J. OPERATOR THEORY
44(2000), 139–150

SEMIGROUP CROSSED PRODUCTS AND THE STRUCTURE OF TOEPLITZ ALGEBRAS

SRIWULAN ADJI

Communicated by Norberto Salinas

ABSTRACT. Suppose Γ is a totally ordered discrete abelian group, and I is an ordered ideal in Γ. We show that the crossed product $A \times \Gamma^+$ by an action inflated from one Γ/I is isomorphic to the induced algebra $\mathrm{Ind}_{I^\perp}^{\widehat{\Gamma}} A \times (\Gamma/I)^+$. Using this we show how the $\widehat{\Gamma}$-invariant ideals in the Toeplitz algebra of Γ are determined by the order ideals in Γ.

KEYWORDS: *$C*$-algebra, endomorphism, ordered group, covariant representation, crossed product, semigroup of isometries, Toeplitz algebra.*

MSC (2000): Primary 46L55; Secondary 06F15, 47B35.

1. INTRODUCTION

Suppose N is a normal subgroup of a group G. Attempts to understand the relationship between the group C^*-algebra $C^*(G)$ and the smaller group algebras $C^*(N)$ and $C^*(G/N)$ have motivated many theorems about crossed products and twisted crossed products by automorphic actions of groups. Here we consider an analogous problem for the Toeplitz algebra $\mathcal{T}(\Gamma)$ of a totally ordered abelian group Γ: given an ordered ideal I in Γ, how is $\mathcal{T}(\Gamma)$ related to $\mathcal{T}(\Gamma/I)$?

Murphy has shown in [10] that if Γ is a lexicographic direct sum $\Gamma_1 \bigoplus_{\mathrm{lex}} \Gamma_2$ and I is the ideal Γ_2 in Γ, then there is a natural surjection of $\mathcal{T}\Big(\Gamma_1 \bigoplus_{\mathrm{lex}} \Gamma_2\Big)$ onto the algebra $C(\widehat{\Gamma}_2, \mathcal{T}(\Gamma_1)) = C(\Gamma_2, \mathcal{T}(\Gamma/\Gamma_2))$, and he identified generators for the kernel. Our main theorem is a similar structure theorem for an arbitrary order ideal: the surjection takes values in an induced C^*-algebra rather than an algebra of continuous functions, but the kernel has the same generators as Murphy's.

Our strategy is to use the realisation of $\mathcal{T}(\Gamma)$ as a semigroup crossed product $B_{\Gamma^+} \times_\tau \Gamma^+$ from [3], the basic structure theory of semigroup crossed products developed in [2], and a new description of semigroup crossed products by actions inflated from a quotient; this new theorem is modelled on a theorem of Olesen and Pedersen for crossed products by actions of abelian goups ([11]).

We begin with our discussion of the crossed product of an inflated system, and prove that it is isomorphic to an induced algebra (Theorem 2.1). Our proof is quite different from that of Olesen and Pedersen ([11]), and uses a characterisation of induced C^*-algebras due to Echterhoff ([6]), as formulated in [12]. In Section 3, we focus on the semigroup crosssed product $B_{\Gamma^+} \times_\tau \Gamma^+$ which was shown in [3] to be isomorphic to the Toeplitz algebra $\mathcal{T}(\Gamma)$, and use Theorem 2.1 to prove our main structure theorem for $\mathcal{T}(\Gamma)$. In Section 4, we use similar techniques to give a complete classification of the invariant ideals in the commutator ideal of $\mathcal{T}(\Gamma)$.

2. CROSSED PRODUCTS OF INFLATED SYSTEMS

Let Γ be a totally ordered discrete abelian group with positive cone Γ^+ and identity e. Suppose I is an order ideal of Γ: that is, a subgroup of Γ such that $e \leqslant x \leqslant y \in I$ implies $x \in I$. The quotient group of Γ/I is a totally ordered group with

$$[x] < [y] \iff \text{there exists } z \in I \text{ such that } x < y + z;$$

the positive cone$(\Gamma/I)^+$ is the image of Γ^+ under the quotient map $q : x \mapsto [x]$.

An endomorphism ϕ of a C^*-algebra A is called *extendible* if it extends to a strictly continuous endomorphism $\overline{\phi}$ of the multiplier algebra $M(A)$; this happens precisely when there is an approximate identity (e_λ) for A and a projection $p \in M(A)$ such that $\phi(e_\lambda)$ converges strictly to p in $M(A)$. Now let α be an action of $(\Gamma/I)^+$ on A by extendible endomorphisms. Then $\beta := \alpha \circ q$ is an action of Γ^+ by extendible endomorphisms; we say that the system (A, Γ^+, β) is *inflated* from $(A, (\Gamma/I)^+, \alpha)$.

