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ON AUTOMORPHISMS OF C^*-ALGEBRAS ASSOCIATED WITH SUBSHIFTS

KENGO MATSUMOTO

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Abstract. We prove that, for a given one-sided subshift X_Λ, any non-trivial automorphism of the subshift always yields an outer automorphism of the C^*-algebra $\mathcal{O}_\Lambda$ associated with the subshift. In particular, any non-trivial automorphism of the one-sided topological Markov shift X_A for a $\{0,1\}$-matrix A yields an outer automorphism of the Cuntz-Krieger algebra $\mathcal{O}_A$. We also determine the form of the automorphisms of the C^*-algebra $\mathcal{O}_\Lambda$ arising from automorphisms of the subshift X_Λ.

Keywords: *C^*-algebras, automorphisms, subshifts, Cuntz-Krieger algebras.*

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1. INTRODUCTION

In [22], the author has introduced and studied a class of C^*-algebras associated with subshifts in the theory of symbolic dynamics. Each of the C^*-algebras associated with subshifts has canonical generators of partial isometries with mutually orthogonal ranges. It also has universal properties subject to some operator relations among the generators ([22], [25]) so that it becomes purely infinite and simple in many cases including Cuntz-Krieger algebras. For a subshift (Λ, σ), we denote by X_Λ the set of all right-infinite sequences that appear in Λ. The dynamical system (X_Λ, σ), simply written as X_Λ, is called the one-sided subshift for Λ. The C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ is essentially constructed by the dynamics (X_Λ, σ). Many dynamical property for (X_Λ, σ) reflects on algebraic structure on the C^*-algebra $\mathcal{O}_\Lambda$ as in [22], [24].

We will in this paper study relationships between automorphisms of the dynamics X_Λ and automorphisms of the algebra $\mathcal{O}_\Lambda$. A homeomorphism h of X_Λ satisfying $h = \sigma \circ h \circ \sigma^{-1}$ is called an automorphism of X_Λ. We denote by $\mathrm{Aut}(X_\Lambda)$ the set of all automorphisms of X_Λ. There have been many studies on automorphisms of subshifts especially of topological Markov shifts (cf. [3], [19], ...). Their studies are closely related to classification of subshifts (cf. [19], [29]).

Let (Λ, σ) be a subshift over a finite set $\Sigma = \{1, 2, \dots, n\}$ with shift transformation σ. Then the C^*-algebra $\mathcal{O}_\Lambda$ associated with the subshift is generated by n canonical partial isometries $S_1, S_2, \dots, S_n$. One typical example of automorphisms of $\mathcal{O}_\Lambda$ is defined by a mapping for $t \in \mathbb{R} : S_j \to e^{\sqrt{-1}t} S_j$, $j = 1, 2, \dots, n$. These automorphisms are called the gauge automorphisms. The fixed point algebra of the C^*-algebra $\mathcal{O}_\Lambda$ under the gauge automorphisms is an AF-algebra which is written as $\mathcal{F}_\Lambda$ ([22]). We denote by $\mathcal{D}_\Lambda$ the C^*-algebra of all diagonal elements of $\mathcal{F}_\Lambda$, which is commutative. The commutative C^*-algebra $C(X_\Lambda)$, denoted by $\mathfrak{D}_\Lambda$, of all continuous functions on X_Λ is naturally embedded into the algebra $\mathcal{D}_\Lambda$. Hence each automorphism h of X_Λ yields an automorphism h^* of the subalgebra $\mathfrak{D}_\Lambda$ of $\mathcal{O}_\Lambda$. The induced endomorphism of $\mathfrak{D}_\Lambda$ from the shift σ of X_Λ is uniquely extended to an endomorphism φ_Λ of $\mathcal{D}_\Lambda$ that is defined by $\varphi_\Lambda(X) = \sum_{j=1}^{n} S_j X S_j^*$ for $X \in \mathcal{D}_\Lambda$. They satisfy the relation $h^* \circ \varphi_\Lambda = \varphi_\Lambda \circ h^*$ on $\mathfrak{D}_\Lambda$. We first see the following:

PROPOSITION 1.1. (Proposition 4.2) *For an automorphism h of X_Λ, there exists an automorphism α_h of $\mathcal{O}_\Lambda$ such that $\alpha_h(x) = h^*(x)$, $x \in \mathfrak{D}_\Lambda$ and the correspondence $h \in \mathrm{Aut}(X_\Lambda) \to \alpha_h \in \mathrm{Aut}(\mathcal{O}_\Lambda)$ gives rise to a homomorphism.*

Let $\mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$ be the set of all automorphisms of $\mathcal{O}_\Lambda$ whose restrictions to the algebra $\mathfrak{D}_\Lambda$ give rise to automorphisms of X_Λ. Namely that is the group of automorphisms of $\mathcal{O}_\Lambda$ coming from $\mathrm{Aut}(X_\Lambda)$. The extension of $h \in \mathrm{Aut}(X_\Lambda)$ to an automorphism of $\mathcal{O}_\Lambda$ is not necessarily unique. By proving the result:

$$\mathfrak{D}_\Lambda{}' \cap \mathcal{O}_\Lambda = \mathcal{D}_\Lambda$$

as in Proposition 3.3, we see that any automorphism of X_Λ may be uniquely extended to an automorphism of $\mathcal{O}_\Lambda$ modulo unitaries in $\mathcal{D}_\Lambda$. That is, for an automorphism h of X_Λ, if two automorphisms α^h, β^h of $\mathcal{O}_\Lambda$ coincide with h^* on X_Λ, then $\alpha^h = \beta^h \circ \lambda(U)$ for some unitary U in $\mathcal{D}_\Lambda$ where $\lambda(U) \in \mathrm{Aut}(\mathcal{O}_\Lambda)$ is defined to be $\lambda(U)(S_i) = US_i$ (Corollary 4.10). We denote by $\mathcal{U}(\mathcal{D}_\Lambda)$ the group of all unitaries in $\mathcal{D}_\Lambda$. Let $Z_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda))$ ($\cong \mathcal{U}(\mathcal{D}_\Lambda)$) be the set of all unitary one-cocycles for φ_Λ of $\mathcal{U}(\mathcal{D}_\Lambda)$ that are defined to be $\mathcal{U}(\mathcal{D}_\Lambda)$-valued functions U from $\mathbb{N}$ such that $U(k+l) = U(k)\varphi_\Lambda^k(U(l))$, $k, l \in \mathbb{N}$. We will in fact prove

THEOREM 1.2. (Theorem 4.9) *There exists a natural short exact sequence:*

$$0 \to Z_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda)) \to \mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) \to \mathrm{Aut}(X_\Lambda) \to 0$$

that splits.

We will next study outerness for the automorphisms of $\mathcal{O}_\Lambda$ coming from automorphisms of X_Λ. We introduce a condition for an automorphism of X_Λ called condition (I). The condition is considered as a relative version to the condition (I) for the original dynamics (X_Λ, σ). We will show that if a non-trivial automorphism of X_Λ satisfies the condition (I), its extension to an automorphism of $\mathcal{O}_\Lambda$ is outer (Theorem 5.2). We will also prove that any extension as an automorphism of $\mathcal{O}_\Lambda$ of a non-trivial automorphism of X_Λ is always outer if X_Λ satisfies a certain aperiodicity condition called (D) (Theorem 5.12). In particular, any extension of a non-trivial automorphism of a topological Markov shift X_A for an aperiodic matrix A to an automorphism of the Cuntz-Krieger algebra $\mathcal{O}_A$ is outer. We see that the automorphism $\lambda(u)$ of $\mathcal{O}_\Lambda$ for a unitary u in $\mathcal{D}_\Lambda$ is inner if and only if u gives rise to a coboundary for φ_Λ in $\mathcal{U}(\mathcal{D}_\Lambda)$. Let $B^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda))$ be the subgroup of all coboundaries in $Z^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda))$. Set $H^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda)) = Z^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda))/B^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda))$ the one-cohomology group for φ_Λ of $\mathcal{U}(\mathcal{D}_\Lambda)$.

THEOREM 1.3. (Theorem 5.16) *There exists a natural short exact sequence:*

$$0 \to H^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda)) \to \mathrm{Out}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) \to \mathrm{Aut}(X_\Lambda) \to 0$$

that splits, where $\mathrm{Out}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$ *means the group of all outer automorphisms of* $\mathcal{O}_\Lambda$ *in* $\mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$.

We will, in the final section, present certain examples of automorphisms of the C^*-algebra coming from some subshifts. We will further see that if a subshift X_Λ has a fixed point, then the non-trivial gauge automorphisms are outer (Corollary 6.5).

Slightly similar exact sequences to the above two exact sequences have appeared in a discussion of classification of von Neumann algebras arising from non-singular ergodic transformation (cf. [20]). The classification has exactly corresponds to orbit equivalences of such ergodic transformations (cf. [5], [10], [20]). C^*-algebraic analogies have also been discussed in [4], [15], [27], etc. If a subshift Λ is a topological Markov shift and, in particular, a full shift, the associated C^*-algebra $\mathcal{O}_\Lambda$ becomes a Cuntz-Krieger algebra and a Cuntz algebra respectively. Hence our study, in this paper, includes studies of automorphisms of these algebras from a view point of symbolic dynamical systems. Studies of automorphisms of Cuntz-Krieger algebras and Cuntz algebras are seen in many papers as in [1], [8], [12], [13], [18], [26], [28], ... The author has recently received a preprint [18] by Katayama-Takehana in which outerness of automorphisms of Cuntz-Krieger algebras are discussed by using a technique of Hilbert C^*-bimodules (cf. [17]).

2. BASIC NOTATION AND THE C^*-ALGEBRA $\mathcal{O}_\Lambda$

Let Σ be a finite set $\{1, 2, \dots, n\}$ for $n > 1$. Let $\Sigma^{\mathbb{Z}}$, $\Sigma^{\mathbb{N}}$ be the infinite product spaces $\prod_{i=-\infty}^{\infty} \Sigma_i$, $\prod_{i=1}^{\infty} \Sigma_i$ where $\Sigma_i = \Sigma$, endowed with the product topology respectively. The transformation σ on $\Sigma^{\mathbb{Z}}, \Sigma^{\mathbb{N}}$ given by $(\sigma(x))_i = x_{i+1}$, $i \in \mathbb{Z}$, $\mathbb{N}$ for $x = (x :)$ is called the (full) shift. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|\Lambda)$ is called a subshift. We denote $\sigma|\Lambda$ by σ for simplicity. This class of the subshifts includes the class of the topological Markov shifts (cf. [19], [21]).

