

ON THE PUNCTURED NEIGHBOURHOOD THEOREM

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ABSTRACT. Let X, Y, Z be Banach spaces and $X \xrightarrow{S(z)} Y \xrightarrow{T(z)} Z$ an analytically dependent sequence of operators satisfying $T(z)S(z) = 0$. We study the properties of the function $z \mapsto \dim \text{Ker } T(z) / \text{Im } S(z)$.

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Let X, Y be complex Banach spaces. Denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y . If $Y = X$ then we write for short $\mathcal{L}(X) = \mathcal{L}(X, X)$.

Recall the well-known punctured neighbourhood theorem:

THEOREM 0.1. *Let $T \in \mathcal{L}(X)$ be a Fredholm operator. Then there exist $\varepsilon > 0$ and constants $k_1 \leq \dim \text{Ker } T$, $k_2 \leq \text{codim Im } T$ such that $\dim \text{Ker}(T - z) = k_1$ and $\text{codim Im}(T - z) = k_2$ for all z , $0 < |z| < \varepsilon$.*

In this paper we study a more general situation. Let X, Y, Z be Banach spaces, let U be an open subset of \mathbb{C}^n , let $S : U \rightarrow \mathcal{L}(X, Y)$ and $T : U \rightarrow \mathcal{L}(Y, Z)$ be analytic operator-valued functions satisfying $T(z)S(z) = 0$ for all $z \in U$. For $z \in U$ write $\alpha(z) = \dim \text{Ker } T(z) / \text{Im } S(z)$.

The aim of the paper is to study the behaviour of the function $z \mapsto \alpha(z)$.

The main result of the first section is the following generalization of Theorem 0.1: if $U \subset \mathbb{C}$, $w \in U$, $\text{Im } T(w)$ is closed and $\alpha(w) < \infty$, then $\alpha(z) = k$ is constant in a punctured neighbourhood of w .

Clearly the classical punctured neighbourhood theorem follows easily from this generalization for sequences $0 \rightarrow X \xrightarrow{T-z} Y$ and $X \xrightarrow{T-z} Y \rightarrow 0$, respectively.

In the second section we study the case $n \geq 2$. This situation has been studied mainly in connection with the Koszul complex of an n -tuple of commuting operators.

I

For $T \in \mathcal{L}(X, Y)$ denote by $\gamma(T)$ the Kato reduced minimum modulus, $\gamma(T) = \inf\{\|Tx\| : \text{dist}\{x, \text{Ker } T\} = 1\}$ (formally we set $\gamma(0) = \infty$). Clearly $\gamma(T) > 0$ if and only if $\text{Im } T$ is closed.

If M, L are closed subspaces of X then write

$$\delta(M, L) = \sup_{\substack{x \in M \\ \|x\| \leq 1}} \text{dist}\{x, L\}$$

and the gap between M and L is defined by $\widehat{\delta}(M, L) = \max\{\delta(M, L), \delta(L, M)\}$. For the properties of the reduced minimum modulus and the gap see [6].

The following result is due to Markus, cf. [13], Theorem 1.4.

THEOREM 1.1. *Let U be an open subset of \mathbb{C}^n , let $T : U \rightarrow \mathcal{L}(X, Y)$ be a norm-continuous function, let $w \in U$ and $\text{Im } T(w)$ be closed. The following conditions are equivalent:*

- (i) *the function $z \mapsto \gamma(T(z))$ is continuous at w ;*
- (ii) $\liminf_{z \rightarrow w} \gamma(T(z)) > 0$;
- (iii) $\lim_{z \rightarrow w} \delta(\text{Ker } T(w), \text{Ker } T(z)) = 0$;
- (iv) $\lim_{z \rightarrow w} \widehat{\delta}(\text{Ker } T(w), \text{Ker } T(z)) = 0$;
- (v) $\lim_{z \rightarrow w} \delta(\text{Im } T(z), \text{Im } T(w)) = 0$;
- (vi) $\lim_{z \rightarrow w} \widehat{\delta}(\text{Im } T(z), \text{Im } T(w)) = 0$.

The equivalences (iii) \Leftrightarrow (iv) and (v) \Leftrightarrow (vi) follow from the fact that automatically $\lim_{z \rightarrow w} \delta(\text{Ker } T(z), \text{Ker } T(w)) = 0$ and $\lim_{z \rightarrow w} \delta(\text{Im } T(w), \text{Im } T(z)) = 0$.

A continuous function $T : U \rightarrow \mathcal{L}(X, Y)$ is called regular at w if $\text{Im } T(w)$ is closed and T satisfies any of equivalent conditions (i)–(vi). In particular, condition (ii) implies that the set of all regularity points of T is open. Also, T is regular at w if and only if the adjoint function $z \mapsto T(z)^*$ is regular at w .

