

$$\varphi' \cdot \psi \cdot \varphi_\lambda^{-1}|_{\varphi_\lambda(\mathrm{Dom}(\psi))} \in [\mathcal{S}|_{A_\lambda}]_*,$$

and

$$\sigma(\varphi' \cdot \psi \cdot \varphi_\lambda^{-1} x, x) = \theta' \cdot \theta_{\lambda,\lambda_0}^{-1}, \quad \forall\, x \in \varphi_\lambda(\mathrm{Dom}(\psi)).$$

So, by Lemma 2.2, we see $\theta' \cdot \theta_{\lambda,\lambda_0}^{-1} \in \mathbf{r}_\lambda(\mathcal{S})$. Thus, $\sigma(y,x) \in \mathbf{r}_\lambda(\mathcal{S}) \cdot \theta_{\lambda,\lambda_0}$ for a.e. $(y,x) \in \mathcal{S}$ with $y \in A_\lambda$ and $x \in A_{\lambda_0}$. Similarly, we see that $\sigma(y,x) \in \mathbf{r}_{\lambda_0}(\mathcal{S}) \cdot \theta_{\lambda,\lambda_0}^{-1}$ for a.e. $(y,x) \in \mathcal{S}$ with $y \in A_{\lambda_0}$ and $x \in A_\lambda$.

Next if $h \in \mathbf{r}_\lambda(\mathcal{S})$ then we choose a partial transformation $\varphi \in [\mathcal{S}]_*$ such that

$$\mathrm{Dom}(\varphi), \mathrm{Im}(\varphi) \subset \mathrm{Im}(\varphi_\lambda), \quad \sigma(\varphi x, x) = h, \quad \forall\, x \in \mathrm{Dom}(\varphi).$$

So,

$$\varphi_\lambda^{-1} \cdot \varphi \cdot \varphi_\lambda|_{\varphi_\lambda^{-1}(\mathrm{Dom}(\psi))} \in [\mathcal{S}|_{A_{\lambda_0}}]_*,$$

and

$$\sigma(\varphi_\lambda^{-1} \cdot \varphi \cdot \varphi_\lambda x, x) = \theta_{\lambda,\lambda_0}^{-1} \cdot h \cdot \theta_{\lambda,\lambda_0}, \quad \forall\, x \in \varphi_\lambda^{-1}(\mathrm{Dom}(\varphi)).$$

By Lemma 2.2, $\theta_{\lambda,\lambda_0}^{-1} h \theta_{\lambda,\lambda_0} \in \mathbf{r}_{\lambda_0}(\mathcal{S})$. Thus, $\theta_{\lambda,\lambda_0}^{-1} \cdot \mathbf{r}_\lambda(\mathcal{S}) \cdot \theta_{\lambda,\lambda_0} \subset \mathbf{r}_{\lambda_0}(\mathcal{S})$. Similarly we see that if $h \in \mathbf{r}_\lambda(\mathcal{R})$, then $\theta_{\lambda,\lambda_0}^{-1} h \theta_{\lambda,\lambda_0} \in \mathbf{r}_{\lambda_0}(\mathcal{R})$ and that $\theta_{\lambda,\lambda_0}^{-1} \cdot \mathbf{r}_\lambda(\mathcal{R}) \cdot \theta_{\lambda,\lambda_0} \subset \mathbf{r}_{\lambda_0}(\mathcal{R})$.

Replacing λ_0 by λ in the above argument, we see $\theta_{\lambda,\lambda_0} \cdot \mathbf{r}_{\lambda_0}(\mathcal{S}) \cdot \theta_{\lambda,\lambda_0}^{-1} \subset \mathbf{r}_\lambda(\mathcal{S})$ and $\theta_{\lambda,\lambda_0} \cdot \mathbf{r}_{\lambda_0}(\mathcal{R}) \cdot \theta_{\lambda,\lambda_0}^{-1} \subset \mathbf{r}_\lambda(\mathcal{R})$.

We define

$$\theta_{\lambda_0,\lambda} = \theta_{\lambda,\lambda_0}^{-1}, \quad \theta_{\lambda,\mu} = \theta_{\lambda,\lambda_0} \cdot \theta_{\lambda_0,\mu}, \quad \lambda, \mu \in \Lambda.$$

Then,

$$\theta_{\lambda,\mu} \cdot \mathbf{r}_\mu(\mathcal{S}) \cdot \theta_{\lambda,\mu}^{-1} = \mathbf{r}_\lambda(\mathcal{S}), \quad \theta_{\lambda,\mu} \cdot \mathbf{r}_\mu(\mathcal{R}) \cdot \theta_{\lambda,\mu}^{-1} = \mathrm{r}_\lambda(\mathcal{R}),$$
$$\sigma(y,x) \in \mathbf{r}_\mu(\mathcal{S}) \cdot \theta_{\mu,\lambda} \quad \text{a.e. } (y,x) \in \mathcal{S} \text{ with } y \in A_\mu, x \in A_\lambda.$$

Finally, we will show that for a.e. $(y,x) \in \mathcal{R}$ with $y \in A_\mu$ and $x \in A_\lambda$, (y,x) is in $\mathcal{S}$ if $\sigma(y,x) \in \mathbf{r}_\mu(\mathcal{S}) \cdot \theta_{\mu,\lambda}$. To see this, let $\lambda, \mu \in \Lambda$, $\varphi \in [\mathcal{R}]_*$ and $h \in \mathbf{r}_\mu(\mathcal{S})$ be such that

$$\mathrm{Dom}(\varphi) \subset A_\lambda, \quad \mathrm{Im}(\varphi) \subset A_\mu, \quad \sigma(\varphi x, x) = h\theta_{\mu,\lambda}, \quad \forall\, x \in \mathrm{Dom}(\varphi).$$

Since $\mathrm{Ker}(\sigma)|_{A_\mu}$ is ergodic, we get a partial transformation $\psi \in [\mathcal{S}]_*$ such that

$$\mathrm{Dom}(\psi) \subset \mathrm{Im}(\varphi), \quad \mathrm{Im}(\psi) \subset \mathrm{Im}(\varphi_\mu), \quad \sigma(\psi x, x) = h^{-1}, \quad \forall\, x \in \mathrm{Dom}(\psi).$$

Similarly, we have a partial transformation $\psi' \in [\mathrm{Ker}(\sigma)|_{A_{\lambda_0}}]_*$ such that

$$\mathrm{Dom}(\psi') \subset \mathrm{Im}(\varphi_\mu^{-1}(\mathrm{Im}(\psi))), \quad \mathrm{Im}(\psi') \subset \mathrm{Dom}(\varphi_\lambda).$$

Then, by setting $\psi'' = \varphi_\lambda \cdot \psi' \cdot \varphi_\mu^{-1} \cdot \psi \cdot \varphi \in [\mathcal{R}]_*$, we have $\sigma(\psi'' x, x) = e$, $\forall\, x \in \mathrm{Dom}(\psi'') \subset A_\lambda$, so that $\psi'' \in [\mathrm{Ker}(\sigma)\,|_{A_\lambda}]_*$. Thus, $\varphi = \psi^{-1} \cdot \varphi_\mu \cdot \psi'^{-1} \cdot \varphi_\lambda^{-1} \cdot \psi''^{-1} \in [\mathcal{S}]_*$. ∎

By Lemma 2.3, the conjugacy class of the group $\mathbf{r}_\lambda(\mathcal{R})$ and the subgroup $\mathbf{r}_\lambda(\mathcal{S})$ does not depend on a choice of $\lambda \in \Lambda$. So, we have

DEFINITION 2.4. We call the conjugacy class of the pair of the finite group $\mathbf{r}_\lambda(\mathcal{R})$ and the subgroup $\mathbf{r}_\lambda(\mathcal{S})$ the *index ratio set* and denote it by $\{\mathbf{r}(\mathcal{R}),\, \mathbf{r}(\mathcal{S})\}$.

LEMMA 2.5. *The pair of the index ratio sets* $\mathbf{r}(\mathcal{R})$ *and* $\mathbf{r}(\mathcal{S})$ *does not depend on a choice of the set of choice functions* $\{\varphi_i\}_{0\leqslant i\leqslant N-1}$ *and in fact depends only on the orbit equivalence class of* $\mathcal{R}$ *and* $\mathcal{S}$.

Proof. If σ' is the index cocycle determined by another set of choice functions φ_i' of $\mathcal{S} \subset \mathcal{R}$, then σ and σ' are cohomologous ([3]), that is, there exists a measurable function $x \in X \mapsto v(x) \in \Sigma_N$ such that

$$\sigma'(x,y) = v(x)\sigma(x,y)v(y)^{-1}, \quad (x,y) \in \mathcal{R}.$$

Let $\lambda_0 \in \Lambda$ and let $\{B_i \mid i \in I\}$ be the finite partition of X consisting of all $\mathrm{Ker}(\sigma')$-ergodic components. Let $i \in I$ be such that $m(A_{\lambda_0} \cap B_i) > 0$, then we have a γ in Σ_N such that $m(\{x \in X \mid v(x) = \gamma\} \cap A_{\lambda_0} \cap B_i) > 0$. Applying Lemma 2.2, we see that except on a null set, the range of σ and σ' of the restriction of $\mathcal{S}$ to this intersection coincide with $\mathbf{r}_{\lambda_0}(\mathcal{S})$ and $\mathbf{r}_i(\mathcal{S})$ respectively. So, we have $\mathbf{r}_i(\mathcal{S}) = \gamma \cdot \mathbf{r}_{\lambda_0}(\mathcal{S}) \cdot \gamma^{-1}$. Similarly we have $\mathbf{r}_i(\mathcal{R}) = \gamma \cdot \mathbf{r}_{\lambda_0}(\mathcal{R}) \cdot \gamma^{-1}$. ∎

We let $\widetilde{\mathcal{R}}$ be the measured discrete equivalence relation on $(X \times \mathbb{R}, m \times e^u \mathrm{d}u)$ defined by

$$((x,u),(y,v)) \in \widetilde{\mathcal{R}}$$

if $(x,y) \in \mathcal{R}$ and $v = u - \log \delta(y,x)$, where (x,u),$(y,v) \in X \times \mathbb{R}$. Here $\delta(x,y)$ means the Radon-Nikodym derivative. Then $\widetilde{\mathcal{R}}$ is of type II_∞. By $X^{\mathcal{R}}$ we denote the quotient space of $X \times \mathbb{R}$ by the measurable partition consisting of all ergodic components of $\widetilde{\mathcal{R}}$. We let $\pi^{\mathcal{R}}$ be the natural surjection from $X \times \mathbb{R}$ to $X^{\mathcal{R}}$. By $\{T_t \mid t \in \mathbb{R}\}$, we denote the flow $T_t(x,u) = (x, u+t)$ for $(x,u) \in X \times \mathbb{R}$, $t \in \mathbb{R}$. By $\{F_t^{\mathcal{R}} \mid t \in \mathbb{R}\}$, we denote the factor flow of $\{T_t \mid t \in \mathbb{R}\}$ to the quotient space $X^{\mathcal{R}}$ through the factor map $\pi^{\mathcal{R}}$, that is, $\pi^{\mathcal{R}} T_t = F_t^{\mathcal{R}} \pi^{\mathcal{R}}$, $\forall\, t \in \mathbb{R}$. The flow $\{F_t^{\mathcal{R}} \mid t \in \mathbb{R}\}$ is called the associated flow of $\mathcal{R}$ ([4]) and simply denoted by $F^{\mathcal{R}}$.

It is known that $\mathcal{R}$ is ergodic and of type II if and only if $F^{\mathcal{R}}$ is the translation $u \in \mathbb{R} \mapsto u + t \in \mathbb{R}$, $t \in \mathbb{R}$ ([4]).

Lemma 2.6. (i) *$\mathcal{R}$ is of type* II_1 *if and ouly if $\mathcal{S}$ is of type* II_1.
(ii) *$\mathcal{R}$ is of type* II_∞ *if and ouly if $\mathcal{S}$ is of type* II_∞.