A finite sequence $\mu = (\mu_1, \dots, \mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word. We denote by $|\mu|$ the length k of μ. A block $\mu = (\mu_1, \dots, \mu_k)$ is said to occur in $x = (x_i) \in \Sigma^{\mathbb{Z}}$ if $x_m = \mu_1, \dots, x_{m+k-1} = \mu_k$ for some $m \in \mathbb{Z}$. For $x = (x_i) \in \Sigma^{\mathbb{Z}}$ or $\Sigma^{\mathbb{N}}$ and $i \leqslant j$, we write

$$x_{[i,j]} = (x_i, x_{i+1}, \dots, x_j), \quad x_{[i,\infty)} = (x_i, x_{i+1}, \dots) \in \Sigma^{\mathbb{N}}.$$

For a subshift (Λ, σ), let Λ^k be the set of all words with length k in $\Sigma^{\mathbb{Z}}$ occurring in some $x \in \Lambda$. Put $\Lambda_l = \bigcup_{k=0}^{l} \Lambda^k$ for $l \in \mathbb{N}$ and $\Lambda^* = \bigcup_{k=0}^{\infty} \Lambda^k$ where Λ^0 denotes the empty word $\emptyset$. Let X_Λ be the set of all right-infinite sequences that appear in Λ. The dynamical system (X_Λ, σ) is called the one-sided subshift for Λ.

Put

$$\Lambda^l(x) = \{\mu \in \Lambda^l \mid \mu x \in X_\Lambda\} \quad \text{for } x \in X_\Lambda,\ l \in \mathbb{N}.$$

We define equivalence relations in the space X_Λ. For $l \in \mathbb{N}$, two points $x, y \in X_\Lambda$ are said to be *l-past equivalent* if $\Lambda^l(x) = \Lambda^l(y)$. We write this equivalence as $x \sim_l y$ (cf. [24]).

DEFINITION. ([24]) (i) A subshift (X_Λ, σ) satisfies *condition* (I) if for any $l \in \mathbb{N}$ and $x \in X_\Lambda$, there exists $y \in X_\Lambda$ such that $y \neq x$ and $y \sim_l x$.

(ii) A subshift (X_Λ, σ) is *irreducible in past equivalence* if for any $l \in \mathbb{N}$, $y \in X_\Lambda$ and a sequence $(x^k)_{k\in\mathbb{N}}$ of X_Λ with $x^k \sim_k x^{k+1}$ for $k \in \mathbb{N}$, there exist a number N and a word $\mu \in \Lambda^N$ such that $y \sim_l \mu x^{l+N}$.

(iii) A subshift (X_Λ, σ) is *aperiodic in past equivalence* if for any $l \in \mathbb{N}$, there exists a number N such that for any pair $x, y \in X_\Lambda$, there exists a word $\mu \in \Lambda^N$ such that $y \sim_l \mu x$.

If a subshift (X_Λ, σ) is aperiodic in past equivalence or irreducible in past equivalence with an aperiodic point, then it satisfies condition (I) ([24]). If a subshift (X_Λ, σ) is a topological Markov shift (X_A, σ) determined by a square matrix A with entries in $\{0, 1\}$, the above aperiodicity, irreducibility and condition (I) as a subshift coincide with the aperiodicity, irreducibility and condition (I) let it stand as it is in [9] for the matrix A respectively.

Now we will review the construction of the C^*-algebras associated with subshifts along [22]. We henceforth fix an arbitrary subshift (Λ, σ).

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of n-dimensional Hilbert space $\mathbb{C}^n$. We put

$F_\Lambda^0 = \mathbb{C}e_0$ (e_0: vacuum vector);

F_Λ^k = the Hilbert space spanned by the vectors $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}$, $\mu = (\mu_1, \dots, \mu_k) \in \Lambda^k$;

$F_\Lambda = \bigoplus_{k=0}^{\infty} F_\Lambda^k$ (Hilbert space direct sum).

We denote by $T_\nu, (\nu \in \Lambda^*)$ the creation operator on F_Λ of $e_\nu, \nu \in \Lambda^* (\nu \neq \emptyset)$ defined by

$$T_\nu e_0 = e_\nu \quad \text{and} \quad T_\nu e_\mu = \begin{cases} e_\nu \otimes e_\mu & (\nu\mu \in \Lambda^*), \\ 0 & \text{else,} \end{cases}$$

which is a partial isometry. We put $T_\nu = 1$ for $\nu = \emptyset$. Let $\mathbb{P}_0$ be the rank one projection onto the vacuum vector e_0. It immediately follows that $\sum_{i=1}^{n} T_i T_i^* + \mathbb{P}_0 = 1$. We then easily see that for $\mu, \nu \in \Lambda^*$, the operator $T_\mu \mathbb{P}_0 T_\nu^*$ is the rank one partial isometry from the vector e_ν to e_μ. Hence, the C^*-algebra generated by elements of the form $T_\mu \mathbb{P}_0 T_\nu^*$, $\mu, \nu \in \Lambda^*$ is nothing but the C^*-algebra $\mathcal{K}(F_\Lambda)$ of all compact operators on F_Λ. Let $\mathcal{T}_\Lambda$ be the C^*-algebra on F_Λ generated by the elements T_ν, $\nu \in \Lambda^*$.

DEFINITION. ([22]) The C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift (Λ,σ) is defined as the quotient C^*-algebra $\mathcal{T}_\Lambda/\mathcal{K}(F_\Lambda)$ of $\mathcal{T}_\Lambda$ by $\mathcal{K}(F_\Lambda)$.

We denote by S_i, S_μ the quotient images of the operators T_i, $i \in \Sigma$, T_μ, $\mu \in \Lambda^*$ respectively. Hence $\mathcal{O}_\Lambda$ is generated by partial isometries $S_1,\dots,S_n$ with relation $\sum_{i=1}^{n} S_i S_i^* = 1$.

If (Λ,σ) is a topological Markov shift, the C^*-algebra $\mathcal{O}_\Lambda$ is nothing but the Cuntz-Krieger algebra associated with the topological Markov shift (cf. [9], [11], [13]).

We will present notation and basic facts for studying the C^*-algebra $\mathcal{O}_\Lambda$.

Put $a_\mu = S_\mu^* S_\mu$, $\mu \in \Lambda^*$. Since $T_\nu T_\nu^*$ commutes with $T_\mu^* T_\mu$, $\mu,\nu \in \Lambda^*$, the following identities hold

$$(*) \qquad a_\mu S_\nu = S_\nu a_{\mu\nu}, \quad \mu,\nu \in \Lambda^*.$$

We notice that for $\mu,\nu \in \Lambda^*$ with $|\mu| = |\nu|$,

$$S_\mu^* S_\nu \neq 0 \quad \text{if and only if} \quad \mu = \nu.$$

We will use the following notation. Let k, l be natural numbers with $k \leqslant l$.

$\mathcal{A}_l =$ The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by a_μ, $\mu \in \Lambda_l$.

$\mathcal{A}_\Lambda =$ The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by a_μ, $\mu \in \Lambda^*$.

$\mathfrak{D}_\Lambda =$ The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu S_\mu^*$, $\mu \in \Lambda^*$.

$\mathcal{D}_\Lambda =$ The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu a S_\mu^*$, $\mu \in \Lambda^*$, $a \in \mathcal{A}_\Lambda$.

$\mathcal{F}_k^l =$ The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu a S_\nu^*$, $\mu,\nu \in \Lambda^k$, $a \in \mathcal{A}_l$.

$\mathcal{F}_k^\infty =$ The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu a S_\nu^*$, $\mu,\nu \in \Lambda^k$, $a \in \mathcal{A}_\Lambda$.

$\mathcal{F}_\Lambda =$ The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu a S_\nu^*$, $\mu,\nu \in \Lambda^*, |\mu| = |\nu|$, $a \in \mathcal{A}_\Lambda$.

The projections $\{T_\mu^* T_\mu;\, \mu \in \Lambda^*\}$ are mutually commutative so that the C^*-algebras $\mathcal{A}_l$, $l \in \mathbb{N}$ are commutative. Thus we easily see the following lemma (cf. [22], Section 3).

LEMMA 2.1. (i) *$\mathcal{A}_l$ is finite dimensional and commutative.*

(ii) *$\mathcal{A}_l$ is naturally embedded into $\mathcal{A}_{l+1}$ so that $\mathcal{A}_\Lambda = \varinjlim \mathcal{A}_l$ is a commutative AF-algebra.*

(iii) *Each element of $\mathcal{F}_k^l$ is a finite linear combination of elements of the form $S_\mu a S_\nu^*$, $\mu,\nu \in \Lambda^k$, $a \in \mathcal{A}_l$. Hence $\mathcal{F}_k^l$ is finite dimensional.*

(iv) *There are two embeddings in $\{\mathcal{F}_k^l\}_{k \leqslant l}$:*

(a) *$\iota_l : \mathcal{F}_k^l \subset \mathcal{F}_k^{l+1}$ through the embedding $\mathcal{A}_l \subset \mathcal{A}_{l+1}$ and,*

(b) *$\eta_k : \mathcal{F}_k^l \subset \mathcal{F}_{k+1}^{l+1}$ through the identity*

$$S_\mu a S_\nu^* = \sum_{j=1}^{n} S_{\mu j} S_j^* a S_j S_{\nu j}^*, \quad \mu,\nu \in \Lambda^k, a \in \mathcal{A}_l.$$

(v) *Both $\mathcal{F}_k^\infty = \lim_{l\to\infty} \mathcal{F}_k^l$ and $\mathcal{F}_\Lambda = \lim_{k\to\infty} \mathcal{F}_k^\infty$ are AF-algebras.*

In the preceding Hilbert space F_Λ, the transformation $e_\mu \to z^k e_\mu$, $\mu \in \Lambda^k$, $z \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ on each base e_μ yields a unitary representation which leaves $\mathcal{K}(F_\Lambda)$ invariant. Thus it gives rise to an action α of $\mathbb{T}$ on the C^*-algebra $\mathcal{O}_\Lambda$. It is called the gauge action and satisfies $\alpha_z(S_i) = zS_i$, $i = 1, 2, \ldots, n$.

Each element X of the $*$-subalgebra of $\mathcal{O}_\Lambda$ algebraically generated by S_μ, S_ν^*, $\mu, \nu \in \Lambda^*$ is written as a finite sum

$$X = \sum_{|\nu| \geqslant 1} X_{-\nu} S_\nu^* + X_0 + \sum_{|\mu| \geqslant 1} S_\mu X_\mu \quad \text{for some } X_{-\nu}, X_0, X_\mu \in \mathcal{F}_\Lambda \tag{2.1}$$

because of the relation $(*)$. The map $E(X) = \int_{z \in \mathbb{T}} \alpha_z(X)\,\mathrm{d}z$, $X \in \mathcal{O}_\Lambda$ defines a projection of norm one onto the fixed point algebra $\mathcal{O}_\Lambda^\alpha$ under α. We then have (cf. [22], Proposition 3.11)

LEMMA 2.2. $\mathcal{F}_\Lambda = \mathcal{O}_\Lambda^\alpha$.

Note that the C^*-algebra $\mathfrak{D}_\Lambda$ is isomorphic to the commutative C^*-algebra $C(X_\Lambda)$ of all complex valued continuous functions on the one-sided subshift X_Λ for Λ. Put

$$\varphi_\Lambda(X) = \sum_{j=1}^{n} S_j X S_j^*, \quad X \in \mathfrak{D}_\Lambda \text{ (or } X \in \mathcal{O}_\Lambda)$$

which corresponds to the shift σ of X_Λ.