Regular functions are closely related to the exactness:

THEOREM 1.2. ([13], Theorem 2) *Let U be an open subset of \mathbb{C}^n , $w \in U$ and let $T : U \rightarrow \mathcal{L}(X, Y)$ be an analytic function. The following conditions are equivalent:*

- (i) T is regular at w ;
- (ii) there exists a neighbourhood $U_0 \subset U$ of w , a Banach space E and an analytic function $S : U_0 \rightarrow \mathcal{L}(E, X)$ such that $\text{Im } S(z) = \text{Ker } T(z)$ ($z \in U_0$);
- (iii) there exists a neighbourhood $U_0 \subset U$ of w , a Banach space E' and an analytic function $S' : U_0 \rightarrow \mathcal{L}(Y, E')$ such that $\text{Im } T(z) = \text{Ker } S'(z)$ ($z \in U_0$).

In particular, if $T : U \rightarrow \mathcal{L}(X, Y)$ is regular at w and $x \in \text{Ker } T(w)$ then there exist a neighbourhood U_0 of w and an analytic function $f : U_0 \rightarrow X$ such that $f(w) = x$ and $T(z)f(z) = 0$ ($z \in U_0$). Indeed, let $S : U_0 \rightarrow \mathcal{L}(E, X)$ be an analytic function satisfying the properties of (ii). Choose $e \in E$ with $S(w)e = x$ and set $f(z) = S(z)e$.

LEMMA 1.3. *Let U be an open subset of \mathbb{C}^n , let $S : U \rightarrow \mathcal{L}(X, Y)$ and $T : U \rightarrow \mathcal{L}(Y, Z)$ be functions regular in U . Suppose that $T(z)S(z) = 0$ for all $z \in U$. Then $\alpha(z)$ is constant on each connected subset of U .*

Proof. Let $w \in U$ satisfy $\alpha(w) = \dim \text{Ker } T(w) / \text{Im } S(w) < \infty$. By Theorem 1.1 (iv) and (vi),

$$\lim_{z \rightarrow w} \widehat{\delta}(\text{Ker } T(w), \text{Ker } T(z)) = 0 \quad \text{and} \quad \lim_{z \rightarrow w} \widehat{\delta}(\text{Im } T(w), \text{Im } T(z)) = 0.$$

Thus there exists $\varepsilon > 0$ such that $\widehat{\delta}(\text{Ker } T(z), \text{Ker } T(w)) < 1/3$ and $\widehat{\delta}(\text{Im } S(z), \text{Im } S(w)) < 1/3$ for $z \in U$, $\text{dist}\{z, w\} < \varepsilon$. By [1] this implies that

$$\alpha(z) = \dim \text{Ker } T(z) / \text{Im } S(z) = \dim \text{Ker } T(w) / \text{Im } S(w) = \alpha(w)$$

for all $z \in U$, $\text{dist}\{z, w\} < \varepsilon$.

Thus $\alpha(z)$ is locally constant and a standard argument gives that $\alpha(z)$ is constant on the component of connectivity of U containing w .

If U_0 is a component of U and there is no $w \in U_0$ with $\alpha(w) < \infty$, then clearly $\alpha(z) = \infty$ on U_0 . ■

An operator $T \in \mathcal{L}(X)$ with the property that the function $z \mapsto T - z$ is regular at 0 is called semi-regular (sometimes Kato regular). Semi-regular operators exhibit very nice properties and have been studied intensely, see e.g. [9], [10], [12].

An essential version of semi-regular operators has been also studied. Recall that if M, L are closed subspaces of X then we write $M \overset{\circ}{\subset} L$ (M is essentially contained in L) if $\dim M / (M \cap L) < \infty$. We summarize some of equivalent conditions characterizing essentially semi-regular operators.

THEOREM 1.4. ([10], Theorem 3.1) *Let $T \in \mathcal{L}(X)$ be an operator with closed range. The following conditions are equivalent:*

(i) (Kato decomposition) *there exists a decomposition $X = X_1 \oplus X_2$ such that $TX_1 \subset X_1$, $TX_2 \subset X_2$, $\dim X_1 < \infty$, $T|_{X_1}$ is nilpotent and $T|_{X_2}$ is an semi-regular operator;*

$$(ii) \bigcap_{z \neq 0} \overline{\text{Im}(T - z)} \stackrel{e}{\subset} \text{Im } T;$$

(iii) $\dim \text{Ker } T / N^*(T) < \infty$, *where $N^*(T)$ is the set of all $x \in X$ such that there are complex numbers z_i ($i = 1, 2, \dots$) tending to 0 and elements $x_i \in \text{Ker}(T - z_i)$ such that $x = \lim_{i \rightarrow \infty} x_i$ (clearly $N^*(T) \subset \text{Ker } T$);*

(iv) $\dim R^*(T) / \text{Im } T < \infty$ *where $R^*(T)$ is the set of all $x \in X$ such that $x = \lim_{i \rightarrow \infty} x_i$ for some $x_i \in \text{Im}(T - z_i)$ and some $z_i \rightarrow 0$ (clearly $\text{Im } T \subset R^*(T)$).*

Note that condition (i) implies that the function $z \mapsto T - z$ is regular in a punctured neighbourhood $\{z : 0 < |z| < \varepsilon\}$ for some $\varepsilon > 0$.