Proof. For almost all $\widetilde{\mathcal{S}}$-ergodic component there exists a uniquely determined $\widetilde{\mathcal{R}}$-ergodic component containing it. By $\pi^{\mathcal{S}}_{\mathcal{R}}$, we denote this map. Then,

$$\pi^{\mathcal{S}}_{\mathcal{R}} F^{\mathcal{S}}_t = F^{\mathcal{R}}_t \pi^{\mathcal{S}}_{\mathcal{R}}, \quad \forall\, t \in \mathbb{R}.$$

The number of $\widetilde{\mathcal{S}}$-ergodic components contained in the $\widetilde{\mathcal{R}}$-ergodic component containing a point (x, u) is at most N a.e. (x, u). Since $\pi^{\mathcal{S}}_{\mathcal{R}}$ is a finite to 1 factor map, the flow $F^{\mathcal{R}}$ is the translation if and only if so is $F^{\mathcal{S}}$, that is, $\mathcal{R}$ is of type II if and only if so is $\mathcal{S}$. In this case let μ be an invariant measure for $\mathcal{S}$. Then the uniqueness of invariant measure (up to constant) implies that μ is $\mathcal{R}$-invariant, too. ∎

From now on in this section, we denote by G and $H \subset G$ the finite group $\mathbf{r}_{\lambda_0}(\mathcal{R})$ and the subgroup $\mathbf{r}_{\lambda_0}(\mathcal{S})$. By $\{A_\lambda \mid \lambda \in \Lambda\}$, we denote the finite partition of X consisting of all $\mathrm{Ker}(\sigma)$-ergodic components.

Lemma 2.7. *There exists an action $g \in G \mapsto \alpha_g \in [\mathcal{R}] \cap N[\mathrm{Ker}(\sigma)]$ satisfying the following conditions:*

(i) *α_G is outer over $\mathrm{Ker}(\sigma)$, that is, if $\alpha_g \in [\mathrm{Ker}(\sigma)]$ then $g = e$;*
(ii) *$\alpha_h \in [\mathcal{S}]$, $\forall\, h \in H$;*
(iii) *$\sigma(\alpha_g x, x) = \theta_{\lambda,\lambda_0} g \theta_{\lambda_0,\lambda}$, $\forall\, x \in A_\lambda$, $\forall\, \lambda \in \Lambda$;*
(iv) *$\alpha_g(A_\lambda) = A_\lambda$, $\forall\, g \in A_\lambda$, $\forall\, \lambda \in \Lambda$.*

Proof. It is enough to construct such an action α_G on each $\mathrm{Ker}(\sigma)$-ergodic component. Let us first assume that $\mathcal{R}$ is of type II_1, and let $\lambda \in \Lambda$.

We choose a subset $E_\lambda \subset A_\lambda$ with $m(E_\lambda) = \frac{m(A_\lambda)}{\#(G)}$. The ergodicity of $\mathrm{Ker}(\sigma)|_{A_\lambda}$ allows us to get partial transformations $\eta_g \in [\mathcal{R}|_{A_\lambda}]_*$, $g \in G$, such that

$$\eta_e = \mathrm{id}|_{E_\lambda}, \quad \mathrm{Dom}(\eta_g) = E_\lambda, \quad \sigma(\eta_g x, x) = \theta_{\lambda,\lambda_0} g \theta_{\lambda_0,\lambda}, \qquad \text{a.e. } x \in E_\lambda.$$

By (i) of Lemma 2.3, if $g \in H$ then $\eta_g \in [\mathcal{S}|_{A_\lambda}]_*$.

We choose a finite partition $\{K_g \mid g \in G\}$ of A_λ with $m(K_g) = \frac{m(A_\lambda)}{\#(G)}$, $g \in G$. Since $\mathrm{Im}(\eta_g)$ and K_g are $\mathrm{Ker}(\sigma)$-Hopf equivalent, there exists a v_g in $[\mathrm{Ker}(\sigma)|_{A_\lambda}]_*$ such that $\mathrm{Dom}(v_g) = K_g$, $\mathrm{Im}(v_g) = \mathrm{Im}(\eta_g)$.

We define the transformation α_f, $f \in G$, on each A_λ by

$$\alpha_f x = v_{fg}^{-1} \cdot \eta_{fg} \cdot \eta_g^{-1} \cdot v_g x, \quad x \in K_g,\ g \in G.$$

Then, obviously $\alpha_f \in [\mathcal{R}]$ and (iv) is satisfied.

To see (ii) and (iii),

$$\begin{aligned}\sigma(\alpha_f x, x) &= \sigma(\alpha_f x, \eta_{fg}\cdot\eta_g^{-1}\cdot v_g x)\sigma(\eta_{fg}\cdot\eta_g^{-1}\cdot v_g x, \eta_g^{-1}\cdot v_g x)\\ &\quad\cdot\sigma(\eta_g^{-1}v_g x, v_g x)\sigma(v_g x, x)\\ &= e\cdot\theta_{\lambda,\lambda_0} fg\theta_{\lambda,\lambda_0}^{-1}\cdot\theta_{\lambda,\lambda_0} g^{-1}\cdot\theta_{\lambda,\lambda_0}^{-1}\cdot e = \theta_{\lambda,\lambda_0} f\theta_{\lambda,\lambda_0}, \qquad x\in K_g.\end{aligned}$$

In particular, if $f\in H$, then $\sigma(\alpha_f x, x)\in \mathbf{r}_\lambda(\mathcal{S})$, and hence by Lemma 2.3, $\alpha_f\in[\mathcal{S}]$.

Finally, let us check that α_G is an outer action of $\mathrm{Ker}(\sigma)$. If $(x,y)\in\mathrm{Ker}(\sigma)$ and if $(x,y)\in A_\lambda$ then for all $g\in G$

$$\sigma(\alpha_g x,\alpha_g y) = \sigma(\alpha_g x, x)\sigma(x,y)\sigma(y,\alpha_g y) = \theta_{\lambda,\lambda_0} g\theta_{\lambda_0,\lambda}\cdot e\cdot\theta_{\lambda,\lambda_0} g^{-1}\theta_{\lambda_0,\lambda} = e.$$

Thus, $\alpha_g\in N[\mathrm{Ker}(\sigma)]$. Let $g\in G$ be such that for some set E of positive measure, $(\alpha_g x, x)\in\mathrm{Ker}(\sigma)$, $\forall\, x\in E$, then since $\sigma(\alpha_g x, x) = \theta_{\lambda,\lambda_0} g\theta_{\lambda_0,\lambda}$, $\forall\, x\in E$, we see that $g=e$.

In the case that $\mathcal{R}$ is of type II_∞, if $m(A_\lambda)=\infty$ then we replace the requirement $m(E_\lambda)=\frac{m(A_\lambda)}{\#(G)}$ in the above sequel by $m(E_\lambda)=\infty$ and $m(K_g)=\frac{m(A_\lambda)}{\#(G)}$ by $m(K_g)=\infty$, $\forall\, g\in G$ respectively. Then the proof is done by the similar argument. In the case that $\mathcal{R}$ is of type III, so is $\mathrm{Ker}(\sigma)$ by Lemma 2.6. In this case we do not need the requirement $m(K_g)=\frac{m(A_\lambda)}{\#(G)}$ anymore. ∎

DEFINITION 2.8. For a measured discrete equivalence relation $\mathcal{T}$ and an action $\gamma\in\Gamma\mapsto\beta_\gamma\in N[\mathcal{T}]$ of a countable group Γ, the relation $\mathcal{U}$ defined by

$$(x,y)\in\mathcal{U}\quad\text{if}\quad y\in\bigcup_{\gamma\in\Gamma}\mathcal{T}(\beta_\gamma x)$$

is called the *cross product of* $\mathcal{T}$ by β_Γ, and denote it by $\mathcal{T}\rtimes_\beta\Gamma$.

We notice that the action α_G in Lemma 2.7 is free, that is, if $g\in G$ is not e then $\alpha_g x\neq x$ a.e. x. So, it allows us to get a Rholin set for the action α_G. Namely, there exists for each $\lambda\in\Lambda$ a measurable set $F_\lambda\subset A_\lambda$ such that

$$A_\lambda = \bigcup_{\lambda\in\Lambda}\alpha_g(F_\lambda)\quad\text{(disjoint union)}.$$

In the sequel of this section we will fix the subsets $F_\lambda\in\Lambda$, $\lambda\in\Lambda$.

LEMMA 2.9. *Let $\lambda \in \Lambda$. Then,*

$$\mathrm{Ker}(\sigma)|_{A_\lambda} \rtimes_\alpha G = \mathcal{R}|_{A_\lambda} \quad \textit{and} \quad \mathrm{Ker}(\sigma)|_{A_\lambda} \rtimes_\alpha H = \mathcal{S}|_{A_\lambda}$$

Proof. If $\lambda \in \Lambda$, $(x, y) \in \mathcal{R}$ and $x, y \in A_\lambda$, then

$$\sigma(\alpha_{\sigma(y,x)}x, y) = \sigma(\alpha_{\sigma(y,x)}x, x)\sigma(x, y) \overset{\text{Lemma 2.7(iii)}}{=} \sigma(y, x)\sigma(x, y) = e.$$

Hence,

$$y \in \mathrm{Ker}(\sigma)(\alpha_{\sigma(y,x)}x) = \alpha_{\sigma(y,x)}(\mathrm{Ker}(\sigma)(x)).$$

Thus, $y \in (\mathrm{Ker}(\sigma)|_{A_\lambda} \rtimes_\alpha G)(x)$, if $(x, y) \in \mathcal{R}$ with $x, y \in A_\lambda$ and $y \in (\mathrm{Ker}(\sigma)|_{A_\lambda} \rtimes_\alpha H)(x)$, if $(x, y) \in \mathcal{S}$ with $x, y \in A_\lambda$. ∎

We took the partial transformations $\varphi_\lambda, \lambda \in \Lambda$ in the proof of Lemma 2.3. Since $\mathrm{Ker}(\sigma)|_{A_{\lambda_0}}$ is ergodic, we may and do assume that these have the same domain. So, we define the partial transformations $\psi_{\mu,\lambda}$ in $[\mathcal{S}]_*$ by

$$\begin{aligned} \psi_{\lambda,\lambda_0} &= \varphi_\lambda \\ \psi_{\lambda_0,\lambda} &= \varphi_\lambda^{-1} \\ \psi_{\lambda,\mu} &= \psi_{\lambda,\lambda_0}\psi_{\lambda_0,\mu}, \quad \lambda, \mu \in \Lambda. \end{aligned}$$

DEFINITION 2.10. We define the subrelation $\mathcal{P}$ of $\mathcal{S}$ as follows: Let $x \in A_\mu$, $y \in A_\lambda$, where λ, $\mu \in \Lambda$. Then $(x, y) \in \mathcal{P}$ if either $\mu = \lambda$ and $(x, y) \in \mathrm{Ker}(\sigma)$, or, $\mu \neq \lambda$ and $(u, y), (x, \psi_{\mu,\lambda}u) \in \mathrm{Ker}(\sigma)$ for some $u \in A_\lambda$.

THEOREM 2.11. *The equivalence relation $\mathcal{P}$ is ergodic, and the system $\{\mathcal{P}, H \subset G, \alpha_G\}$ in* Lemma 2.7 *satisfies the following properties:*

(i) *G is a finite group and H is a subgroup which does not include any normal subgroup $\neq \{e\}$ of G;*

(ii) *The action $\alpha_G \subset N[\mathcal{P}]$ is outer;*

(iii) *$\mathcal{R} = \mathcal{P} \rtimes_\alpha G$, $\mathcal{S} = \mathcal{P} \rtimes_\alpha H$.*

We notice that the collection $\{\mathcal{P}, H \subset G, \alpha_G\}$ satisfying the conditions (ii) and (iii) was obtained by C. Sutherland ([10]). It will be made clear in the next section that the condition (i) in this theorem is a key for proving the uniqueness of $\{\mathcal{P}, H \subset G, \alpha_G\}$.

Proof. Each restriction $\mathcal{P}|_{A_\lambda} = \mathrm{Ker}(\sigma)|_{A_\lambda}$, is ergodic. For each λ and $\mu \in \Lambda$ the partial transformation $\psi_{\mu,\lambda}$ in $[\mathcal{P}]_*$ hits the set A_μ from its domain in A_λ. So, $\mathcal{P}$ is ergodic.

