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ON THE COMMUTANT OF THE DIRECT SUM OF OPERATORS OF MULTIPLICATION BY THE INDEPENDENT VARIABLE

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Abstract. Let $\mathcal{B}$ be a direct sum of spaces of functions on each of which the operator M_z of multiplication by z ($f \to zf$) is bounded. We determine the commutant of the direct sum of the operators of multiplication by z on certain Hilbert spaces of functions (Banach spaces of functions). Also we characterize the commutant of M_z and multipliers of Lipschitz algebras. Let μ be a compactly supported measure on $\mathbb{C}$ and $t \geqslant 1$. We determine the commutant of the operator M_z on $P^t(\mu)$, the closure of polynomials in $L^t(\mu)$, thus extending a result of M. Raphael for the case $t = 2$.

Keywords: *Commutant, muliplication by z, bounded point evaluation, Lipschitz algebra, direct sum of spaces.*

MSC (2000): Primary 47E37; Secondary 47B48.

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space of functions defined on a set G in the plane and $B(\mathcal{H})$ be the subset of all $\lambda \in G$ such that the linear functional of evaluation at λ, e_λ, is bounded on $\mathcal{H}$. For every $\lambda \in B(\mathcal{H})$ we find an element k_λ in $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for every $f \in \mathcal{H}$. We call k_λ the *reproducing kernel* for the point λ. Furthermore assume that $1 \in \mathcal{H}$, $z\mathcal{H} \subset \mathcal{H}$ and $\bigvee\{k_\lambda : \lambda \in B(\mathcal{H})\} = \mathcal{H}$. The latter assumption says that if $f \in \mathcal{H}$ and $f = 0$ on $B(\mathcal{H})$, then $f = 0$ as an element of $\mathcal{H}$. By the closed graph theorem, the operator M_z of multiplication by z on $\mathcal{H}$ given by $f \mapsto zf$ is bounded. We also assume that $\dim \ker(M_z - \lambda)^* = 1$ for each $\lambda \in B(\mathcal{H})$. This assumption is equivalent to saying that $\ker e_\lambda = \operatorname{ran}(M_z - \lambda)^-$,

$\lambda \in B(\mathcal{H})$. By a *Hilbert space of functions* we mean one satisfying the above conditions. A subset S of $B(\mathcal{H})$ is called *dense* in $\mathcal{H}$ if $\bigvee\{k_\lambda : \lambda \in S\} = \mathcal{H}$.

Several examples of such Hilbert spaces $\mathcal{H}$ are $L^2_{\mathrm{a}}(G)$, the Bergman space on a bounded open set G in the plane, D_α the Dirichlet space for $-\infty < \alpha < \infty$ and Hardy spaces on a bounded region G.

A complex valued function φ on G for which $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$ is called a *multiplier* of $\mathcal{H}$ and the collection of all these multipliers is denoted by $\mathcal{M}(\mathcal{H})$. Each multiplier φ of $\mathcal{H}$ determines a multiplication operator M_φ on $\mathcal{H}$ by $M_\varphi f = \varphi f$, $f \in \mathcal{H}$. It is well known that each multiplier is a bounded function on $B(\mathcal{H})$. In fact $\|\varphi\|_{B(\mathcal{H})} \leqslant \|M_\varphi\|$. If $\mathcal{H}$ consists of functions that are analytic on $B(\mathcal{H})$ and φ is a multiplier, then $\varphi \in \mathcal{H}$ because $1 \in \mathcal{H}$. Hence $\varphi \in H^\infty(B(\mathcal{H}))$. Morever X is in the commutant $\{M_z\}'$ of M_z if and only if there exists $\varphi \in \mathcal{M}(\mathcal{H})$ such that $X = M_\varphi$ ([11]). The present work is a continuation of our work ([5]).

Let X be a compact subset of the plane. The algebra of all continuous functions on X that are analytic in the interior of X is denoted by $A(X)$. The set of all complex valued functions f defined on a compact subset K of the complex plane $\mathbb{C}$ such that

$$M_f^\alpha = \sup\left\{\frac{|f(x) - f(y)|}{|x-y|^\alpha} : x, y \in K,\, x \neq y\right\} < \infty$$

for $\alpha > 0$ is an algebra which is denoted by $\mathrm{Lip}(\alpha, K)$ and called the *Lipschitz algebra of order* α. Clearly $\mathrm{Lip}(\alpha, K) \subset C(K)$. For $0 < \alpha \leqslant 1$ the algebra $\mathrm{Lip}(\alpha, K)$ with the norm defined by $\|f\|_\alpha = \|f\| + M_f^\alpha$ is a Banach algebra.

