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INVERSE SPECTRAL THEORY: NOWHERE DENSE SINGULAR CONTINUOUS SPECTRA AND HAUSDORFF DIMENSION OF SPECTRA

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Abstract. Let S be a symmetric operator in a Hilbert space $\mathcal{H}$. Suppose that the deficiency indices of S are infinite and S has some gap J. Then for every topological support T of an absolutely continuous (with respect to the Lebesgue measure) measure there exists a self-adjoint extension H^T of S such that $\sigma_{\rm sc}(H^T) \cap J = T \cap J$. Moreover for every $\alpha \in [0,1]$ there exists a self-adjoint extension H_α of S such that $\dim(\sigma_{\rm sc}(H_\alpha) \cap J) = \alpha$ and another self-adjoint extension H'_α and an α-dimensional singular continuous measure μ_α such that $H'_\alpha \simeq Q_{\mu_\alpha} \oplus R$ for some self-adjoint operator R without spectrum within J. Here Q_{μ_α} denotes the operator of multiplication by the identity function in $L^2(\mathbb{R}, \mu_\alpha)$.

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1. INTRODUCTION

The classical extension theories due to von Neumann and Krein only give little information about the spectral properties of self-adjoint extensions. In [1], [3], [4], [5] and [6] Albeverio, Brasche, Neidhardt and Weidmann have made an attempt to investigate the following problems:

- What kind of spectral properties can the self-adjoint extensions of a symmetric operator have?
- How to represent self-adjoint extensions with preassigned spectral properties?

Apparently it is not possible to control the spectra of the self-adjoint extensions on the whole real axis, or, more precisely, the spectral properties of all self-adjoint extensions of a symmetric operator S strongly depend on S. For this reason one has concentrated on the spectral properties within a gap J (cf. Definition 1) of S. One has derived affirmative results on the point spectra (cf. [6]) and the absolutely continuous spectra (cf. [1], [3], [4]) of self-adjoint extensions but only got little information about singular continuous spectra (cf. [1], [3], [5]); cf. Section 2 for the definition of the various kinds of spectra.

It is the goal of the present note to improve our understanding of the singular continuous spectra of self-adjoint extensions. We shall consider a symmetric operator S in a Hilbert space $\mathcal{H}$ and suppose that the deficiency indices (cf. Section 2 for the definition) of S are infinite and S has some gap J. For the first time we shall give explicitly non-empty nowhere dense sets which equal the singular continuous spectrum within the gap J of some self-adjoint extension H of S. More precisely we shall show that for every topological support T of an absolutely continuous (with respect to the Lebesgue measure) measure there exists a self-adjoint extension H^T of S such that

$$\sigma_{\mathrm{sc}}(H^T) \cap J = T \cap J.$$

Here σ_{sc} denotes the singular continuous spectrum. Note that the topological support of an absolutely continuous measure might be non-empty and nowhere dense, e.g. it might be a generalized Cantor set.

Moreover we shall show that for every $\alpha \in [0,1]$ there exists a self-adjoint extension H_α of S such that

$$\dim(\sigma_{\mathrm{sc}}(H_\alpha) \cap J) = \alpha$$

and another self-adjoint extension H'_α and an α-dimensional singular continuous measure μ_α such that

$$H'_\alpha \simeq Q_{\mu_\alpha} \oplus R$$

for some self-adjoint operator R without spectrum within J. Here Q_{μ_α} denotes the operator of multiplication by the identity function in $L^2(\mathbb{R}, \mu_\alpha)$ and "$\simeq$" means "unitarily equivalent". Cf. Section 3 for the definition of "dim" and "α-dimensional".

The main tools we shall use are as follows:

- The theory of rank one perturbations and singular continuous spectra, cf., the articles [7], [8], [9], [13], [15] and [16] by del Rio, Jitomirskaya, Last, Makarov, Simon and Wolff and references given therein.

• A new representation theorem for symmetric extensions, cf. [1], Lemma 2.1, or [3], Lemma 15.

The organization of the paper is as follows. In Sections 2 and 3 we shall recall notions and results from the theory of rank-1-perturbations and singular continuous spectra and the mentioned representation theorem for symmetric extensions for the convenience of the reader. In Section 4 we shall show that for every set T which is the topological support of an absolutely continuous measure there exists a self-adjoint extension H^T of H such that

$$\sigma_{\mathrm{sc}}(H^T)\cap J = T\cap J.$$

Section 5 contains the mentioned results on the dimensions of singular continuous spectra and spectral measures. In Section 6 we shall present new results on mixed types of spectra, i.e. singular continuous, absolutely continuous and point spectra.

2. SYMMETRIC EXTENSIONS

ran (S), $\ker(S)$, $\mathrm{D}(S)$ and S^* denote the range, the kernel, the domain and the adjoint of the operator S, respectively.

Let S be a symmetric operator in a Hilbert space $\mathcal{H}$.

Definition 2.1. The open interval $J=(a,b), -\infty\leqslant a<b<\infty$, is a gap of S, if and only if

$$\left\|\left(S-\frac{a+b}{2}\right)f\right\|\geqslant\frac{b-a}{2}\|f\|,\quad f\in\mathrm{D}(S),\quad \text{if } a>-\infty$$

and

$$(Sf,f)\geqslant b\|f\|^2,\quad f\in\mathrm{D}(S),\quad \text{if } a=-\infty.$$

It is a classical result by Friedrichs ([11]) and Krein ([12]) that $\sigma(H_0)\cap J=\emptyset$ for some self-adjoint extension H_0 of S if and only if the open interval J is a gap of S in the sense of the above definition.

The dimension of the spaces $\ker(S^*\mp \mathrm{i})$ are called the deficiency indices of S. It is well known that

$$\dim\ker(S^*+\mathrm{i})=\dim\ker(S^*-\mathrm{i})=\dim\ker(S^*-E)$$

for every $E\in J$ provided J is a gap of S. It is also well known that $\sigma(H)\cap J$ is a discrete set for every self-adjoint extension H of S provided J is a gap of S and the deficiency indices of S are finite. Here $\sigma(H)$ and $\sigma_{\mathrm{p}}(H)$ denote the spectrum and the set of eigenvalues of H, respectively. Thus we are merely interested in symmetric operators with infinite deficiency indices.

A measure μ on the Borel-σ-algebra of $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, will be called a Borel measure on $\mathbb{R}$. The complement of the largest open set U such that $\mu(U) = 0$ will be called the topological support of μ, $\mathrm{supp}(\mu)$. By Lebesgues decomposition theorem, every σ-finite Borel measure μ on $\mathbb{R}$ can be uniquely represented as

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{pp}}$$

where the Borel measure μ_{ac} is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}\lambda$, the Borel measure μ_{sc} is singular with respect to $\mathrm{d}\lambda$ and continuous in the sense that $\mu_{\mathrm{sc}}(\{a\}) = 0$ for every $a \in \mathbb{R}$ and the Borel measure μ_{pp} is a pure point measure, i.e. $\mu_{\mathrm{pp}}(\mathbb{R} \setminus D) = 0$ for some countable set D, respectively. The Lebesgue measure of a set B will also be denoted by $|B|$.