Let $(A \times_\beta \Gamma^+, i_A, i_{\Gamma^+})$ and $(A \times_\alpha (\Gamma/I)^+, j_A, j_{(\Gamma/I)^+})$ denote the crossed products, as in [3]. Both carry dual actions; for example, $\widehat{\beta} : \widehat{\Gamma} \to \operatorname{Aut}(A \times_\beta \Gamma^+)$ is characterised by $\widehat{\beta}_\gamma(i_A(a)) = i_A(a)$ and $\widehat{\beta}_\gamma(i_{\Gamma^+}(x)) = \gamma(x) i_{\Gamma^+}(x)$ for $\gamma \in \widehat{\Gamma}$. The pair $(j_A, j_{(\Gamma/I)^+} \circ q)$ is a covariant representation of (A, Γ^+, β) in $A \times_\alpha (\Gamma/I)^+$, so there is a nondegenerate homomorphism $Q : A \times_\beta \Gamma^+ \to A \times_\alpha (\Gamma/I)^+$ such that $Q \circ i_A = j_A$ and $Q \circ i_{\Gamma^+} = j_{(\Gamma/I)^+} \circ q$. Since the range of Q is a C^*-subalgebra of $A \times_\alpha (\Gamma/I)^+$ containing all the generators, Q is surjective.

Our first theorem shows that we can realise $A \times_\beta \Gamma^+$ as the induced C^*-algebra $\operatorname{Ind}_{I^\perp}^{\widehat{\Gamma}} A \times_\alpha (\Gamma/I)^+$, which by definition consists of the continuous functions $f : \widehat{\Gamma} \to A \times_\alpha (\Gamma/I)^+$ satisfying $f(\gamma\mu) = \widehat{\beta}_\mu^{-1}(f(\gamma))$ for $\mu \in I^\perp$. The induced algebra carries a natural action $\operatorname{Ind} \widehat{\alpha}$ of $\widehat{\Gamma}$ such that $(\operatorname{Ind} \widehat{\alpha})_\gamma(f)(\chi) = f(\gamma^{-1}\chi)$. For a discussion of induced C^*-algebras and their properties, we refer to Section 6.3 of [13].

THEOREM 2.1. *There is an isomorphism* Ψ *of the dual system* $(A\times_\beta\Gamma^+,\widehat{\Gamma},\widehat{\beta})$ *onto the induced system* $(\mathrm{Ind}_{I^\perp}^{\widehat{\Gamma}}(A\times_\alpha(\Gamma/I)^+),\widehat{\Gamma},\mathrm{Ind}\,\widehat{\alpha})$ *such that* $\Psi(\xi)(\gamma) = Q(\widehat{\beta}_\gamma^{-1}(\xi))$ *for* $\xi\in A\times_\beta\Gamma^+$ *and* $\gamma\in\widehat{\Gamma}$.

We are going to apply a theorem of Echterhoff ([6]) to $(A\times_\beta\Gamma^+,\widehat{\Gamma},\widehat{\beta})$. For this we need a $\widehat{\Gamma}$-equivariant map from $\mathrm{Prim}\,A\times_\beta\Gamma^+$ onto $\widehat{I}$, and we use Lemma 3.3 of [12] to obtain such a map. Note that the evaluation maps $\{\varepsilon_x : x\in I\}$ span a dense subalgebra of $C(\widehat{I}\,)$, and that the action of $\widehat{\Gamma}$ by translation on $C(\widehat{I})$ satisfies $\gamma\cdot\varepsilon_x=\gamma(x)\varepsilon_x$.

LEMMA 2.2. *There is a* $\widehat{\Gamma}$*-equivariant nondegenerate homomorphism* ψ *of* $C(\widehat{I}\,)$ *into the center* $ZM(A\times_\beta\Gamma^+)$ *of* $M(A\times_\beta\Gamma^+)$ *such that* $\psi(\varepsilon_x)=i_{\Gamma^+}(x)$ *for all* $x\in I^+$.

Proof. Consider $U=i_{\Gamma^+}|I^+ : I^+\to M(A\times_\beta\Gamma^+)$. Then for $x\in I^+$, we have

$$U_xU_x^*=i_{\Gamma^+}(x)i_{\Gamma^+}(x)^*=\overline{i}_A(\overline{\beta}_x(1))=\overline{i}_A(\overline{\alpha}_{[e]}(1))=1,$$

and U extends to a homomorphism $U: I\to UM(A\times_\beta\Gamma^+)$ by setting $U_x=U_{-x}^*$ for $x<e$ (see Lemma 1.1 of [7]). Thus the universal property of $C^*(I)$ gives a nondegenerate homomorphism $\pi : C^*(I)\to M(A\times_\beta\Gamma^+)$ such that $\pi(\delta_x)=U_x= i_{\Gamma^+}(x)$ for all $x\in I^+$ (where $\delta : I\to C^*(I)$ is the canonical homomorphism).

We claim that the range of π is contained in $ZM(A\times_\beta\Gamma^+)$. To see this, we first show that for $x\in I^+$ and $y\in\Gamma^+$, the element U_x commutes with $i_{\Gamma^+}(y)^*$. By writing $y=x+(y-x)$ for $y>x$, and $x=(x-y)+y$ for $y\leqslant x$, we have

$$U_xi_{\Gamma^+}(y)^*=\begin{cases} i_{\Gamma^+}(y-x)^* & y>x,\\ i_{\Gamma^+}(y-x)i_{\Gamma^+}(y)i_{\Gamma^+}(y)^* & y\leqslant x.\end{cases}$$