Consider the following condition called (I_Λ) for the C^*-algebra $\mathcal{O}_\Lambda$ (cf. [22]).

(I_Λ): For any $l, k \in \mathbb{N}$ with $l \geqslant k$, there exists a projection q_k^l in $\mathfrak{D}_\Lambda$ such that

(i) $q_k^l a \neq 0$ for any nonzero $a \in \mathcal{A}_l$;

(ii) $q_k^l \varphi_\Lambda^m(q_k^l) = 0$, $1 \leqslant m \leqslant k$.

As in [24], the subshift (X_Λ, σ) satisfies condition (I) if and only if the C^*-algebra $\mathcal{O}_\Lambda$ satisfies condition (I_Λ). Hence we may describe structure theorems for the C^*-algebra $\mathcal{O}_\Lambda$ proved in [22].

LEMMA 2.3. ([22], Theorems 4.9 and 5.2) *Let $\mathfrak{A}$ be a unital C^*-algebra. Suppose that there is a unital $*$-homomorphism π from $\mathcal{A}_\Lambda$ to $\mathfrak{A}$ and there are n partial isometries $s_1, \ldots, s_n \in \mathfrak{A}$ satisfying the following relations*

$$\sum_{j=1}^{n} s_j s_j^* = 1, \quad s_\mu^* s_\mu s_\nu = s_\nu s_{\mu\nu}^* s_{\mu\nu}, \quad \mu, \nu \in \Lambda^*,$$
$$s_\mu^* s_\mu = \pi(S_\mu^* S_\mu), \qquad \mu \in \Lambda^*$$

where $s_\mu = s_{\mu_1} \cdots s_{\mu_k}$, $\mu = (\mu_1, \ldots, \mu_k)$. Then there exists a unital $*$-homomorphism $\widetilde{\pi}$ from $\mathcal{O}_\Lambda$ to $\mathfrak{A}$ such that $\widetilde{\pi}(S_i) = s_i$, $i = 1, \ldots, n$ and its restriction to $\mathcal{A}_\Lambda$ coincides with π. In addition, if the subshift X_Λ satisfies condition (I), *this extended homomorphism $\widetilde{\pi}$ becomes injective whenever π is injective.*

LEMMA 2.4. ([22], Theorem 6.3 and Theorem 7.5 and [24], Theorem 5.8) *If a subshift X_Λ is irreducible in past equivalence and has an aperiodic point, then $\mathcal{O}_\Lambda$ is simple. In addition, if a subshift X_Λ is aperiodic in past equivalence, the C^*-algebra $\mathcal{O}_\Lambda$ is simple and purely infinite.*

We notice the following lemma.

Lemma 2.5. ([22], Proposition 5.8 and [24], Lemma 4.5, cf. [9], 2.17 Proposition) *Suppose that both subshifts* (X_{Λ_1},σ) *and* (X_{Λ_2},σ) *satisfy condition* (I). *If they are topologically conjugate, then there exists an isomorphism* Φ *from* $\mathcal{O}_{\Lambda_1}$ *onto* $\mathcal{O}_{\Lambda_2}$ *such that* $\Phi\circ\alpha_z^1=\alpha_z^2\circ\Phi$, $z\in\mathbb{T}$ *where* α^i *is the gauge action on* $\mathcal{O}_{\Lambda_i}$, $i=1,2$ *respectively.*

3. THE COMMUTANT OF $\mathfrak{D}_\Lambda$ IN $\mathcal{O}_\Lambda$

We henceforth fix an arbitrary subshift (X_Λ,σ) which satisfies condition (I). We denote by $\mathcal{D}_\Lambda$ the C^*-subalgebra of $\mathcal{F}_\Lambda$ consisting of all diagonal elements of $\mathcal{F}_\Lambda$ as in the previous section. In this section, we will show that the commutant of the commutative C^*-algebra $\mathfrak{D}_\Lambda$ in $\mathcal{O}_\Lambda$ is exactly the algebra $\mathcal{D}_\Lambda$.

Lemma 3.1.
$$\mathfrak{D}_\Lambda'\cap\mathcal{O}_\Lambda\subset\mathcal{F}_\Lambda.$$

Proof. Assume that $X\in\mathcal{O}_\Lambda$ commutes with each element of $\mathfrak{D}_\Lambda$. For a non empty word $\mu\in\Lambda^*$, put $X_\mu=E(S_\mu^*X)$, $X_{-\mu}=E(XS_\mu)$. We will show that $X_\mu=X_{-\mu}=0$. For $f\in\mathfrak{D}_\Lambda$, we see $E(S_\mu^*Xf)=E(S_\mu^*fS_\mu S_\mu^*X)$ so that
$$X_\mu f=S_\mu^*fS_\mu X_\mu.$$
We in particular have
$$X_\mu=X_\mu S_\mu S_\mu^*,\quad X_\mu S_\mu fS_\mu^*=fX_\mu.$$
Let i be the length of μ. It then follows that
$$X_\mu\varphi_\Lambda^i(f)=X_\mu S_\mu S_\mu^*\sum_{\nu\in\Lambda^i}S_\nu fS_\nu^*=X_\mu S_\mu fS_\mu^*.$$
Thus we obtain
$$X_\mu\varphi_\Lambda^i(f)=fX_\mu,\quad f\in\mathfrak{D}_\Lambda.$$

Now suppose that $X_\mu\neq0$. For any $\varepsilon>0$, take $X_\mu(m)\in\mathcal{F}_{m_k}^{m_l}$ such that $\|X_\mu-X_\mu(m)\|<\varepsilon$ for some $m_l\geqslant m_k\geqslant i$ and assume that $\|X_\mu\|=\|X_\mu(m)\|=1$. We then have
$$\|fX_\mu(m)-X_\mu(m)\varphi_\Lambda^i(f)\|\leqslant2\|f\|\varepsilon.$$
Since $\mathcal{O}_\Lambda$ satisfies condition (I_Λ), for $m_l\geqslant m_k$, there exists a projection $q_{m_k}^{m_l}\in\mathfrak{D}_\Lambda$ satisfying the condition (i), (ii) in condition (I_Λ). Put $Q(m)=\varphi_\Lambda^{m_k}(q_{m_k}^{m_l})\in\mathfrak{D}_\Lambda$. It is easy to see that $Q(m)$ commutes with $X_\mu(m)$. Hence we get
$$\|X_\mu(m)Q(m)-X_\mu(m)\varphi_\Lambda^i(Q(m))\|\leqslant2\varepsilon.$$
As $Q(m)$ is orthogonal to $\varphi_\Lambda^i(Q(m))$ because of condition (I_Λ), the correspondence $Y\in\mathcal{F}_{m_k}^{m_l}\to Q(m)YQ(m)\in Q(m)\mathcal{F}_{m_k}^{m_l}Q(m)$ yields an isomorphism and hence isometric by [22], Corollary 5.4. Hence we have $\|X_\mu(m)Q(m)\|=\|X_\mu(m)\|=1$ so that
$$\begin{aligned}\|X_\mu(m)Q(m)-X_\mu(m)\varphi_\Lambda^i(Q(m))\|&=\mathrm{Max}\{\|X_\mu(m)Q(m)\|,\|X_\mu(m)\varphi_\Lambda^i(Q(m))\|\}\\&\geqslant\|X_\mu(m)Q(m)\|=1.\end{aligned}$$
This is a contradiction for a sufficiently small ε. Thus we conclude $X_\mu=0$. We similarly have $X_{-\mu}=0$. This mean that $X=E(X)\in\mathcal{F}_\Lambda$. ∎

LEMMA 3.2.
$$\mathfrak{D}'_\Lambda \cap \mathcal{F}_\Lambda = \mathcal{D}_\Lambda.$$

Proof. It suffices to show the inclusion relation $\mathfrak{D}'_\Lambda \cap \mathcal{F}_\Lambda \subset \mathcal{D}_\Lambda$. Set the algebras:

$\mathcal{D}_k^l$ = The C^*-subalgebra of $\mathcal{D}_\Lambda$ generated by $S_\mu a S_\mu^*$, $\mu \in \Lambda^k$, $a \in \mathcal{A}_l$.

$\mathcal{D}_k^\infty$ = The C^*-subalgebra of $\mathcal{D}_\Lambda$ generated by $S_\mu a S_\mu^*$, $\mu \in \Lambda^k$, $a \in \mathcal{A}_\Lambda$.

$\mathfrak{D}_k$ = The C^*-subalgebra of $\mathfrak{D}_\Lambda$ generated by $S_\mu S_\mu^*$, $\mu \in \Lambda^k$.

Put $P_\mu = S_\mu S_\mu^*$ for $\mu \in \Lambda^*$. The map $\mathcal{E}_k^l$ defined by $\mathcal{E}_k^l(X) = \sum_{\mu\in\Lambda^k} P_\mu X P_\mu$ for $X \in \mathcal{F}_k^l$ yields an expectation from $\mathcal{F}_k^l$ to $\mathcal{D}_k^l$. Since the restriction of $\mathcal{E}_k^{l+1}$ to $\mathcal{F}_k^l$ coincides with $\mathcal{E}_k^l$, the sequence of the expectations $\{\mathcal{E}_k^l\}_{l\in\mathbb{N}}$ gives rise to an expectation $\mathcal{E}_k$ from $\mathcal{F}_k^\infty$ onto $\mathcal{D}_k^\infty$ such that $\mathcal{E}_k|\mathcal{F}_k^l = \mathcal{E}_k^l$. Similarly the sequence of the expectations $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ gives rise to an expectation $\mathcal{E}_\Lambda$ from $\mathcal{F}_\Lambda$ onto $\mathcal{D}_\Lambda$ such that $\mathcal{E}|\mathcal{F}_k^\infty = \mathcal{E}_k$. Now let X be an element of $\mathcal{F}_\Lambda$ which commutes with $\mathfrak{D}_\Lambda$. Since we have $\mathcal{E}_k(X) = X$ for all $k \in \mathbb{N}$, we see $\mathcal{E}(X) = X$ so that X belongs to $\mathcal{D}_\Lambda$. ∎

Therefore we obtain

PROPOSITION 3.3.
$$\mathfrak{D}'_\Lambda \cap \mathcal{O}_\Lambda = \mathcal{D}_\Lambda.$$

We also see

PROPOSITION 3.4. (i) $\mathcal{D}_\Lambda$ *is a maximal abelian* $*$*-subalgebra of* $\mathcal{O}_\Lambda$.
(ii) *There exists a faithful conditional expectation* $\mathcal{E}_\Lambda$ *from* $\mathcal{O}_\Lambda$ *onto* $\mathcal{D}_\Lambda$.