Let H be a self-adjoint operator in the Hilbert space $\mathcal{H}$. For every $f \in H$ the spectral measure of f with respect to H will be denoted by μ_f, i.e. μ_f is the unique finite Borel measure such that

$$\int g(t)\mu_f(\mathrm{d}t) = (g(H)f, f)$$

for every bounded Borel measurable function g.

Along with the decomposition of measures μ one also has a unique decomposition of the Hilbert space $\mathcal{H}$ and the self-adjoint operator H:

$$\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(H) \oplus \mathcal{H}_{\mathrm{sc}}(H) \oplus \mathcal{H}_{\mathrm{pp}}(H)$$

where $\mathcal{H}_{\mathrm{ac}}(H), \mathcal{H}_{\mathrm{sc}}(H)$ and $\mathcal{H}_{\mathrm{pp}}(H)$ denote the set of all $f \in \mathcal{H}$ such that $\mu_f = \mu_{f\mathrm{ac}}$, $\mu_f = \mu_{f\mathrm{sc}}$ and $\mu_f = \mu_{f\mathrm{pp}}$, respectively.

$$H = H_{\mathrm{ac}} \oplus H_{\mathrm{sc}} \oplus H_{\mathrm{pp}}$$

where $H_{\mathrm{ac}}, H_{\mathrm{sc}}$ and H_{pp} is a self-adjoint operator in $\mathcal{H}_{\mathrm{ac}}(H), \mathcal{H}_{\mathrm{sc}}(H)$ and $\mathcal{H}_{\mathrm{pp}}(H)$, respectively. The absolutely continuous spectrum of $H, \sigma_{\mathrm{ac}}(H)$, the singular continuous spectrum of $H, \sigma_{\mathrm{sc}}(H)$, and the pure point spectrum of $H, \sigma_{\mathrm{pp}}(H)$ are defined by

$$\sigma_{\mathrm{ac}}(H) := \sigma(H_{\mathrm{ac}}), \quad \sigma_{\mathrm{sc}}(H) := \sigma(H_{\mathrm{sc}}), \quad \sigma_{\mathrm{pp}}(H) := \sigma(H_{\mathrm{pp}}).$$

Note that $\sigma_{\mathrm{pp}}(H)$ equals the closure $\overline{\sigma_{\mathrm{p}}(H)}$ of the set $\sigma_{\mathrm{p}}(H)$ of eigenvalues of H.

For every subset B of $\mathbb{R}$ let $1_B : \mathbb{R} \to \mathbb{R}$ be the characteristic function of B, i.e.

$$1_B(x) := \begin{cases} 1, & \text{if } x \in B; \\ 0, & \text{if } x \in \mathbb{R} \setminus B. \end{cases}$$

We put

$$1 := 1_{\mathbb{R}}.$$

By the spectral theorem, for every Borel set B in $\mathbb{R}$ the range $\operatorname{ran} 1_B(H)$ of the operator $1_B(H)$ is a closed subspace of $\mathcal{H}$ and a reducing subspace of H, i.e.

$$H = H_B \oplus H_{\mathbb{R}\setminus B} \tag{2.1}$$

where H_B is a self-adjoint operator in $\operatorname{ran} 1_B(H)$ and $H_{\mathbb{R}\setminus B}$ a self-adjoint operator in

$$\operatorname{ran} 1_B(H)^{\perp} = \operatorname{ran} 1_{\mathbb{R}\setminus B}(H).$$

Here

$$\mathcal{M}^{\perp} := \{f \in \mathcal{H} : (f, g) = 0,\ g \in \mathcal{M}\}.$$

Remark 2.2. In general, the operator H_B stores only little information about the spectral properties of H in B. However, if B is open, then it stores all the information about the spectral properties of H in B. In particular, if B is open, then the spectral measures with respect to H and the spectral measures with respect to H_B coincide on the Borel-σ-algebra of B and

$$\sigma_{\mathrm{ac}}(H)\cap B = \sigma_{\mathrm{ac}}(H_B)\cap B, \quad \sigma_{\mathrm{sc}}(H)\cap B = \sigma_{\mathrm{sc}}(H_B)\cap B, \quad \sigma_{\mathrm{p}}(H)\cap B = \sigma_{\mathrm{p}}(H_B)\cap B.$$

The following representation theorem for symmetric extensions from [1] (cf. [1], Lemma 2.1) will play a crucial role in this paper. Here

$$N + M := \{f + g : f \in N,\ g \in M\}.$$

Theorem 2.3. *Let S be a symmetric operator in a Hilbert space $\mathcal{H}$. Suppose that S has a gap J. Let M be a self-adjoint operator in some closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $M \subset S^*$, i.e., M is a restriction of S^*, and $\sigma(M) \subset \overline{J}$. Then there exists a symmetric operator G_0 in $\mathcal{H}_0^{\perp}$ such that J is a gap of G_0 and*

$$S_M := S^*|(\mathrm{D}(S) + \mathrm{D}(M)) = M \oplus G_0.$$

In particular, S has a self-adjoint extension H such that

$$H_J = M_J.$$

Also the following Corollary 2.5 of the representation theorem will be very useful. Before we state and prove the corollary we shall introduce one more notation.

Definition 2.4. Let μ be a Borel measure on $\mathbb{R}$. Q_μ denotes the operator of multiplication by the independent variable in $L^2(\mathbb{R},\mu)$, i.e. the operator Q_μ in $L^2(\mathbb{R},\mu)$ is defined by

$$\mathrm{D}(Q_\mu) := \Big\{ f \in L^2(\mathbb{R},\mu) : \int x^2 |f(x)|^2 \mu(\mathrm{d}x) < \infty \Big\},$$

$$Q_\mu f(x) := x f(x), \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}.$$