Since I is an order ideal of Γ, $i_{\Gamma^+}(y)i_{\Gamma^+}(y)^*=1$ for $y\leqslant x$. Therefore

$$\begin{aligned} U_xi_{\Gamma^+}(y)^* &= \begin{cases} i_{\Gamma^+}(y-x)^* & y>x,\\ i_{\Gamma^+}(x-y) & y\leqslant x;\end{cases}\\ &= \begin{cases} i_{\Gamma^+}(y-x)^*i_{\Gamma^+}(x)^*i_{\Gamma^+}(x) & y>x,\\ i_{\Gamma^+}(y)^*i_{\Gamma^+}(y)i_{\Gamma^+}(x-y) & y\leqslant x;\end{cases}\\ &= i_{\Gamma^+}(y)^*U_x. \end{aligned}$$

Thus for $x\in I^+$, $a\in A$, and $y,z\in\Gamma^+$, we have

$$\begin{aligned} \pi(\delta_x)&(i_{\Gamma^+}(y)^*i_A(a)i_{\Gamma^+}(z)) = i_{\Gamma^+}(y)^*i_{\Gamma^+}(x)i_A(a)i_{\Gamma^+}(z)\\ &= i_{\Gamma^+}(y)^*i_A(\beta_x(a))i_{\Gamma^+}(x)i_{\Gamma^+}(z) = i_{\Gamma^+}(y)^*i_A(\alpha_{[e]}(a))i_{\Gamma^+}(x)i_{\Gamma^+}(z)\\ &= i_{\Gamma^+}(y)^*i_A(a)i_{\Gamma^+}(z)i_{\Gamma^+}(x) = (i_{\Gamma^+}(y)^*i_A(a)i_{\Gamma^+}(z))\pi(\delta_x). \end{aligned}$$

For negative x in I, we have $\pi(\delta_x)=\pi(\delta_{-x})^*$, and $\pi(\delta_{-x})^*$ commutes with all $i_{\Gamma^+}(y)^*i_A(a)i_{\Gamma^+}(z)$. So $\pi(C^*(I))\subset ZM(A\times_\beta\Gamma^+)$, as claimed.

Since I is abelian, there is an isomorphism ϕ of $C(\widehat{I})$ onto $C^*(I)$ such that $\phi(\varepsilon_x)=\delta_x$ for all $x\in I$. So the homomorphism $\pi\circ\phi : C(\widehat{I}\,)\to ZM(A\times_\beta\Gamma^+)$ is nondegenerate, and it is $\widehat{\Gamma}$-equivariant because for $\gamma\in\widehat{\Gamma}$ and $x\in I^+$ we have

$$\begin{aligned} \pi\circ\phi(\gamma\cdot\varepsilon_x) &= \pi\circ\phi(\gamma(x)\varepsilon_x)=\pi(\gamma(x)\delta_x)\\ &= \gamma(x)i_{\Gamma^+}(x)=\widehat{\beta}_\gamma(\pi\circ\phi(\varepsilon_x)). \quad\blacksquare \end{aligned}$$

Proof of Theorem 2.1. Applying Lemma 3.3 of [12] to the $\widehat{\Gamma}$-equivariant homomorphism $\psi : C(\widehat{I}) \to ZM(A \times_\beta \Gamma^+)$ of Lemma 2.2 gives a contionous $\widehat{\Gamma}$-equivariant map $\phi : \operatorname{Prim} A \times_\beta \Gamma^+ \to \widehat{I}$, which is characterized as follows. If $P = \ker \rho \times V$ is kernel of an irreducible representation $\rho \times V$ of $A \times_\beta \Gamma^+$, then there is a nonzero homomorphism $\varphi_P : C(\widehat{I}) \to \mathbb{C}$ such that $\varphi_P(f) = \theta^{-1}(\psi(f))(P)$, where $\theta : C_{\mathrm{b}}(\operatorname{Prim} A \times_\beta \Gamma^+) \to ZM(A \times_\beta \Gamma^+)$ is the isomorphism given by the Dauns-Hoffmann Theorem; thus there is a unique $\gamma \in \widehat{I}$ such that $\varphi_P = \varepsilon_\gamma$, and $\phi(P)$ is by definition γ. Let $g \in C_{\mathrm{b}}(\operatorname{Prim} A \times_\beta \Gamma^+)$ and $\xi \in A \times_\beta \Gamma^+$. Then $\theta(g)\xi - g(P)\xi$ is in P, which means

$$\rho \times V(\theta(g)\xi) - g(P)\rho \times V(\xi) = 0 \quad \text{for all } g \text{ and } \xi.$$

This implies tht $\rho \times V(\theta(g)) = g(P) \cdot 1$ for all g. So by taking $g = \theta^{-1}(\psi(\varepsilon_x))$ for $x \in I$, we see that our equivariant map ϕ is characterised by

$$\rho \times V(\psi(\varepsilon_x)) = \theta^{-1}(\psi(\varepsilon_x))(P) \cdot 1 = \varphi_P(\varepsilon_x) \cdot 1 = \gamma(x) \cdot 1 = \phi(P)(x) \cdot 1. \tag{2.1}$$