4. AUTOMORPHISMS OF $\mathcal{O}_\Lambda$ COMING FROM X_Λ

Put
$$U_\mu = \{(x_1, x_2, \dots,) \in X_\Lambda \mid x_1 = \mu_1,\, x_2 = \mu_2, \dots,\, x_k = \mu_k\}$$
the cylinder set for $\mu = \mu_1 \cdots \mu_k \in \Lambda^k$. We denote by χ_{U_μ} the characteristic function of U_μ on X_Λ. The correspondence $S_\mu S_\mu^* \to \chi_{U_\mu}$ yields an isomorphism from $\mathfrak{D}_\Lambda$ onto $C(X_\Lambda)$.

LEMMA 4.1. *Let* H_Λ *be the Hilbert space with complete orthonormal basis* $\{e_x \mid x \in X_\Lambda\}$. *Let* $T_1, \dots, T_n$ *be the operators on* H_Λ *defined by*
$$T_j e_x = \begin{cases} e_{jx} & \text{if } jx \in X_\Lambda; \\ 0 & \text{otherwise.} \end{cases}$$
Then $T_1, \dots, T_n$ *are partial isometries such that the correspondence* $S_j \to T_j$ *yields a faithful nondegenerate representation of* $\mathcal{O}_\Lambda$ *onto the* C^**-algebra generated by* $T_1, \dots, T_n$.

Proof. The assertion is easily shown from Lemma 2.3. ∎

Now suppose that $\mathcal{O}_\Lambda$ is represented on the Hilbert space H_Λ. For words $\mu \in \Lambda^k$, $\nu \in \Lambda^l$ with $k \leqslant l$, the projection $S_\mu a_\nu S_\mu^*$ exactly corresponds to the orthogonal projection onto the subspace spanned by the vectors: e_x for $x \in U_\mu \cap \sigma^{-k}(\sigma^l(U_\nu))$. In particular, for a word $\nu = \widetilde{\nu}\mu \in \Lambda^*$ with $|\widetilde{\nu}| = m$, the projection $S_\mu a_\nu S_\mu^*$ is represented by the orthogonal projection onto the subspace spanned by the vectors: e_x for $x \in \sigma^m(U_\nu)$.

For an automorphism h of X_Λ, we denote by h^* the induced automorphism of the algebra $\mathfrak{D}_\Lambda$ defined by $h^*(f) = f \circ h^{-1}$ for $f \in \mathfrak{D}_\Lambda = C(X_\Lambda)$. By Lemma 2.5, we know that the automorphism h^* of $\mathfrak{D}_\Lambda$ may be extended to an automorphism of the C^*-algebra $\mathcal{O}_\Lambda$. In the following proposition, we will give another proof of this fact and show that an extension can be taken in a homomorphic way.

Proposition 4.2. *For an automorphism h of X_Λ, there exists an automorphism α_h of $\mathcal{O}_\Lambda$ such that $\alpha_h(x) = h^*(x)$, $x \in \mathfrak{D}_\Lambda$ and the correspondence $h \in \mathrm{Aut}(X_\Lambda) \to \alpha_h \in \mathrm{Aut}(\mathcal{O}_\Lambda)$ gives rise to a homomorphism.*

Proof. We assume that $\mathcal{O}_\Lambda$ is represented on the Hilbert space H_Λ. For an automorphism h of X_Λ, put a unitary V_h on H_Λ:

$$V_h e_x = e_{h(x)}, \quad x \in X_\Lambda.$$

We will show that $\mathrm{Ad}(V_h)(\mathcal{O}_\Lambda) = \mathcal{O}_\Lambda$. Put $S_i' = \mathrm{Ad}(V_h)(S_i)$, $i = 1, \ldots, n$ so that we see for $x \in X_\Lambda$

$$S_i' e_x = \begin{cases} e_{h(ih^{-1}(x))} & \text{if } ih^{-1}(x) \in X_\Lambda; \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$Y_i = \{x \in X_\Lambda \mid ih^{-1}(x) \in X_\Lambda\}$$

so that $Y_i = h(\sigma(U_i))$. As h is a sliding block code (cf. [21]), $h(U_i)$ is a finite disjoint union of cylinder sets ([16]). Hence Y_i is of the form: $Y_i = \bigcup_{m=1}^{p} \sigma(U_{\nu_i(m)})$ for some $\nu_i(m) \in \Lambda^*$. Let P_i be the orthogonal projection on H_Λ onto the subspace corresponding to the set Y_i. Since the projection for the subset $\sigma(U_{\nu_i(m)})$ is written as $S_{\mu_i(m)} a_{\nu_i(m)} S_{\mu_i(m)}^*$ where $\nu_i(m) = \widetilde{\nu}_i(m)\mu_i(m)$ with $|\widetilde{\nu}_i(m)| = 1$, the projection P_i belongs to the algebra $\mathcal{D}_\Lambda$. For $y \in Y_i$, we denote by $h(ih^{-1}(y))_1$ the first coordinate of $h(ih^{-1}(y))$. Set

$$Y_i(j) = \{y \in Y_i \mid h(ih^{-1}(y))_1 = j\} \quad \text{for } j = 1, \ldots, n.$$

We see that

$$h^{-1}(Y_i(j)) = \{x \in X_\Lambda \mid ix \in h^{-1}(U_{\{j\}})\} \cap h^{-1}(Y_i).$$

The set $Y_i(j)$ is the intersection between Y_i and a finite union of cylinder sets. Hence the orthogonal projection corresponding to the set $Y_i(j)$ belongs to $\mathcal{D}_\Lambda$, that we denote by $P_i(j)$. For an element $x \in X_\Lambda$, x belongs to $Y_i(j)$ if and only if $e_{h(ih^{-1}(x))} = e_{jx}$ as vectors in H_Λ. Hence we have $S_i' P_i(j) = S_j P_i(j)$. Since we have $P_i = \sum_{j=1}^{n} P_i(j)$ and $P_i = S_i'^* S_i'$, it follows that $S_i' = \sum_{j=1}^{n} S_j P_i(j)$ so that $\mathrm{Ad}(V_h)(S_i)$ belongs to the algebra $\mathcal{O}_\Lambda$. We then write $\alpha_h = \mathrm{Ad}(V_h)$. It defines an automorphism of $\mathcal{O}_\Lambda$. This correspondence $h \in \mathrm{Aut}(X_\Lambda) \to \alpha_h \in \mathrm{Aut}(\mathcal{O}_\Lambda)$ gives rise to a homomorphism. ∎

We set

$$\mathrm{Aut}(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) = \{\alpha \in \mathrm{Aut}(\mathcal{O}_\Lambda) \mid \alpha(\mathfrak{D}_\Lambda) = \mathfrak{D}_\Lambda\},$$
$$\mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) = \{\alpha \in \mathrm{Aut}(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) \mid \alpha \circ \sigma^* = \sigma^* \circ \alpha \text{ on } \mathfrak{D}_\Lambda\}$$

where σ^* denotes the endomorphism $\varphi_\Lambda \left(= \sum_{j=1}^n S_j \cdot S_j^* \right)$ of $\mathfrak{D}_\Lambda$ induced by the shift σ.

As an extension on $\mathcal{O}_\Lambda$ of an automorphism h of X_Λ commutes with shift on $\mathfrak{D}_\Lambda$, we will study the group $\mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$. We first see a difference between $\mathrm{Aut}(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$ and $\mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$ as follows:

LEMMA 4.3. *An automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$ belongs to $\mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$ if and only if $\alpha(S_\mu^*)S_\nu$ belongs to $\mathcal{D}_\Lambda$ for all words $\mu, \nu \in \Lambda^*$ with $|\mu| = |\nu|$.*

Proof. We see that α commutes with φ_Λ if and only if the following equalities hold:

$$\alpha\bigg(\sum_{\mu \in \Lambda^k} S_\mu S_\gamma S_\gamma^* S_\mu^* \bigg) = \sum_{\nu \in \Lambda^k} S_\nu \alpha(S_\gamma S_\gamma^*) S_\nu^*$$

for any word $\gamma \in \Lambda^*$. The above equality is equivalent to the equality:

$$\alpha(S_\mu^* S_\mu S_\gamma S_\gamma^* S_\mu^*) S_\nu = \alpha(S_\mu)^* S_\nu \alpha(S_\gamma S_\gamma^*) S_\nu^* S_\nu$$

that is equivalent to the condition that $\alpha(S_\mu)^* S_\nu$ commutes with $\alpha(S_\gamma S_\gamma^*)$. This means that $\alpha(S_\mu)^* S_\nu$ belongs to the algebra $\mathcal{D}_\Lambda$ by Proposition 3.3. ∎

Thus we see

PROPOSITION 4.4. *For an automorphism $\alpha \in \mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$, we have*

(i) *$\alpha \circ \alpha_t = \alpha_t \circ \alpha$ for all $t \in \mathbb{R}$, where α_t is the gauge automorphism of $\mathcal{O}_\Lambda$.*

(ii) *$\alpha(\mathcal{D}_\Lambda) = \mathcal{D}_\Lambda$.*

(iii) *$\alpha \circ \lambda_\Lambda = \lambda_\Lambda \circ \alpha$ on $\mathcal{D}_\Lambda$ where λ_Λ is defined by $\lambda_\Lambda(X) = \sum_{j=1}^n S_j^* X S_j$ for $X \in \mathcal{O}_\Lambda$.*

Proof. (i) For $j, k = 1, \ldots, n$, put $f_{j,k} = \alpha(S_j)^* S_k$ that belongs to $\mathcal{D}_\Lambda$ by the previous lemma. Since $\alpha(S_j) = \sum_{k=1}^n S_k f_{j,k}^*$, it follows that

$$\alpha_t(\alpha(S_j)) = \sum_{k=1}^n \mathrm{e}^{\sqrt{-1}t} S_k f_{j,k}^* = \mathrm{e}^{\sqrt{-1}t} \alpha(S_j) = \alpha(\alpha_t(S_j)).$$

(ii) For $\mu, \nu \in \Lambda^k$ and $\gamma \in \Lambda^*$, we put $f_{\mu,\nu} = \alpha(S_\mu)^* S_\nu$, $g_\gamma = \alpha(S_\gamma^* S_\gamma) \in \mathcal{D}_\Lambda$. As the algebra $\mathfrak{D}_\Lambda$ is invariant under α, it commutes with $\alpha(\mathcal{D}_\Lambda)$. Hence we have for $\nu \neq \xi$

$$f_{\mu,\nu}^* g_\gamma f_{\mu,\xi} = S_\nu^* S_\nu S_\nu^* \alpha(S_\mu S_\gamma^* S_\gamma S_\mu^*) S_\xi S_\xi^* S_\xi = 0.$$

It follows that

$$\alpha(S_\mu a_\gamma S_\mu^*) = \sum_{\nu,\xi \in \Lambda^k} S_\nu f_{\mu,\nu}^* g_\gamma f_{\mu,\xi} S_\xi^* = \sum_{\nu \in \Lambda^k} S_\nu f_{\mu,\nu}^* g_\gamma f_{\mu,\nu} S_\nu^*.$$

This shows that $\alpha(\mathcal{D}_\Lambda) = \mathcal{D}_\Lambda$.