We will show (ii). In order to see $\alpha_G \subset N[\mathcal{P}]$, we let $\mu, \lambda \in \Lambda$ and $g \in G$. Since both of the restrictions of $\mathrm{Ker}(\sigma)$ to the sets A_λ and A_μ are ergodic, one can get for a.e. $(x, y) \in \mathcal{P}$ with $x \in A_\lambda$ and $y \in A_\mu$, points u and $z \in \mathrm{Dom}(\psi_{\mu,\lambda})$ satisfying

$$(x, u) \in \mathrm{Ker}(\sigma), \quad (\psi_{\mu,\lambda}u, y) \in \mathrm{Ker}(\sigma) \quad \text{and} \quad (\alpha_g u, z) \in \mathrm{Ker}(\sigma).$$

Then,

$$\begin{aligned}\sigma(\psi_{\mu,\lambda}z, \alpha_g y) =& \sigma(\psi_{\mu,\lambda}z, z)\sigma(z, \alpha_g u)\sigma(\alpha_g u, u)\sigma(u, \psi_{\mu,\lambda}u)\sigma(\psi_{\mu,\lambda}u, y)\sigma(y, \alpha_g y)\\ =&\theta_{\mu,\lambda} \cdot e \cdot \theta_{\lambda,\lambda_0} g \theta_{\lambda_0,\lambda} \cdot \theta_{\lambda,\mu} \cdot e \cdot \theta_{\mu,\lambda_0} g^{-1} \theta_{\lambda_0,\mu} = e.\end{aligned}$$

Hence, $(\psi_{\mu,\lambda}z, \alpha_g y) \in \mathrm{Ker}(\sigma)$. On the other hand, $(\alpha_g x, \alpha_g u) \in \mathrm{Ker}(\sigma)$. Thus $(\alpha_g x, \alpha_g y) \in \mathcal{P}$. To see that α_G is outer, we let $E \subset X$ be of positive measure and $g \in G$ be such that $\alpha_g x \in \mathcal{P}(x)$, $x \in E$. Since x and $\alpha_g x$ sit on the same $\mathrm{Ker}(\sigma)$-ergodic component, and since the restriction of $\mathcal{P}$ to this set is just the same as the restriction of $\mathrm{Ker}(\sigma)$, we see that $\alpha_g x \in \mathrm{Ker}(\sigma)(x), x \in E$. Hence,

$$\theta_{\lambda,\lambda_0} g \theta_{\lambda_0,\lambda} = \sigma(\alpha_g x, x) = e.$$

Thus, $g = e$.

Next we will show (iii). We get for a.e. $(x, y) \in \mathcal{S}$ (respectively $(x, y) \in \mathcal{R}$) with $x \in A_\lambda$ and $y \in A_\mu$, a point u in $\mathrm{Dom}(\psi_{\mu,\lambda})$ such that $(u, x) \in \mathrm{Ker}(\sigma)$. Since $(y, \psi_{\mu,\lambda}u) \in \mathcal{S}$ (respectively $(y, \psi_{\mu,\lambda}u) \in \mathcal{R}$), it follows from Lemma 2.9 that

$$\begin{aligned}(\mathrm{Ker}(\sigma)|_{A_\mu} \rtimes_\alpha H)(\psi_{\mu,\lambda}u) \subset (\mathcal{P} \rtimes_\alpha H)(u) = (\mathcal{P} \rtimes_\alpha H)(x)\\ (\text{respective } (\mathrm{Ker}(\sigma)|_{A_\mu} \rtimes_\alpha G)(\psi_{\mu,\lambda}u) \subset (\mathcal{P} \rtimes_\alpha G)(u) = (\mathcal{P} \rtimes_\alpha G)(x)).\end{aligned}$$

Thus, $y \in (\mathcal{P} \rtimes_\alpha H)(x)$ (respectively $y \in (\mathcal{P} \rtimes_\alpha G)(x)$).

Finally, we will show (i). Let K be a normal subgroup of G such that $K \subset H$. Let G/K be the quotient group of G by the subgroup K and denote each coset $gK(= Kg)$ by $[g]$, for $g \in G$. We choose and fix representatives $g_j \in G$, $j \in J$, so that $G/K = \{[g_j] \mid j \in J\}$.

Consider the coset space G/H of G by the subgroup H, that is, g and $g' \in G$ are equivalent if $gg'^{-1} \in H$ and we denote for $g \in G$ its equivalence class by $[g]_H$. Then there exist a subset $I \subset J$ such that

$$\begin{aligned}&\#(I) = \#(G/H)\\ &G/H = \{[g_i]_H \mid i \in I\}.\end{aligned}$$

In order to see this, we notice that if an element g in G satisfies $[g] = [h]$ for some $h \in H$ then $g \in H$. So, we may set

$$H_K = \{[g] \in G/K \mid g \in H\}.$$

Obviously, H_K is a subgroup of G/K. So, consider the coset space of G/K by H_K defined by that if $[g]$ and $[f] \in G/K$ then $[g]$ is equivalent with $[f]$ if $[g][f]^{-1} \in H_K$. Then we see that the above equivalence relation is just the same as the equivalence $[g]_H = [f]_H$. Hence, we get a subset $I \subset J$ with $\#(I) = \#(G/H)$ so that $[g_i]_H, i \in I$, are all equivalence classes of G/H.

We see that $\{\alpha_{g_i} \mid i \in I\}$ is the set of choice functions of $\mathcal{S} \subset \mathcal{R}$. In fact, $\{\alpha_{g_i} \mid i \in I\}$ satisfies

$$\bigcup_{i \in I} \mathcal{S}(\alpha_{g_i} x) = \bigcup_{i \in I} \bigcup_{h \in H} \mathcal{P}(\alpha_h \alpha_{g_i} x) = \mathcal{P}\Big(\bigcup_{g \in G} \alpha_g x\Big) = \mathcal{R}(x) \quad \text{a.e. } x.$$

By $\overline{\sigma}$, we denote the index cocycle corresponding to these choice functions. We will show that if $i, j, i', j' \in J$ and $[g_j g_i^{-1}] = [g_{j'} g_{i'}^{-1}]$, then

$$\begin{aligned} \overline{\sigma}(\alpha_{kg_i} x, \alpha_{lg_j} x) &= \overline{\sigma}(\alpha_{k'g_{i'}} x, \alpha_{l'g_{j'}} x) \\ &= \text{a constant} \qquad \text{a.e. } x, \text{ and } k, l, k', l' \in K. \end{aligned} \tag{2.1}$$

Let $x, x' \in X$, $k, l, k', l' \in K$, $m \in I$, and let

$$\begin{aligned} g_n &= \overline{\sigma}(\alpha_{kg_i} x, \alpha_{lg_j} x)(g_m) \\ g_{n'} &= \overline{\sigma}(\alpha_{k'g_{i'}} x', \alpha_{l'g_{j'}} x')(g_m). \end{aligned}$$

This means $(\alpha_{g_m l g_j} x, \alpha_{g_n k g_i} x) \in \mathcal{S}$ and hence $[g_m l g_j]_H = [g_n k g_i]_H$. On the other hand, $[g_m l g_j] = [g_m g_j]$ and $[g_n k g_i] = [g_n g_i]$. Therefore, $[g_m g_j]_H = [g_n g_i]_H$ and $[g_{m'} g_{j'}]_H = [g_{n'} g_{i'}]_H$. By the assumption that $[g_j g_i^{-1}] = [g_{j'} g_{i'}^{-1}]$, there is an element q in K such that

$$g_m^{-1} H g_n \cap g_m^{-1} H g_{n'} k \neq \emptyset.$$

Choose elements h and $h' \in H$ so that $g_m^{-1} h g_n = g_m^{-1} h' g_{n'} k$, then $h g_n = h' g_{n'} k = h' k' g_{n'}$ for some $k' \in K$. Thus, $g_n = g_{n'}$. Hence we may write

$$\overline{\sigma}(\alpha_{kg_i} x, \alpha_{lg_j} x) = \text{a constant } = \theta([g_j g_i^{-1}]) \quad \text{a.e. } x, k, l \in K.$$

We will show that for each λ in Λ

$$\overline{\sigma}(y, x) = e \quad \text{if } (y, x) \in \operatorname{Ker}(\sigma) \quad \text{with } y, x \in A_\lambda. \tag{2.2}$$

Let $(y,x) \in \mathrm{Ker}(\sigma)$ and $y, x \in A_\lambda$, and let $g_n = \overline{\sigma}(y,x)(g_m)$, where n and $m \in I$. This means $(\alpha_{g_m}x, \alpha_{g_n}y) \in \mathcal{S}$. Using $\mathcal{S}|_{A_\lambda} = \mathrm{Ker}(\sigma)|_{A_\lambda} \rtimes_\alpha H$, we have $\alpha_{g_n}y \in \mathrm{Ker}(\sigma)(\alpha_h(\alpha_{g_m}x))$ for some $h \in H$. Since $\alpha_{g_m} \in N[\mathrm{Ker}(\sigma)]$, $(\alpha_{g_m}x, \alpha_{g_m}y) \in \mathrm{Ker}(\sigma)$. Hence, $(\alpha g_n y, \alpha_h \alpha_{g_m} y) \in \mathrm{Ker}(\sigma)$. Since the action α_G is outer, $g_n = hg_m$, and hence $g_n = g_m$.

Finally, by (i) and (ii), we see that

$$\{\overline{\sigma}(y,x) \mid (y,x) \in \mathcal{R} \text{ and } y, x \in A_\lambda\} \subset \theta(G/K).$$

By Lemma 2.2, the set in the left hand side is $\mathbf{r}(\mathcal{R}) = G$. Obviously, $\#(\theta(G/K)) \leqslant \#(G/K)$. Thus,

$$\#(G) \leqslant \#(G/K).$$

This implies $K = \{e\}$. ∎

REMARK 2.12. The collection $\{\mathcal{P}, H \subset G, \alpha_G\}$ satisfying the conditions (ii) and (iii) was obtained by C. Sutherland ([11]). It will be made clear in the next section that the condition (i) in this theorem is a key for proving the uniqueness of $\{\mathcal{P}, H \subset G, \alpha_G\}$.

3. THE CONJUGACY CLASS OF $H \subset G$

Throughout this section, we let $\mathcal{R}$ and $\mathcal{S} \subset \mathcal{R}$ be an ergodic measured discrete equivalence relation and an ergodic subrelation admitting an ergodic subrelation $\mathcal{P} \subset \mathcal{S}$ together with a finite group G and a subgroup $H \subset G$ and an action $g \in G \mapsto \alpha_g \in N[\mathcal{P}]$ satisfying the following conditions:

(i) H does not contain any normal subgroup $\neq \{e\}$ of G;
(ii) α_G is outer;
(iii) $\mathcal{R} = \mathcal{P} \rtimes_\alpha G$, and $\mathcal{S} = \mathcal{P} \rtimes_\alpha H$.

In the previous section we showed that every ergodic measured discrete equivalence relation and an ergodic subrelation with finite index admits the system $\{\mathcal{P}, H \subset G, \alpha_G\}$ satisfying the above conditions (i)–(iii). In this section we will show that the conjugacy class of the pair of the group G and the subgroup H is uniquely determined by the inclusion data $\mathcal{S} \subset \mathcal{R}$.

THOEREM 3.1. *The conjugacy class of the pair of the finite group G and the subgroup H depends only on the orbit equivalence class of the pair of $\mathcal{R}$ and $\mathcal{S}$.*

In fact, we will prove that the pair H and G is conjugate with the pair of $\mathbf{r}(\mathcal{S})$ and $\mathbf{r}(\mathcal{R})$. Then in view of Lemma 2.5, the latter pair depends only on the orbit equivalence class of the pair of $\mathcal{R}$ and $\mathcal{S}$. After preparing several lemmas and a proposition, we will prove this.

Throughout this section, we choose and fix a Rohlin set F for the free action α_G, that is,

$$\bigcup_{g\in G} \alpha_g F = X \quad \text{(disjoint union)}.$$

We let G/H be the coset space of the group G by the subgroup H, and choose and fix representatives $g_i \in G$, $i \in I$, so that $G = \{[g_i]_H \mid i \in I\}$, where $g_0 = e$ and $[g]_H = Hg$ for $g \in G$. We define the transformations $\varphi_i \in [\mathcal{R}]$, $i \in I$ by

$$\varphi_i x = \alpha_{hg_ih^{-1}} x, \qquad \text{for } x \in \bigcup_{j\in I} \alpha_{hg_j}(F), \quad h \in H.$$

We set for each $h \in H$,

$$F_h = \bigcup_{i\in I} \alpha_{hg_i}(F).$$

We note that $\varphi_i(\alpha_{hg_j}(F)) = \alpha_{hg_ig_j}(F)$.

LEMMA 3.2. *$\{\varphi_i \mid i \in I\}$ is the set of choice functions of $\mathcal{S} \subset \mathcal{R}$.*

Proof. We will show that if $(\varphi_i x, \varphi_k x) \in \mathcal{S}$, then $i = k$ for a.e. x. Let $u \in F$, $j \in I$ and let $h \in H$ be such that $x = \alpha_{hg_j} u$. Then $(\varphi_i x, \varphi_k x) = (\alpha_{hg_ig_j} u, \alpha_{hg_kg_j} u) \in \mathcal{S}$. So, by (iii) there is an element h' in H such that $(\alpha_{h'} \alpha_{hg_ig_j} u, \alpha_{hg_kg_j} u) \in \mathcal{P}$. By (ii), $h'hg_ig_j = hg_kg_j$. Hence $g_i = g_k$.