For $0 < \alpha < 1$ the subset of $\mathrm{Lip}(\alpha, K)$ consisting of functions f for which

$$\lim_{h \to 0} \frac{|f(t+h) - f(t)|}{|h|^\alpha} = 0$$

is denoted by $\mathrm{lip}(\alpha, K)$ and is a closed subalgebra of $\mathrm{Lip}(\alpha, K)$. If we further assume that $\mathrm{lip}_A(\alpha, K)$ and $\mathrm{Lip}_A(\alpha, K)$ denote respectively $\mathrm{lip}(\alpha, K) \cap A(K)$, $\mathrm{Lip}(\alpha, K) \cap A(K)$ we can see that $\mathrm{lip}_A(\alpha, K)$ ($\mathrm{Lip}_A(\alpha, K)$) is a closed subalgebra of $\mathrm{lip}(\alpha, K)$ ($\mathrm{Lip}(\alpha, K)$).

We also need some properties of T-invariant algebras, for this we refer the reader to [4], Chapter 5, Section 6.

2. THE COMMUTANT OF 2×2 OPERATOR MATRICES

Every operator X acting on the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ can be written in the form

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{2.1}$$

where $X_{ij} : \mathcal{H}_j \to \mathcal{H}_i,\ i, j = 1, 2$ is defined by $X_{ij} = P_i X|_{\mathcal{H}_j}$ and $P_i : \mathcal{H} \to \mathcal{H}_i$ is the projection onto $\mathcal{H}_i$, $i = 1, 2$.

THEOREM 2.1. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces of functions on G and W respectively. If $M = M_z^1 \oplus M_z^2$, then every member X of the commutant $\{M\}'$ of M is of the form* (2.1) *in which $X_{11} \in \{M_z^1\}'$, and $X_{22} \in \{M_z^2\}'$ and $X_{21}(pf) = pX_{21}f$ for all polynomials p and all $f \in \mathcal{H}_1$. If $\sigma(M_z^2) \subset \mathbb{C} \setminus S$ where S is a subset of $B(\mathcal{H}_1)$ dense in $\mathcal{H}_1$ then $X_{12} = 0$. Furthermore if $\sigma(M_z^1) \subset \mathbb{C} \setminus T$, where T is a subset of $B(\mathcal{H}_2)$ dense in $\mathcal{H}_2$ then $X_{21} = 0$. In particular, if $B(\mathcal{H}_1) \cap B(\mathcal{H}_2) = \emptyset$, then $X_{12} = X_{21} = 0$.*

Proof. Let X commute with M and represent X as in (2.1). Then we have the following relations:

$$\begin{aligned} X_{11}M_z^1 &= M_z^1 X_{11}, \quad X_{22}M_z^2 = M_z^2 X_{22}, \\ X_{12}M_z^2 &= M_z^1 X_{12}, \quad X_{21}M_z^1 = M_z^2 X_{21}. \end{aligned}$$

We only need to show that $X_{12} = 0$ and X_{21} has the required form. Because $X_{12}M_z^2 = M_z^1 X_{12}$, we conclude that $X_{12}^* M_z^{1^*} = M_z^{2^*} X_{12}^*$. Applying the latter on every $k_\lambda, \lambda \in S$, we get $X_{12}^* M_z^{1^*} k_\lambda = M_z^{2^*} X_{12}^* k_\lambda = \overline{\lambda} X_{12}^* k_\lambda$. If $X_{12}^* k_\lambda \neq 0$, then it is an eigenvector for $M_z^{2^*}$ corresponding to the eigenvalue $\overline{\lambda}$. Because $\sigma(M_z^2) \subseteq \mathbb{C} \setminus S$, we get a contradiction. Hence $X_{12}^* k_\lambda = 0$ for all $\lambda \in S$. Because S is dense in $\mathcal{H}_1$ we conclude that $X_{12} = 0$. To characterize X_{21}, note that if p is a polynomial and $f \in \mathcal{H}_1$, then $X_{21}(pf) = X_{21}p(M_z^1)f = p(M_z^2)X_{21}f = pX_{21}f$. ∎

Let W and G be open subsets of the complex plane and assume $\mathcal{H}_1 = L_{\mathrm{a}}^2(W)$ and $\mathcal{H}_2 = L_{\mathrm{a}}^2(G)$ such that $W \cap G = \emptyset$. It is well known that $\sigma(M_z^1) = \overline{W}$ and $\sigma(M_z^2) = \overline{G}$, also $B(\mathcal{H}_1) = W$ and $B(\mathcal{H}_2) = G$. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $M = M_z^1 \oplus M_z^2$. It may be that $\overline{W} \cap \overline{G} \neq \emptyset$ but by Theorem 2.1 if $X \in \{M\}'$ then

$$X = \begin{bmatrix} M_\phi & 0 \\ 0 & M_\psi \end{bmatrix},$$

since $\mathcal{M}(L_{\mathrm{a}}^2(W)) = H^\infty(W)$ and $\mathcal{M}(L_{\mathrm{a}}^2(G)) = H^\infty(G)$, $\phi \in H^\infty(W)$ and $\psi \in H^\infty(G)$. Furthermore assume W and G are regions in the plane and $(\mathbb{C} \setminus \overline{W}) \cap G \neq \emptyset$ and $(\mathbb{C} \setminus \overline{G}) \cap W \neq \emptyset$, then $\{M\}'$ splits, that is $\{M\}' = \{M_z^1\}' \oplus \{M_z^2\}'$.