Now suppose $J = \cap\{P \in \operatorname{Prim} A \times_\beta \Gamma^+ : \phi(P) = 1\}$ and $D = (A \times_\beta \Gamma^+)/J$. Then Echterchoff's Theorem ([6]) says that $(A \times_\beta \Gamma^+, \widehat{\Gamma}, \widehat{\beta})$ is $\widehat{\Gamma}$-equivariantly isomorphic to $(\operatorname{Ind}_{I^\perp}^{\widehat{\Gamma}} D, \widehat{\Gamma}, \operatorname{Ind} \widehat{\beta})$: the isomorphism $\Psi : A \times_\beta \Gamma^+ \to \operatorname{Ind}_{I^\perp}^{\widehat{\Gamma}} D$ is given by $\Psi(\xi)(\chi) = \widehat{\beta}_\chi^{-1}(\xi) + J$ for $\xi \in A \times_\beta \Gamma^+$ and $\chi \in \widehat{\Gamma}$. The result will follow if we can show that $\ker Q = J$, so that Q induces an isomorphism of D onto $A \times_\alpha (\Gamma/I)^+$.

Suppose $\xi \in \ker Q$ and $P = \ker \rho \times V$ satisfies $\phi(P) = 1$. Then from (2.1) we have

$$V_x = \rho \times V(i_{\Gamma^+}(x)) = \rho \times V(\psi(\varepsilon_x)) = \phi(P)(x) \cdot 1 = 1$$

for all $x \in I$. Thus V factors through an isometric representation W of $(\Gamma/I)^+$, and we have an irreducible representation $\rho \times W$ of $A \times_\alpha (\Gamma/I)^+$ such that $\rho \times V = (\rho \times W) \circ Q$. Thus $\rho \times V(\xi) = (\rho \times W) \circ Q(\xi) = 0$, and we have proved that $\ker Q \subset J$. Conversely, let $\xi \in J$, and suppose that $\rho \times W$ is an irreducible representation of $A \times_\alpha (\Gamma/I)^+$. Then $\rho \times (W \circ q) = (\rho \times W) \circ Q$ is an irreducible representation of $A \times_\beta \Gamma^+$ with $\phi(\ker \rho \times (W \circ q)) = 1$, so we have $\rho \times W(Q(\xi)) = 0$. This is true for every such $\rho \times W$, so we must have $\xi \in \ker Q$, as required. ∎

3. THE STRUCTURE OF TOEPLITZ ALGEBRAS

The Toeplitz algebra of a totally ordered group Γ is the C^*-subalgebra $\mathcal{T}(\Gamma)$ of $B(H^2(\Gamma^+))$ generated by the Toeplitz operators T_f with continuous symbol f in $C(\widehat{\Gamma})$; since $C(\widehat{\Gamma})$ is spanned by the functions $\varepsilon_x : \gamma \mapsto \gamma(x)$, $\mathcal{T}(\Gamma)$ is generated by the Toeplitz operators $T_x := T_{\varepsilon_x}$ for $x \in \Gamma^+$, which form a semigroup of isometries on $H^2(\Gamma^+)$ ([8]). The functions

$$\mathbb{1}_x(y) = \begin{cases} 1 & \text{if } y \geqslant x, \\ 0 & \text{otherwise;} \end{cases}$$

for $x \in \Gamma^+$ span a C^*-subalgebra B_{Γ^+} of $l^\infty(\Gamma)$; the action τ of Γ^+ by translation on $l^\infty(\Gamma)$ leaves B_{Γ^+} invariant and satisfies $\tau_x(\mathbb{1}_y) = \mathbb{1}_{x+y}$. It was shown in [3] that there is an isomorphism of $B_{\Gamma^+} \times_\tau \Gamma^+$ onto the Toeplitz algebra $\mathcal{T}(\Gamma)$ taking $i_{\Gamma^+}(x)$ to T_x (apply Theorem 2.4 of [3] to the isometric representation $x \mapsto T_x$). Since any isometric representation V of Γ^+ extends to a covariant representation (π_V, V) of $(B_{\Gamma^+}, \Gamma^+, \tau)$ (Proposition 2.2 of [3]), it follows that for any ordered ideal I there is a canonical surjection R^I of $\mathcal{T}(\Gamma)$ onto $\mathcal{T}(\Gamma/I)$ such that $R^I(T_x) = T_{[x]}$.

We can now state our main theorem.

THEOREM 3.1. *Let I be an order ideal in a totally ordered abelian group Γ, let $R^I : \mathcal{T}(\Gamma) \to \mathcal{T}(\Gamma/I)$ be the canonical surjection taking T_x to $T_{[x]}$, and let $\widehat{\tau}$ be the dual action of $\widehat{\Gamma}$ on $\mathcal{T}(\Gamma)$ characterised by $\widehat{\tau}_\gamma(T_x) = \gamma(x)T_x$ for $\gamma \in \widehat{\Gamma}$. Then there is a short exact sequence*

$$0 \longrightarrow \mathcal{C}_I \longrightarrow \mathcal{T}(\Gamma) \xrightarrow{\Theta} \operatorname{Ind}_{I^\perp}^{\widehat{\Gamma}} \mathcal{T}(\Gamma/I) \longrightarrow 0,$$

in which $\mathcal{C}_I$ is the ideal in $\mathcal{T}(\Gamma)$ generated by $\{T_uT_u^* - T_vT_v^* : v - u \in I^+\}$, and Θ is defined by $\Theta(a)(\gamma) := R^I(\widehat{\tau}_\gamma^{-1}(a))$.