Next, let $g \in G$ and $x = \alpha_{hg_j} u$, where $u \in F, h \in H$. Then, $\alpha_g x = \alpha_{ghg_j} u = \alpha_{h'g_lg_j} u$, where $l \in I, h' \in H$ with $gh = h'g_l$. Hence,

$$\alpha_g x = \alpha_{h'h^{-1}} \cdot \alpha_{hg_lg_j} u = \alpha_{h'h^{-1}} \varphi_l(x) \in \mathcal{S}(\varphi_l(x)).$$

By the condition (iii), $\mathcal{R}(x) = \bigcup_{l\in I} \mathcal{S}(\varphi_l x)$ a.e. x. ∎

By σ, we denote the index cocycle corresponding to the choice functions φ_i, $i \in I$.

LEMMA 3.3. $\sigma(\alpha_{h'}y, \alpha_h x) = \sigma(y, x)$ *a.e.* $(x, y) \in \mathcal{R}$, $h, h' \in H$.

Proof. Let $(y, x) \in \mathcal{R}$, $k \in I$ and h and $h' \in H$. Set $l = \sigma(y, x)(k)$. It is easy to see

$$\varphi_l \alpha_g = \alpha_g \varphi_l \quad \forall\, g \in H \text{ and } \forall\, l \in I.$$

Therefore by the condition (iii), $(\varphi_k(\alpha_h x), \varphi_k(x)) = (\alpha_h \varphi_k(x), \varphi_k x) \in \mathcal{S}$ and $(\varphi_l(\alpha_{h'} x), \varphi_l(x)) = (\alpha_{h'} \varphi_l(x), \varphi_l x) \in \mathcal{S}$. Hence

$$(\varphi_k x, \varphi_l y) \in \mathcal{S} \quad \text{if and only if} \quad (\varphi_k \alpha_h x, \varphi_l \alpha_{h'} y) \in \mathcal{S}.$$

Thus, $l = \sigma(\alpha_{h'} y, \alpha_h x))(k)$. ∎

LEMMA 3.4. *For each $h \in H$,*

$$\mathcal{P}|_{F(h)} = \operatorname{Ker}(\sigma)|_{F(h)}.$$

Proof. To see the inclusion $\mathcal{P}|_{F(h)} \subset \operatorname{Ker}(\sigma)|_{F(h)}$, we let $x = \alpha_{hg_i} u$ and $y = \alpha_{hg_j} v$, where $h \in H$, $u, v \in F$, and suppose $(x, y) \in \mathcal{P}$. Since $\alpha_{hg_l h^{-1}} \in N[\mathcal{P}]$, we have $(\varphi_l x, \varphi_l y) = (\alpha_{hg_l g_i} u, \alpha_{hg_l g_j} v) \in \mathcal{P}$, $\forall\, l \in I$. Thus, $l = \sigma(y, x)(l), \forall\, l \in I$. We will show the converse inclusion. We let $h \in H$, $u \in F$, set $x = \alpha_{hg_i} u$, $y = \alpha_{hg_j} v$ and suppose $\sigma(x, y) = e$. By Lemma 3.3,

$$e = \sigma(x, y) = \sigma(\alpha_{hg_i} u, \alpha_{hg_j} v) = \sigma(\alpha_{g_i} u, \alpha_{g_j} v).$$

This implies $(\varphi_l(\alpha_{g_i} u), \varphi_l(\alpha_{g_j} v)) \in \mathcal{S}$, $\forall\, l \in I$ and hence, $(\alpha_{g_l g_i} u, \alpha_{g_l g_j} v) \in \mathcal{S}$, $\forall\, l \in I$. So, there exist elements h_l in H, $l \in I$ such that $(\alpha_{g_l g_i} u, \alpha_{h_l} \alpha_{g_l g_j} v) \in \mathcal{P}$. In particular, $(\alpha_{g_i} u, \alpha_{g_j} v) \in \mathcal{S}$, so we have an element $\overline{h}$ in H such that

$$(\alpha_{g_i} u, \alpha_{\overline{h} g_i} u) \in \mathcal{S} \quad \text{and} \quad (\alpha_{\overline{h} g_i} u, \alpha_{g_j} v) \in \mathcal{P}.$$

Therefore, $\alpha_{g_l \overline{h}^{-1}}$ and $\alpha_{h_l g_l}$ map the $\mathcal{P}$-orbit $\mathcal{P}(\alpha_{\overline{h} g_i} u) = \mathcal{P}(\alpha_{g_j} v)$ onto the $\mathcal{P}$-orbit $\mathcal{P}(\alpha_{g_l g_i} u) = \mathcal{P}(\alpha_{h_l g_l g_j} v)$. By the condition (ii), we see $h_l g_l = g_l \overline{h}^{-1} \in H$, $\forall\, l \in I$. Namely, $g_l \overline{h}^{-1} g_l^{-1}$, $\forall\, l \in I$. Then, by the condition (i), we see $\overline{h} = e$. Thus, $(\alpha_{g_i} u, \alpha_{hg_j} v) \in \mathcal{P}$. Hence, $(x, y) = (\alpha_{hg_i} u, \alpha_{hg_j} v) \in \mathcal{P}$. ∎

LEMMA 3.5. $\operatorname{Ker}(\sigma)$ *is ergodic.*

Proof.

$$\operatorname{Ker}(\sigma)|_{F(h)} = \mathcal{P}|_{F(h)}, \qquad \forall\, h \in H \text{ (use Lemma 3.4)},$$
$$\sigma(\alpha_h x, x) = e, \qquad \text{a.e. } x,\ \forall\, h \in H \text{ (use Lemma 3.3)}.$$

Hence, $\operatorname{Ker}(\sigma)$ is ergodic. ∎

LEMMA 3.6. *The measurable function* $u \in F \mapsto \sigma(\alpha_{g_j}u, u) \in \Sigma\#(I)$ *is constant a.e.,* $\forall j \in I$.

Proof. Let $u, v \in F$ and suppose $(u, v) \in \mathcal{P}$. Then, since $\alpha_{g_j} \in N[\mathcal{P}]$, $(\alpha_{g_j}u, \alpha_{g_j}v) \in \mathcal{P}$. Applying Lemma 3.4, $\sigma(u, v) = \sigma(\alpha_{g_j}v, \alpha_{g_j}u) = e$. On the other hand, $\sigma(\alpha_{g_j}v, v) = \sigma(\alpha_{g_j}v, \alpha_{g_j}u)\sigma(\alpha_{g_j}u, u)\sigma(u, v)$. So, $\sigma(\alpha_{g_j}v, v) = \sigma(\alpha_{g_j}u, u)$. In other words, the function $u \in F \mapsto \sigma(\alpha_{g_j}u, u)$ is $\mathcal{P}|_F$-invariant. The ergodicity of $\mathcal{P}|_F$ implies that

$$\sigma(\alpha_{g_j}u, u) = \text{constant} \quad \text{a.e. } u \in F. \quad \blacksquare$$

By θ_{g_j}, we denote the constant $\sigma(\alpha_{g_j}u, u)$, $u \in F$.

PROPOSITION 3.7. *Let* h *and* $h' \in H$. *Then,* $\sigma(u, v) =$ a constant *a.e.* $(u, v) \in \mathcal{P}$ *with* $u \in F(h)$ *and* $v \in F(h')$. *Moreover, this constant depends only on* $h^{-1}h'$. *Denoting this constant by* $\theta_{h^{-1}h'}$, *then the map* $h \in H \mapsto \theta_h \in \Sigma_{\sharp(I)}$ *gives a group into isomorphism.*

Proof. Let $h, h' \in H$, $u, u_1 \in F(h)$ and $v, v_1 \in F(h')$ and suppose that $(u, v), (v, v_1)$ and $(v_1, u_1) \in \mathcal{P}$. Then, by lemma 3.5, we see $\sigma(v_1, v) = e$ and $\sigma(u, u_1) = e$. So, $\sigma(u, v) = \sigma(u, u_1)\sigma(u_1, v_1)\sigma(v_1, v) = \sigma(u_1, v_1)$. Since both of $\mathcal{P}|_{F(h)}$ and $\mathcal{P}|_{F(h')}$ are ergodic, $\sigma(u, v) =$ constant a.e. $(u, v) \in \mathcal{P}$ with $u \in F(h)$ and $v \in F(h')$. By $\theta_{h,h'}$, we denote this constant $\sigma(u, v)$, where $u \in F(h)$, $v \in F(h')$. Since σ is a cocycle, θ satisfies the cocycle property

$$\theta_{h,h'} \cdot \theta_{h',h''} = \theta_{h,h''}, \quad h, h' \in H. \tag{3.1}$$

Let us choose $u, z \in F(e)$, $v \in F(h')$ and $w \in F(h)$ so that $(u, v), (v, w)$ and $(w, z) \in \mathcal{P}$. Then,

$$e = \sigma(z, u) = \sigma(z, w)\sigma(w, v)\sigma(v, u) = \theta_{e,h}\theta_{h,h'}\theta_{h',e}.$$

Therefore, $\theta_{h,h'} = \theta_{h,e}\theta_{e,h'}$. Here, set $\theta_{h^{-1}} = \theta_{h,e}$, then, $\theta_{e,h} = \theta_h$, and hence $\theta_h^{-1} = \theta_{h^{-1}}$ and $\theta_{h,h'} = \theta_h^{-1}\theta_{h'}$.

We will show the left invariance of $\theta_{h,h''}$ in the sense that

$$\theta_{h,e} = \theta_{\overline{h}h,\overline{h}}, \quad h, \overline{h} \in H. \tag{3.2}$$

Choose $(u, v) \in \mathcal{P}$ with $u \in F(e)$ and $v \in F(h)$. Then, $(\alpha_{\overline{h}}u, \alpha_{\overline{h}}v) \in \mathcal{P}$ and hence,

$$\theta_{\overline{h}h,\overline{h}} = \sigma(\alpha_{\overline{h}}v, \alpha_{\overline{h}}u) = \sigma(\alpha_{\overline{h}}v, v)\sigma(v, u)\sigma(u, \overline{h}u) \overset{\text{(Lemma 3.3)}}{=} \theta_{h,e}.$$

This makes θ an homomorphism. In fact, if $h,\overline{h}\in H$, then

$$\theta_h\cdot\theta_{\overline{h}}=\theta_{h^{-1},e}\cdot\theta_{\overline{h}^{-1},e}\overset{(3.2)}{=}\theta_{\overline{h}^{-1}h^{-1},\overline{h}^{-1}}\cdot\theta_{\overline{h}^{-1},e}\overset{\text{cocycle property (iii)}}{=}\theta_{\overline{h}^{-1}h^{-1},e}=\theta_{h\overline{h}}.$$

Finally, we will show that the map $h\in H\mapsto\theta_h\in\mathcal{S}_{\sharp(I)}$ is injective. Let $h\in H$ and suupose $\theta_h=e$. Choose a point $(u,v)\in\mathcal{P}$ such that $u\in F(e)$ and $v\in\mathcal{P}(h)$. Then,

$$\sigma(\alpha_h^{-1}v,u)=\sigma(\alpha_h^{-1}v,v)\sigma(v,u)=e\cdot\theta_h^{-1}=e.$$

Since u and $\alpha_h^{-1}v\in F(e)$, it follows from Lemma 3.4 that $(u,\alpha_h^{-1}v)\in\mathcal{P}$. Hence, $(\alpha_h^{-1}v,v)\in\mathcal{P}$. Since the action α_G is outer over $\mathcal{P}$, we see $h=e$. ∎

By θ_h, we denote the constant $\sigma(u,v)$, $u\in F(e)$, $v\in F(h)$, $h\in H$.