Now we consider a Banach space $\mathcal{B}$ consisting of functions defined on a set G in the plane such that $1 \in \mathcal{B}$ and $z\mathcal{B} \subset \mathcal{B}$. For every $\lambda \in B(\mathcal{B})$ the functional of evaluation at λ, e_λ, is bounded and $\bigvee\{e_\lambda : \lambda \in B(\mathcal{B})\} = \mathcal{B}^*$. We also assume that $\dim \ker(M_z - \lambda)^* = 1$ for $\lambda \in B(\mathcal{B})$. It is easy to see that for $\lambda \in B(\mathcal{B})$, $\ker(M_z - \lambda)^* = [e_\lambda]$, the linear span of e_λ. Such a Banach space is called a *Banach space of functions* on G. In particular, $\{M_z\}' = \{M_\phi : \phi \in \mathcal{M}(\mathcal{B})\}$. It is easy to see that $H^\infty(G)$ satisfies these conditions. A subset S of $B(\mathcal{B})$ is called *dense in* $\mathcal{B}^*$ if $\bigvee\{e_\lambda : \lambda \in S\} = \mathcal{B}^*$.

In [3], Cole and Gamelin proved that if $\mathcal{A}$ is a T-invariant algebra on K, then for each $\lambda \in K$, $\operatorname{ran}(M_z - \lambda)^- = \ker e_\lambda$. Hence $\dim \ker(M_z - \lambda)^* = 1$ and every T-invariant algebra is a Banach space of functions.

THEOREM 2.2. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces of functions on G. If $B(\mathcal{B}_1) \cap B(\mathcal{B}_2)$ is dense in $\mathcal{B}_2^*$ and $A : \mathcal{B}_1 \to \mathcal{B}_2$ is such that $M_z^2 A = A M_z^1$ then there is a function $\varphi \in \mathcal{M}(\mathcal{B}_1, \mathcal{B}_2)$ such that $A = M_\varphi$. In particular, $\{M_z\}' = \{M_\phi : \phi \in \mathcal{M}(\mathcal{B})\}$. If $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$ and $M = M_z^1 \oplus M_z^2$, then $X \in \{M\}'$ if and only if*

$$X = \begin{bmatrix} M_\Phi & M_\phi \\ M_\psi & M_\Psi \end{bmatrix} \tag{2.2}$$

in which $\Phi \in \mathcal{M}(\mathcal{B}_1)$, $\psi \in \mathcal{M}(\mathcal{B}_1, \mathcal{B}_2)$, $\phi \in \mathcal{M}(\mathcal{B}_2, \mathcal{B}_1)$ and $\Psi \in \mathcal{M}(\mathcal{B}_2)$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces of functions on W and G respectively. If $\sigma(M_z^1) \subset \mathbb{C} \setminus S$ where S is dense in $\mathcal{B}_2^*$ and $X : \mathcal{B}_1 \to \mathcal{B}_2$ such that $M_z^2 X = X M_z^1$ then $X = 0$.

Proof. Let e_λ^i denote the functional of evaluation at λ in $\mathcal{B}_i$, $i = 1, 2$. For each $\lambda \in B(\mathcal{B}_1) \cap B(\mathcal{B}_2)$, $(M_z^1 - \lambda)^* A^* = A^*(M_z^2 - \lambda)^*$. Hence $A^* \ker(M_z^2 - \lambda)^* \subset \ker(M_z^1 - \lambda)^*$ and therefore $A^*(e_\lambda^2) = \phi(\lambda) e_\lambda^1$. Now we have

$$(Af)(\lambda) = \langle Af, e_\lambda^2 \rangle = \langle f, A^* e_\lambda^2 \rangle = \langle f, \phi(\lambda) e_\lambda^1 \rangle = \phi(\lambda) f(\lambda).$$

Hence $Af = \varphi f$ on $B(\mathcal{B}_1) \cap B(\mathcal{B}_2)$. Because $B(\mathcal{B}_1) \cap B(\mathcal{B}_2)$ is dense in $\mathcal{B}_2^*$ we get $Af = \varphi f$. ∎

The weighted Bergman spaces are defined by

$$L_a^p(G, w\mathrm{d}A) = \{g \in L^p(G, w\mathrm{d}A) : g \text{ is analytic on } G\}$$

where w is a positive continuous function defined on a region G. Assume $p < s$ and $\mathcal{B} = L_a^p(G, w\mathrm{d}A) \oplus L_a^s(W, w\mathrm{d}A)$ and $M = M_z^1 \oplus M_z^2$. Then $X \in \{M\}'$ is of the form (2.2) in which $\Phi \in H^\infty(G)$, $\psi = 0$ ([2], Theorem 4), $\Psi \in H^\infty(W)$ and information about ϕ can be found in [1] and [8].

3. MULTIPLIERS OF LIPSCHITZ ALGEBRAS

Lipschitz algebras have been studied by Sherbert ([10]) and O'Farrell ([7]). We now study the commutant of M_z on these spaces and apply the result of the previous section to Lipshitz algebras.