(iii) For $\mu, \gamma \in \Lambda^*$ with $\mu = \mu_1\mu'$, $\mu_1 = 1, \dots, n$, it follows that

$$\alpha \circ \lambda_\Lambda(S_\mu a_\gamma S_\mu^*) = \alpha(S_{\mu'} a_\gamma S_{\mu'}^*)\alpha(S_{\mu_1}^* S_{\mu_1}).$$

On the other hand, we have

$$\begin{aligned}\lambda_\Lambda \circ \alpha(S_\mu a_\gamma S_\mu^*) &= \sum_{j=1}^n S_j^* \alpha(S_{\mu_1})\alpha(S_{\mu'} a_\gamma S_{\mu'}^*)\alpha(S_{\mu_1}^*)S_j \\ &= \sum_{j=1}^n \alpha(S_{\mu'} a_\gamma S_{\mu'}^*)\alpha(S_{\mu_1}^*)S_j S_j^* \alpha(S_{\mu_1})\end{aligned}$$

because both the elements $S_j^*\alpha(S_{\mu_1}), \alpha(S_{\mu_1}^*)S_j$ belong to $\mathcal{D}_\Lambda$ by the previous lemma. Hence we have the assertion. ∎

Lemma 4.5. *If $\alpha \in \mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$ is the identity on $\mathfrak{D}_\Lambda$, it is also the identity on $\mathcal{D}_\Lambda$. Hence an extension of an automorphism of X_Λ to an automorphism of $\mathcal{D}_\Lambda$ is unique.*

Proof. Suppose that α is the identity on $\mathfrak{D}_\Lambda$. As α commutes with λ_Λ, we see for $\mu \in \Lambda^l$,

$$\alpha(S_\mu^* S_\mu) = \alpha \circ \lambda_\Lambda^l(S_\mu S_\mu^*) = \lambda_\Lambda^l \circ \alpha(S_\mu S_\mu^*) = S_\mu^* S_\mu.$$

For $\nu \in \Lambda^k$ with $k \leqslant l$, it follows that by Lemma 4.3

$$\begin{aligned}\alpha(S_\nu a_\mu S_\nu^*) &= \sum_{\xi \in \Lambda^k} S_\xi S_\xi^* \alpha(S_\nu) a_\mu \alpha(S_\nu^*) \\ &= \sum_{\xi \in \Lambda^k} S_\xi a_\mu \alpha(S_\nu^*)\alpha(S_\nu) S_\xi^* \alpha(S_\nu)\alpha(S_\nu^*) = S_\nu a_\mu S_\nu^*.\end{aligned}$$

Hence we obtain that α is the identity on $\mathcal{D}_\Lambda$. ∎

Lemma 4.6. *For an automorphism α of $\mathcal{O}_\Lambda$, its restriction to $\mathfrak{D}_\Lambda$ is the identity if and only if there exists a unitary $U_\alpha \in \mathcal{O}_\Lambda$ such that*

$$\alpha(S_i) = U_\alpha S_i, \quad i = 1, 2, \dots, n \text{ and } U_\alpha \in \mathcal{D}_\Lambda.$$

Proof. Suppose that the restriction of an automorphism α of $\mathcal{O}_\Lambda$ to the subalgebra $\mathfrak{D}_\Lambda$ is the identity. Set $U_\alpha = \sum_{i=1}^n \alpha(S_i)S_i^*$. Since the extension of an automorphism of X_Λ to an automorphism of the algebra $\mathcal{D}_\Lambda$ is unique, we see $\alpha(S_i^* S_i) = S_i^* S_i$, $i = 1, 2, \dots, n$. It follows that $U_\alpha S_i = \alpha(S_i)$ and

$$U_\alpha U_\alpha^* = \sum_{i=1}^n U_\alpha S_i S_i^* U_\alpha^* = \sum_{i=1}^n \alpha(S_i S_i^*) = 1.$$

We also have

$$U_\alpha^* U_\alpha = \sum_{i,j=1}^n S_i \alpha(S_i^*)\alpha(S_j) S_j^* = \sum_{i=1}^n S_i S_i^* = 1.$$

For a word $\mu = (\mu_1, \dots, \mu_l) \in \Lambda^*$, put $\mu' = (\mu_2, \dots, \mu_l)$. It then follows that

$$\begin{aligned} U_\alpha S_\mu S_\mu^* U_\alpha^* &= \sum_{j,k=1}^{n} \alpha(S_j) S_j^* S_\mu S_\mu^* S_k \alpha(S_k^*) \\ &= \alpha(S_{\mu_1}) S_{\mu_1}^* S_{\mu_1} S_{\mu'} S_{\mu'}^* S_{\mu_1}^* S_{\mu_1} \alpha(S_{\mu_1}^*) \\ &= \alpha(S_{\mu_1}) S_{\mu'} S_{\mu'}^* \alpha(S_{\mu_1}^*) = \alpha(S_{\mu_1} S_{\mu'} S_{\mu'}^* S_{\mu_1}^*) = S_\mu S_\mu^*. \end{aligned}$$

Hence U_α commutes with every element of $\mathfrak{D}_\Lambda$ so that it belongs to $\mathcal{D}_\Lambda$ by Proposition 3.3. The converse implication is easy. ∎

For an automorphism α of $\mathcal{O}_\Lambda$, put

$$U_\alpha(k) = \sum_{\mu \in \Lambda^k} \alpha(S_\mu) S_\mu^* \quad \text{for } k = 1, 2, \dots.$$

COROLLARY 4.7. *For an automorphism α of $\mathcal{O}_\Lambda$, its restriction to $\mathfrak{D}_\Lambda$ is the identity if and only if $U_\alpha(k)$ is a unitary in $\mathcal{D}_\Lambda$ for each $k = 1, 2, \dots$. In this case, we have*

$$U_\alpha(k+l) = U_\alpha(k) \varphi_\Lambda^k(U_\alpha(l)) \quad \textit{for } k, l = 1, 2, \dots \tag{4.1}$$

and

$$\alpha(S_\mu) = U_\alpha(k) S_\mu \quad \textit{for } \mu \in \Lambda^k,\ k = 1, 2, \dots. \tag{4.2}$$

Proof. Suppose that α is the identity on $\mathfrak{D}_\Lambda$. As in the proof of the previous lemma, we see that $U_\alpha(k)$ commutes with every element of the algebra $\mathfrak{D}_\Lambda$ so that it belongs to $\mathcal{D}_\Lambda$ by Proposition 3.3. The converse implication is direct. The identities (4.1) and (4.2) are straightforward. ∎

Let $\mathcal{U}(\mathcal{D}_\Lambda)$ be the set of all unitaries in $\mathcal{D}_\Lambda$. A unitary one-cocycle for φ_Λ is defined as a $\mathcal{U}(\mathcal{D}_\Lambda)$-valued function U from $\mathbb{N}$ satisfying

$$U(k+l) = U(k) \varphi_\Lambda^k(U(l)) \quad \text{for } k, l = 1, 2, \dots.$$

We denote by $Z_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda))$ the set of all unitary one-cocycles for φ_Λ in $\mathcal{U}(\mathcal{D}_\Lambda)$. It is an abelian group in natural way. For $U \in Z_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda))$, put

$$\lambda(U)(S_\mu) = U(k) S_\mu \quad \text{for } \mu \in \Lambda^k,\ k = 1, 2, \dots.$$

By Lemma 2.3, we see that $\lambda(U)$ yields an automorphism of the C^*-algebra $\mathcal{O}_\Lambda$ that acts identically on $\mathfrak{D}_\Lambda$. Hence λ gives rise to a map from $Z_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda))$ to $\mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$. We notice that $Z_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda))$ is regarded as the unitary group $\mathcal{U}(\mathcal{D}_\Lambda)$ by corresponding to the value at 1. We sometimes identify them.

LEMMA 4.8. *The map $\lambda : U \in Z_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda)) \to \lambda(U) \in \mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$ gives rise to an injective homomorphism.*

Proof. Since $\lambda(U)(v) = v$ for $v \in \mathcal{U}(\mathcal{D}_\Lambda)$, λ gives rise to a homomorphism. Suppose that $\lambda(U) = \mathrm{id}$ on $\mathcal{O}_\Lambda$. It follows that

$$U(1) = \sum_{j=1}^{n} \lambda(U)(S_j) S_j^* = \sum_{j=1}^{n} S_j S_j^* = 1.$$

Hence U is the unit of $Z_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda))$. ∎

Thus we have the following theorem.

Theorem 4.9. *Suppose that* (X_Λ, σ) *satisfies condition* (I). *There exists a natural short exact sequence:*

$$0 \to Z^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda)) \to \mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) \to \mathrm{Aut}(X_\Lambda) \to 0$$

that splits. Hence we have a semidirect product:

$$\mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) = \mathrm{Aut}(X_\Lambda) \cdot \mathcal{U}(\mathcal{D}_\Lambda).$$

Namely we have

Corollary 4.10. *Any automorphism of* X_Λ *is uniquely extended to an automorphism of* $\mathcal{O}_\Lambda$ *modulo unitaries in* $\mathcal{D}_\Lambda$. *That is, for an automorphism* h *of* X_Λ, *if two automorphisms* α^h, β^h *of* $\mathcal{O}_\Lambda$ *coincide with* h^* *on* X_Λ, *then* $\alpha^h = \beta^h \circ \lambda(u)$ *for some unitary* u *in* $\mathcal{D}_\Lambda$ *where* $\lambda(u) \in \mathrm{Aut}(\mathcal{O}_\Lambda)$ *is defined to be* $\lambda(u)(S_i) = uS_i$.

Now we refer a connection to the K-theory for $\mathcal{O}_\Lambda$ and $\mathcal{F}_\Lambda$.

Corollary 4.11. *Any automorphism* h *of* X_Λ *induces an automorphism* h_* *of the* K*-groups* $\mathrm{K}_*(\mathcal{O}_\Lambda)$ *and* $\mathrm{K}_0(\mathcal{F}_\Lambda)$ *such that the maps* $h \in \mathrm{Aut}(X_\Lambda) \to h_* \in \mathrm{Aut}(\mathrm{K}_*(\mathcal{O}_\Lambda))$ *and* $h \in \mathrm{Aut}(X_\Lambda) \to h_* \in \mathrm{Aut}(\mathrm{K}_0(\mathcal{F}_\Lambda))$ *give rise to homomorphisms respectively. In particular,* $h_* \in \mathrm{Aut}(\mathrm{K}_0(\mathcal{F}_\Lambda))$ *commutes with the induced automorphism* $\lambda_{\Lambda *}$ *of* $\mathrm{K}_0(\mathcal{F}_\Lambda)$.