Lemma 3.8. *$\theta_{g_j}\neq e$ if $g_j\neq e$.*

Proof. Let $j\in I$ be such that $\theta_{g_j}=e$. If $u\in F$ then since $\alpha_{g_j}u\in F(e)$, it follows from Lemma 3.4 that $(u,\alpha_{g_j}u)\in\mathcal{P}$. Hence, $g_j=e$, because α_G is outer. ∎

Lemma 3.9. *Let $h,\overline{h}\in H$ and let $i,k\in I$. If $g_ih=\overline{h}g_k$ then $\theta_{g_i}\theta_h=\theta_{\overline{h}}\theta_{g_k}$.*

Proof. Let $h,\overline{h}\in H$ and let $i,k\in I$. Suppose $g_ih=\overline{h}g_k$. Using the ergodicity of $\mathcal{P}$, we have for a.e. $\omega\in F(h)$, a point $u\in F$ such that $(u,\omega)\in\mathcal{P}$. Then, $(\alpha_{g_i}u,\alpha_{\overline{h}g_kh^{-1}}\omega)\in\mathcal{P}$. Hence,

$$\begin{aligned}\sigma(\alpha_{g_i}u,\alpha_h^{-1}\omega)&=\sigma(\alpha_{g_i}u,\alpha_{\overline{h}g_kh^{-1}}\omega)\sigma(\alpha_{\overline{h}g_kh^{-1}}\omega,\alpha_{g_kh^{-1}}\omega)\sigma(\alpha_{g_kh^{-1}}\omega,\alpha_{h^{-1}}\omega)\\&=\theta_{e,\overline{h}}\cdot e\cdot\theta_{g_k}=\theta_{\overline{h}}\cdot\theta_{g_k}.\end{aligned}$$

On the other hand,

$$\sigma(\alpha_{g_i}u,\alpha_h^{-1}\omega)=\sigma(\alpha_{g_i}u,u)\sigma(u,\omega)\sigma(\omega,\alpha_h^{-1}\omega)=\theta_{g_i}\cdot\theta_{e,h}\cdot e=\theta_{g_i}\theta_h.\quad ∎$$

Lemma 3.10. *Let $i,j,k\in I$ and $h\in H$. If $g_ig_j=hg_k$ then $\theta_{g_i}\theta_{g_j}=\theta_h\theta_{g_k}$.*

Proof. Suppose $g_ig_j=hg_k$. Using the ergodicity of $\mathcal{P}$, we have for a.e. $u\in F$ a point $\omega\in F$ such that $(\alpha_{g_j}u,\omega)\in\mathcal{P}$. Then, since $\alpha_{g_i}\in N[\mathcal{P}]$, we see $(\alpha_{hg_k}u,\alpha_{g_i}\omega)=(\alpha_{g_i}\alpha_{g_j}u,\alpha_{g_i}\omega)\in\mathcal{P}$. Since $\alpha_{hg_k}u=\alpha_{g_i}(\alpha_{g_j}u)\in F(h)$ and $\alpha_{g_i}\omega\in F(e)$,

$$\sigma(\alpha_{hg_k}u,\alpha_{g_i}\omega)=\theta_{h,e}=\theta_{h^{-1}}.$$

The cocycle equation of σ implies

$$\begin{aligned}\theta_h\cdot\theta_{g_k}&=\sigma(\alpha_{g_i}\omega,\alpha_{hg_k}u)\sigma(\alpha_{hg_k}u,\alpha_{g_k}u)\sigma(\alpha_{g_k}u,u)=\sigma(\alpha_{g_i}\omega,u)\\&=\sigma(\alpha_{g_i}\omega,\omega)\sigma(\omega,\alpha_{g_j}u)\sigma(\alpha_{g_j}u,u)=\theta_{g_i}\theta_{g_j}.\quad ∎\end{aligned}$$

This lemma allows us to define the map $g\in G\mapsto\theta_g\in\Sigma_{\#(I)}$ as follows.

DEFINITION 3.11. For $h, \overline{h} \in H$ and $i, j \in I$, we define

$$\theta_{hg_j} = \theta_h \theta_{g_j} \quad \text{and} \quad \theta_{g_i\overline{h}} = \theta_{g_i}\theta_{\overline{h}}.$$

We note that $\theta_{g_i^{-1}} = \theta_{g_i}^{-1}$. Because, g_i^{-1} is of the form hg_k for some $h \in H$ and $k \in I$. $g_k g_i = h^{-1}$ implies $\theta_{g_k}\theta_{g_i} = \theta_h^{-1} = \theta_h^{-1}$, and hence $\theta_{g_i}^{-1} = \theta_h\theta_{g_k} = \theta_{hg_k} = \theta_{g_i^{-1}}$.

LEMMA 3.12. (i) $\sigma(u,v) \in \theta_G$ *a.e.* $(u,v) \in \mathcal{R}$;
(ii) $\sigma(u,v) \in \theta_H$ *a.e.* $(u,v) \in \mathcal{S}$.

Proof. (i) Since $\mathcal{R} = \mathcal{P} \rtimes_\alpha G$, it is enough to see that if $h, \overline{h} \in H$ and $i, j \in I$ and if $u \in F$ and $(v, \alpha_{\overline{h}g_i hg_j} u) \in \mathcal{P}$ then $\sigma(v, \alpha_{g_j} u) \in \theta_G$. In fact,

$$\begin{aligned}\sigma(v, \alpha_{g_j} u) = \sigma(v, \alpha_{\overline{h}g_i}(\alpha_{g_j} u))\sigma(\alpha_{\overline{h}}(\alpha_{g_i g_j} u), \alpha_{g_i g_j} u)\sigma(\alpha_{g_i g_j} u, \alpha_{g_j} u) \\ \in \theta_H \cdot \sigma(\alpha_{g_i g_j} u, \alpha_{g_j} u)\end{aligned}$$

and, since $g_i g_j = hg_k$ for some $h \in H$,

$$\sigma(\alpha_{g_i g_j} u, \alpha_{g_j} u) = \sigma(\alpha_h(\alpha_{g_k} u), \alpha_{g_j} u) = \sigma(\alpha_{g_k} u, \alpha_{g_j} u) = \sigma(\alpha_{g_k} u, u)\sigma(u, \alpha_{g_j} u) \in \theta_G.$$

(ii) In the proof of (i), consider the case where $i = 0$, that is $g_i = e$. Then $g_k = g_j$ and $\sigma(v, \alpha_{hg_j} u) = \theta_{h'^{-1}\overline{h}} \in \theta_H$. ∎

LEMMA 3.13. *The map* $\theta : g \mapsto \theta_g \in \Sigma_{\#(I)}$ *is an into group isomorphism.*

Proof. Let $h, h' \in H$ and $i, i' \in I$, and set $g = hg_j, g' = h'g_i$. By the definition of θ_G,

$$\begin{aligned}\theta_{g'g} &= \theta_{h'\overline{h}}\theta_{g_k g_j} \quad (\text{where } g_i h = \overline{h}g_k,\ \overline{h} \in H) \\ &= \theta_{h'}\theta_{\overline{h}}\theta_{g_k}\theta_{g_j} \quad (\text{use Proposition 3.7 and Lemma 3.10}) \\ &= \theta_{h'}\theta_{g_i}\theta_h\theta_{g_j} \quad (\text{use Lemma 3.9}) \\ &= \theta_{g'}\theta_g.\end{aligned}$$

In order to see that the map θ is injective, let $h \in H$ and $j \in I$ and suppose $\theta_{hg_j} = e$. Since $\mathcal{P}$ is ergodic, we obtain for a.e. $u \in F$ a point $v \in \alpha_h(F)$ with $(u,v) \in \mathcal{P}$. Then,

$$\sigma(\alpha_h^{-1} v, \alpha_{g_j} u) = \sigma(\alpha_h^{-1} v, v)\sigma(v,u)\sigma(u, \alpha_{g_j} u) = \theta_h^{-1}\theta_{g_j}^{-1}.$$

Since $\alpha_h^{-1} v$ and $\alpha_{g_j} u \in F(e)$, it follows from Lemma 3.4 that $(\alpha_h^{-1} v, \alpha_{g_j} u) \in \mathcal{P}$. Hence $(\alpha_{g_j}^{-1}\alpha_h^{-1} v, u) \in \mathcal{P}$. On the other hand, since $(u,v) \in \mathcal{P}$, we have $(\alpha_{g_j}^{-1}\alpha_h^{-1} v, v) \in \mathcal{P}$. Since α_G is outer, $g_j^{-1}h^{-1} = e$. Therefore, $g_j = e$ and $h = e$. Thus $hg_j = e$. ∎

Proof of Theorem 3.1. In fact, by Lemma 3.11 and Lemma 3.12 we see that $\theta(G) = \mathbf{r}(\mathcal{R})$ and $\theta(H) = \mathbf{r}(\mathcal{S})$. ∎

4. CANONICAL SYSTEM $\{\mathcal{P}, H \subset G, \alpha_G\}$

Continued to the previous section, we are going to show that the subrelation $\mathcal{P}$ and the action α_G depend only on the orbit equivalence class of the pair of $\mathcal{R}$ and $\mathcal{S}$.

Theorem 4.1. *Suppose ergodic measured discrete equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ and ergodic subrelations $\mathcal{S} \subset \mathcal{R}$ and $\mathcal{S}' \subset \mathcal{R}'$ admit the collection $\{\mathcal{P}, \alpha_G, H \subset G\}$ and $\{\mathcal{P}', \alpha'_{G'}, H' \subset G'\}$ respectively satisfying the conditions* (i), (ii) *and* (iii) *in Section* 3. *If the pairs $\{\mathcal{R}, \mathcal{S}\}$ and $\{\mathcal{R}', \mathcal{S}'\}$ are orbit equivalence then there exists a measure isomorphism $\varphi : X \mapsto X'$ and a group isomoprphism $\gamma : G \mapsto G'$ such that:*

(i) $\gamma(H) = H'$;
(ii) $\varphi[\mathcal{P}]\varphi^{-1} = [\mathcal{P}']$;
(iii) $\varphi\alpha_g\varphi^{-1} = \alpha'_{\gamma(g)}, \ \forall g \in G$.

After preparing the Propositions 4.2 and 4.3, we will prove this theorem.

Throughout this section we assume that $\mathcal{R}$ and $\mathcal{S}$ (respectively $\mathcal{R}'$ and $\mathcal{S}'$) satisfy the conditions in Theorem 4.1.

Proposition 4.2. *Let $\{\mathcal{P}, H \subset G, \alpha_G\}$ satisfy the conditions* (i), (ii) *and* (iii) *for the pair $\{\mathcal{R}, \mathcal{S}\}$. Then there exists an index cocycle σ of $\mathcal{S} \subset \mathcal{R}$, an action $g \in \mapsto \beta_g \in N[\mathrm{Ker}(\sigma)]$ and a $\varphi \in [\mathcal{S}]$ satisfying the following conditions:*

(i) $\mathrm{Ker}(\sigma)$ *is ergodic;*
(ii) β_G *is outer;*
(iii) $\sigma(\beta_g x, x) = g, \ \forall g \in G$, *a.e. x;*
(iv) $\varphi[\mathcal{P}]\varphi^{-1} = [\mathrm{Ker}(\sigma)]$;
(v) $\varphi\alpha_g\varphi^{-1} = \beta_g, \ g \in G$.

Proof. We choose and fix representatives $g_j, \ j \in I$, from the coset space G/H, where $g_0 = e$ and let σ be the index cocycle of $\mathcal{S} \subset \mathcal{R}$ constructed in the proof of Theorem 3.1. We choose and fix a Rohlin set F for the free action α_G and define the sets $F(h)$, $h \in H$, as in the proof of Theorem 3.1. By Lemma 3.3 and Lemma 3.4,

$$\mathcal{P}|_{F(e)} = \mathrm{Ker}(\sigma)|_{F(e)}$$

and

$$(\alpha_h x, x) \in \mathrm{Ker}(\sigma), \qquad \text{a.e. } x, \ h \in H.$$

Since $\alpha_h(F(e)) = F(h)$ and $X = \bigcup_{h \in H} F(h)$, and since $\mathcal{P}|_{F(e)}$ is ergodic, we see that $\mathrm{Ker}(\sigma)$ is ergodic.