For the proof we need to recall some results from [2]. Suppose α is an action of Γ^+ by extendible endomorphisms of a C^*-algebra A. For any ideal F of A, there is a canonical nondegenerate homomorphism $\psi : A \to M(F)$. We say that F is *extendibly α-invariant* if it is α-invariant and has an approximate identity (i_λ) such that $\alpha(i_\lambda)$ converges strictly to $\overline{\psi}(\overline{\alpha}(1))$ in $M(F)$.

This concept is more subtle than it might appear at first sight: in particular, extendibility of the endomorphisms $\alpha_x|F$ does not automatically imply extendibility of the ideal F. (Take $A = c_0(\mathbb{Z})$, $\Gamma = \mathbb{N}$, define α by $\alpha_m(f)(n) = f(n-m)$ and take $F := \{f \in A : f(n) = 0 \text{ for } n \leqslant 0\}$. Then $\alpha_m|F$ is extendible, but for any approximate identity (e_λ) in F, the net $\alpha_1(e_\lambda)$ converges strictly to $(0, 1, 1, 1, \ldots)$ in $M(F)$ which is not equal to $\overline{\psi}(\overline{\alpha}(1)) = (1, 1, 1, \ldots)$.)

Theorem 3.1 of [2] says that if F is an extendibly α-invariant ideal of A, then there is a short exact sequence of C^*-algebras

$$0 \longrightarrow F \times_\alpha \Gamma^+ \longrightarrow A \times_\alpha \Gamma^+ \longrightarrow A/F \times_{\tilde{\alpha}} \Gamma^+ \longrightarrow 0.$$

Since B_{Γ^+} has identity $\mathbb{1}_e$, the endomorphism τ_x are trivially extendible. We claim that the span $B_{\Gamma^+,\infty}$ of $\{\mathbb{1}_x - \mathbb{1}_y : x, y \in \Gamma^+, x < y\}$ is an extendibly invariant ideal. To see this, note that $(\mathbb{1}_e - \mathbb{1}_y)_{y \in \Gamma^+}$ is an approximate identity for $B_{\Gamma^+,\infty}$, and $\tau_x(\mathbb{1}_e - \mathbb{1}_y)_{y \in \Gamma^+}$ converges strictly to the projection $\mathbb{1}_x = \tau_x(\mathbb{1}_e)$ in $M(B_{\Gamma^+,\infty})$. Becausse $\sum \lambda_i \mathbb{1}_{x_i}(x) = \sum \lambda_i$ when $x \geqslant x_i$ for all i, the functions in B_{Γ^+} have limits as $x \to \infty$ in Γ^+, and we can view $B_{\Gamma^+,\infty}$ as the collection of functions $f \in B_{\Gamma^+}$ such that $f(x) \to 0$ as $x \to \infty$. In other words, if we define $\varepsilon : B_{\Gamma^+} \to \mathbb{C}$ by $\varepsilon(f) = \lim\limits_{x \to \infty} f(x)$, then we have a short exact sequence

$$0 \longrightarrow B_{\Gamma^+,\infty} \longrightarrow B_{\Gamma^+} \xrightarrow{\varepsilon} \mathbb{C} \longrightarrow 0$$

to which we can apply the result from [2]:

LEMMA 3.2. *There is a short exact sequence of C^*-algebras:*

$$0 \longrightarrow B_{\Gamma^+,\infty} \times_\tau \Gamma^+ \longrightarrow B_{\Gamma^+} \times_\tau \Gamma^+ \xrightarrow{\sigma} C(\widehat{\Gamma}) \longrightarrow 0; \tag{3.1}$$

it follows that $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$ is a maximal $\widehat{\Gamma}$-invariant ideal of $B_{\Gamma^+} \times_\tau \Gamma^+$.

Proof. Since translating by $y \in \Gamma^+$ does not affect limits as $x \to \infty$, the action $\widetilde{\tau}$ on $\mathbb{C} = B_{\Gamma^+}/B_{\Gamma^+,\infty}$ is trivial. Thus the quotient crossed product is $\mathbb{C} \times_{\mathrm{id}} \Gamma^+$. Since $\mathbb{C}$ has only trivial nondegenerate representations $z \mapsto z \cdot 1$, the covariance condition says that the isometric part of a covariant representation of $(\mathbb{C}, \Gamma^+, \mathrm{id})$ consists of unitaries, and we have $\mathbb{C} \times_{\mathrm{id}} \Gamma^+ \simeq C^*(\Gamma) \simeq C(\widehat{\Gamma})$. Thus [2] gives the desired exact sequence. Finally, $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$ is a maximal $\widehat{\Gamma}$-invariant ideal of B_{Γ^+} becuase there is no nontrivial ideal in $C(\widehat{\Gamma})$. ∎

REMARK 3.3. Since the isomorphism of $B_{\Gamma^+} \times_\tau \Gamma^+$ onto $\mathcal{T}(\Gamma)$ takes the generator $\mathbb{1}_x - \mathbb{1}_y$ for $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$ to the difference of commutators $(T_xT_x^* - T_x^*T_x) - (T_yT_y^* - T_y^*T_y)$, and since the quotient $B_{\Gamma^+} \times_\tau \Gamma^+/B_{\Gamma^+,\infty} \times_\tau \Gamma^+$ is commutative, it carries $B_{\Gamma^+,\infty \times_\tau \Gamma^+}$ onto the commutator ideal $\mathcal{C}$ of $\mathcal{T}(\Gamma)$. The exact sequence of Lemma 3.2 becomes the usual one in which σ takes a Toeplitz operator T_f to its symbol f ([8]).