General analytic operator-valued functions of one variable can be reduced to the linear case by the method of linearization, see [2], Theorem 2.6.

THEOREM 1.5. *Let $U \subset \mathbb{C}$ be an open set, $T : U \rightarrow \mathcal{L}(X, Y)$ an analytic function and $w \in U$. Then there exist a neighbourhood U_0 of w , Banach spaces Z and M , an operator $V \in \mathcal{L}(M)$ and analytic functions $A : U_0 \rightarrow \mathcal{L}(M, X \oplus Z)$, $B : U_0 \rightarrow \mathcal{L}(Y \oplus Z, M)$ such that $A(z)$ and $B(z)$ are invertible operators and*

$$B(z)(T(z) \oplus I_Z)A(z) = V - zI_M \quad (z \in U_0).$$

Let $U \subset \mathbb{C}$ be an open set and let $T : U \rightarrow \mathcal{L}(X, Y)$ be an analytic operator-valued function. Let $w \in U$. Write

$$\begin{aligned} R^*(T(w)) &= \{y \in Y : \text{there exist } z_k \in U, z_k \rightarrow w \\ &\quad \text{and } y_k \in \text{Im } T(z_k) \text{ with } y_k \rightarrow y\}, \\ R^{**}(T(w)) &= \{y \in Y : \lim_{z \rightarrow w} \text{dist}\{y, \text{Im } T(z)\} = 0\}. \end{aligned}$$

Clearly $\text{Im } T(w) \subset R^{**}(T(w)) \subset R^*(T(w))$ and $R^*(T(w))$, $R^{**}(T(w))$ are closed subspaces of Y .

Similarly write

$$\begin{aligned} N^*(T(w)) &= \{x \in X : \text{there are } z_k \in U, x_k \in \text{Ker } T(z_k) \\ &\quad \text{with } z_k \rightarrow w \text{ and } x_k \rightarrow x\}, \\ N^{**}(T(w)) &= \{x \in X : \lim_{z \rightarrow w} \text{dist}\{x, \text{Ker } T(z)\} = 0\}. \end{aligned}$$

Clearly $N^{**}(T(w)) \subset N^*(T(w)) \subset \text{Ker } T(w)$ and $N^*(T(w))$, $N^{**}(T(w))$ are closed subspaces of X .

THEOREM 1.6. *Let $U \subset \mathbb{C}$ be an open set, $T : U \rightarrow \mathcal{L}(X, Y)$ an analytic function and $w \in U$. The following statements are equivalent:*

- (i) $\dim R^*(T(w))/\text{Im } T(w) < \infty$;
- (ii) $\dim R^{**}(T(w))/\text{Im } T(w) < \infty$;
- (iii) $\dim \text{Ker } T(w)/N^*(T(w)) < \infty$ and $\text{Im } T(w)$ is closed;
- (iv) $\dim \text{Ker } T(w)/N^{**}(T(w)) < \infty$ and $\text{Im } T(w)$ is closed.

Any of these conditions implies that there exists $\varepsilon > 0$ such that the function T is regular in the punctured neighbourhood $\{z \in U : 0 < |z - w| < \varepsilon\}$. Further $N^(T(w)) = N^{**}(T(w))$, $R^*(T(w)) = R^{**}(T(w))$ and $\dim \text{Ker } T(w)/N^*(T(w)) = \dim R^*(T(w))/\text{Im } T(w)$.*

Proof. A. Suppose first that $Y = X$ and $T(z) = V - zI_X$ for some operator $V \in \mathcal{L}(X)$. We show that in this case conditions (i)–(iv) are equivalent to

- (v) $V - w$ is essentially semi-regular.

Clearly (i) \Rightarrow (ii) and (iv) \Rightarrow (iii).

By Theorem 1.4, (i) \Leftrightarrow (iii) \Leftrightarrow (v).

(ii) \Rightarrow (v): Clearly (ii) implies that $\text{Im } T(w)$ is closed. Further

$$\bigcap_{z \neq w} \overline{\text{Im}(V - z)} \subset R^{**}(V - w)$$

so that, by Theorem 1.4, $V - w$ is essentially semi-regular.

Suppose now that $V - w$ is essentially semi-regular. Let $X = X_1 \oplus X_2$ be the Kato decomposition of $V - w$, i.e., $VX_1 \subset X_1$, $VX_2 \subset X_2$, $\dim X_1 < \infty$, $(V - w)|_{X_1}$ is nilpotent and $(V - w)|_{X_2}$ is semi-regular. It is easy to see that, for $z \neq w$, $\text{Ker}(V - z) = \text{Ker}((V - z)|_{X_2})$ and $\text{Im}(V - z) = X_1 + \text{Im}((V - z)|_{X_2})$. Thus

$$N^*(V - w) = N^{**}(V - w) = \text{Ker}((V - w)|_{X_2})$$

and

$$R^*(V - w) = R^{**}(V - w) = X_1 + \text{Im}((V - w)|_{X_2}).$$

Hence (v) implies (iv). Further

$$\begin{aligned} \dim \text{Ker}(V - w)/N^*(V - w) &= \dim \text{Ker}((V - w)|_{X_1}) = \dim X_1/(V - w)X_1 \\ &= \dim R^*(V - w)/\text{Im}(V - w). \end{aligned}$$

Also the Kato decomposition implies that the function $z \mapsto V - z$ is regular in a certain punctured neighbourhood of w .