What we are going to do is to define the action β_G in the theorem. For this, set

$$\beta_{g_j} x = \alpha_{g_j} x, \quad x \in F,\ j \in J.$$

then $\sigma(\beta_{g_j} x, x) = \theta_{g_j}$, $x \in F$. Using the ergodicity of $\mathcal{P}$, we get for each $h \in H$ a $\beta_h \in [\mathcal{P}]_*$ such that $\beta_h(\alpha_{g_j} F) = \alpha_{hg_j}(F)$, $\forall\, j \in I$. We set for $h \in H$, and $j \in I$,

$$\beta_{hg_j} u = \beta_h \beta_{g_j} u, \quad u \in F.$$

then

$$\sigma(\beta_{hg_j} u, u) = \sigma(\beta_{hg_j} u, \beta_{g_j} u)\sigma(\beta_{g_j} u, u) = \theta_h \theta_{g_j} = \theta_{hg_j}.$$

We define the transformations $\beta_g \in [\mathcal{S}]$, $g \in G$, by

$$\beta_g x = \beta_{g\overline{g}} \beta_{\overline{g}}^{-1} x, \quad x \in \alpha_{\overline{g}}(F),\ \forall\, g \in G.$$

It is easy to see that β_G is an action (i.e. $\beta_{gg'} = \beta_g \beta_{g'}$). If $x \in \alpha_{\overline{g}}(F)$ then

$$\begin{aligned} \sigma(\beta_g x, x) &= \sigma(\beta_{g\overline{g}}(\beta_{\overline{g}}^{-1} x), \beta_{\overline{g}}^{-1} x)\sigma(\beta_{\overline{g}}^{-1} x, x) \\ &= \theta_{g\overline{g}} \cdot \sigma(\beta_{\overline{g}} y, y)^{-1}, \qquad\qquad \text{where } y = \beta_{\overline{g}}^{-1} x \\ &= \theta_{g\overline{g}} \cdot \theta_{\overline{g}}^{-1} = \theta_{g\overline{g}\cdot\overline{g}^{-1}} = \theta_g. \end{aligned}$$

Hence, if $(x, y) \in \mathrm{Ker}(\sigma)$ then $\sigma(\beta_g x, \beta_g y) = e$, $\forall\, g \in G$, so that $\beta_G \subset \mathrm{Ker}(\sigma) \cap [\mathcal{R}]$. Here we note that the subrelation $\mathrm{Ker}(\sigma)$ is characterized as follows. If $h, \overline{h} \in H$ and $(x, y) \in \mathcal{R}$ with $x \in F(h)$ and $y \in F(\overline{h})$ then $(x, y) \in \mathrm{Ker}(\sigma)$ if and only if $(\alpha_{\overline{h}h^{-1}} x, y) \in \mathcal{P}$. In fact,

$$\sigma(y, x) = \sigma(y, \alpha_{\overline{h}h^{-1}} x)\sigma(\alpha_{\overline{h}h^{-1}} x, x) = \sigma(y, \alpha_{\overline{h}h^{-1}} x),$$

(use Lemma 3.3) and hence,

$$(y, x) \in \mathrm{Ker}(\sigma) \Leftrightarrow (y, \alpha_{\overline{h}h^{-1}} x) \in \mathcal{P} \quad \text{(use Lemma 3.5)}.$$

Finally, we define the transformation $\varphi \in [\mathcal{S}]$ by

$$\begin{aligned} \varphi x &= x, & x &\in F(e), \\ \varphi \alpha_h x &= \beta_h x, & x &\in F(e),\ h \in H. \end{aligned}$$

We are going to prove

$$\varphi \alpha_g = \beta_g \varphi \qquad \forall\, g \in G, \tag{4.1}$$

$$\varphi [\mathrm{Ker}(\sigma)] \varphi^{-1} = [\mathcal{P}]. \tag{4.2}$$

To see (4.1), if $u \in F$, $h \in H$, $j \in I$ and $g = \overline{h} g_i$, with $i \in I$ and $\overline{h} \in H$, and if $x = \alpha_{hg_j} u \in F(h)$ then

$$\begin{aligned} \alpha_g x &= \alpha_{\overline{h} g_i h g_j} \\ &= \alpha_{\overline{h} h' g_k g_j} u \quad (\text{where } h' \in H \text{ and } g_i h = h' g_k) \\ &= \alpha_{\overline{h} h' h'' g_l} u \quad (\text{where } h'' \in H \text{ and } g_k g_j = h'' g_l). \end{aligned}$$

Hence

$$\begin{aligned} \varphi \alpha_g x &= \varphi \alpha_{\overline{h} h' h'' g_l} u = \beta_{\overline{h} h' h''} \alpha_{g_l} u = \beta_{\overline{h} h' h'' g_l} u \\ &= \beta_{\overline{h} g_i} \beta_{h g_j} u = \beta_g \varphi \alpha_{h g_j} u = \beta_g \varphi x. \end{aligned}$$

To see (4.2), if $h, \overline{h} \in H$, $i, j \in I$ and $u, v \in F$ and if $x = \alpha_{hg_i} u$, $y = \alpha_{\overline{h} g_j} v$, then

$$\begin{aligned} (x, y) \in \operatorname{Ker}(\sigma) &\Leftrightarrow (\alpha_{\overline{h} h^{-1}} x, y) \in \mathcal{P} \\ &\Leftrightarrow (\alpha_{g_i} u, \alpha_{g_j} v) \in \mathcal{P} \\ &\Leftrightarrow (\varphi x, \varphi y) = (\beta_h \alpha_{g_i} u, \beta_{\overline{h}} \alpha_{g_j} v) \in \mathcal{P}. \quad ▮ \end{aligned}$$

Proposition 4.3. *Let σ and σ' be index cocycles of $\mathcal{S} \subset \mathcal{R}$ having ergodic kernels* $\operatorname{Ker}(\sigma)$ *and* $\operatorname{Ker}(\sigma')$ *respectively. Assume that the outer actions $g \in G \mapsto \beta_g \in N[\operatorname{Ker}(\sigma)]$ and $g \in G \mapsto \beta'_g \in N[\operatorname{Ker}(\sigma)]$ satisfy the following conditions:*

(i) $\mathcal{R} = \operatorname{Ker}(\sigma) \rtimes_\beta G, \qquad \mathcal{S} = \operatorname{Ker}(\sigma) \rtimes_\beta H,$
$\mathcal{R} = \operatorname{Ker}(\sigma') \rtimes_{\beta'} G, \qquad \mathcal{S} = \operatorname{Ker}(\sigma') \rtimes_\beta H;$

(ii) $\sigma(\beta_g x, x) =$ *a constant* $= \theta_g, \qquad \forall g \in G,$ *a.e.* x,
$\sigma'(\beta'_g x, x) =$ *a constant* $= \theta'_g, \qquad \forall g \in G,$ *a.e.* x.

Then, there exists an invertible non-singular transformation φ and a group automorphism γ in $\operatorname{Aut}(G) \cap \operatorname{Aut}(H)$ *such that*

$$\begin{aligned} &\varphi[\operatorname{Ker}(\sigma)]\varphi^{-1} = [\operatorname{Ker}(\sigma)] \\ &\varphi \beta_g \varphi^{-1} = \beta'_{\gamma(g)}, \qquad\qquad g \in G. \end{aligned}$$

Here, $\operatorname{Aut}(G)$ means the set of all group automorphisms of G. We note that the transformation φ is in $[\mathcal{S}]$. After preparing several lemmas, we will show the proposition.

As both of the σ and σ' are index cocycles of $\mathcal{S} \subset \mathcal{R}$, it is known ([3]) that they are cohomologous, that is, there exists a measurable function $x \in X \mapsto v(x) \in \Sigma_N$ satisfying

$$\sigma'(x, y) = v(x)\sigma(x, y)v(y)^{-1}, \quad \text{a.e. } (x, y) \in \mathcal{R}.$$

Lemma 4.4. *There exist an element ζ in Σ_N, a group automorphism γ in* $\operatorname{Aut}(G) \cap \operatorname{Aut}(H)$, *Rohlin sets F and F' of the action β_G and $\beta'_{G'}$ respectively with* *their intersection of positive measure and a subset E of $F \cap F'$ of positive measure such that*

$$\begin{aligned} &\mathrm{Ker}(\sigma)|_E = \mathrm{Ker}(\sigma')|_E \\ &v(x) = \zeta, && \forall\, x \in E \\ &\theta'_{\gamma(g)} = \zeta \cdot \theta_g \cdot \zeta^{-1}, && \forall\, g \in G. \end{aligned}$$

Here $N = [\mathcal{R} : \mathcal{S}]$.

Proof. Since $\sigma(\beta_g x, x)$ is constant a.e., the cocycle property of σ implies that the map $g \in G \mapsto \theta_g \in \Sigma_N$ is a homomorphism, and $\beta_G \subset N[\mathrm{Ker}(\sigma)]$. Moreover, since β_G is outer, the map $g \in G \mapsto \theta_g \in \Sigma_N$ is a group isomorphism. Since β_G and $\beta'_{G'}$ are free respectively, we can obtain Rohlin sets F and F' for each so that the set $F \cap F'$ is of positive measure. We may choose and fix an element ζ in Σ_N such that

$$m\{x \in F \cap F' \mid v(x) = \zeta\} > 0,$$

and set $E = \{x \in F \cap F' \mid v(x) = \zeta\}$. Applying Lemma 3.12 for the index cocycles σ and σ' with ergodic kernels, we see that

$$\begin{aligned} &\mathbf{r}^{\sigma}(\mathcal{R}) = \theta_G, && \mathbf{r}^{\sigma}(\mathcal{R}) = \theta_H \\ &\mathbf{r}^{\sigma'}(\mathcal{R}) = \theta'_G, && \mathbf{r}^{\sigma'}(\mathcal{R}) = \theta'_H. \end{aligned}$$

Here, we use the symbol $\mathbf{r}^{\sigma}(\mathcal{R})$ etc, instead of $\mathbf{r}(\mathcal{R})$, because we need to show the dependence of the ratio sets on the choice of index cocycles. We note that if $x, y \in E$ then $\sigma'(x,y) = \zeta\sigma(x,y)\zeta^{-1}$ a.e. $(x,y) \in \mathcal{R}$. So, for such a point (x,y) in $\mathcal{R}$

$$\sigma'(x,y) = e \Leftrightarrow \sigma(x,y) = e.$$

Lemma 2.2 says that the index ratio set $\{\mathbf{r}^{\sigma}(\mathcal{R}), \mathbf{r}^{\sigma}(\mathcal{S})\}$ is the pair of the image of $\sigma(x,y)$ for a.e. $(x,y) \in \mathcal{R}$ with $x, y \in E$ and the image of $\sigma(x,y)$ for a.e. $(x,y) \in \mathcal{S}$ with $x, y \in E$. Therefore,

$$\theta'_{G'} = \zeta \cdot \theta_G \cdot \zeta^{-1}, \quad \theta'_{H'} = \zeta \cdot \theta_H \cdot \zeta^{-1}.$$

So, we can define $\gamma(g) \in G'$, $g \in G$, by

$$\theta'_g = \zeta\theta_{\gamma(g)}\zeta^{-1}.$$

Then, we easily see that $\gamma \in \mathrm{Aut}(G) \cap \mathrm{Aut}(H)$. ∎

Since $\mathrm{Ker}(\sigma)$ and $\mathrm{Ker}(\sigma')$ ergodically act respectively, we can construct finite partitions $\{E_i \mid i \in \Lambda\}$ of F and $\{E'_i \mid i \in \Lambda\}$ of F' and $e_{i,j} \in [\mathrm{Ker}(\sigma)]_*$ and $e'_{i,j} \in [\mathrm{Ker}(\sigma')]_*$, $i, j \in \Lambda$, satisfying

$$\begin{aligned} &E_0 = E'_0 = E, \\ &E_i = \mathrm{Dom}(e_{j,i}) = \mathrm{Im}(e_{i,j}), \quad E'_i = \mathrm{Dom}(e'_{j,i}) = \mathrm{Im}(e'_{i,j}), \\ &e_{i,j}e_{j,k} = e_{i,k} \end{aligned}$$

where F, F' and E are the sets in Lemma 4.4 and $0 \in \Lambda$ is the specified index.

We define the invertible non-singular transformation φ by

$$\begin{aligned} &\varphi x = x, && x \in E, \\ &\varphi e_{j,0} x = e'_{j,0}\varphi x, && x \in E,\ j \in \Lambda, \\ &\varphi \beta_g x = \beta'_{\gamma(g)}\varphi x, && x \in F,\ g \in G. \end{aligned}$$

Then,

$$\begin{aligned} &\varphi(E_i) = E'_i, && i \in \Lambda, \\ &\varphi\beta_g\varphi^{-1} = \beta_{\gamma(g)}, && g \in G, \\ &\varphi\beta_g(F) = \beta'_{\gamma(g)}(F), && g \in G. \end{aligned}$$

Proof of Proposition 4.3. The fact that $\varphi\beta_g\varphi^{-1} = \beta'_{\gamma(g)}$, $g \in G$ is obvious. Let $g \in G$, $x \in E_0$ and $y \in \beta_g(F)$ and assume $(x, y) \in \mathrm{Ker}(\sigma)$. Set $z = \beta_g^{-1} y$ and let $i \in \Lambda$ be such that $z \in E_i$. Set $x' = \varphi x$, $y' = \varphi y$ and $u = e_{0,i} z \in E$ and $u' = \varphi u$. Here, E is the set E in Lemma 4.4 and we take the $\zeta \in \Sigma$ in Lemma 4.4. Then,

$$\begin{aligned} \sigma'(y', x') &= \sigma'(y', z')\sigma'(z', x') = \theta'_{\gamma(g)}\sigma'(z', u')\sigma(u', x') \\ &= \theta'_{\gamma(g)} v(u')\sigma(u', x')v(x')^{-1} = \theta'_{\gamma(g)}\zeta\sigma(u, x)\zeta^{-1} \end{aligned}$$

and

$$\sigma(u, x) = \sigma(u, z)\sigma(z, y)\sigma(y, x) = \theta_{g^{-1}}.$$

Hence,

$$\sigma'(y', x') = \theta'_{\gamma(g)}\zeta\theta_{g^{-1}}\zeta^{-1} = \theta'_{\gamma(g)\gamma(g^{-1})} = e,$$

that is $(y', x') = (\varphi y, \varphi x) \in \mathrm{Ker}(\sigma')$. Next consider the case that $x \in E_i$, $y \in \alpha_g(F)$. Assume $(x, y) \in \mathrm{Ker}(\sigma)$. Set $x_0 = e_{0,i}x$, then $(x_0, y) \in \mathrm{Ker}(\sigma)$ and $x_0 \in E$. So, $(\varphi x_0, \varphi y) \in \mathrm{Ker}(\sigma')$. By the definition of φ, $\varphi x = e'_{i,0}\varphi x_0$. Hence, $(\varphi x, \varphi y) \in \mathrm{Ker}(\sigma')$.