Corollary 2.5. *Suppose that the symmetric operator S with gap J has a self-adjoint extension H such that*

$$H_J \simeq Q_\nu$$

for some measure ν. Let $\rho : \mathbb{R} \to [0,\infty)$ be a locally ν-integrable function. Then there exists a self-adjoint extension $\widehat{H}$ of S such that

$$\widehat{H}_J \simeq Q_{\rho\nu}.$$

Proof. Let $V : \operatorname{ran} 1_J(H) \to L^2(\mathbb{R},\nu)$ be any unitary transformation such that

$$H_J = V^{-1} Q_\nu V.$$

Obviously

$$U : L^2(\mathbb{R},\rho\nu) \to L^2(\mathbb{R},\nu),$$
$$Uf := \sqrt{\rho} f, \quad f \in L^2(\mathbb{R},\rho\nu),$$

defines a partial isometry from $L^2(\mathbb{R},\rho\nu)$ onto the closed subspace $\operatorname{ran}(U)$ of $L^2(\mathbb{R},\nu)$ and

$$Q_\nu | \operatorname{ran}(U) = U Q_{\rho\nu} U^{-1}.$$

Moreover

$$\mathcal{H}_0 := V^{-1} \operatorname{ran}(U)$$

is a reducing subspace for H_J and the operator

$$M := V^{-1} U Q_{\rho\nu} U^{-1} V | \mathcal{H}_0$$

is a self-adjoint operator in $\mathcal{H}_0$, unitarily equivalent to $Q_{\rho\nu}$ and a restriction of H_J. Thus, by Theorem 2.3, S has a self-adjoint extension $\widehat{H}$ such that

$$\widehat{H}_J = M_J \simeq Q_{\rho\nu}. \quad \blacksquare$$

3. RANK–1–PERTURBATIONS

Let B be a Borel set in $\mathbb{R}$ and $\alpha \in [0,1]$. For $\delta > 0$ let

$$h_\delta^\alpha(B) := \inf\left\{ \sum_{n=1}^{\infty} |b_n - a_n|^\alpha : B \subset \bigcup_{n=1}^{\infty} (a_n, b_n),\ |b_n - a_n| < \delta \right\}$$

and

$$h^\alpha(B) := \lim_{\delta \downarrow 0} h_\delta^\alpha(B).$$

Then $h^\alpha(B)$ is called the *α-dimensional Hausdorff measure* of B. There is a unique $\alpha_0 \in [0,1]$ such that

$$h^\alpha(B) = \infty \qquad \text{for } \alpha < \alpha_0$$

and

$$h^\alpha(B) = 0 \qquad \text{for } \alpha > \alpha_0.$$

Then α_0 is called *the Hausdorff dimension*, $\dim(B)$, *of* B. The following definition is due to Rogers-Taylor ([14]).

Definition 3.1. A *Borel measure* μ on $\mathbb{R}$ is said to be of *exact dimension* α for $\alpha \in [0,1]$ if and only if:

(i) for any $\beta \in [0,1]$ with $\beta < \alpha$ and B a Borel set of dimension β, $\mu(B) = 0$;
(ii) there is a Borel set B_0 of dimension α such that $\mu(\mathbb{R} \setminus B_0) = 0$.

For $B \subset \mathbb{R}$ we put

$$B^{-1} := \left\{ \frac{1}{E} : E \in B,\ E \neq 0 \right\}. \tag{3.1}$$

Definition 3.2. Let μ be a finite Borel measure on $\mathbb{R}$ and, as above, let Q_μ denote the operator of multiplication by the independent variable in $L^2(\mathbb{R},\mu)$, i.e. the operator Q_μ in $L^2(\mathbb{R},\mu)$ is defined by

$$\mathrm{D}(Q_\mu) := \left\{ f \in L^2(\mathbb{R},\mu) : \int x^2 |f(x)|^2 \mu(\mathrm{d}x) < \infty \right\},$$

$$Q_\mu f(x) := x f(x), \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}.$$

(i) For every $\lambda \in \mathbb{R}$, let μ^λ be the unique finite Borel measure on $\mathbb{R}$ such that

$$Q_\mu + \lambda(1,\cdot)1 \simeq Q_{\mu^\lambda}.$$

(ii) For every $\lambda \in \mathbb{R}$, let ν_λ be the finite Borel measure on $\mathbb{R}$ such that

$$\nu_\lambda(B) = \mu^\lambda(B^{-1}), \quad B \in \mathcal{B}(\mathbb{R}).$$

Remark 3.3. (i) For the existence and uniqueness of the measure μ^λ cf., e.g., [7];

(ii) $\nu_\lambda(\mathbb{R}) = \mu^\lambda(\mathbb{R})$ if and only if $\mu^\lambda(\{0\}) = 0$;

(iii) let $\mu^\lambda(\{0\}) = 0$; note that μ^λ is absolutely continuous, singular continuous, pure point and α-dimensional, respectively, if and only if ν_λ is absolutely continuous, singular continuous, pure point and α-dimensional, respectively.

The following three theorems will play a crucial role in the following sections.

Theorem 3.4. *Let μ be a finite Borel measure on $\mathbb{R}$.*

(i) *There exists a $\lambda \in \mathbb{R}$ such that*

$$\sigma_{\mathrm{p}}(Q_{\mu^\lambda}) \cap \sigma(Q_\mu) = \emptyset; \tag{3.2}$$

(ii) *if $\mu_{\mathrm{ac}}([a,b]) = 0$ for some interval $[a,b]$, $a < b$, and $[a,b] \subset \operatorname{supp} \mu_{\mathrm{pp}}$ then*

$$[a,b] \subset \operatorname{supp} \mu^\lambda_{\mathrm{sc}}$$

for some $\lambda \in \mathbb{R}$.

The assertion (i) is due to N. Aronszajn ([2]) and W. Donogue ([10]). Cf. [8] and [7] for a discussion of the question about "how big" the set of coupling parameters λ with the property (3.2) is.

(ii) is a trivial consequence of (i) since essential and absolutely continuous spectra are stable under rank-1-perturbations.

Modifying the proofs of Theorem 6.2, Example 2 in Section 6 and Theorem 6.5 of [7] in an obvious way we get the following:

Theorem 3.5. *Let J be an open interval and I a compact subinterval of J. Let $\alpha \in [0,1]$. Then there exists a finite pure point Borel measure μ_0 on $\mathbb{R}$ such that the following holds:*

(i) $\operatorname{supp} \mu_0 = I$;

(ii) *for every finite Borel measure μ on $\mathbb{R}$ such that*

$$1_J \mu = \mu_0$$

there exists a real number λ such that

$$\operatorname{supp} 1_I \mu_0^\lambda = I$$

and the measure $1_I \mu^\lambda$ is α-dimensional;

(iii) *let $\alpha > 0$; then for every finite Borel measure μ on $\mathbb{R}$ such that*

$$1_J \mu = \mu_0$$

the measure $1_I\mu^\lambda$ *is purely singular continuous for* $\mathrm{d}\lambda$ *a.e.* $\lambda \in \mathbb{R}$.

THEOREM 3.6. ([9], Theorem 0) *Let* μ *be a normalized finite Borel measure on* $\mathbb{R}$. *Then*

$$\int \mu^\lambda(B)\,\mathrm{d}\lambda = |B|$$

for every Borel set B.

4. NOWHERE DENSE SINGULAR CONTINUOUS SPECTRA

Let S be a symmetric operator. Suppose that S has some gap J and its deficiency indices are infinite. In this section we shall give, among others, large classes of nowhere dense sets T such that

$$\sigma_{\mathrm{sc}}(H) \cap J = T$$

for some self-adjoint extension H of S.