B. Let now $T(z)$ be a general analytic operator-valued function. By Theorem 1.5 there exist a neighbourhood U_0 of w , Banach spaces Z, M , an operator $V \in \mathcal{L}(M)$ and analytic functions $A : U_0 \rightarrow \mathcal{L}(M, X \oplus Z)$, $B : U_0 \rightarrow \mathcal{L}(Y \oplus Z, M)$ whose values are invertible operators, such that

$$B(z)(T(z) \oplus I_Z)A(z) = V - zI_Z \quad (z \in U_0).$$

For $z \in U_0$ we have

$$\text{Ker}(V - zI) = \text{Ker}((T(z) \oplus I_Z)A(z)) = A(z)^{-1} \text{Ker}(T(z) \oplus I_Z) = A(z)^{-1} \text{Ker} T(z)$$

and

$$\text{Im}(V - zI) = \text{Im}(B(z)(T(z) \oplus I_Z)) = B(z)(\text{Im} T(z) + Z).$$

Thus

$$\begin{aligned} N^*(V - wI) &= A(w)^{-1}N^*(T(w)), \\ N^{**}(V - wI) &= A(w)^{-1}N^{**}(T(w)), \\ R^*(V - wI) &= B(w)(R^*(T(w)) + Z) \quad \text{and} \\ R^{**}(V - wI) &= B(w)(R^{**}(T(w)) + Z). \end{aligned}$$

Hence all the statements for the function $T(z)$ are equivalent to the corresponding statements for $V - zI$ and the general case reduces to the previous case. ■

REMARK 1.7. Let $U \subset \mathbb{C}$, $w \in U$ and let $T : U \rightarrow \mathcal{L}(X, Y)$ be an analytic function. Then $\dim \text{Ker} T(w)/N^*(T(w))$ can be interpreted as the ‘‘jump’’ in the kernel of $T(z)$; similarly $\dim R^*(T(w))/\text{Im} T(w)$ signifies the jump in the range of $T(z)$. It is interesting to note that these two numbers are always equal.

THEOREM 1.8. *Let U be an open subset of \mathbb{C} and $w \in U$. Suppose that $S : U \rightarrow \mathcal{L}(X, Y)$, $T : U \rightarrow \mathcal{L}(Y, Z)$ are analytic functions satisfying $T(z)S(z) = 0$ ($z \in U$), $\alpha(w) < \infty$ and $\text{Im} T(w)$ is closed. Then there exist $\varepsilon > 0$ and a constant $c \leq \alpha(w)$ such that $\alpha(z) = c$ for all z , $0 < |z - w| < \varepsilon$.*

Proof. By [14], Lemma 2.1, $\alpha(z) \leq \alpha(w)$ for all z in a neighbourhood of w . Using the previous theorem, both $z \mapsto S(z)$ and $z \mapsto T(z)$ are regular in a certain punctured neighbourhood of w so that, by Lemma 4, $\alpha(z)$ is constant in this punctured neighbourhood. ■

II

In this section we study analytic operator-valued functions of n -variables.

It is not possible to expect the punctured neighbourhood theorem for $n \geq 2$; the proper generalization seems to be

CONJECTURE 2.1. Let $U \subset \mathbb{C}^n$ be open, let $S : U \rightarrow \mathcal{L}(X, Y)$ and $T : U \rightarrow \mathcal{L}(X, Y)$ be analytic on U . Suppose that $T(z)S(z) = 0$, $\text{Im } T(z)$ is closed and $\alpha(z) = \dim \text{Ker } T(z) / \text{Im } S(z) < \infty$ ($z \in U$). Let $k \in \mathbb{N}$. Then the set $\{z \in U : \alpha(z) \geq k\}$ is analytic in U .

Recall that a set $M \subset U$ is called analytic in U if for each $w \in U$ there exist a neighbourhood U_0 of w and analytic (scalar-valued) functions f_1, \dots, f_r such that $M \cap U_0 = \{z \in U_0 : f_1(z) = \dots = f_r(z) = 0\}$.

The conjecture is true in the following particular cases:

A. If the ranges and kernels of $S(z)$ and $T(z)$ are complemented subspaces, see Theorem 2.5 below. In particular, the conjecture is true for operators in Hilbert spaces.

B. If either $S(z) \equiv 0$ or $T(z) \equiv 0$; this means that the other function is upper (lower) semi-Fredholm-valued and the conjecture reduces to the statement about defect indices of semi-Fredholm-valued functions, see [5].