In the general case, let $x \in \beta_g(F)$, $y \in \beta_f(F)$ and assume $(x, y) \in \mathrm{Ker}(\sigma)$. Set $s = \beta_g^{-1} y$ and $t = \beta_g^{-1} x \in F$. Then $\sigma(s, t) = \sigma(s, \beta_g s)\sigma(y, x)\sigma(\beta_g t, t) = e$. By the previous argument we see that $(\varphi s, \varphi t) \in \mathrm{Ker}(\sigma')$. Thus, $(\varphi y,\ \varphi x) = (\beta'_{\gamma(g)}\varphi s, \beta'_{\gamma(g)}\varphi t) \in \mathrm{Ker}(\sigma')$. ∎

Proof of Theorem 4.1. Combining Theorem 3.1, Proposition 4.2 and Proposition 4.3, we immediately have Theorem 4.1. ∎

Thus we proved that if an ergodic subrelation $\mathcal{S}$ of an ergodic measured discrete equivalence relation $\mathcal{R}$ has a finite index then the pair $(\mathcal{R}, \mathcal{S})$ admits the uniquely determined system $\{\mathcal{P}, H \subset G, \alpha_G\}$ satisfying the conditions (i), (ii) and (iii) in Theorem 4.1. We call this system the canonical system for $\mathcal{S} \subset \mathcal{R}$.

Next we will show a generalization of Dye's theorem on orbit equivalence of finite measure preserving tranformations to orbit equivalence of pairs of an amenable ergodic measured discrete type II_1 equivalence relation and an ergodic subrelation of finite index.

DEFINITION 4.5. (i) A *tower* $\xi = (\mathcal{P}_\xi, \mathcal{T}_\xi)$ on a measurable subset $E \subset X$ consists of a finite partition $\mathcal{P}_\xi = \{E_i \mid i \in \Lambda\}$ of E, and a finite family of partial transformations $\mathcal{T}_\xi = \{e_{i,j} \mid i, j \in \Lambda\} \subset [\mathcal{R}]_*$ satisfying

$$\mathrm{Dom}(e_{i,j}) = E_j, \qquad \mathrm{Im}(e_{i,j}) = E_i$$
$$e_{i,j} \cdot e_{j,k} = e_{i,k}, \qquad e_{i,i} = \mathrm{Id}|_{E_i}.$$

The tower ξ is also considered as the finite subrelation $\{(e_{i,j}x, x) \mid x \in E_j,\ i, j \in \Lambda\}$ on E. We simply write $\xi = \{e_{i,j} \mid i, j \in \Lambda\}$.

(ii) Let ξ_i, $i = 1, 2$ be towers on a measurable subset E, and let $\mathcal{P}_{\xi_i} = \{E_\alpha \mid \alpha \in \Lambda_i\}$ and $\mathcal{T}_{\xi_i} = \{e_{\alpha,\beta} \mid \alpha, \beta \in \Lambda_i\}$. We say that ξ_2 *refines* ξ_1 if

$$\begin{aligned} &\Lambda_2 = \Lambda_1 \rtimes \Gamma, && (\Gamma \text{a finite set}),\\ &E_\alpha = \bigcup_{\gamma\in\Gamma} E_{(\alpha,\gamma)}, && (\alpha \in \Lambda_1) \text{ and},\\ &e_{(\alpha,\gamma),(\beta,\gamma)} = e_{\alpha,\beta} \text{ on } E_{(\beta,\gamma)}, && (\alpha, \beta \in \Lambda_1, \gamma \in \Gamma). \end{aligned}$$

Choose and fix an $\alpha \in \Lambda_1$, and define the tower $\eta = (\mathcal{P}_\eta, \mathcal{T}_\eta)$ on E_α by setting

$$\mathcal{P}_\eta = \{E_{(\alpha,\gamma)} \mid \gamma \in \Gamma\}, \quad \mathcal{T}_\eta = \{e_{(\alpha,\gamma),(\alpha,\gamma')} \mid \gamma, \gamma' \in \Gamma\}$$

then we denote ξ_2 by $\xi_1 \rtimes \eta$ and call it a product tower.

THEOREM 4.6. *The mapping* $\{(\mathcal{R}, \mathcal{S}) \mid \mathcal{R}$ *an ergodic measured discrete amenable type* II_1 *equivalence relation and* $\mathcal{S}$ *an ergodic subrelation of finite index*$\}$ $\ni (\mathcal{R}, \mathcal{S}) \to (\mathbf{r}(\mathcal{R}), \mathbf{r}(\mathcal{S})) \in \{(G, H) \mid G$ *a finite group and* H *a subgroup which does not contain any normal subgroup* $\neq \{e\}$ *of* $G\}$ *is a bijection up to orbit equivalence and conjugacy of a group and a subgroup.*

Proof. First of all we note that Theorem 3.1 shows that the mapping defined as above is well defined up to orbit equivalence and conjugacy of a group and a subgroup. Next we show the mapping is surjective. So, we let G, H be a finite group and a subgroup which does not contain any normal subgroup $\neq \{e\}$ of G. Set

$$Y = \prod_{n=-\infty}^{\infty} G$$

where Y is equipped with the infinite product measure of the uniform measure on each coordinate space G. On Y the left shift mapping is defined in a measure preserving way. We denote it by S. Then we construct the product space $X = Y \rtimes G$ equipped with the product measure whose second coordinate marginal measure is the uniform measure of G. We then define a skew product measure preserving transformation T on X by setting for $y = (y_n) \in Y$ and $g \in G$

$$T(y, g) = (Sy, y_0 \cdot g).$$

We also define a G-action α_G on X by

$$\alpha_l(y, g) = (y, g \cdot l^{-1}), \quad l \in G.$$

We let $\mathcal{R}$ (respective $\mathcal{S}$) be the equivalence relation generated by T and α_l's, $l \in G$ (respectively T and α_l's, $l \in H$). Since the action α_G commutes with T, $\mathcal{R}$ is an amenable equivalence relation. Since the left shift mapping is ergodic, thus we have a pair of an ergodic measured discrete amenable type II_1 equivalence relation $\mathcal{R}$ and an ergodic subrelation $\mathcal{S}$.

If we let $\mathcal{P}$ be the equivalence relation generated by T, then it is easily seen that $\{\mathcal{P}, H \subset G, \alpha_G\}$ gives the canonical system for the inclusion $\mathcal{R} \supset \mathcal{S}$ and that

$$(\mathbf{r}(\mathcal{R}), \mathbf{r}(\mathcal{S})) = (G, H).$$

Finally, we show the injectivity of our mapping up to orbit equivalence and conjugacy of a group and a subgroup. We are given inclusions $\mathcal{R} \supset \mathcal{S}$ on $(X, \mathcal{B}, m)$ and $\mathcal{R}' \supset \mathcal{S}'$ on $(X', \mathcal{B}', m')$ which are orbit equivalent. As usual we denote their canonical systems by $\{\mathcal{P},\ H \subset G,\ \alpha_G\}$ and $\{\mathcal{P}',\ H' \subset G',\ \alpha'_{G'}\}$. We may assume that $G = G', H = H'$ and that m (respective m') is $\mathcal{R} = \mathcal{P} \rtimes_\alpha G$-invariant (respective $\mathcal{R}' = \mathcal{P}' \rtimes_{\alpha'} G$-invariant) probability measure.

Firstly, we take a $\mathcal{P} \rtimes_\alpha G$-tower $\{e_{i,j} \mid i, j \in \Lambda\}$of the set X. We put

$$E_j = \mathrm{Dom}(e_{i,j}).$$

Corresponding to this tower, we choose a finite partition $\{E'_i \mid i \in \Lambda\}$ of X' of equal measure.

We are going to show that for an arbitrary fixed index $i_0 \in \Lambda$ and for any measure preserving isomorphism $\varphi : E_{i_0} \to E'_{i_0}$, there exists a $\mathcal{P}\rtimes_\alpha G$-tower $\{e'_{i,j} \mid i, j \in \Lambda\}$ of the set X' and an extended invertible measure preserving map $\varphi : X \to X'$ such that

$$\operatorname{Dom}(e'_{i,j}) = E'_j, \quad \operatorname{Im}(e'_{i,j}) = E'_i, \quad \varphi \cdot e_{i,j}(x) = e'_{i,j} \cdot \varphi(x), \quad (x \in E_j)$$
$$\alpha_g \cdot e_{i,j} \cdot \operatorname{Id}|_A \in [\mathcal{P}]_* \Leftrightarrow \alpha'_g \cdot e'_{i,j} \cdot \operatorname{Id}|_{\varphi(A)} \in [\mathcal{P}']_*$$

where $A \subset E_j$ and $g \in G$. We note that if $\alpha_g \cdot e_{i,j} \cdot \operatorname{Id}|_A \in [\mathcal{P}]_*$ then g is uniquely determined.

Each e_{j,i_0} is of the form :

$$e_{j,i_0} x = \alpha_g \cdot \gamma x, \qquad (x \in E_{i_0})$$

where $\gamma \in [\mathcal{P}]_*$ with $\operatorname{Dom}(\gamma) = E_{i_0}$, and $g = g(x, j) \in G$. As if necessary one can decompose the set E_{i_0} into at most countable number of disjoint sets on which $g(x, j)$ is constant, we may and do assume that $g(x, j)$ is a function of only j and write

$$g(j) = g(x, j), \quad (x \in E_{i_0}).$$

Since $m'(E'_{i_0}) = m(E_{i_0})$, we have a $m - m'$ preserving isomorphism $\varphi : E_{i_0} \to E'_{i_0}$. We note

$$m'(\alpha'_{g(j)}(E'_{i_0})) = m'(E'_j).$$

So, using Hopf-equivalence by $\mathcal{P}'$, we obtain $h'_j \in [\mathcal{P}']_*$ such that

$$\begin{cases} \operatorname{Dom}(h'_j) = \alpha'_{g(j)}(E'_{i_0}), \\ \operatorname{Im}(h'_j) = E'_j. \end{cases}$$

These partial transformations h'_j give us partial transformations $e'_{j,i_0} : E'_{i_0} \to E'_j$ by setting

$$e'_{j,i_0} x' = h'_j \cdot \alpha'_{g(j)} x', \quad (x' \in E'_{i_0}).$$

Then,

$$\begin{cases} e'_{j,i_0} \in [\mathcal{P}' \rtimes_{\alpha'} G]_* \\ \operatorname{Dom}(e'_{j,i_0}) = E'_{i_0} \\ \operatorname{Im}(e'_{j,i_0}) = E'_j \\ \alpha_g \cdot e_{j,i_0} \cdot \operatorname{Id}|_A \in [\mathcal{P}_m]_* \Leftrightarrow \alpha'_g \cdot e'_{j,i_0} \cdot \operatorname{Id}|_{\varphi(A)} \in [\mathcal{P}'_{m'}]_*, \end{cases}$$

where $A \subset E_{i_0}$ and $g \in G$. We note that

$$e_{j,i_0} \in [\mathcal{P} \rtimes_\alpha H]_* \Leftrightarrow g(j) \in H \Leftrightarrow e'_{j,i_0} \in [\mathcal{P}' \rtimes_{\alpha'} H]_*.$$

Now let us extend φ to a $m - m'$ preserving measure isomorphism $X \to X'$ by setting for each j

$$\varphi x = e'_{j,i_0} \cdot \varphi \cdot e_{i_0,j} x \qquad (x \in E_j).$$

Set

$$\begin{cases} e'_{i_0,j} = {e'_{j,i_0}}^{-1} \\ e'_{j,l} = e'_{j,i_0} \cdot e'_{i_0,l} \\ \xi' = \{e'_{j,l} \mid j, l \in \Lambda\}. \end{cases}$$

Thus we have constructed the desired $\mathcal{P}' \rtimes_{\alpha'} G$-tower $\xi' = \{e'_{i,j} \mid i, j \in \Lambda\}$ of the set X'.