LEMMA 4.1. *Let* S *be a symmetric operator in the Hilbert space* $\mathcal{H}$. *Suppose that* S *has a gap* J *such that* $0 \in J$. *Let* C *be a compact subset of* J *and* P *the orthogonal projection from* $\mathcal{H}$ *onto* $\ker(S^*)$. *Then*

$$\|Pf\| \geqslant c\|f\|, \quad f \in \ker(S^* - E),\, E \in C,$$

for some strictly positive constant c.

Proof. We choose numbers a and b such that

$$0 < a < \frac{t-E}{t} < b < \infty, \quad t \in \mathbb{R} \setminus J,\, E \in C.$$

We choose any self-adjoint extension H_0 of S such that J is a gap of H_0.

Let $E \in C$ and $f \in \ker(S^* - E)$, $f \neq 0$. We put

$$\widehat{f} := (H_0 - E)H_0^{-1} f.$$

Then

$$(\widehat{f}, Sg) = (f, (S-E)g) = 0, \quad g \in \mathrm{D}(S),$$

i.e. $\widehat{f} \in \ker(S^*)$. Thus

$$\|Pf\| \geqslant \left(f, \frac{\widehat{f}}{\|\widehat{f}\|}\right) = \frac{\int \frac{t-E}{t}\mu_f(\mathrm{d}t)}{\sqrt{\int |\frac{t-E}{t}|^2 \mu_f(\mathrm{d}t)}} \geqslant \frac{a}{b}\|f\|. \quad \blacksquare$$

The following lemma makes it possible to apply results from the theory of rank-1-perturbations for the investigation of singular continuous spectra of self-adjoint extensions.

Lemma 4.2. *Let S be a symmetric operator in a Hilbert space $\mathcal{H}$. Suppose that S has some gap J, $0 \in J$ and the deficiency indices of S are infinite. Let μ_0 be a finite pure point Borel measure on $\mathbb{R}$ such that the topological support of μ_0 is a compact subset of*

$$J^{-1} := \left\{ \frac{1}{E} : E \in J, E \neq 0 \right\}.$$

Then there exists a finite Borel measure μ on $\mathbb{R}$ with the following properties:

(i) $1_{J^{-1}} \mu = \mu_0$;

(ii) *for every $\lambda \in \mathbb{R}$ there exists a self-adjoint extension H^λ of S such that*

$$H^\lambda_J \simeq Q_{1_J \nu_\lambda}$$

(cf. (2.1) *and Definition* 3.2 *for the definition of H_J, Q_μ and ν_λ).*

Proof. There exist a finite or countable infinite index set N, strictly positive real numbers b_n, $n \in N$, and points $\eta_n \in J^{-1}$, $n \in N$, such that

$$\mu_0 = \sum_{n \in N} b_n \delta_{\eta_n} \quad \text{and} \quad \sum_{n \in N} b_n < \infty.$$

Let

$$E_n := \frac{1}{\eta_n}, \quad n \in N,$$

and $P : \mathcal{H} \to \ker(S^*)$ the orthogonal projection from $\mathcal{H}$ onto the kernel of the adjoint of S. By Lemma 4.1, for every $E \in J$, the restriction of P to $\ker(S^* - E)$ is injective. Moreover for every $E \in J$ the kernel of $S^* - E$ is infinite dimensional since the deficiency indices of S are infinite. Thus we can choose, by induction, an orthonormal system $\{e_n\}$ such that

$$S^* e_n = E_n e_n, \quad n \in N, \tag{4.1}$$

and

$$(Pe_n, Pe_j) = 0, \quad \text{if } n \neq j. \tag{4.2}$$

Let $\mathcal{H}_0$ be the closure of the span of the e_n, $n \in N$, and M the unique self-adjoint operator in $\mathcal{H}_0$ such that E_n is an eigenvalue of M and e_n a corresponding eigenvector for every $n \in N$. Clearly

$$M \subset S^* \quad \text{and} \quad \sigma(M) \subset \overline{J}.$$

By Theorem 2.3, this implies that

$$H_J = M_J = M$$

for some self-adjoint extension H of S.

Note that the operator H has the following properties:

$$H \text{ has pure point spectrum in } J; \tag{4.3}$$

$$\sigma_{\mathrm{p}}(H) \cap J = \{E_n : n \in N\}; \tag{4.4}$$

$$\ker(H - E_n) = \operatorname{span}\{e_n\}, \quad n \in N. \tag{4.5}$$

By hypothesis, $\inf_{n \in N} |E_n| > 0$. Thus it follows from (4.3) and (4.4), that the operator H is invertible and its inverse H^{-1} is bounded.

By hypothesis, J is a gap of S, $0 \in J$ and $\{E_n : n \in N\}$ is a relatively compact subset of J. By Lemma 4.1, this implies that

$$\inf_{n \in N} \|Pe_n\| > 0.$$

Thus

$$\sum_{n \in N} \frac{b_n}{\|Pe_n\|^2} < \infty.$$

Put

$$a_n := \frac{\sqrt{b_n}}{\|Pe_n\|}, \quad n \in N.$$

Then

$$g := \sum_{n \in N} a_n \frac{Pe_n}{\|Pe_n\|}$$

belongs to the kernel of S^*.

Let $\mathcal{H}_g$ be the closure of the span of the set

$$\{(H^{-1})^n g : n \in \mathbb{N} \text{ or } n = 0\}.$$

Obviously $H^{-1} f \in \mathcal{H}_g$ for every $f \in \mathcal{H}_g$,

$$H^{-1} = R_0 \oplus R \tag{4.6}$$

for some self-adjoint operator R in the Hilbert space $\mathcal{H}_g$ and some self-adjoint operator R_0 in $\mathcal{H}_g^\perp$ and there exists a unitary transformation $U : \mathcal{H}_g \to L^2(\mathbb{R}, \mu)$ such that

$$R = U^{-1} Q_\mu U \quad \text{and} \quad Ug = 1.$$

Here μ denotes the spectral measure of g with respect to R.

Let P_g be the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_g$. Let $n \in N$. $P_g e_n \in \mathrm{D}(H^{-1})$ and

$$H^{-1} P_g e_n = P_g H^{-1} e_n = \eta_n P_g e_n$$

since $\mathcal{H}_g$ is a reducing subspace for H^{-1} and η_n is an eigenvalue of H^{-1} and e_n a corresponding eigenvector. Since η_n is a simple eigenvalue of H^{-1} this implies that $P_g e_n = 0$ or $P_g e_n = e_n$. Since

$$|(g, P_g e_n)| = |(g, e_n)| = |a_n| \, \|P e_n\| = \sqrt{b_n} > 0, \tag{4.7}$$

it follows that $P_g e_n = e_n \in \mathcal{H}_g$.