C. If the sequence $X \xrightarrow{S(z)} Y \xrightarrow{T(z)} Z$ is a part of a Fredholm complex vanishing at the ends, see [7], [8], [11] or Theorem 2.9 below.

We start with the following lemma:

LEMMA 2.2. Let $U \subset \mathbb{C}^n$ be an open subset, let $T : U \rightarrow \mathcal{L}(X, Y)$ be an analytic function, let $k \in \mathbb{N}$. Then the set $\{z \in U : \dim \text{Im } T(z) < k\}$ is analytic.

Proof. If $x_1, \dots, x_k \in X$, $y_1^*, \dots, y_k^* \in Y^*$, $z \in U$ and $\dim \text{Im } T(z) < k$ then the vectors $T(z)x_1, \dots, T(z)x_k$ are linearly dependent and $\det(\langle T(z)x_i, y_j^* \rangle) = 0$.

On the other hand, if $\dim \text{Im } T(z) \geq k$ then there are vectors $x_1, \dots, x_k \in X$, $y_1^*, \dots, y_k^* \in Y^*$ such that $\det(\langle T(z)x_i, y_j^* \rangle) \neq 0$. Thus

$$\begin{aligned} & \{z \in U : \dim \text{Im } T(z) < k\} \\ &= \{z \in U : \det(\langle T(z)x_i, y_j^* \rangle) = 0 \text{ for all } x_1, \dots, x_k \in X, y_1^*, \dots, y_k^* \in Y^*\} \end{aligned}$$

which is an analytic set, see [3], p. 86. ■

COROLLARY 2.3. Let $S : U \rightarrow \mathcal{L}(X, Y)$ and $T : U \rightarrow \mathcal{L}(Y, Z)$ be analytic functions and let $k \in \mathbb{N}$. Then the set

$$\{z \in U : \dim \operatorname{Im} S(z) / (\operatorname{Im} S(z) \cap \operatorname{Ker} T(z)) < k\}$$

is analytic.

Proof. Clearly $\dim \operatorname{Im} S(z) / (\operatorname{Im} S(z) \cap \operatorname{Ker} T(z)) = \dim \operatorname{Im}(T(z)S(z))$ so that the corollary follows from the previous lemma. ■

LEMMA 2.4. Let U be an open subset of \mathbb{C}^n , let $S : U \rightarrow \mathcal{L}(X, Y)$ and $T : U \rightarrow \mathcal{L}(Y, Z)$ be analytic functions satisfying $T(z)S(z) = 0$ ($z \in U$). Suppose that there are Banach spaces X_1 and Z_1 and regular analytic functions $S_1 : U \rightarrow \mathcal{L}(X_1, Y)$, $T_1 : U \rightarrow \mathcal{L}(Y, Z_1)$ satisfying

$$\operatorname{Ker} T_1(z) \subset \operatorname{Im} S(z) \subset \operatorname{Ker} T(z) \subset \operatorname{Im} S_1(z)$$

and $\dim \operatorname{Im} S_1(z) / \operatorname{Ker} T_1(z) < \infty$ ($z \in U$). Then the set

$$\{z \in U : \dim \operatorname{Ker} T(z) / \operatorname{Im} S(z) \geq k\}$$

is analytic in U .

Proof. The situation is illustrated by the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{S(z)} & Y & \xrightarrow{T(z)} & Z \\ & \nearrow_{S_1(z)} & & \searrow_{T_1(z)} & \\ X_1 & & & & Z_1 \end{array}$$

Figure 1.

We can assume that U is connected. For each j set

$$A_j = \{z \in U : \dim \operatorname{Im} S(z) / \operatorname{Ker} T_1(z) \leq j\}$$

and

$$B_j = \{z \in U : \dim \operatorname{Im} S_1(z) / \operatorname{Ker} T(z) \leq j\}.$$

By Corollary 2.3, A_j and B_j are analytic sets. As in the proof of Lemma 1.3 (or using Theorem 1.2) it is easy to see that there is a constant c such that $\dim \operatorname{Im} S_1(z) / \operatorname{Ker} T_1(z) = c$ in U . Thus

$$\begin{aligned} & \{z \in U : \dim \operatorname{Ker} T(z) / \operatorname{Im} S(z) \geq k\} \\ &= \{z \in U : \dim \operatorname{Im} S_1(z) / \operatorname{Ker} T(z) + \dim \operatorname{Im} S(z) / \operatorname{Ker} T_1(z) \leq c - k\} \\ &= \bigcup_{i=0}^{c-k} A_i \cap B_{c-k-i}. \end{aligned}$$

The last set is clearly analytic. ■

Let $T \in \mathcal{L}(X, Y)$. An operator $S \in \mathcal{L}(Y, X)$ is called a generalized inverse of T if $TST = T$ and $STS = S$. If S is a generalized inverse of T then TS and ST are projections satisfying $\text{Im}(TS) = \text{Im} T$ and $\text{Ker}(ST) = \text{Ker} T$. Thus T has a generalized inverse if and only if both $\text{Ker} T$ and $\text{Im} T$ are complemented subspaces of X and Y , respectively.