Lemma 3.1. *Let* $\alpha \in (0,1)$ *and* M_z *denote the multiplication operator on* $\mathrm{lip}_A(\alpha, K)$. *Then* $\{M_z\}' = \mathcal{M}(\mathrm{lip}_A(\alpha, K)) = \mathrm{lip}_A(\alpha, K)$.

Proof. By [6], Corollary 1, dim $\ker(M_z - \lambda)^* = 1$ for each $\lambda \in K$. Hence $\mathrm{lip}_A(\alpha, K)$ is a Banach space of functions and $\{M_z\}' = \mathcal{M}(\mathrm{lip}_A(\alpha, K))$. Since $\mathrm{lip}_A(\alpha, K)$ is an algebra, $\mathcal{M}(\mathrm{lip}_A(\alpha, K)) = \mathrm{lip}_A(\alpha, K)$. ∎

Let α and β be in $(0,1)$ and $\alpha > \beta$; it is easy to see that

$$\mathcal{M}(\mathrm{lip}_A(\alpha, K), \mathrm{lip}_A(\beta, K)) = \mathrm{lip}_A(\beta, K),$$
$$\mathcal{M}(\mathrm{Lip}_A(\alpha, K), \mathrm{Lip}_A(\beta, K)) = \mathrm{Lip}_A(\beta, K).$$

Now we show that if $\alpha < \beta$, $\mathcal{M}(\mathrm{lip}_A(\alpha, K), \mathrm{lip}_A(\beta, K)) = \{0\}$. Before doing this we need the following result.

Theorem 3.2. *If* $\phi \in \mathrm{lip}_A(\alpha, K)$ *and* $\alpha > \beta$ *then* $M_\phi : \mathrm{lip}_A(\alpha, K) \to \mathrm{lip}_A(\beta, K)$ *is compact.*

Proof. It is easy to see that for each $f \in \mathrm{lip}_A(\alpha, K)$, $\phi f \in \mathrm{lip}_A(\alpha, K)$ and M_ϕ is continuous. Assume $0 < c < 1$ is a constant such that $c\alpha > \beta$. Now let $\{f_n\}$ be a sequence in $\mathrm{lip}_A(\alpha, K)$ such that $\|f_n\|_\alpha \leqslant 1$. Then $\|f_n\| \leqslant 1$ and $|f_n(x) - f_n(y)| \leqslant |x-y|^\alpha$ for all $n \in \mathbb{N}$ and $x, y \in K$. Hence by Arzela–Ascoli Theorem there is a subsequence $\{f_{n_k}\}$ and a function $g \in C(K)$ such that $f_{n_k} \to g$ uniformly on K. We can see that the uniform limit of a bounded sequence in $\mathrm{Lip}_A(\alpha, K)$ is in $\mathrm{Lip}_A(\alpha, K)$. Hence $\phi g \in \mathrm{Lip}_A(\alpha, K)$. Now we have

$$\begin{aligned}
&\frac{|(f_{n_k}\phi(x) - g\phi(x)) - (f_{n_k}\phi(y) - g\phi(y))|}{|x-y|^\beta} \\
&= \frac{|f_{n_k}\phi(x) - g\phi(x) - (f_{n_k}\phi(y) - g\phi(y))|^c}{|x-y|^\beta |f_{n_k}\phi(x) - g\phi(x) - (f_{n_k}\phi(y) - g\phi(y))|^{c-1}} \\
&\leqslant \frac{(2M_\phi^\alpha + 2\|\phi\|)^c |x-y|^{\alpha c}}{|x-y|^\beta} |f_{n_k}\phi(x) - g\phi(x) - (f_{n_k}\phi(y) - g\phi(y))|^{1-c} \\
&\leqslant (2M_\phi^\alpha + 2\|\phi\|)^c (\operatorname{diam} K)^{\alpha c - \beta} |f_{n_k}\phi(x) - g\phi(x) - (f_{n_k}\phi(y) - g\phi(y))|^{1-c} \\
&\to 0.
\end{aligned}$$

The proof is now complete. ∎

The idea of the proof of the next theorem is from [2], Theorem 4, on multipliers of Bergman spaces.

Theorem 3.3. *If α and β are in $(0,1)$, $\alpha < \beta$ and $K = \overline{\mathrm{int}(K)}$ then* $\mathcal{M}(\mathrm{lip}_A(\alpha,K),\mathrm{lip}_A(\beta,K)) = \{0\}$.

Proof. Without loss of generality we can assume $\mathrm{int}(K)$ is connected, otherwise let W be a component of $\mathrm{int}(K)$ and $\phi \in \mathcal{M}(\mathrm{lip}_A(\alpha,K),\ \mathrm{lip}_A(\beta,K))$ then $\phi|_W$ is in $\mathcal{M}(\mathrm{lip}_A(\alpha,\overline{W}),\mathrm{lip}_A(\beta,\overline{W}))$ and we can prove that $\phi|_W = 0$.