Since $e_n \in \mathcal{H}_g$ for every $n \in N$ and by the properties (4.3), (4.4), (4.5) and (4.6) of the operator H the following holds:

$$\sigma(R_0) \cap J^{-1} = \emptyset,$$

R has pure point spectrum in J^{-1},

$$\sigma_{\mathrm{p}}(R) \cap J^{-1} = \{\eta_n : n \in N\}$$

and

$$\ker(R - \eta_n) = \mathrm{span}\{e_n\}, \quad n \in N.$$

Thus the spectral measure μ of g with respect to R satisfies

$$1_{J^{-1}} \mu = \sum_{n \in N} d_n \delta_{\eta_n} \tag{4.8}$$

for some strictly positive real numbers d_n, $n \in N$.

For every $n \in N$ the eigenspace $\ker(Q_\mu - \eta_n)$ is spanned by the normalized vector

$$\widetilde{e}_n := \frac{1}{\sqrt{\mu(\{\eta_n\})}} 1_{\{\eta_n\}}$$

and the scalar product of $\widetilde{e}_n$ and 1 in $L^2(\mathbb{R}, \mu)$ equals

$$(\widetilde{e}_n, 1) = \sqrt{\mu(\{\eta_n\})}.$$

Since $R = U^{-1}Q_\mu U$ and $Ug = 1$ for the unitary transformation U and the eigenspace $\ker(R-\eta_n)$ is spanned by the normalized vector e_n, this implies that

$$Ue_n = c_n\widetilde{e}_n$$

for some normalized constant c_n. Thus

$$|(g,e_n)| = |(\widetilde{e}_n,1)| = \sqrt{\mu(\{\eta_n\})}, \quad n \in N.$$

By (4.7) and (4.8), this implies that

$$1_{J^{-1}}\mu = \mu_0.$$

Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. The operator $H^{-1} + \lambda(g,\cdot\,)g$ is also invertible. In fact, if

$$H^{-1}f + \lambda(g,f)g = 0,$$

then $(g,f)\,g \in \mathrm{D}(H)$ and $H\left((g,f)\,g\right) = S^*\left((g,f)\,g\right) = 0$. Thus $(g,f)\,g = 0$ and $f = 0$. Since $g \in \ker(S^*) = \operatorname{ran}(S)^\perp$, along with H^{-1} also the operator $H^{-1} + \lambda\,(g,\cdot)\,g$ is a self-adjoint extension of S^{-1}. Thus its inverse

$$H^\lambda := \left(H^{-1} + \lambda\,(g,\cdot)\,g\right)^{-1} \tag{4.9}$$

is a self-adjoint extension of S.

Note that

$$H^\lambda_J = \left((R + \lambda(g,\cdot)g^{-1}\right)_J \tag{4.10}$$

since

$$\sigma(R_0) \cap J^{-1} = \emptyset.$$

Since

$$R = U^{-1}Q_\mu U \quad \text{and} \quad Ug = 1$$

for the unitary transformation U from $\mathcal{H}_g$ onto $L^2(\mathbb{R},\mu)$, we have

$$R + \lambda\,(g,\cdot)\,g \simeq Q_\mu + \lambda\,(1,\cdot)\,1 \simeq Q_{\mu^\lambda} \tag{4.11}$$

where the last equality is just the definition of the measure μ^λ.

Along with μ^λ, the measure ν_λ, defined by

$$\nu_\lambda(B) := \mu^\lambda\left(\left\{\frac{1}{\eta} : \eta \in B,\, \eta \neq 0\right\}\right)$$

for every Borel set B in $\mathbb{R}$, is a finite Borel measure on $\mathbb{R}$. By the general transformation theorem for integrals, the mapping

$$V : L^2(\mathbb{R}, \mu^\lambda) \to L^2(\mathbb{R}, \nu_\lambda),$$

$$Vf(x) := f\left(\frac{1}{x}\right) \quad \text{for } \nu_\lambda\text{-a.e. } x \in \mathbb{R},\ f \in L^2(\mathbb{R}, \mu^\lambda),$$

is unitary. Obviously

$$\left(Q_{\mu^\lambda}\right)^{-1} = V^{-1} Q_{\nu_\lambda} V.$$

By (4.10) and (4.11), this implies that

$$H_J^\lambda \simeq Q_{1_J \nu_\lambda}. \quad ▮$$

Lemma 4.3. *Let I be a compact interval. There exists a finite pure point Borel measure μ_0 such that the following holds:*

(i) $\operatorname{supp}(\mu_0) = I$*;*

(ii) *if μ is a finite Borel measure such that*

$$1_I \mu = \mu_0$$

then

$$\operatorname{supp}(1_I \mu^\lambda) = I$$

and the measure $1_I \mu^\lambda$ is purely singular continuous for every $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Proof. We may assume that the diameter of I is strictly positive because the other case is trivial.

We choose $a_n > 0$ and $\eta_n \in I$ such that $\eta_n \neq \eta_k$ for $k \neq n$,

$$\sum_{n \in \mathbb{N}} a_n < \infty$$

and

$$\sum_{n \in \mathbb{N}} \frac{a_n}{|\eta - \eta_n|^2} = \infty, \quad \eta \in I. \tag{4.12}$$

For instance we may put

$$a_n = n^{-1-\varepsilon}, \quad n \in \mathbb{N},$$

for some $0 < \varepsilon \leqslant 1/2$, successively subdivide I in $1, 2, 3, \ldots$ subintervals of length $|I|, |I|/2, |I|/3, \ldots$ and choose successively different points from these subintervals.

We put

$$\mu_0 := \sum_{n\in\mathbb{N}} a_n \delta_{\eta_n}.$$

Let μ be a finite Borel measure such that

$$1_I \mu = \mu_0.$$

Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. By definition of the measure μ^λ,

$$Q_{\mu^\lambda} \simeq Q_\mu + \lambda(1,\cdot)1.$$

Since I is contained in the essential spectrum of Q_μ and Q_μ has no absolutely continuous spectrum in I we have only to show that the operator $Q_\mu + \lambda(1,\cdot)1$ has no eigenvalue in I.

Let $\eta \in I$ and

$$Q_\mu f + \lambda(1,f)1 = \eta f. \tag{4.13}$$

It remains to show that

$$(1,f) = 0$$

because then $\eta = \eta_n$ and $f = c1_{\eta_n}$ for some $n \in \mathbb{N}$ and some constant c and therefore the equality $(1,f) = 0$ implies $f = 0$.