The next result shows that Conjecture 2.1 is true for operators with generalized inverses. We adopt the method of [4].

THEOREM 2.5. *Let U be an open subset of \mathbb{C}^n , let $S : U \rightarrow \mathcal{L}(X, Y)$ and $T : U \rightarrow \mathcal{L}(Y, Z)$ be analytic functions. Suppose that $T(z)S(z) = 0$, $\dim \text{Ker} T(z) / \text{Im} S(z) < \infty$ and the operators $S(z)$ and $T(z)$ have generalized inverses for $z \in U$. Let $k \in \mathbb{N}$. Then the set $\{z \in U : \alpha(z) \geq k\}$ is analytic in U .*

Proof. Let $w \in U$. Let V be a generalized inverse of $S(w)$, i.e., $VS(w)V = V$ and $S(w)VS(w) = S(w)$. Set $P = I - S(w)V$. Then P is a projection, $\text{Ker} P = \text{Im} S(w)$.

For z close to w , the operator $I + (S(z) - S(w))V$ is invertible. Define $P(z) \in \mathcal{L}(Y)$ by $P(z) = P(I + (S(z) - S(w))V)^{-1} \in \mathcal{L}(Y)$. Clearly the function $z \mapsto P(z)$ is regular at w since $\text{Im} P(z) = \text{Im} P$ is constant. We prove that $\text{Ker} P(z) \subset \text{Im} S(z)$. Let $y \in \text{Ker} P(z)$, i.e., $0 = P(z)y = P(I + (S(z) - S(w))V)^{-1}y$. Then

$$(I + (S(z) - S(w))V)^{-1}y \in \text{Ker} P = \text{Im} S(w).$$

For some $x \in X$ we have

$$y = (I + (S(z) - S(w))V)S(w)x = S(z)VS(w)x \in \text{Im} S(z).$$

Similarly, let W be a generalized inverse of $T(w)$. Set $Q = I - WT(w)$. Then Q is a projection with $\text{Im} Q = \text{Ker} T(w)$. For z close to w define $Q(z) \in \mathcal{L}(Y)$ by $Q(z) = (I + W(S(z) - S(w)))^{-1}Q$. Clearly the function $z \mapsto Q(z)$ is regular since $\text{Ker} Q(z) = \text{Ker} Q$ is constant. We have

$$WT(z) = WT(w) + W(T(z) - T(w)) = I - Q + W(T(z) - T(w))$$

so that

$$(I + W(T(z) - T(w)))^{-1}WT(z) = I - (I + W(T(z) - T(w)))^{-1}Q = I - Q(z).$$

Consequently, $\text{Ker} T(z) \subset \text{Im} Q(z)$.

Thus we have $\text{Ker} P(z) \subset \text{Im} S(z) \subset \text{Ker} T(z) \subset \text{Im} Q(z)$ and

$$\dim \text{Im} Q(z) / \text{Ker} P(z) = \dim \text{Im} Q / \text{Ker} P = \dim \text{Ker} T(w) / \text{Im} S(w) < \infty.$$

As in Lemma 1.3 we have that $\dim \text{Im} Q(z) / \text{Ker} P(z) < \infty$ in a neighbourhood of w . The rest follows from Lemma 2.4. ■

COROLLARY 2.6. *Conjecture 2.1 is true for operators in Hilbert spaces.*

In the following we consider a complex

$$(2.1) \quad 0 \longrightarrow X_0 \xrightarrow{\delta_0(z)} X_1 \xrightarrow{\delta_1(z)} \dots \xrightarrow{\delta_{n-1}(z)} X_n \longrightarrow 0,$$

where X_0, \dots, X_n are Banach spaces, the operators $\delta_j(z)$ satisfy $\delta_j(z)\delta_{j-1}(z) = 0$ and depend analytically on a parameter $z \in U$, where U is an open subset of \mathbb{C}^n .

Suppose that the complex (2.1) is Fredholm, i.e., $\dim \text{Ker } \delta_j(z) / \text{Im } \delta_{j-1}(z) < \infty$ for all $j = 0, \dots, n$ and $z \in U$ (formally we set $\delta_{-1}(z) = 0$ and $\delta_n(z) = 0$).

Let $k \in \mathbb{N}$. It is a folklore among specialists in the sheaf theory that the set $\{z \in U : \dim \text{Ker } \delta_j(z) / \text{Im } \delta_{j-1}(z) \geq k\}$ is analytic. This result is stated without proof (for the Koszul complex of a commuting n -tuple of operators) in [7] and [8]; cf. also [11]. Since apparently there is no elementary proof of this result, we include the proof here.