Given $\phi \in \mathcal{M}(\mathrm{lip}_A(\alpha,K),\mathrm{lip}_A(\beta,K))$, the operator $M_\phi : \mathrm{lip}_A(\alpha,K) \to \mathrm{lip}_A(\beta,K)$ is bounded. Assume $h \in \mathrm{lip}_A(\beta,K)$ is not equal to zero, then by Theorem 3.2, $M_h : \mathrm{lip}_A(\beta,K) \to \mathrm{lip}_A(\alpha,K)$ is compact. Therefore $M_{h\phi} = M_h M_\phi : \mathrm{lip}_A(\alpha,K) \to \mathrm{lip}_A(\alpha,K)$ is compact. Choose $z \in \mathrm{int}(K)$ such that $h(z)$ and $\phi(z)$ are not equal to zero. We can see that $M_{h\phi} - h(z)\phi(z)$ is not onto because each function in the range of $M_{h\phi} - h(z)\phi(z)$ is equal to zero at z. Now by Fredholm alternative $M_{h\phi} - h(z)\phi(z)$ is not injective and hence there is a function $f \neq 0$ in $\mathrm{lip}_A(\alpha,K)$ such that $(h\phi - h(z)\phi(z))f = 0$ on $\mathrm{int}(K)$. Hence $h\phi - h(z)\phi(z) = 0$ and $h\phi$ is constant on K. Since $h\phi$ is constant for each $h \in \mathrm{lip}_A(\beta,K)$ we have $\phi = 0$. ∎

Remark. In Theorem 3.2 and Theorem 3.3 we can replace lip_A with Lip_A.

Theorem 3.4. *Let $\mathcal{B} = \mathrm{lip}_A(\alpha,K) \oplus \mathrm{lip}_A(\beta,K)$ and $K = \overline{\mathrm{int}(K)}$, $\alpha,\beta \in (0,1)$, $\alpha > \beta$. If $M = M_z^1 \oplus M_z^2$ then $X \in \{M\}'$ if and only if*

$$X = \begin{bmatrix} M_\Phi & 0 \\ M_\psi & M_\Psi \end{bmatrix} \tag{3.1}$$

in which Φ is in $\mathrm{lip}_A(\alpha,K)$ and ψ and Ψ are in $\mathrm{lip}_A(\beta,K)$.

Proof. Apply Theorem 2.2, Lemma 3.1 and Theorem 3.3. ∎

Assume $f \in \mathrm{Lip}_A(\alpha,K)$. We apply Proposition 1.4 of [10] to the real and the imaginary parts of f to obtain an extension F of f in $\mathrm{Lip}(\alpha,\mathbb{C})$ and multiply this function by a function $h \in C_c^\infty$ defined by $h = 1$ on K and equal to 0 on W^c, where W is a bounded open subset containing K. Hence we have the following well known extension theorem about functions in $\mathrm{Lip}_A(\alpha,K)$. The second part of the theorem asserts that $\mathrm{Lip}_A(\alpha,K)$ is a T-invariant algebra. A.G. O'Farrell ([7]) shows this for $g \in C_c^\infty$ and gives a bound for $\|T_g F\|_\alpha$. This bound holds if $g \in C_c^1$.

Theorem 3.5. *Each f in $\mathrm{Lip}_A(\alpha,K)$ has an extension in $\mathrm{Lip}(\alpha,\mathbb{C})$ which has compact support. If $f \in \mathrm{Lip}_A(\alpha,K)$, F is the extension of f and $g \in C_c^1$ then $T_g F|_K$ is in $\mathrm{Lip}_A(\alpha,K)$.*

The next lemma is a modification of a lemma proved by Cole and Gamelin in [3].

Lemma 3.6. *Let K be a compact subset of the plane and $\alpha \in (0,1)$. Then $\mathrm{Lip}_A(\alpha,K)$ is a Banach algebra of functions on K. If $K = \overline{\mathrm{int}(K)}$, then $\{M_z\}' = \mathcal{M}(\mathrm{Lip}_A(\alpha,K)) = \mathrm{Lip}_A(\alpha,K)$.*

Proof. For each $f \in \mathrm{Lip}_A(\alpha,K)$ the function $\frac{f-f(\lambda)}{z-\lambda}$, $\lambda \in \mathrm{int}(K)$, is in $\mathrm{Lip}_A(\alpha,K)$. This follows if in the proof of [4], Theorem 6.5, p. 189, we let F be the extension of f in Theorem 3.5. Hence $\mathrm{ran}(M_z-\lambda) = \ker e_\lambda$, $\dim\ker(M_z-\lambda)^* = 1$ for $\lambda \in \mathrm{int}(K)$ and $\sigma_{\mathrm{ap}}(M_z) = \partial K$. ∎

Lemma 3.7. *Let $A : \mathrm{Lip}(\beta,K) \to \mathrm{Lip}(\alpha,K)$ $(\alpha,\beta \in [0,1]$ and $\alpha > \beta)$ be an operator such that $AM_z^1 = M_z^2 A$. If $K = \overline{\mathrm{int}(K)}$, then $A = 0$. Let $\mathcal{B} = \mathrm{Lip}_A(\alpha,K) \oplus \mathrm{Lip}_A(\beta,K)$, $K = \overline{\mathrm{int}(K)}$, α and β are in $[0,1]$, $\alpha > \beta$. If $M = M_z^1 \oplus M_z^2$ then $X \in \{M\}'$ if and only if*

$$X = \begin{bmatrix} M_\Phi & 0 \\ M_\psi & M_\Psi \end{bmatrix}$$

in which ϕ is in $\mathrm{lip}_A(\alpha,K)$ and ψ and Ψ are in $\mathrm{Lip}_A(\beta,K)$.