By (4.13),

$$\eta_n f(\eta_n) + \lambda(1,f)1 = \eta f(\eta_n), \quad n \in \mathbb{N}.$$

If $\eta = \eta_n$ for some $n \in \mathbb{N}$, then $(1,f) = 0$. Otherwise we have

$$\infty > \int |f|^2 \,\mathrm{d}\mu \geqslant \sum_{n\in\mathbb{N}} a_n |f(\eta_n)|^2 = \lambda^2 |(1,f)|^2 \sum_{n\in\mathbb{N}} \frac{a_n}{|\eta-\eta_n|^2}.$$

By (4.12), this implies that $(1,f) = 0$. ∎

Lemma 4.4. *Let $B \subset J$ be a Borel set with strictly positive Lebesgue measure. Then there exist a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$, a self-adjoint operator M in $\mathcal{H}_0$ and a symmetric operator G_0 in $\mathcal{H}_0^\perp$ such that the following holds:*

(i) *$S \subset M \oplus G_0$;*

(ii) *M has a purely singular continuous spectrum and*

$$\emptyset \neq \sigma_{\mathrm{sc}}(M) \subset B;$$

(iii) *J is a gap of G_0 and the deficiency indices of G_0 are infinite.*

Proof. We choose an open interval D such that the closure $\overline{D}$ of D is a compact subset of J^{-1} and the Lebesgue measure of $B^{-1} \cap D$ is strictly positive. By Lemma 4.3, we can choose a finite pure point Borel measure μ_0 on $\mathbb{R}$ with the following properties:

(i) $\mu_0(\mathbb{R} \setminus \overline{D}) = 0$;

(ii) if μ is a finite Borel measure on $\mathbb{R}$ and $1_{J^{-1}}\mu = \mu_0$, then for every $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the measure $1_D \mu^\lambda$ is purely singular continuous and $\mu^\lambda(D) > 0$.

By Lemma 4.2, we can choose a finite Borel measure μ on $\mathbb{R}$ with the following properties:

(i) $1_{J^{-1}}\mu = \mu_0$;

(ii) for every $\lambda \in \mathbb{R}$ there exists a self-adjoint extension H^λ of S such that

$$(H^\lambda)_J \simeq (Q_{\nu_\lambda})_J.$$

By Theorem 3.6, we can choose a λ such that the measure $1_D \mu^\lambda$ is purely singular continuous and

$$\mu^\lambda(B^{-1} \cap D) > 0.$$

Thus the measure $1_{B \cap D^{-1}} \nu_\lambda$ is purely singular continuous and $\nu_\lambda(B \cap D^{-1}) > 0$.

By the inner regularity of ν_λ we can choose disjoint compact subsets C and $\widetilde{C}$ of $B \cap D^{-1}$ such that $\nu_\lambda(C) > 0$ and $\nu_\lambda(\widetilde{C}) > 0$. Then

$$\operatorname{ran} 1_C(Q_{\nu_\lambda}) \perp \operatorname{ran} 1_{\widetilde{C}}(Q_{\nu_\lambda}), \tag{4.14}$$

$$\emptyset \neq \sigma_{\mathrm{sc}}((Q_{\nu_\lambda})_C) \subset C, \tag{4.15}$$

$$\emptyset \neq \sigma_{\mathrm{sc}}((Q_{\nu_\lambda})_{\widetilde{C}}) \subset \widetilde{C}. \tag{4.16}$$

By Lemma 4.2, there exists a unitary transformation W such that

$$H^\lambda_J = W^{-1}(Q_{\nu_\lambda})_J W.$$

Then

$$\mathcal{H}_0 := W^{-1}\operatorname{ran}(1_C(Q_{\nu_\lambda})),$$
$$\widetilde{\mathcal{H}}_0 := W^{-1}\operatorname{ran}(1_{\widetilde{C}}(Q_{\nu_\lambda})),$$

are closed subspaces of $\mathcal{H}$ and, by (4.14),

$$\mathcal{H}_0 \perp \widetilde{\mathcal{H}}_0.$$

Moreover

$$M := W^{-1}(Q_{\nu_\lambda})_C W,$$
$$\widetilde{M} := W^{-1}(Q_{\nu_\lambda})_{\widetilde{C}} W,$$

is a self-adjoint operator in $\mathcal{H}_0$ and $\widetilde{\mathcal{H}}_0$, respectively,

$$M \subset H^\lambda \subset S^*,$$
$$\widetilde{M} \subset H^\lambda \subset S^*,$$

and, by (4.15) and (4.16),

$$\emptyset \neq \sigma_{\rm sc}(M) \subset C \subset B \subset J,$$
$$\emptyset \neq \sigma_{\rm sc}(\widetilde{M}) \subset \widetilde{C} \subset B \subset J.$$

By Theorem 2.3, there exist symmetric operators G_0 and G_0' in $\mathcal{H}_0^\perp$ and $\widetilde{\mathcal{H}}_0^\perp$, respectively, such that

$$S \subset S_M = M \oplus G_0,$$
$$S \subset S_{M \oplus \widetilde{M}} = M \oplus \widetilde{M} \oplus G_0'$$

and J is a gap of G_0 and G_0'. Then G_0 has a self-adjoint extension G such that $\sigma_{\rm sc}(G) \cap J \neq \emptyset$, since $G_0 \subset \widetilde{M} \oplus G_0'$. Thus the deficiency indices of G_0 are infinite. ∎

Theorem 4.5. *Let T be the topological support of an absolutely continuous Borel measure. Then there exists a self-adjoint extension H of S such that*

$$\sigma_{\rm sc}(H) \cap J = T \cap J.$$

Proof. We choose Borel sets B_n, $n \in \mathbb{N}$, with the following properties:

(i) $B_n \cap T \subset J$, $n \in \mathbb{N}$;

(ii) the Lebesgue measure of $B_n \cap T$ is strictly positive for every $n \in \mathbb{N}$;

(iii) for every $x \in T \cap J$ and $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that $x \in B_n$ and the diameter of B_n is less than ε.

By Lemma 4.4, we can choose, by induction, pairwise orthogonal closed subspaces $\mathcal{H}_n$ of $\mathcal{H}$, self-adjoint operators M_n in $\mathcal{H}_n$ and symmetric operators G_n in the orthogonal complement of $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ such that the following holds:

(i) M_n has a purely singular continuous spectrum and

$$\emptyset \neq \sigma_{\rm sc}(M_n) \subset B_n \cap T, \quad n \in \mathbb{N};$$

(ii) $S \subset M_1 \oplus G_1$ and

$$G_n \subset M_{n+1} \oplus G_{n+1}, \quad n \in \mathbb{N};$$

(iii) J is a gap of G_n and the deficiency indices of G_n are infinite for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ we have

$$(M_1 \oplus \cdots \oplus M_n \oplus G_n)^* = M_1^* \oplus \cdots \oplus M_n^* \oplus G_n^* = M_1 \oplus \cdots \oplus M_n \oplus G_n^* \subset S^*$$

since $S \subset M_1 \oplus \cdots \oplus M_n \oplus G_n$. Thus

$$M := \bigoplus_{n=1}^{\infty} M_n \subset S^*,$$

M is a self-adjoint operator in

$$\mathcal{H}_0 := \bigoplus_{n=1}^{\infty} \mathcal{H}_n,$$

has a purely singular continuous spectrum and

$$\sigma_{\text{sc}}(M) \cap J = \overline{\bigcup_{n=1}^{\infty} \sigma_{\text{sc}}(M_n)} \cap J.$$

By our choice of the operators M_n and the sets B_n, it follows that

$$\sigma_{\text{sc}}(M) \cap J = T \cap J.$$

By Theorem 2.3, S has a self-adjoint extension H such that

$$H_J = M_J$$

and, in particular,

$$\sigma_{\text{sc}}(H) \cap J = T \cap J. \quad \blacksquare$$

5. HAUSDORFF-DIMENSION OF SINGULAR CONTINUOUS SPECTRA AND SPECTRAL MEASURES

In this section we shall show that within a gap J of S the self-adjoint extensions of S can have singular continuous spectral measures and singular continuous spectra of any dimension.