We need the following modification of Lemma 2.4:

LEMMA 2.7. *Let U be an open subset of \mathbb{C}^n , let $S : U \rightarrow \mathcal{L}(X, Y)$ and $T : U \rightarrow \mathcal{L}(Y, Z)$ be analytic functions satisfying $T(z)S(z) = 0$ ($z \in U$). Suppose that there are Banach spaces X_1, Z_1 , finite dimensional Banach spaces F, G and regular analytic functions $S_1 : U \rightarrow \mathcal{L}(X_1, Y \oplus F)$ and $T_1 : U \rightarrow \mathcal{L}(Y \oplus G, Z_1)$ such that $\text{Im } S_1(z) \supset \text{Ker } T(z) \supset \text{Im } S(z)$, $\text{Im } S(z) + G \supset \text{Ker } T_1(z)$ and $\dim(\text{Im } S_1(z) + G) / \text{Ker } T_1(z) < \infty$ ($z \in U$), see Figure 2. Let $k \in \mathbb{N}$. Then the set $\{z \in U : \alpha(z) \geq k\}$ is analytic in U .*

$$\begin{array}{ccccc}
 X_1 & & & & \\
 & \searrow^{S_1(z)} & & & \\
 & & \left\{ \begin{array}{c} F \\ \oplus \\ Y \\ \oplus \\ G \end{array} \right\} & & \\
 X & \xrightarrow{S(z)} & & \xrightarrow{T(z)} & Z \\
 & & & \searrow^{T_1(z)} & \\
 & & & & Z_1
 \end{array}$$

Figure 2.

Proof. Set $Y' = Y \oplus F \oplus G$. For $z \in U$ define the operators $S'(z) : X \oplus G \rightarrow Y'$, $T'(z) : Y' \rightarrow Z \oplus F$, $S'_1(z) : X_1 \oplus G \rightarrow Y'$ and $T'_1(z) : Y' \rightarrow Z_1 \oplus F$ by

$$\begin{aligned}
 S'(z)(x \oplus g) &= S(z)x + g, \\
 T'(z)(y \oplus f \oplus g) &= T(z)y + f, \\
 S'_1(z)(x_1 \oplus g) &= S_1(z)x_1 + g, \\
 T'_1(z)(y \oplus f \oplus g) &= T_1(z)(y \oplus g) + f
 \end{aligned}$$

for all $x \in X$, $f \in F$, $g \in G$ and $x_1 \in X_1$. Thus $\text{Im } S'(z) = \text{Im } S(z) + G$, $\text{Ker } T'(z) = \text{Ker } T(z) + G$, $\text{Im } S'_1(z) = \text{Im } S_1(z) + G$ and $\text{Ker } T'_1(z) = \text{Ker } T_1(z)$. We have

$$\text{Im } S'_1(z) \supset \text{Ker } T'(z) \supset \text{Im } S'(z) \supset \text{Ker } T'_1(z)$$

and

$$\dim \text{Im } S'_1(z) / \text{Ker } T'_1(z) = \dim (\text{Im } S_1(z) + G) / \text{Ker } T_1(z) < \infty.$$

By Lemma 2.4, the set $\{z \in U : \dim \text{Ker } T'(z) / \text{Im } S'(z) \geq k\}$ is analytic in U . This set, however, is equal to the set $\{z \in U : \alpha(z) \geq k\}$. ■

LEMMA 2.8. *Let U be an open subset of \mathbb{C}^n , let $S : U \rightarrow \mathcal{L}(X, Y)$ and $T : U \rightarrow \mathcal{L}(Y, Z)$ be analytic functions satisfying $T(z)S(z) = 0$ and $\alpha(z) < \infty$ ($z \in U$). Let $w \in U$. Suppose that there are finite dimensional spaces G, H , a neighbourhood U_1 of w and a regular analytic function $T_1 : U_1 \rightarrow \mathcal{L}(Y \oplus G, Z \oplus H)$ such that $T_1(z)|_Y = T(z)$. Then there exist a finite dimensional space F , a neighbourhood U_2 of w and a regular analytic function $S_1 : U_2 \rightarrow \mathcal{L}(X \oplus F, Y \oplus G)$ such that $S_1(z)|_X = S(z)$ and $\text{Im } S_1(z) = \text{Ker } T_1(z) \supset \text{Ker } T(z)$, see Figure 3.*

$$\left. \begin{array}{c} X \\ \oplus \\ F \end{array} \right\} \begin{array}{c} \xrightarrow{S(z)} \\ \xrightarrow{S_1(z)} \end{array} \left. \begin{array}{c} Y \\ \oplus \\ G \end{array} \right\} \begin{array}{c} \xrightarrow{T(z)} \\ \xrightarrow{T_1(z)} \end{array} \left. \begin{array}{c} Z \\ \oplus \\ H \end{array} \right\}$$

Figure 3.