LEMMA 3.4. *The set $C_{I^+} = \overline{\mathrm{sp}}\,\{\mathbb{1}_x - \mathbb{1}_y : x, y \in \Gamma^+, y - x \in I^+\}$ is an extendibly τ-invariant ideal of B_{Γ^+}.*

Proof. Since $\mathbb{1}_z(\mathbb{1}_x - \mathbb{1}_y) = \mathbb{1}_{z\vee x} - \mathbb{1}_{z\vee y}$ and $e \leqslant z \vee y - z \vee x \leqslant y - x$, C_{I^+} is an ideal in B_{Γ^+}. Thus we have to produce an approximate identity (i_λ) for C_{I^+} such that for each $x \in \Gamma^+$, the net $\tau_x(i_\lambda)$ converges srictly to $\mathbb{1}_x = \tau_x(\mathbb{1}_e)$ in $M(C_{I^+})$. Let D be the set of pairs (F, t) in which F is a finite subset of Γ^+ with at most one point in each coset $[y] \in \Gamma/I$, and $t \in I^+$. Define a relation on D by

$$(F, t) \leqslant (G, s) \iff [u, u + t) \subseteq \bigcup_{v \in G} [v, v + s) \quad \text{for all } u \in F.$$

We claim that this relation directs D. Reflexivity and transitivity are obvious. Suppose (F, t) and (G, s) are given; we want (E, r) which dominates both (F, t) and (G, s). Write $F = \{u_1, u_2, \ldots, u_n\}$ and $G = \{v_1, v_2, \ldots, v_m\}$. If $[u_i] \cap G = \emptyset$, then take u_i as an element of E. If $[v_i] \cap F = \emptyset$, then take v_i as an element of E. Otherwise $[u_i] \cap G \neq \emptyset$, and hence there must be exactly one $v_{j(i)} \in G$ such that $[v_{j(i)}] = [u_i]$, in this case we put $\min\{u_i, v_{j(i)}\}$ in E. It is not so obvious that the last condition is symmetric, but starting with j such that $[v_j] \cap F \neq \emptyset$ would give $u_{j(i)}$ such that $[v_j] = [u_{i(j)}]$, and then we would have $j = j(i_j)$ because G hits each coset at most once. We now have a finite set E such that for each $[y] \in \Gamma/I$ the intersection $E \cap [y]$ is empty or a single point. Let r be the largest of the three elements t, s and

$$\{\max(v_{j(i)} + s, u_i + t) - \min(u_i, v_{j(i)}) : i \text{ satisfies } [u_i] \cap G \neq \emptyset\}.$$

Then r is in I^+, and we have $(E, r) \geqslant (G, s)$ and $(E, r) \geqslant (F, t)$. Thus D is directed.

We claim that the elements $\mathbb{1}_{(F,t)} := \sum_{u\in F}(\mathbb{1}_u - \mathbb{1}_{u+t})$ form an approximate identity for C_{I^+}. To see this, it is enough to show that if $f = \sum_{i=1}^{n}\lambda_i(\mathbb{1}_{y_i} - \mathbb{1}_{z_i})$ and $z_i - y_i \in I^+$, then there exists $(F,t)\in D$ such that

$$(G,s) \geqslant (F,t) \Rightarrow \mathbb{1}_{(G,s)}f = f. \tag{3.2}$$

First we suppose that all the y_i are in different I-cosets. Let $F = \{y_1,\ldots,y_n\}$ and $t = \max_{1\leqslant i\leqslant n}(z_i - y_i)$. Then each interval $[y_i, z_i)$ is contained in $[y_i, y_i + t)$, hence $\mathbb{1}_{(F,t)}f = f$. So (3.2) is satisfied, because $(G,s) \geqslant (F,s) \Rightarrow \mathbb{1}_{(G,s)} \geqslant \mathbb{1}_{(F,t)}$. If not all of the y_i are in different I-cosets, say $\{y_{i1}, y_{i2},\ldots,y_{ik}\}$ are in the same coset as y_i, and z_{ij} is the point that corresponds to y_{ij}, then we let $w_i = \min\{y_{i1}, y_{i2},\ldots,y_{ik}\}$ and $r_i = \max_{1\leqslant j\leqslant k}(z_{ij}) - w_i$. Since there are only finitely many y_i involved, this gives a finite set $F = \{w_1,\ldots,w_m\}$, and we can take $t = \max_{1\leqslant i\leqslant m} r_i$. Then $\mathbb{1}_{(F,t)}f = f$, as required.