Lemma 5.1. *Let S be a symmetric operator in a Hilbert space $\mathcal{H}$. Suppose that S has a gap J such that $0 \in J$ and the deficiency indices of S are infinite. Let $I \subset J \setminus \{0\}$ be a compact interval such that $|I| > 0$, $\alpha \in [0,1]$ and $0 < b < \infty$. Then there exist a finite Borel measure ν on $\mathbb{R}$ and a self-adjoint extension $\widehat{H}$ of S with the following properties:*

(i) *ν is purely singular continuous, i.e.*

$$\nu = \nu_{\mathrm{sc}},$$

and α-dimensional;

(ii) $\operatorname{supp}(\nu) = I$ *and* $\nu(I) = b$;

(iii) $\widehat{H}_J \simeq Q_\nu$.

Proof. By Theorem 3.5, we can choose a pure point finite Borel measure μ_0 on $\mathbb{R}$ with the following properties:

(i) $\operatorname{supp}(\mu_0) = I^{-1}$;

(ii) for every finite Borel measure μ on $\mathbb{R}$ such that

$$1_{J^{-1}}\mu = \mu_0 \tag{5.1}$$

there exists a real number λ such that the measure $1_{I^{-1}}\mu^\lambda$ is α-dimensional (cf. Definition 3.2 for the definition of μ^λ; recall that, by Theorem 3.4,

$$\operatorname{supp} 1_{I^{-1}}\mu^\lambda = I^{-1} \tag{5.2}$$

for every $\lambda \in \mathbb{R}$ provided μ satisfies (5.1)).

By Lemma 4.2, we can choose a finite Borel measure μ on $\mathbb{R}$, such that (5.1) holds and for every $\lambda \in \mathbb{R}$ there exists a self-adjoint extension H^λ of S such that

$$H^\lambda_J \simeq Q_{1_J\nu_\lambda}. \tag{5.3}$$

We choose $\lambda \in \mathbb{R}$ such that the measure $1_I\nu_\lambda$ is α-dimensional. By (5.2),

$$\operatorname{supp}(1_I\nu_\lambda) = I.$$

Thus the measure

$$\nu := \frac{b}{\nu_\lambda(I)} 1_I\nu_\lambda$$

has the required properties. It follows from (5.3), Theorem 2.3 and Corollary 2.5, that

$$\widehat{H}_J \simeq Q_\nu$$

for some suitably chosen self-adjoint extension $\widehat{H}$ of S. ∎

Corollary 5.2. *Let S, J, I, α and b be as in Lemma 5.1. Then there exist a Borel measure ν' on $\mathbb{R}$ and a self-adjoint extension H' of S with the following properties:*

(i) *ν' is purely singular continuous and* $\dim \operatorname{supp}(\nu') = \alpha$;

(ii) $\operatorname{supp}(\nu') \subset I$ *and* $\nu'(I) = b$;

(iii) $H'_J \simeq Q_{\nu'}$.

In particular, the following holds:

(i) $\emptyset \neq \sigma_{\mathrm{sc}}(H') \cap J \subset I$;

(ii) $\dim(\sigma_{\mathrm{sc}}(H') \cap J) = \alpha$.

Proof. Choose a Borel measure ν and a self-adjoint extension $\widehat{H}$ of S as in Lemma 5.1. Since ν is α-dimensional, there exists a Borel set $B_\alpha \subset I$ such that

$$\nu(\mathbb{R} \setminus B_\alpha) = 0 \quad \text{and} \quad \dim B_\alpha = \alpha.$$

By the inner regularity of ν we can choose a compact subset K_α of B_α such that $\nu(K_\alpha) > 0$. We put

$$\nu' := \frac{b}{\nu(K_\alpha)} 1_{K_\alpha} \nu.$$

Along with ν, also ν' is purely singular continuous and, α-dimensional and we have

$$\operatorname{supp}(\nu') \subset K_\alpha \subset I \quad \text{and} \quad \nu'(I) = b.$$

Since the support of ν' is contained in the α-dimensional set B_α and ν' is α-dimensional we have

$$\dim \operatorname{supp}(\nu') = \alpha.$$

By Theorem 5.1 (iii) and Corollary 2.5, there exists a self-adjoint extension H' of S such that

$$H'_J \simeq Q_{\nu'}. \quad \blacksquare$$

Corollary 5.3. *Let $S, \mathcal{H}, J, I, \alpha, b$ be as in Lemma 5.1. Then there exist a Borel measure ν on $\mathbb{R}$, a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$, a self-adjoint operator M in $\mathcal{H}_0$ and a symmetric operator G_0 in $\mathcal{H}_0^\perp$ with the following properties:*

(i) *ν is purely singular continuous and α-dimensional;*

(ii) $\operatorname{supp}(\nu) = I$ *and* $\nu(I) = b$;

(iii) $M \simeq Q_\nu$;

(iv) *J is a gap of G_0 and the deficiency indices of G_0 are infinite;*

(v) $S \subset M \oplus G_0$.

Proof. We choose $a_1 < a_2 < a_3$ such that

$$I = [a_1, a_2]$$

and $\widetilde{I} := [a_1, a_3] \subset J \backslash \{0\}$. By Lemma 5.1, we can choose a positive Radon measure $\widetilde{\nu}$ and a self-adjoint extension $\widetilde{H}$ of S such that the following holds:

(i) $\widetilde{\nu}$ is purely singular continuous and α-dimensional;
(ii) $\operatorname{supp}(\widetilde{\nu}) = \widetilde{I}$ and $\widetilde{\nu}(I) = b$;
(iii)

$$\widetilde{H}_J \simeq Q_{\widetilde{\nu}}. \tag{5.4}$$

We put

$$\nu := 1_I \widetilde{\nu}.$$

Clearly the measure ν satisfies the claims (i) and (ii). By (5.4),

$$\widetilde{H}_J = M \oplus M'$$

for some self-adjoint operators M and M' such that

$$M \simeq Q_\nu \quad \text{and} \quad M' \simeq Q_{\widetilde{\nu}-\nu}.$$

By Theorem 2.3 and Corollary 2.5, there exist symmetric operators G_0 and G_1 such that

$$S \subset S_M = M \oplus G_0$$
$$S \subset S_{M\oplus M'} = M \oplus M' \oplus G_1$$

and J is a gap of G_0 and G_1. Then G_0 has a self adjoint extension G such that

$$\sigma_{\mathrm{sc}}(G) \cap J \neq \emptyset$$

since $G_0 \subset M' \oplus G_1$, J is a gap of G_1 and $M' \simeq Q_{\widetilde{\nu}-\nu}$. Thus the deficiency indices of G_0 are infinite. ∎