Proof. For $U \in \mathcal{U}(\mathcal{D}_\Lambda)$, as $\lambda(U) = \mathrm{id}$ on $\mathcal{D}_\Lambda$ and hence on $\mathcal{A}_\Lambda$, the induced homomorphism $\lambda(U)_*$ on $\mathrm{K}_*(\mathcal{O}_\Lambda)$ is trivial because of [23]. Hence the assertion is clear by Theorem 4.9 with Lemma 4.5. ∎

5. OUTER AUTOMORPHISMS

If a subshift Λ is the full n-shift Λ_n, the C^*-algebra $\mathcal{O}_{\Lambda_n}$ is nothing but the Cuntz algebra $\mathcal{O}_n$ of order n. Outerness of some types of automorphisms of $\mathcal{O}_n$ have been discussed in several papers (cf. [1], [2], [8], [12], [13], [26], [28], etc.)

In this section, we will discuss on outerness of automorphisms of $\mathcal{O}_\Lambda$ coming from automorphisms of X_Λ. Let $\mathrm{Inn}(\mathcal{O}_\Lambda)$ be the set of all inner automorphisms of $\mathcal{O}_\Lambda$. We set

$$\begin{aligned}\mathrm{Inn}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) &= \mathrm{Inn}(\mathcal{O}_\Lambda) \cap \mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) \\ &= \{\mathrm{Ad}(v) \in \mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) \mid v \in \mathcal{O}_\Lambda, \text{ unitary}\}\end{aligned}$$

and

$$\mathrm{Out}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) = \mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)/\mathrm{Inn}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda).$$

Lemma 5.1. *For an automorphism* $\alpha \in \mathrm{Aut}(\mathcal{O}_\Lambda)$, *if there exists a unitary* $v \in \mathcal{O}_\Lambda$ *such that* $\alpha = \mathrm{Ad}(v)$, *then we have* $U_\alpha(k) = v\varphi^k_\Lambda(v^*)$ *for* $k \in \mathbb{N}$.

Proof. For a unitary $v \in \mathcal{O}_\Lambda$ with $\alpha = \mathrm{Ad}(v)$, it follows that for $\mu \in \Lambda^k$,

$$U_\alpha(k) S_\mu S^*_\mu = v S_\mu v^* S^*_\mu.$$

Hence we get the assertion. ∎

Now we introduce the notion of condition (I) for an automorphism of X_Λ.

DEFINITION. An automorphism $h \in \mathrm{Aut}(X_\Lambda)$ satisfies *condition* (I) if it satisfies the following condition: For any $l, k \in \mathbb{N}$ with $l \geqslant k$, there exists a projection q_k^l in $\mathfrak{D}_\Lambda$ such that:

(i) $h^*(q_k^l)a \neq 0$ for any nonzero $a \in \mathcal{A}_l$;

(ii) $h^*(q_k^l)\varphi_\Lambda^m(q_k^l) = 0$, $1 \leqslant m \leqslant k$.

Hence we see that a subshift (X_Λ, σ) satisfies condition (I) if and only if the trivial automorphism id $\in \mathrm{Aut}(X_\Lambda)$ satisfies condition (I) in the above sense.

We will first verify the following theorem.

THEOREM 5.2. *If a non-trivial automorphism $h \in \mathrm{Aut}(X_\Lambda)$ satisfies condition* (I)*, then any extension of h to an automorphism of $\mathcal{O}_\Lambda$ is always outer.*

We fix an automorphism $h \in \mathrm{Aut}(X_\Lambda)$ satisfying condition (I) and its arbitrary extension $\alpha \in \mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$ to $\mathcal{O}_\Lambda$. Suppose that α is inner in $\mathcal{O}_\Lambda$ that is implemented by a unitary $v \in \mathcal{O}_\Lambda$.

In order to prove the above theorem, we provide some lemmas.

LEMMA 5.3. *For $k, l = 1, 2, \ldots, n$, we put $X = S_k^* v S_l$. Then we have $Xf = vfv^*X$ for all $f \in \mathfrak{D}_\Lambda$.*

Proof. By Lemma 4.3, $\alpha^{-1}(S_k^*)S_l$ commutes with $\mathfrak{D}_\Lambda$. This implies that $v^*Xf = fv^*X$ for all $f \in \mathfrak{D}_\Lambda$. ∎

LEMMA 5.4. *We have $X \in \mathcal{F}_\Lambda$ and hence $v \in \mathcal{F}_\Lambda$.*

Proof. Although the proof given here is parallel to the proof of Lemma 3.1, we give it for the sake of completeness. Put $X_\mu = E(S_\mu^* X)$, $X_{-\mu} = E(XS_\mu)$ $\mu \in \Lambda^*$. We will show that $X_\mu = X_{-\mu} = 0$ for any non-empty word μ. For $f \in \mathfrak{D}_\Lambda$, as $Xf = h^*(f)X$ by the above lemma, it follows that

$$X_\mu f = E(S_\mu^* X f) = S_\mu^* h^*(f) S_\mu X_\mu.$$

Put $i = |\mu|$ so that we see

$$X_\mu \varphi_\Lambda^i(f) = S_\mu^* \varphi_\Lambda^i(h^*(f)) S_\mu X_\mu = S_\mu^* S_\mu h^*(f) S_\mu^* S_\mu X_\mu = h^*(f) X_\mu.$$

Now suppose that $X_\mu \neq 0$. For $\varepsilon > 0$, take $X_\mu(m) \in \mathcal{F}_{k_m}^{l_m}$ with $l_m \geqslant k_m \geqslant i$ such that $\|X_\mu - X_\mu(m)\| < \varepsilon$. We may assume $\|X_\mu\| = \|X_\mu(m)\| = 1$. It then follows that

$$\|h^*(f)X_\mu(m) - X_\mu(m)\varphi_\Lambda^i(f)\| \leqslant 2\varepsilon\|f\|.$$

As h satisfies condition (I), there exists a projection q_m in $\mathfrak{D}_\Lambda$ such that

(i) $h^*(q_m)a \neq 0$ for any nonzero $a \in \mathcal{A}_{l_m}$;

(ii) $h^*(q_m)\varphi_\Lambda^j(q_m) = 0$, $1 \leqslant j \leqslant k_m$.

Put $Q_m = \varphi_\Lambda^{k_m}(q_m)$. Both of the projections $h^*(Q_m), \varphi_\Lambda^i(Q_m)$ belong to $\varphi_\Lambda^{k_m}(\mathfrak{D}_\Lambda)$ so that $h^*(Q_m), \varphi_\Lambda^i(Q_m)$ commute with $\mathcal{F}_{k_m}^{l_m}$. Since we see

$$h^*(Q_m)\varphi_\Lambda^i(Q_m) = 0,$$

it follows that

$$\begin{aligned}\|h^*(Q_m)X_\mu(m) - X_\mu(m)\varphi_\Lambda^i(Q_m)\| &= \mathrm{Max}\{\|h^*(Q_m)X_\mu(m)\|, \|X_\mu(m)\varphi_\Lambda^i(Q_m)\|\} \\ &\geqslant \|h^*(Q_m)X_\mu(m)\|.\end{aligned}$$

By using a similar manner to the proof of [22], Corollary 5.4, we see the mapping

$$X \in \mathcal{F}_{k_m}^{l_m} \to h^*(Q_m)Xh^*(Q_m) \in h^*(Q_m)\mathcal{F}_{k_m}^{l_m}h^*(Q_m)$$

is an isomorphism. Hence we have $\|h^*(Q_m)X_\mu(m)\| = \|X_\mu(m)\| = 1$. This is a contradiction for sufficiently small ε. Thus we conclude that $X_\mu = 0$ and similarly $X_{-\mu} = 0$ so that $X \in \mathcal{F}_\Lambda$. We also see that $v \in \mathcal{F}_\Lambda$ because of the identity $v = \sum_{k,l=1}^{n} S_k S_k^* v S_l S_l^*$. ∎

Lemma 5.5. *For any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for any word $\mu \in \Lambda^k$ we have*

$$\|vS_\mu^* S_\mu - b_\mu\| < \varepsilon \quad \textit{for some } b_\mu \in \mathcal{D}_\Lambda.$$

Proof. By the above lemma, for $\varepsilon > 0$, take $v_m \in \mathcal{F}_{k_m}^{l_m}$ with $l_m \geqslant k_m$ such that $\|v - v_m\| < \varepsilon$. Put $k = k_m$. For any $\mu, \nu \in \Lambda^k$, we have $S_\nu^* v_m S_\mu \in \mathcal{A}_\Lambda$. As $\alpha(S_\mu^*)S_\nu \in \mathcal{D}_\Lambda$, we see $vS_\mu^* v^* S_\nu \cdot S_\nu^* v_m S_\mu$ belongs to $\mathcal{D}_\Lambda$. Put $b_\mu = vS_\mu^* v^* v_m S_\mu$ that belongs to $\mathcal{D}_\Lambda$. Hence we get $\|vS_\mu^* S_\mu - b_\mu\| < \varepsilon$. ∎

Proof of Theorem 5.2. Keep the above notation. It suffices to show that the unitary v belongs to the algebra $\mathcal{D}_\Lambda$. For any $\varepsilon > 0$, take $k \in \mathbb{N}$ such that for a word $\mu \in \Lambda^k$ there exists an element $b_\mu \in \mathcal{D}_\Lambda$ as above. For any $a \in \mathfrak{D}_\Lambda$, we have

$$\|(av - va)S_\mu^* S_\mu\| \leqslant \|a(vS_\mu^* S_\mu - b_\mu)\| + \|(b_\mu - vS_\mu^* S_\mu)a\| \leqslant 2\varepsilon.$$

Let $f_1^k, \dots, f_{n(k)}^k$ be the set of all nonzero minimal projections in the commutative C^*-algebra generated by projections a_μ for $\mu \in \Lambda^k$. As $\sum_{i=1}^{n(k)} f_i^k = 1$, we have

$$f_i^k(av - va)^*(av - va)f_j^k = 0 \quad \text{for } i \neq j$$

so that we see

$$\|av - va\|^2 = \left\| \sum_{i=1}^{n(k)} (av - va)f_i^k \right\|^2 = \max_{1 \leqslant i \leqslant n(k)} \|(av - va)f_i^k\|^2.$$

Since f_i^k is majorized by a projection of the form $S_\mu^* S_\mu$ for some $\mu \in \Lambda^k$. We obtain that

$$\|av - va\| \leqslant 2\varepsilon.$$

Now $a \in \mathcal{D}_\Lambda$ is independent of ε and hence $v \in \mathfrak{D}_\Lambda' \cap \mathcal{F}_\Lambda$. This implies $v \in \mathcal{D}_\Lambda$ and the homeomorphism h is trivial. ∎

We next introduce some condition, called (D), for subshifts that guarantee condition (I) for all non-trivial automorphism of X_Λ. A subshift (X_Λ, σ) satisfies *condition* (D) if for any $l \in \mathbb{N}$, there exists $N_l \in \mathbb{N}$ such that for any $x \in X_\Lambda$, there exists $y \in X_\Lambda$ such that $y \neq x$, $y \sim_l x$ and $\sigma^{N_l}(x) = \sigma^{N_l}(y)$.

This condition is clearly a stronger condition than condition (I) for subshifts. But the following proposition shows that it is not a so strong condition.

PROPOSITION 5.6. *Suppose that X_Λ is not a single point. If X_Λ is aperiodic in past equivalence, then it satisfies condition* (D).