Next, we show that for $z\in\Gamma^+$, $\tau_z(\mathbb{1}_{(F,t)})_{(F,t)\in D}$ converges strictly to $\mathbb{1}_z$ in $M(C_{I^+})$. Again, it is enough to show that $\tau_z(\mathbb{1}_{(F,t)})f$ converges to f in C_{I^+} for all f of the form $\sum_i \lambda_i(\mathbb{1}_{y_i} - \mathbb{1}_{z_i})$ where $y_i - z_i \in I^+$. Choose (F,t) satisfying (3.2), and let $(G,s)\geqslant(F,t)$. We write $G' = \{v - z : v\in G\}\cap\Gamma^+$, and add the point e to G' if z belongs to some interval $[v, v+s)$. Then $\tau_z(\mathbb{1}_{(G',s)})f = \mathbb{1}_z\mathbb{1}_{(G,s)}f = \mathbb{1}_z f$, and hence $\tau_z(\mathbb{1}_{(E,r)})f = \mathbb{1}_z f$ for all $(E,r)\geqslant(G',s)$. Thus $\tau_z(\mathbb{1}_{(F,t)})f$ converges to $\mathbb{1}_z f$ in C_{I^+}. Since $\tau_z(\mathbb{1}_e) = \mathbb{1}_z$, this proves that C_{I^+} is extendibly τ-invariant, and proves the lemma. ∎

Proof of Theorem 3.1. From Lemma 3.4 and Theorem 3.1 of [2] we obtain an exact sequence

$$0 \longrightarrow C_{I^+}\times_\tau \Gamma^+ \longrightarrow B_{\Gamma^+}\times_\tau\Gamma^+ \longrightarrow (B_{\Gamma^+}/C_{I^+})\times_{\tilde{\tau}}\Gamma^+ \longrightarrow 0.$$

Since the isomorphism of $B_{\Gamma^+}\times_\tau\Gamma^+$ onto $\mathcal{T}(\Gamma)$ carries $C_{I^+}\times\Gamma^+$ onto the ideal $\mathcal{C}_I$, it remains to identify $(B_{\Gamma^+}/C_{I^+})\times\Gamma^+$ with $\operatorname{Ind}\mathcal{T}(\Gamma/I)$ in a compatible way.

We begin by noting that $\sum\lambda_i\mathbb{1}_{u_i}\mapsto\sum\lambda_i\mathbb{1}_{[u_i]}$ is norm-decreasing, and hence extends to well-defined surjection l of B_{Γ^+} onto $B_{(\Gamma/I)^+}$ such that $l(f)([u]) = \lim_{\substack{z\to\infty\\ z\in I^+}} f(u+z)$ for $f\in B_{\Gamma^+}$. Since $l(\mathbb{1}_u\mathbb{1}_v) = l(\mathbb{1}_{\max(u,v)}) = \mathbb{1}_{[\max(u,v)]} = \mathbb{1}_{\max([u],[v])} = \mathbb{1}_{[u]}\mathbb{1}_{[v]}$, l is a homomorphism. A messy computation shows that

$$C_{I^+} = \{f\in B_{\Gamma^+} : f(x+z)\to 0 \text{ as } z\to\infty \text{ in } I^+\} = \ker l,$$

so l induces an isomorphism of B_{Γ^+}/C_{I^+} onto $B_{(\Gamma/I)^+}$. It is easy to check that this isomorphism intertwines the action $\tilde{\tau}$ induced by translation on B_{Γ^+} and the action $\tau\circ q$ inflated from the action τ of $(\Gamma/I)^+$ by translation on $B_{(\Gamma/I)^+}$. We therefore have an isomorphism of $(B_{\Gamma^+}/C_{I^+})\times_{\tilde{\tau}}\Gamma^+$ onto $B_{(\Gamma/I)^+}\times_{\tau\circ q}\Gamma^+$, which is the identity on the generating copies of Γ^+.

We can apply Theorem 2.1 to the inflated system $(B_{(\Gamma/I)^+}, \Gamma^+, \tau\circ q)$, and obtain an isomorphism Ψ of $B_{(\Gamma/I)^+}\times_{\tau\circ q}\Gamma^+$ onto $\operatorname{Ind}_{I^\perp}^{\widehat{\Gamma}} B((\Gamma/I)^+\times_\tau(\Gamma/I)^+$. If we now put our isomorphisms together and identify $B_{(\Gamma/I)^+}\times_\tau(\Gamma/I)^+$ with the

Toeplitz algebra $\mathcal{T}(\Gamma/I)$, we obtain an isomorphism of $(B_{\Gamma^+}/C_{I^+}) \times_{\tilde{\tau}} \Gamma^+$ onto $\mathrm{Ind}_{I^\perp}^{\widehat{\Gamma}} \mathcal{T}(\Gamma/I)$; a quick look at the formula for Ψ in Theorem 2.1 shows that this isomorphism takes the generator $i_{\Gamma^+}(x)$ to the function $\gamma \mapsto \gamma^{-1}(x)T_{[x]}$. Since the isomorphism of $B_{\Gamma^+} \times_\tau \Gamma^+$ onto $\mathcal{T}(\Gamma)$ takes $i_{\Gamma^+}(x)$ to T_x and $R^I(T_x) = T_{[x]}$, this shows that the induced surjection $\mathcal{T}(\Gamma)$ onto $\mathrm{Ind}_{I^\perp}^{\widehat{\Gamma}} \mathcal{T}(\Gamma/I)$ takes T_x to the function $\gamma \mapsto R^I(\widehat{\tau}_\gamma^{-1}(T_x))$. Thus this surjection agrees on generators with the homomorphism Θ described in the theorem, and hence agrees an all of $\mathcal{T}(\Gamma)$. This completes the proof of Theorem 3.1. ∎