COROLLARY 5.4. *Let $S, \mathcal{H}, J, I, \alpha, b$ be as in Lemma* 5.1. *Then there exist a Borel measure ν, a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$, a self-adjoint operator M in $\mathcal{H}_0$ and a symmetric operator G_0 in $\mathcal{H}_0^\perp$ with the following properties:*

(i) *ν is purely singular continuous and α-dimensional;*
(ii) $\operatorname{supp}(\nu) \subset I, \nu(I) = b$ *and* $\dim \operatorname{supp}(\nu) = \alpha$*;*
(iii) $M \simeq Q_\nu$*;*
(iv) *J is a gap of G_0 and the deficiency indices of G_0 are infinite;*
(v) $S \subset M \oplus G_0$.

Proof. This corollary can be proven as the previous one. Instead of Lemma 5.1 one uses Corollary 5.3 for the proof. ∎

Theorem 5.5. *Let S be a symmetric operator in a Hilbert space $\mathcal{H}$. Suppose that S has a gap J and the deficiency indices of S are infinite. Let N_1 and N_2 be disjoint, empty, finite or countable infinite index sets such that*

$$N := N_1 \cup N_2 \neq \emptyset.$$

For every $n \in N$, let I_n be a closed subinterval of J, $\alpha_n \in [0,1]$ and $b_n \in (0,\infty)$. Then there exist a self-adjoint extension H of S and Borel measures ν_n on $\mathbb{R}$, $n \in N$, with the following properties:

(i) $H_J \simeq \bigoplus_{n\in N} Q_{\nu_n}$;

(ii) *for every $n \in N$ the measure ν_n is purely singular continuous and α_n-dimensional,* $\operatorname{supp}(\nu_n) \subset I_n$ *and* $\nu_n(I_n) = b_n$;

(iii) *for every $n \in N_1$*

$$\operatorname{supp}(\nu_n) = I_n;$$

(iv) *for every $n \in N_2$*

$$\dim \operatorname{supp}(\nu_n) = \alpha_n.$$

Proof. Without loss of generality we may assume that $0 \in J$ and I_n is a compact subset of $J \setminus \{0\}$ for every $n \in N$. By Corollaries 5.3 and 5.4 and by induction, we can choose measures ν_n, $n \in N$, with the required properties, pairwise orthogonal closed subspaces $\mathcal{H}_n$, $n \in N$, of $\mathcal{H}$ and operators M_n in $\mathcal{H}_n$, $n \in N$, such that the following holds:

(i) for every $n \in N$

$$M_n \simeq Q_{\nu_n};$$

(ii) for every $n \in N$

$$M_n \subset S^*.$$

Then

$$\mathcal{H}' := \bigoplus_{n\in N} \mathcal{H}_n \quad \text{and} \quad M := \bigoplus_{n\in N} M_n$$

is a closed subspace of $\mathcal{H}$ and a self-adjoint operator in $\mathcal{H}'$, respectively. Moreover

$$M = M_J \simeq \bigoplus_{n\in N} Q_{\nu_n}$$

and $\sigma(M) \subset \overline{J}$. By Theorem 2.3, there exists a self-adjoint extension H of S such that

$$H_J = M,$$

and the theorem is proved. ∎

6. MIXED TYPES OF SPECTRA

New results on singular continuous spectra automatically yield new results on mixed types of spectra, cf. the considerations in [1], Section 6. Thus, in particular, the following Theorems 6.3 and 6.4 hold. For the formulation of these theorems we need the following:

Definition 6.1. ([1]) A closed symmetric operator S with gap J is called *significantly deficient* if and only if

$$P_{\ker(S^*-E)}\mathrm{D}(S) \neq \ker(S^* - E)$$

for one (and therefore every) $E \in J$.

It is known (cf. [1]) that this definition does not depend on the special choice of the gap J of S. Moreover we have the following:

Example 6.2. ([1]) The closed symmetric operator S with gap J is significantly deficient provided its deficiency indices are infinite and the operator $(S - E)^{-1}$ is compact for one (and therefore every) $E \in J$.

In particular, the minimal Laplacian on a bounded domain D in $\mathbb{R}^d$, $d > 1$, is significantly deficient.

Theorem 6.3. *Let S be a symmetric operator in the separable Hilbert space $\mathcal{H}$. Suppose that S has some gap J. Moreover suppose that S is significantly deficient in the sense of the Definition* 6.1 *or that $\mathcal{H}$ is complex and S is the orthogonal sum of infinitely many operators with strictly positive deficiency indices.*

Then for every set T which is the topological support of an absolutely continuous Borel measure and every self-adjoint operator M' in $\mathcal{H}$ there exists a self-adjoint extension H of S with the following properties:

(i) $H_{\mathrm{ac}J} \simeq M'_{\mathrm{ac}J}$;

(ii) $H_{\mathrm{pp}J} \simeq M'_{\mathrm{pp}J}$;

(iii) $\sigma_{\mathrm{sc}}(H) \cap J = T \cap J$.

Theorem 6.4. *Let S, J and $\mathcal{H}$ be as in the Theorem* 6.3. *Let M' be a self-adjoint operator in $\mathcal{H}$. Let N_1 and N_2 be disjoint, empty, finite or countable infinite index sets such that*

$$N := N_1 \cup N_2 \neq \emptyset.$$

For every $n \in N$, let I_n be a closed subinterval of J, $\alpha_n \in [0, 1]$ and $b_n \in (0, \infty)$. Then there exist a self-adjoint extension H of S and Borel measures ν_n on $\mathbb{R}$, $n \in N$, with the following properties:

(i) $H_{\mathrm{sc}J} \simeq \bigoplus_{n\in N} Q_{\nu_n}$;

(ii) *for every* $n \in N$ *the measure* ν_n *is purely singular continuous and* α_n-*dimensional,* $\mathrm{supp}(\nu_n) \subset I_n$ *and* $\nu_n(I_n) = b_n$;

(iii) *for every* $n \in N_1$

$$\mathrm{supp}(\nu_n) = I_n;$$

(iv) *for every* $n \in N_2$

$$\dim \mathrm{supp}(\nu_n) = \alpha_n;$$

(v) $H_{\mathrm{ac}J} \simeq M'_{\mathrm{ac}J}$;

(vi) $H_{\mathrm{pp}J} \simeq M'_{\mathrm{pp}J}$.

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