We take a $\mathcal{P}' \rtimes_{\alpha'} G$-tower η' of the set E'_{i_0} such that the product tower $\xi' \rtimes \eta'$ approximates $\mathcal{R}'$-orbits and the measurable subsets of X' in some fixed precision. Again take a corresponding partition of the set E_{i_0} and copy the tower η' into this set in the same way as previous argument. Apply again this procedure and contiune back and forth in this fashion. In the limit we obtain a $m - m'$ preserving measure isomorphism $\varphi : X \to X'$ satisfying that for a.e. x,

$$\begin{cases} \varphi(\mathcal{P} \rtimes_\alpha G(x)) = \mathcal{P}' \rtimes_{\alpha'} G(\varphi(x)) \\ \varphi(\mathcal{P} \rtimes_\alpha H(x)) = \mathcal{P}' \rtimes_{\alpha'} H(\varphi(x)). \end{cases} \quad \blacksquare$$

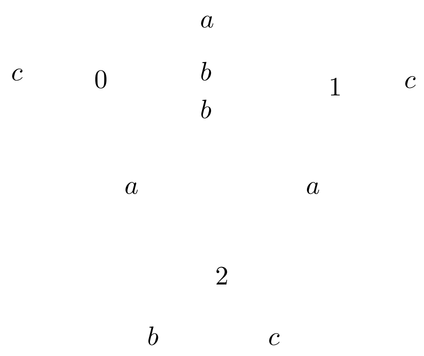
5. COMPUTATION OF INDEX RATIO SETS

Let us take a finite-to-one factor map φ from an ergodic finite measure preserving transformation T on a Lebesgue measure space $(X, \mathfrak{B}_X, m_X)$ to an ergodic finite measure preserving transformation S on a Lebesgue measure space $(Y, \mathfrak{B}_Y, m_X)$, that is, $\pi T = S\pi$, $\pi^{-1}(\mathfrak{B}_Y) \subset \mathfrak{B}_X$, $m_X(\pi^{-1} \cdot) = m_Y(\cdot)$. By $\mathcal{S}$, we denote the ergodic measured discrete equivalence relation $\{(T^n x, x) \mid n \in \mathbb{Z}, x \in X\}$. Let us define an ergodic equivalence relation $\mathcal{R}$ by

$$\mathcal{R} = \mathcal{S}_X \vee \{(x, x') \mid \pi(x) = \pi(x')\}.$$

Here, the right hand side means the equivalence relation generated by both of the relation $\mathcal{S}$ and $\{(x, x') \mid \pi(x) = \pi(x')\}$. We remark that $\mathcal{R}$ is amenable. Under this setup, we are going to show a computation of the pair of index ratio sets of

$\mathcal{S} \subset \mathcal{R}$, when the factor map is arising from a sofic system. For this, let us take the following labeled graph:



Construct the set X of all possible two sided infinite concatenation of edges and the set Y of all possible two sided infinite concatenation of labels respectively. Shifts T on X and S on Y are called a topological Markov shift and a sofic system respectively. A natural map from $(x_n)_{n\in\mathbb{Z}} \in X$ to $(y_n)_{n\in\mathbb{Z}} \in Y$ is induced by defining that each y_n is the label of an edge x_n. Introducing the maximal measures m for T and μ for S respectively, we obtain a measure preserving factor map between T and S (i.e. $\pi \cdot T = S \cdot \pi$). We notice that since the directed graph is irreducible, both of T and S are ergodic and that they have the unique maximal measures, because the directed graph is aperiodic.

Define the permutations $\varphi_a, \varphi_b, \varphi_c \in \Sigma_3$ acting on the set $\{0, 1, 2\}$ by

$$\varphi_c = (0\ 1\ 2)\,, \quad \varphi_b = (1\ 0\ 2)\,, \quad \varphi_a = (1\ 2\ 0)\,.$$

Every path $x = (x_n)_{n\in\mathbb{Z}} \in X$ is identified with $(y, i) \in Y \times \{0, 1, 2\}$, where $y = (y_n)_{n\in\mathbb{Z}}$ and i the initial vertex of the edge x_0 and y_n is the label of x_n. So, we may and do assume $X = Y \times \{0, 1, 2\}$. Through this identification, T is of the form $T(y, i) = (Sy, \varphi_{y_0}(i))$, $(y, i) \in X$. The maximal measures m_X and m_Y are given by

$$m_X = m_Y \times \mu, \qquad \mu(0) = \mu(1) = \mu(2) = \frac{1}{3},$$
$$m_Y = \prod_{i=1}^{\infty} P, \qquad P(a) = P(b) = P(c) = \frac{1}{3}.$$

Set

$$\varphi(n, y) = \varphi_{y_{n-1}} \cdots \varphi_{y_1} \cdot \varphi_{y_0}, \qquad n > 0$$
$$\varphi(0, y) = \mathrm{id}$$
$$\varphi(n, y) = \varphi_{y_n}^{-1} \cdots \varphi_{y_{-2}}^{-1} \cdot \varphi_{y_{-1}}^{-1}, \qquad n < 0.$$

The φ is a cocycle of T and satisfies $T^n(y,i) = (S^n y, \varphi(n,y)(i))$.

Define the transformation ψ by $\psi(y,i) = (y, i+1 (\mathrm{mod}\, 3))$, $(y,i) \in X$, $n \in \mathbb{Z}$. Then we easily see that $\{\mathrm{id}, \psi, \psi^2\}$ is the set of choice functions of $\mathcal{S} \subset \mathcal{R}$. Here we notice $[\mathcal{R} : \mathcal{S}] = 3$.

By σ, we denote the index cocycle corresponding to the above choice functions, that is, if $i, j \in \{0,1,2\}$ and $((y,k),(y',k')) \in \mathcal{R}$ then

$$j = \sigma((y',k'),(y,k))(i) \quad \text{if and only if} \quad ((y', k'+j), ((y, k+i)) \in \mathcal{S}.$$

LEMMA 5.1. *The restriction* $\mathrm{Ker}(\sigma)|_{Y\times\{0\}}$ *of the subrelation* $\mathrm{Ker}(\sigma)$ *to the set* $Y \times \{0\}$ *is ergodic.*

Proof. Set $X_1 = X \times \{0,1,2\} = Y \times \{0,1,2\} \times \{0,1,2\}$, and define the measure preserving transformation T_1 on X_1 by

$$T_1(y,i,j) = (Sy, \varphi_{y_0}(i), \varphi_{y_0}(j)), \quad (y,i,j) \in X_1.$$

Later, we will show that the number of the ergodic components of T_1 is 2. If so, one of them is the subset $Y \times \{(i,j) \mid i \neq j\} \subset X_1$, and hence the induced transformation of T_1 to the subset $Y \times \{0\} \times \{1\}$ is ergodic, too. Take any measurable subsets $E, F \subset Y$ of positive measure. Then, there exist $k, l \in \mathbb{Z}$ and subsets $E_0 \subset E$, $F_0 \subset F$ of positive measure satisfying

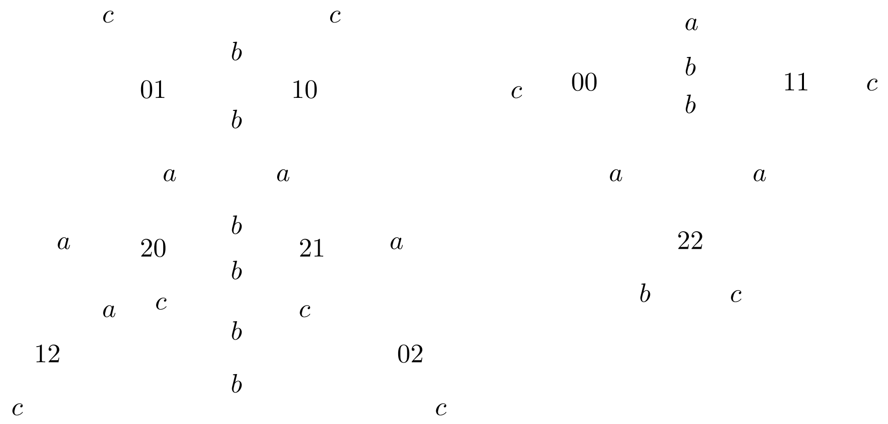
$$\begin{aligned} &T_1|^k_{Y\times\{0\}\times\{1\}}(y,0,1) = (S^l y, \varphi(l,y)(0), \varphi(l,y)(1)), \quad y \in E_0 \\ &S^l(E_0) = F_0. \end{aligned}$$

Hence,

$$\sigma((S^l y, 0),(y,0))(0) = 0, \quad \sigma((S^l y,0),(y,0))(1) = 1.$$

That is, $((S^l y,0),(y,0)) \in \mathrm{Ker}(\sigma)$, $y \in E_0$. Moreover, $T^l(E_0 \times \{0\}) = F_0 \times \{0\}$ and $\mathrm{Ker}(\sigma)|_{Y\times\{0\}}$ is ergodic.

To see that T_1 has only two ergodic components, we consider the following labeled graph.



In fact, the natural map obtained from this labeled graph which has two irreducible components, is the factor map π_1 from T_1 to S, that is, $\pi_1(y, i, j) = y$. So, the ergodic components of T_1 are these two disjoint path spaces consisting of infinite concatenation of edges arising from each irreducible component. ∎

LEMMA 5.2. *The index ratio set of the* $\{\mathcal{R}, \mathcal{S}\}$ *is*

$$\{\mathbf{r}(\mathcal{R}), \mathbf{r}(\mathcal{S})\} = \{\Sigma_3, \Sigma_2\}.$$

Proof. We saw that $\mathrm{Ker}(\sigma)|_{Y\times\{0\}}$ is ergodic. So, by Lemma 2.3, it is enough to compute the images $\{\sigma(x, z) \mid x, z \in Y \times \{0\}, (x, z) \in \mathcal{R}\}$ and $\{\sigma(x, z) \mid x, z \in Y \times \{0\}, (x, z) \in \mathcal{S}\}$. If $((y, 0), (u, 0)) \in \mathcal{S}$, then $\varphi(n, y)(0) = 0$, where $u = S^n y$. In this case, we see from the above figure that $\varphi(n, y)(1) \in \{1, 2\}$, and both of the cases occur. In other words,

$$\sigma((u, 0), (y, 0)) = (0\ 1\ 2)$$

or,

$$\sigma((u, 0), (y, 0)) = (0\ 2\ 1)\,.$$

Thus, we showed $\mathbf{r}(\mathcal{S}) = \Sigma_2$. In order to prove $\mathbf{r}(\mathcal{R}) = \Sigma_3$, it is enough to show that there is a permutation in $\mathbf{r}(\mathcal{R})$ which does not belong to Σ_2. In fact,

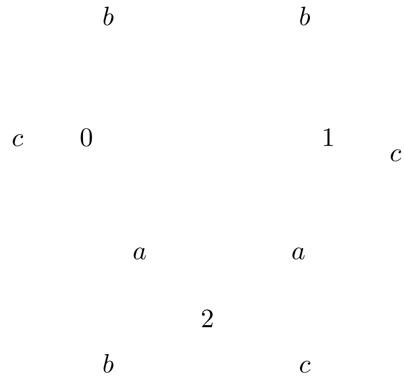
$$\sigma((Sy, 0), (y, 0)) = (1\ 2\ 0)\,, \quad \text{if } y_0 = a.$$

Because, if $y \in Y$ satisfies $y_0 = a$ then

$$\begin{aligned} &((y,0),(y,1)) \in \mathcal{R}, ((y,1), \quad (T(y,1)) = ((y,1),(Sy,2)) \in \mathcal{S} \\ &((y,0),T(y,0)) = (y,0), \qquad (Sy,1)) \in \mathcal{S}. \end{aligned}$$

Hence, $(1,2,0) \in \mathbf{r}(\mathcal{R})$. ∎

Remark 5.3. In amenable type II_1 case, the orbit equivalence classes of relations-subrelations of index 3 are only two. In fact by Theorem 4.6, all possible index ratio sets are the pairs $\{\mathbb{Z}_3, \{e\}\}$ and $\{\Sigma_3, \Sigma_2\}$. The first case appears in the previous example. About the second case, for instance it is enough to consider the following labeled graph.



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