To prove the above proposition, we need the following lemma.

LEMMA 5.7. *Suppose that X_Λ is not a single point. If X_Λ is aperiodic in past equivalence, there exists $K \in \mathbb{N}$ such that for any $z \in X_\Lambda$ there are words $\mu, \nu \in \Lambda^K$ satisfying*

$$\mu \neq \nu \quad \textit{and} \quad \mu z, \nu z \in X_\Lambda.$$

Proof. If Λ is a full shift, the assertion is clear. Suppose that Λ is not a full shift. Take $l \in \mathbb{N}$ and $a, b \in X_\Lambda$ such that a is not l-past equivalent to b. Since X_Λ is aperiodic in past equivalence, we find $K \in \mathbb{N}$ such that for any $z \in X_\Lambda$, there are words $\mu_a, \mu_b \in \Lambda^K$ satisfying $\mu_a z \sim_l a$, $\mu_b z \sim_l b$ so that we see $\mu_a \neq \mu_b$. ∎

Proof of Proposition 5.6. For any $l \in \mathbb{N}$, as X_Λ is aperiodic in past equivalence, take $N \in \mathbb{N}$ as in the property of aperiodicity in past equivalence and $K \in \mathbb{N}$ as in the above lemma. Set $N_l = N + K$. For any $x \in X_\Lambda$, put $\gamma = x_{[1,N]}$, $\xi = x_{[N+1,N+K]}$ and $x' = x_{[N+K+1,\infty)}$. By the above lemma, there exist distinct words $\mu, \nu \in \Lambda^K$ with $\mu x', \nu x' \in X_\Lambda$. We may assume that $\mu \neq \xi$ (otherwise $\nu \neq \xi$). Put $y' = \mu x' \in X_\Lambda$. Since X_Λ is aperiodic in past equivalence, we may find $\eta \in \Lambda^N$ with $x \sim_l \eta y'$. Set $y = \eta y' \in X_\Lambda$. Thus we see that

$$x \neq y, \quad x \sim_l y \quad \text{and} \quad \sigma^{N_l}(x) = \sigma^{N_l}(y). \qquad ∎$$

We will show that every non-trivial automorphism on X_Λ satisfies condition (I) under the condition (D) for the subshift.

The following lemma is direct.

LEMMA 5.8. *A subshift X_Λ satisfies condition* (D) *if and only if it satisfies the following condition:*

For any pair $l, m \in \mathbb{N}$, there exists $N_{l,m} \in \mathbb{N}$ such that for any $x \in X_\Lambda$, there exists $y \in X_\Lambda$ such that

(i) $x_{[1,m]} = y_{[1,m]}$ *and* $x_{[m+N_{l,m}+1,\infty)} = y_{[m+N_{l,m}+1,\infty)}$;

(ii) $x_{[m+1,m+N_{l,m}]} \neq y_{[m+1,m+N_{l,m}]}$;

(iii) $x \sim_l y$.

For $l \in \mathbb{N}$, let $F_1^l, \dots, F_{m(l)}^l$ be the set of all l-past equivalence classes in X_Λ. Hence we have a decomposition of X_Λ: $\bigcup_{i=1}^{m(l)} F_i^l = X_\Lambda$.

LEMMA 5.9. *Suppose that X_Λ satisfies condition* (D). *Then for an automorphism $h \in \mathrm{Aut}(X_\Lambda)$ and a natural number $l \in \mathbb{N}$ and $i = 1, 2, \dots, m(l)$, there exists $y \in F_i^l$ such that*

$$\sigma^m(y) \neq h(y) \quad \textit{for } 1 \leqslant m \leqslant l.$$

Proof. Fix l and $i = 1, \dots, m(l)$. Take $x \in F_i^l$ and suppose that $\sigma(x) = h(x)$. By condition (D), there exists $N_l \in \mathbb{N}$ satisfying the property of (D). Put $\mu = x_{[1,N_l]}$ and take $\mu' \in \Lambda^{N_l}$ such that $\mu \neq \mu'$, and $\mu'\sigma^{N_l}(x)$ is admissible in X_Λ and $\mu'\sigma^{N_l}(x) \sim_l \mu\sigma^{N_l}(x)(= x)$. Put $x' = \mu'\sigma^{N_l}(x) \in X_\Lambda$. As $\mu \neq \mu'$ and $\sigma^{N_l}(x') = \sigma^{N_l}(x)$, we obtain that $\sigma(x') \neq h(x')$. We in fact see that if

$\sigma(x') = h(x')$, $\sigma^{N_l}(x') = h^{N_l}(x')$. Hence $h^{N_l}(x') = h^{N_l}(x)$ because $h(x) = \sigma(x)$. This is a contradiction for $x \neq x'$. Therefore we find an element $x' \in F_i^l$ such that $\sigma(x') \neq h(x')$. Put $x(1) = x'$.

We will next see that there exists $x(2) \in F_i^l$ such that

$$\sigma(x(2)) \neq h(x(2)), \quad \sigma^2(x(2)) \neq h(x(2)).$$

If $\sigma^2(x(1)) \neq h(x(1))$, we may take $x(2)$ as $x(1)$. Suppose that $\sigma^2(x(1)) = h(x(1))$. As h and σ are uniformly continuous on X_Λ, there exists $m_1 \in \mathbb{N}$ such that for $y \in X_\Lambda$, if $x(1)_{[1,m_1]} = y_{[1,m_1]}$, then $\sigma(y) \neq h(y)$. By Lemma 5.8, there exists $N_{l,m_1} \in \mathbb{N}$ and $y \in X_\Lambda$ such that

(i) $x(1)_{[1,m_1]} = y_{[1,m_1]}$ and $x(1)_{[m_1+N_{l,m_1}+1,\infty)} = y_{[m_1+N_{l,m_1}+1,\infty)}$;

(ii) $x(1)_{[m_1+1,m_1+N_{l,m_1}]} \neq y_{[m_1+1,m_1+N_{l,m_1}]}$;

(iii) $x(1) \sim_l y$.

Hence we see $y \in F_i^l$ and $\sigma(y) \neq h(y)$. If $\sigma^2(y) = h(y)$, we have, by the above condition (i) and the condition $\sigma^2(x(1)) = h(x(1))$, $h^{m_1+N_{l,m_1}}(x(1)) = h^{m_1+N_{l,m_1}}(y)$ a contradiction to $x(1) \neq y$. Therefore we obtain $\sigma^2(y) \neq h(y)$. Thus by putting $x(2) = y$, we have

$$x(2) \in F_i^l, \quad \sigma(x(2)) \neq h(x(2)) \quad \text{and} \quad \sigma^2(x(2)) \neq h(x(2)).$$

By continuing similar arguments to the above, we may take, for any $n \in \mathbb{N}$, an element $x(n) \in F_i^l$ such that $\sigma^k(x(n)) \neq h(x(n))$ for all $1 \leqslant k \leqslant n$. ∎

Lemma 5.10. *Suppose that X_Λ satisfies condition* (D). *Then for an automorphism $h \in \mathrm{Aut}(X_\Lambda)$ and natural numbers $l, k \in \mathbb{N}$ with $l \geqslant k$, there exists $y_i^l \in F_i^l$ for each $i = 1, 2, \dots, m(l)$ such that*

$$\sigma^m(y_i^l) \neq h(y_j^l) \quad \textit{for all } 1 \leqslant m \leqslant k \textit{ and } i, j = 1, 2, \dots, m(l).$$

Proof. For $i = 1$, by the previous lemma, we may find $y_1^l \in F_1^l$ such that $\sigma^n(y_1^l) \neq h(y_1^l)$ for all $1 \leqslant n \leqslant k$. Similarly find $x_2^l \in F_2^l$ such that

$$\sigma^n(x_2^l) \neq h(x_2^l) \quad \text{for } 1 \leqslant n \leqslant k. \tag{5.1}$$

By uniformly continuity for h, σ, there exists $K_{2,1} \in \mathbb{N}$ such that if $y \in F_2^l$ satisfies $x^l_{2[1,K_{2,1}]} = y_{[1,K_{2,1}]}$, then $\sigma^n(y) \neq h(y)$ for $1 \leqslant n \leqslant k$. Now the subshift X_Λ satisfies condition (D) so that there exists $z_2^l \in F_2^l$ satisfing $z^l_{2[1,K_{2,1}]} = x^l_{2[1,K_{2,1}]}$ and $(z_2^l)_N \neq (x_2^l)_N$ for some $N > K_{2,1}$. If $\sigma(x_2^l) = h(y_1^l)$, we see $\sigma(z_2^l) \neq h(y_1^l)$. Hence we may find $z_2^l \in F_2^l$ such that

$$\sigma^n(z_2^l) \neq h(z_2^l) \quad \text{for } 1 \leqslant n \leqslant k \quad \text{and} \quad \sigma(z_2^l) \neq h(y_1^l). \tag{5.2}$$

By using (5.2) instead of (5.1), a similar argument to the above one shows that there exists an element $w_2^l \in F_2^l$ such that

$$\sigma^n(w_2^l) \neq h(w_2^l) \quad \text{for } 1 \leqslant n \leqslant k \quad \text{and} \quad \sigma(w_2^l) \neq h(y_1^l), \quad \sigma^2(w_2^l) \neq h(y_1^l).$$

By repeating these procedure, we may find $u_2^l \in F_2^l$ such that

$$\sigma^n(u_2^l) \neq h(u_2^l), \quad \sigma^n(u_2^l) \neq h(y_1^l) \quad \text{for } 1 \leqslant n \leqslant k. \tag{5.3}$$

We next choose an element $v_2^l \in F_2^l$ from (5.3) such that

$$\sigma^n(v_2^l) \neq h(v_2^l), \quad \sigma^n(v_2^l) \neq h(y_1^l) \quad \text{for } 1 \leqslant n \leqslant k \quad \text{and} \quad \sigma(y_1^l) \neq h(v_2^l)$$

by using a similar argument to the preceding one. By repeating these procedure several times, we finally take an element $y_2^l \in F_2^l$ such that

$$\sigma^n(y_i^l) \neq h(y_j^l) \quad \text{for all } 1 \leqslant n \leqslant k,\ i,j = 1,2.$$

Consequently we may find elements $y_i^l \in F_i^l$ for $i = 1, \dots, m(l)$ that satisfy the required condition by similar procedures. ∎

We thus have

PROPOSITION 5.11. *Suppose that X_Λ satisfies condition* (D)*. Then any automorphism $h \in \mathrm{Aut}(X_\Lambda)$ satisfies condition* (I).

Proof. For any $l, k \in \mathbb{N}$ with $l \geqslant k$, we will first find a projection p_k in $\mathfrak{D}_\Lambda$ satisfying the following conditions:

(i) $p_k a \neq 0$ for any nonzero $a \in \mathcal{A}_l$;

(ii) $p_k \varphi_\Lambda^m(h^{*-1} p_k) = 0$, $1 \leqslant m \leqslant k$.