Proof. Let $A1 = \phi$. Then for each $\lambda \in \mathrm{int}(K)$, $Af(\lambda) = \phi(\lambda)f(\lambda)$. Since $Af - \phi f$ is continous on K and equal to 0 on $\mathrm{int}(K)$ we have $Af = \phi f$. Therefore ϕ is a multiplier and by Theorem 3.3 and the remark after that, $\phi = 0$. ∎

4. POLYNOMIAL APPROXIMATION

Let μ be a measure with compact support in the complex plane. A point λ in $\mathbb{C}$ is called a *bounded point evaluation* (bpe) for $P^t(\mu)$, $t \geqslant 1$, if there is a constant C such that

$$|p(\lambda)| \leqslant C\Big(\int |p|^t \mathrm{d}\mu\Big)^{\frac{1}{t}}$$

for every polynomial p.

If λ is a bpe for $P^t(\mu)$, then the linear functional $p \mapsto p(\lambda)$ of evaluation at λ defined on polynomials has a unique extension e_λ to $P^t(\mu)$. We can therefore find a unique element k_λ of $L^s(\mu)$ ($\frac{1}{t} + \frac{1}{s} = 1$) such that $\int f k_\lambda \mathrm{d}\mu = \langle f, e_\lambda\rangle = e_\lambda(f)$ for every f in $P^t(\mu)$. The set of bounded point evaluations for $P^t(\mu)$ is denoted by $B^t(\mu)$. For $f \in P^t(\mu)$ let $\widehat{f}(\lambda) = \int f k_\lambda \mathrm{d}\mu$. It is not hard to see that $f = \widehat{f}$ a.e. $[\mu]$ on the set $B^t(\mu)$. A point λ in $B^t(\mu)$ is an *analytic bounded point evaluation* (abpe) for $P^t(\mu)$, if $\lambda \in B^t(\mu)^\circ$, the interior of $B^t(\mu)$, and for every f in $P^t(\mu)$, the map $z \mapsto \widehat{f}(z)$ is analytic in a neighbourhood of λ. We denote the set of all analytic bpe's for $P^t(\mu)$ by $B^t_{\mathrm{a}}(\mu)$. For more information on bpe's see [4]. The operator of multiplication by z on $P^t(\mu)$ is denoted by S_μ.

Definition 4.1. A measure μ supported on $\overline{\mathbb{D}}$ is called an *m-measure* if the set of abpe's of $P^2(\mu)$, $B_{\rm a}(\mu) = \mathbb{D}$ and the identity mapping on the polynomials extends to an isometric isomorphism that is weak-star homeomorphism from $P^\infty(\mu)$ onto H^∞.

An equivalent condition is that $P^\infty(\mu)$ has no L^∞-summand and the interior of the Sarason hull of μ is $\mathbb{D}$, thus $P^\infty(\mu) = H^\infty$ and therefore $\mu_0 = \mu|\partial\mathbb{D} \ll m$. We are looking for a characterization of the commutant of S_μ, i.e. $P^2(\mu) \cap L^\infty(\mu)$. Recall that $P^2(\mu)$ is a *full analytic subspace* if $B_{\rm a}(\mu)$= int $\Sigma(\mu)$, the interior of the Sarason hull of μ.

Proposition 4.2. *Let μ be a compactly supported measure such that $P^\infty(\mu)$ has no L^∞-summand and $P^2(\mu)$ is a full analytic subspace. Let $\mathcal{H}_1 = \bigvee\{k_\lambda : \lambda \in B_{\rm a}(\mu)\}$ and $\mathcal{H}_0 = \mathcal{H}_1^\perp$. Then $\mathcal{H}_0$ reduces S_μ, $\mathcal{H}_1$ is a full analytic subspace and $N_0 = S_\mu|_{\mathcal{H}}$ is a cyclic normal operator with scalar-valued spectral measure μ_0 absolutely continuous with respect to harmonic measure for $\Sigma(\mu)$. Moreover, $P^2(\mu) = P^2(\mu_1) \oplus L^2(\mu_0)$, where $\mu_1 = \mu - \mu_0$, $\mathcal{H}_0 = L^2(\mu_0)$ and $\mathcal{H}_1 = P^2(\mu_1)$.*

Proof. [4], Proposition 7.7.4. ∎

The next result can be found in [5] for the case of an m-measure. Because the reference [5] is not easily accessible we bring the proof for the benefit of the reader and the sake of completeness.