Proof. For $z \in U_1$ we have

$$\dim \text{Ker } T_1(z) / \text{Im } S(z) = \dim \text{Ker } T_1(z) / \text{Ker } T(z) + \dim \text{Ker } T(z) / \text{Im } S(z) < \infty.$$

Let y_1, \dots, y_r be linearly independent vectors in $\text{Ker } T_1(w)$ such that

$$\text{Im } S(w) \vee \{y_1, \dots, y_r\} = \text{Ker } T_1(w).$$

Since T_1 is regular, for $i = 1, \dots, r$, there exists a $(Y \oplus G)$ -valued analytic function φ_i defined in a neighbourhood of w such that $T_1(z)\varphi_i(z) = 0$ and $\varphi_i(w) = y_i$. Let F be an r -dimensional space with a basis f_1, \dots, f_r and define $S_1(z) : X \oplus F \rightarrow Y \oplus G$ by

$$S_1(z) \left(x \oplus \sum_{i=1}^r \beta_i f_i \right) = S(z)x + \sum_{i=1}^r \beta_i \varphi_i(z) y_i \quad (x \in X, \beta_i \in \mathbb{C}).$$

Clearly $T_1(z)S_1(z) = 0$ and $\text{Im } S_1(w) = \text{Ker } T_1(w)$ so that there is a neighbourhood of w where $\text{Ker } T_1(z) = \text{Im } S_1(z)$, see [14]. Thus S_1 is regular in a neighbourhood of w and satisfies all the required conditions. ■

THEOREM 2.9. *Let X_0, X_1, \dots, X_n be Banach spaces, U an open subset of \mathbb{C}^n . Let*

$$0 \longrightarrow X_0 \xrightarrow{\delta_0(z)} X_1 \xrightarrow{\delta_1(z)} \dots \xrightarrow{\delta_{n-1}(z)} X_n \longrightarrow 0$$

be a Fredholm complex analytically dependent on $z \in U$ (i.e., $\delta_j(z)\delta_{j-1}(z) = 0$ and $\dim \text{Ker } \delta_j(z) / \text{Im } \delta_{j-1}(z) < \infty$ for all $z \in U$ and $j = 0, \dots, n$).

Let $0 \leq j \leq n$ and $k \in \mathbb{N}$. Then the set $\{z \in U : \dim \text{Ker } \delta_j(z) / \text{Im } \delta_{j-1}(z) \geq k\}$ is analytic in U .

Proof. Let $w \in U$. Using Lemma 2.8 repeatedly, it is easy to see by the downward induction that there are finite dimensional spaces F_{j-1}, F_j and a regular analytic function $S(z) : X_{j-1} \oplus F_{j-1} \rightarrow X_j \oplus F_j$ defined in a neighbourhood of w such that $S(z)|_{X_{j-1}} = \delta_{j-1}(z)$ and $\text{Im } S(z) \supset \text{Ker } \delta_j(z)$. In particular, $\dim \text{Im } S(z) / \text{Ker } \delta_{j-1}(z) < \infty$.

Consider the ‘‘adjoint’’ complex

$$0 \longleftarrow X_0^* \xleftarrow{\delta_0^*(z)} X_1^* \xleftarrow{\delta_1^*(z)} \dots \xleftarrow{\delta_{n-1}^*(z)} X_n^* \longleftarrow 0$$

where we write for short $\delta_j^*(z)$ instead of $(\delta_j(z))^*$. Since this complex is also Fredholm, similarly as above there exist finite dimensional spaces G_j and G_{j+1} and a regular analytic function $T(z) : X_{j+1}^* \oplus G_{j+1} \rightarrow X_j^* \oplus G_j$ defined in a neighbourhood of w such that $\text{Im } T(z) \supset \text{Ker } (\delta_{j-1}^*(z))$ and $\dim \text{Im } T(z) / \text{Ker } \delta_{j-1}^*(z) < \infty$. Further the operator $S^*(z) : X_j^* \oplus F^* \rightarrow X_{j-1}^* \oplus F_{j-1}^*$ satisfies

$$\text{Ker } S^*(z) = (\text{Im } S(z))^\perp \subset (\text{Ker } \delta_j(z))^\perp + F_j^* = \text{Im } \delta_j^*(z) + F_j^*.$$

By Lemma 2.7, the set $\{z : \dim \text{Ker } \delta_{j-1}^*(z) / \text{Im } \delta_j^*(z) \geq k\}$ is analytic. Since

$$\dim \text{Ker } \delta_{j-1}^*(z) / \text{Im } \delta_j^*(z) = \dim \text{Ker } \delta_j(z) / \text{Im } \delta_{j-1}(z),$$

this finishes the proof. ■

Let $A = (A_1, \dots, A_n)$ be an n -tuple of commuting operators on a Banach space X . Denote by $\sigma_T(A)$ the Taylor spectrum of A . The essential spectrum $\sigma_{\text{Te}}(A)$ of A is defined as the set of all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that the Koszul complex of the n -tuple $(A_1 - \lambda_1, \dots, A_n - \lambda_n)$ is not Fredholm.

COROLLARY 2.10. ([7], [8]) *Let $A = (A_1, \dots, A_n)$ be an n -tuple of commuting operators on a Banach space X . Then the set $\sigma_T(A) \setminus \sigma_{\text{Te}}(A)$ is analytic in $\mathbb{C}^n \setminus \sigma_{\text{Te}}(A)$.*

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