For any $l, k \in \mathbb{N}$ with $l \geqslant k$, take $y_i^l \in F_i^l$ as in the previous lemma. Put $Y = \{y_i^l \mid i = 1, \dots, m(l)\} \subset X_\Lambda$. As we see $\sigma^{-m}(h(Y)) \cap Y = \emptyset$ for $1 \leqslant m \leqslant k$, there exists a clopen set V, that includes Y, such that $\sigma^{-m}(h(V)) \cap V = \emptyset$ for $1 \leqslant m \leqslant k$. Let p_k be the characteristic function of V on X_Λ. The projection p_k satisfies the above conditions (i), (ii). We then put $q_k^l = h^{*-1}(p_k)$ that satisfies the required conditions for condition (I). ∎

We reach the following theorem

THEOREM 5.12. *Suppose that X_Λ satisfies the condition* (D)*. Then any extension of a non-trivial automorphism of the subshift X_Λ to an automorphism of the C^*-algebra $\mathcal{O}_\Lambda$ is outer.*

Let X_A be the one-sided topological Markov shift determined by an $n \times n$ square matrix A with entries in $\{0, 1\}$. If A is an aperiodic matrix, the subshift X_Λ is aperiodic in past equivalence and hence satisfies condition (D). Thus we have

COROLLARY 5.13. *For an aperiodic matrix A with entries in $\{0, 1\}$, any extension of a non-trivial automorphism of the topological Markov shift X_A to an automorphism of the Cuntz-Krieger algebra $\mathcal{O}_A$ is outer.*

A coboundary U is defined as a $\mathcal{U}(\mathcal{D}_\Lambda)$-valued function U from $\mathbb{N}$ such that there exists $v \in \mathcal{U}(\mathcal{D}_\Lambda)$ such that

$$U(k) = v \varphi_\Lambda^k(v^*) \quad \text{for } k = 1, 2, \dots.$$

We denote by $B_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda))$ the set of all coboundaries in $\mathcal{U}(\mathcal{D}_\Lambda)$. It is a subgroup of $Z_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda))$. If we identify $Z_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda))$ with $\mathcal{U}(\mathcal{D}_\Lambda)$, we can regard $B_\sigma^1(\mathcal{U}(\mathcal{D}_\Lambda))$ as the set of all unitaries U in $\mathcal{U}(\mathcal{D}_\Lambda)$ that is of the form

$$U = v \varphi_\Lambda(v^*) \quad \text{for some unitary } v \in \mathcal{U}(\mathcal{D}_\Lambda).$$

We recall that for a unitary $U \in \mathcal{U}(\mathcal{D}_\Lambda)$, an automorphism $\lambda(U)$ of $\mathcal{O}_\Lambda$ is defined as $\lambda(U)(S_i) = US_i$, $i = 1, \dots, n$ that gives rise to an element of $\mathrm{Aut}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$.

Lemma 5.14. *For a unitary $U \in \mathcal{U}(\mathcal{D}_\Lambda)$, the automorphism $\lambda(U)$ is of the form $\lambda(U) = \mathrm{Ad}(v)$ for some unitary $v \in \mathcal{O}_\Lambda$ if and only if $v \in \mathcal{U}(\mathcal{D}_\Lambda)$ and*

$$U = v\varphi_\Lambda(v^*).$$

Proof. Suppose that $\lambda(U) = \mathrm{Ad}(v)$ for some unitary $v \in \mathcal{O}_\Lambda$. Since $\lambda(U)$ is the identity on $\mathfrak{D}_\Lambda$, v commutes with every element of $\mathfrak{D}_\Lambda$ so that v belongs to the algebra $\mathcal{D}_\Lambda$ by Proposition 3.3. The condition $\lambda(U)(S_i) = \mathrm{Ad}(v)(S_i)$, $i = 1, \ldots, n$ is equivalent to the condition $US_ivS_i^* = vS_iS_i^*$. That is also equivalent to the condition $\sum_{i=1}^{n} S_ivS_i^* = U^*v$ that is nothing but $U = v\varphi_\Lambda(v^*)$. ∎

We thus have

Proposition 5.15. *For a unitary $U \in \mathcal{U}(\mathcal{D}_\Lambda)$, the automorphism $\lambda(U)$ belongs to $\mathrm{Inn}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda)$ if and only if U belongs to $B^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda))$.*

Now we set

$$H^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda)) = Z^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda))/B^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda))$$

the one-cohomology group. Therefore we conclude

Theorem 5.16. *Suppose that a subshift (X_Λ, σ) satisfies condition* (D). *There exists a natural short exact sequence:*

$$0 \to H^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda)) \to \mathrm{Out}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) \to \mathrm{Aut}(X_\Lambda) \to 0$$

that splits. Hence we have a semidirect product:

$$\mathrm{Out}_\sigma(\mathcal{O}_\Lambda, \mathfrak{D}_\Lambda) = \mathrm{Aut}(X_\Lambda) \cdot \mathcal{U}(\mathcal{D}_\Lambda)/B^1_\sigma(\mathcal{U}(\mathcal{D}_\Lambda)).$$

Proof. The above exact sequence is induced by the exact sequence in Theorem 4.9 and Proposition 5.15. ∎

6. EXAMPLES

In this section, we will present some examples of automorphisms of $\mathcal{O}_\Lambda$ coming from automorphisms of certain subshifts X_Λ. In [3], Boyle–Franks–Kitchens have studied automorphisms of one-sided topological Markov shifts. We will use some of their results in [3].

Example 6.1. The full 2-shift Λ_2.

It is known that the automorphism group $\mathrm{Aut}(X_2)$ of the one-sided full 2-shift X_2 is the group $\mathbb{Z}/2\mathbb{Z}$ (cf. [16], [3]). The non-trivial element is the flip-flop s_{12} that interchanges the symbols 1 and 2. Let α_{12} be the automorphism of the Cuntz algebra $\mathcal{O}_2$ defined by

$$\alpha_{12}(S_1) = S_2, \quad \alpha_{12}(S_2) = S_1.$$

It is an extension of s_{12} and hence outer by Corollary 5.13. The outerness of the automorphism was first proved by Archbold in [2]. The discussion has been generalized in [12], [26] and [18].

EXAMPLE 6.2. The topological Markov shift determined by the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

It was proved in [3] that the automorphism group $\mathrm{Aut}(X_A)$ of the one-sided topological Markov shift X_A is isomorphic to the group $\mathfrak{S}_3$ of all permutations of order 3. By the calculation formula for the K_0-group $\mathrm{K}_0(\mathcal{O}_A)$ of the Cuntz-Krieger algebra $\mathcal{O}_A$ in [7], we know that the group $\mathrm{K}_0(\mathcal{O}_A)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ through the correspondences:

$$[1] = (0,0), \quad [S_1 S_1^*] = (1,0), \quad [S_2 S_2^*] = (0,1), \quad [S_3 S_3^*] = (1,1).$$

Let $s_{(ijk)} \in \mathfrak{S}_3$ be the permutation given by $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$. Put

$$\alpha_{(ijk)}(S_1) = S_i, \quad \alpha_{(ijk)}(S_2) = S_j, \quad \alpha_{(ijk)}(S_3) = S_k.$$

Then $\alpha_{(ijk)}$ gives rise to an automorphism of $\mathcal{O}_A$ that is an extension of an automorphism of X_A induced by the permutation $s_{(ijk)}$ of the symbols. It is an outer automorphism of $\mathcal{O}_A$ by Corollary 5.13 or by [18]. Such automorphisms of $\mathcal{O}_A$ yields automorphisms of $\mathrm{K}_0(\mathcal{O}_A)$ so that we see a natural isomorphism between $\mathrm{Aut}(X_A)$ and $\mathrm{Aut}(\mathrm{K}_0(\mathcal{O}_A))$ (cf. Corollary 4.11).

EXAMPLE 6.3 The full 3-shift Λ_3.

Boyle-Franks-Kitchens in [3] showed that, for $n > 2$, the automorphism group $\mathrm{Aut}(X_n)$ of the one-sided full n-shift X_n is infinite. We now treat automorphisms of the full 3-shift X_3. For $k = 1, 2, \dots$, let τ_k be an automorphism of X_3 defined by exchanging words:

$$\tau_k(3\underbrace{2\cdots 2}_{k \text{ times}}) = 1\underbrace{2\cdots 2}_{k \text{ times}}, \quad \tau_k(1\underbrace{2\cdots 2}_{k \text{ times}}) = 3\underbrace{2\cdots 2}_{k \text{ times}}$$

and τ_k identically acts on other words in X_3. Put

$$\begin{aligned} \alpha_{\tau_k}(S_2) &= S_2, \\ \alpha_{\tau_k}(S_3) &= S_1 P_{2^k} + S_3(1 - P_{2^k}), \\ \alpha_{\tau_k}(S_1) &= S_3 P_{2^k} + S_1(1 - P_{2^k}), \end{aligned}$$

where $P_{2^k} = \underbrace{S_2 \cdots S_2}_{k \text{ times}} \underbrace{S_2^* \cdots S_2^*}_{k \text{ times}}$. It is easy to see that α_{τ_k} yields an automorphism of the Cuntz algebra $\mathcal{O}_3$ that is an extension of τ_k. The automorphisms are outer by Corollary 5.13 or by [26], Theorem 1.

We finally remark on outerness of the automorphisms $\lambda(u)$ of $\mathcal{O}_\Lambda$ coming from unitaries u of $\mathcal{U}(\mathcal{D}_\Lambda)$. Suppose that a subshift X_Λ satisfies condition (I). We denote by $\mathrm{Per}_\sigma^n(X_\Lambda)$ the set of all n periodic points of X_Λ under the shift σ. The following proposition is directly seen from Lemma 5.14.

PROPOSITION 6.4. *For a unitary u in $\mathcal{D}_\Lambda$ if there exists a point x in $\mathrm{Per}_\sigma^n(X_\Lambda)$ for some $n \in \mathbb{N}$, such that $u(x) \neq u^*(\sigma^{n-1}(x))u^*(\sigma^{n-2}(x)) \cdots u^*(\sigma(x))$, the automorphism $\lambda(u)$ is outer in $\mathcal{O}_\Lambda$. In particular, if u is a complex number z with modulus one such that $z^n \neq 1$ and $\mathrm{Per}_\sigma^n(X_\Lambda)$ is not empty, then the automorphism $\lambda(z)$ defined by $\lambda(z)(S_i) = zS_i$ is outer.*

COROLLARY 6.5. *If there exists a fixed point in X_Λ for σ, the gauge action α of $\mathcal{O}_\Lambda$ is an outer action of the one dimensional torus group $\mathbb{T}$.*

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KENGO MATSUMOTO
Department of Mathematics
Joetsu University of Education
Joetsu 943–8512
JAPAN

E-mail: matsu@juen.ac.jp

Current address:

Department of Mathematical Sciences
Yokohama City University
22–2 Seto, Kanazawa-ku
Yokohama, 236–0027
JAPAN

E-mail: matsu@math.yokohama-cu.ac.jp

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