Proposition 4.3. *Let μ be a compactly supported measure such that $P^\infty(\mu)$ has no L^∞-summand and $P^2(\mu)$ is a full analytic subspace, moreover $B_{\rm a}(\mu)$ is connected. Assume μ_0 is as in Proposition 4.2. Set $\mu_1 = \mu - \mu_0$. Then*

$$P^2(\mu) \cap L^\infty(\mu) = P^\infty(\mu_1) \oplus L^\infty(\mu_0) \tag{4.1}$$

and $P^2(\mu_1)$ is a full analytic subspace and $P^\infty(\mu_1)$ has no L^∞-summand.

Proof. By Proposition 4.2,

$$P^2(\mu) = P^2(\mu_1) \oplus L^2(\mu_0).$$

Let Δ be the carrier of μ_0 and $\Delta' = \operatorname{supp}\mu \setminus \Delta$ be the carrier of μ_1. Observe that $P^\infty(\mu_1) \oplus L^\infty(\mu_0) \subset P^2(\mu_1) \oplus L^2(\mu_0) = P^2(\mu)$ and $P^\infty(\mu_1) \oplus L^\infty(\mu_0) \subset L^\infty(\mu)$, thus the RHS of (4.1) is a subset of its LHS.

To show the converse, let $X \in \{S_\mu\}'$. Then X can be written in the form (2.1) where, similar to Theorem 2.1, $X_{11} = M_\varphi$, and $X_{22} = M_\psi$ with $\varphi \in \{M_z^1\}' = P^2(\mu_1) \cap L^\infty(\mu_1)$ ([4], Corollary 5.5, p. 52) and $\psi \in L^\infty(\mu_0)$. Also $\varphi \in H^\infty(B_{\rm a}(\mu_1))$. By Proposition 4.2, we have $P^2(\mu_1) = \bigvee\{k_\lambda : \lambda \in B_{\rm a}(\mu)\}$.

Therefore, $B_{\mathrm{a}}(\mu_1) = B_{\mathrm{a}}(\mu)$. Since every $f \in P^2(\mu)$ satisfies $f = \widehat{f}$ a.e. on the set of bpe's and $B_{\mathrm{a}}(\mu)$ is connected, there is no nontrivial characteristic function in $P^2(\mu_1)$. Hence $P^\infty(\mu_1)$ has no L^∞-summand and hence $P^\infty(\mu_1) = H^\infty(B_{\mathrm{a}}(\mu_1))$ ([4], Sarason's Theorem 7.1, p. 301). There exists $\varphi_0 \in P^\infty(\mu_1)$ such that on $B_{\mathrm{a}}(\mu_1)$ we have $\varphi = \varphi_0$. Since $P^2(\mu_1) = \bigvee\{k_\lambda : \lambda \in B_{\mathrm{a}}(\mu)\}$ we have $\varphi \in P^\infty(\mu_1)$.

Now let $f \in P^2(\mu) \cap L^\infty(\mu)$. Then writing $X = M_f$ in the previous paragraph we conclude that $f = \varphi$ on Δ' and $f = h + \psi$ on Δ, where $h = X_{21}1 \in L^2(\mu_0)$. Because $f \in L^\infty(\mu)$ and $\psi \in L^\infty(\mu_0)$ we have $h \in L^\infty(\mu)$. Hence $h \in L^\infty(\mu_0)$. We can now write $f = \chi_{\Delta'} f + \chi_\Delta f$ and therefore $P^2(\mu) \cap L^\infty(\mu) \subset P^\infty(\mu_1) \oplus L^\infty(\mu_0)$, thus (4.1) holds. ∎

As a result of the above theorem we obtain the main result of [9] when μ is an m-measure. This result can be extended by Thomson's Theorem to $P^t(\mu)$. We also investigate the commutant of S_μ for $P^t(\mu)$ ($t \geqslant 1$). First we recall several useful theorems.

Theorem 4.4. *Let μ be a positive measure and let $t \geqslant 1$. Then there exists a Borel partition $\{\Delta_0, \Delta\}$ of the support of μ such that*

$$P^t(\mu) = L^t(\mu|\Delta_0) \oplus P^t(\mu|\Delta)$$

and $P^t(\mu|\Delta)$ contains no L^t summand.

Proof. See [12], Theorem 1.2. ∎

Thomson's Theorem. *If μ is any compactly supported measure on $\mathbb{C}$ and S_μ is multiplication by z on $P^t(\mu)$, then there exists a Borel partition $\{\Delta_i\}_{i=0}^\infty$ of the support of μ such that if $\mu_n = \mu|\Delta_n$ then the following statements are true:*

(a)

$$P^t(\mu) = L^t(\mu_0) \oplus \Big(\bigoplus_{i=1}^\infty P^t(\mu_i)\Big); \tag{4.2}$$

(b) *if $n \geqslant 1$, then $P^t(\mu_n)$ is irreducible; that is, $P^t(\mu_n)$ contains no nontrivial characteristic functions;*

(c) *if $n \geqslant 1$ and $\Omega_n = B^t_{\mathrm{a}}(\mu_n)$, then Ω_n is a simply connected region with* $\operatorname{supp}(\mu_n) \subset \operatorname{cl}\Omega_n$*;*

(d) *if S_μ is pure (that is $\Delta_0 = \emptyset$) and $f \in P^t(\mu)$ such that f vanishes a.e. $[\mu]$ on $B^t_{\mathrm{a}}(\mu)$, then $f = 0$. Equivalently, $\bigvee\{k_\lambda : \lambda \in B^t_{\mathrm{a}}(\mu)\} = P^s(\mu)$ ($\frac{1}{t} + \frac{1}{s} = 1$).*

(e) *If S_μ is pure, then the map $f \to \widehat{f}$ is a dual algebra isomorphism of $P^t(\mu) \cap L^\infty(\mu)$ onto $H^\infty(B^t_{\mathrm{a}}(\mu))$.*

Remark. By part (d) of Thomson's Theorem and the fact that for every $\lambda \in B^t(\mu)$, $\dim \ker(S_\mu - \lambda)^* = 1$, we conclude that if $P^t(\mu)$ is pure it is a Banach space of functions with $B(P^t(\mu)) = B^t(\mu)$.

Theorem 4.5. *Let μ be a measure such that $P^t(\mu)$ is pure. If $\{W_i\}_{i=1}^{\infty}$ are the components of $B_{\mathrm{a}}^t(\mu)$, then there is a partition $\{\Delta_i\}_{i=1}^{\infty}$ of* $\operatorname{supp}\mu$ *such that*

$$P^t(\mu) = \bigoplus_{i=1}^{\infty} P^t(\mu_i) \tag{4.3}$$

where $\mu_i = \mu|\Delta_i$, each $P^t(\mu_i)$ is irreducible and $\Delta_i \subset \overline{W}_i$.

Proof. See the proof of part (a) of Thomson's Theorem in the pure case ([12], Theorem 5.8). ∎

Theorem 4.6. *Let μ be a measure such that $P^t(\mu)$ is pure relative to the decomposition* (4.3). *If $X \in \{S_\mu\}' = P^t(\mu) \cap L^\infty(\mu)$ then*

$$X = \begin{bmatrix} M_{\phi_1} & 0 & \cdots \\ 0 & M_{\phi_2} & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \end{bmatrix}$$

where each ϕ_i, $i = 1, 2, \ldots$, belongs to $P^t(\mu_i) \cap L^\infty(\mu_i)$. If $P^t(\mu)$ is pure and the direct sum in (4.3) *is finite that is*

$$P^t(\mu) = \bigoplus_{i=1}^{n} P^t(\mu_i), \quad n \geqslant 1,$$

then $P^t(\mu) \cap L^\infty(\mu) = \bigoplus_{i=1}^{n} (P^t(\mu_i) \cap L^\infty(\mu_i))$.

Proof. Since $\overline{W}_i \cap W_j = \emptyset$ for $i \neq j$ and $\sigma(M_z^i) \subset \overline{W}_i$ by Theorem 2.2. $X_{ij} = 0$ for $i \neq j$. Since $X_{ii} \in \{M_z^i\}'$, $i \in \mathbb{N}$, then $X_{ii} = M_{\phi_i}$ for some $\phi_i \in P^t(\mu_i) \cap L^\infty(\mu_i)$. ∎

Theorem 4.7. *If the decomposition in* (4.2) *is finite, that is*

$$P^t(\mu) = L^t(\mu_0) \oplus \Big(\bigoplus_{i=1}^{n-1} P^t(\mu_i) \Big), \quad n \geqslant 1,$$

then $P^t(\mu) \cap L^\infty(\mu) = L^\infty(\mu) \oplus \Big(\bigoplus_{i=1}^{n-1} P^t(\mu_i) \cap L^\infty(\mu_i) \Big)$.

Proof. In this case if $X \in \{S_\mu\}'$, then $X = M_\phi$ for some $\phi \in P^t(\mu) \cap L^\infty(\mu)$. On the other hand

$$X = \begin{bmatrix} M_{\phi_0} & X_{01} & X_{02} & X_{03} & \cdots & X_{0n-1} \\ 0 & M_{\phi_1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & M_{\phi_2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & & \\ 0 & 0 & 0 & 0 & \cdots & M_{\phi_{n-1}} \end{bmatrix}.$$

Therefore

$$\phi = (\phi_0 + X_{01}(1) + X_{02}(1) + \cdots + X_{0n-1}(1))\chi_{\Delta_0} + \phi_1\chi_{\Delta_1} + \cdots + \phi_{n-1}\chi_{\Delta_{n-1}}.$$

Since $\phi_0 \in L^\infty(\mu_0)$ and $\phi \in L^\infty(\mu)$ we have $X_{01}(1) + X_{02}(1) + \cdots + X_{0n-1}(1) \in L^\infty(\mu_0)$. Because each $\phi_i \in P^t(\mu_i) \cap L^\infty(\mu_i)$ for $0 < i < n$,

$$P^t(\mu) \cap L^\infty(\mu) \subset L^\infty(\mu) \oplus \Big(\bigoplus_{i=1}^{n-1} P^t(\mu_i) \cap L^\infty(\mu_i)\Big).$$

The reverse inclusion is obvious. ∎

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