When Γ decomposes as a lexicographic direct sum $(\Gamma/I) \bigoplus_{\mathrm{lex}} I$, the dual group $\widehat{\Gamma}$ decomposes as $I^\perp \times \widehat{I}$, and the inducing construction collapses. Therefore we can recover Murphy's description of the Toeplitz algebra of a lexicografic sum:

COROLLARY 3.5. ([10]) *If Γ is a lexicografic direct sum $\Gamma_1 \bigoplus_{\mathrm{lex}} \Gamma_2$ of totally ordered abelian groups Γ_1 and Γ_2, then $\mathcal{T}(\Gamma)/\mathcal{C}_{\Gamma_2} \cong \mathcal{T}(\Gamma_1) \otimes C(\widehat{\Gamma}_2)$.*

Proof. We just need to apply the Theorem 3.1 to the ideal Γ_2 and note that, if $G = H \times K$, then $\mathrm{Ind}_H^G D \cong C(K, D)$. ∎

REMARK 3.6. It is important to note that Theorem 3.1 is more general then Corollary 3.5: Clifford has shown that Γ is not always isomorphic to $\Gamma/I \bigoplus_{\mathrm{lex}} I$ ([4]). In his example, Γ is the subgroup of $\mathbb{Q} \bigoplus_{\mathrm{lex}} \mathbb{Q}$ generated by $(1/p_n, n/p_n)$ where p_n is the n-th prime, and I is the set of all $(0, y)$ with y an integer. In general, $\Gamma \cong \Gamma/I \bigoplus_{\mathrm{lex}} I$ if and only if the short exact sequence $0 \longrightarrow I \longrightarrow \Gamma \longrightarrow \Gamma/I \longrightarrow 0$ splits in the purely group-theoretic sense, and Clifford proves that this does not happen for his example.

4. INVARIANT IDEALS

When Γ is a discrete subgroup of $\mathbb{R}$, the commutator ideal $\mathcal{C}$ in the Toeplitz algebra $\mathcal{T}(\Gamma)$ is simple ([5]). For more general totally ordered groups, the commutator ideal can have many ideals. Here we shall use our techniques to completely determine the ideals in $\mathcal{C}$ which are invariant under the dual action $\widehat{\tau}$.

In the previous section, we showed that an order ideal I in Γ determines an extendibly invariant ideal C_{I^+} in B_{Γ^+}. This ideal is always contained in the ideal $B_{\Gamma^+,\infty}$, and hence Theorem 3.1 of [2] gives an exact sequence

$$0 \longrightarrow C_{I^+} \times_\tau \Gamma^+ \xrightarrow{\imath} B_{\Gamma^+,\infty} \times_\tau \Gamma^+ \xrightarrow{\Phi} (B_{\Gamma^+,\infty}/C_{I^+}) \times_{\tilde{\tau}} \Gamma^+ \longrightarrow 0.$$

The embedding $\imath$ satisfies $\imath(i_{\Gamma^+}(x)) = i_{\Gamma^+}(x)$, and hence respects the dual action $\widehat{\tau}$; thus the range of $\imath$ is an invariant ideal of $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$. Our theorem says that every invariant ideals has this form:

THEOREM 4.1. *Suppose I is an order ideal in a totally ordered abelian group Γ. Then the crossed product $C_{I^+} \times_\tau \Gamma^+$ naturally embeds as a $\widehat{\Gamma}$-invariant ideal of $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$, and the map $I \mapsto C_{I^+} \times_\tau \Gamma^+$ is an isomorphism of the lattice of order ideals of Γ onto the lattice of $\widehat{\Gamma}$-invariant ideals of $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$.*

The homomorphism $l : B_{\Gamma^+} \to B_{(\Gamma/I)^+}$ in the proof of Theorem 3.1 maps $B_{\Gamma^+,\infty}$ to $B_{(\Gamma/I)^+,\infty}$, and hence induces an isomorhism Θ of $(B_{\Gamma^+,\infty}/C_{I^+}) \times_{\tilde{\tau}} \Gamma^+$ onto $B_{(\Gamma/I)^+,\infty} \times_{\tau \circ q} \Gamma^+$. Now Theorem 2.1 implies that there is an isomorphism

$$\Psi \circ \Theta : (B_{\Gamma^+,\infty}/C_{I^+}) \times_{\tilde{\tau}} \Gamma^+ \to \operatorname{Ind}_{I^\perp}^{\widehat{\Gamma}} (B_{(\Gamma/I)^+,\infty} \times_\tau (\Gamma/I)^+),$$

and we have an exact sequence

$$0 \longrightarrow C_{I^+} \times_\tau \Gamma^+ \xrightarrow{\imath} B_{\Gamma^+,\infty} \times_\tau \Gamma^+ \xrightarrow{\Psi\circ\Theta\circ\Phi} \operatorname{Ind}_{I^\perp}^{\widehat{\Gamma}} (B_{(\Gamma/I)^+,\infty} \times_\tau (\Gamma/I)^+) \longrightarrow 0.$$

Therefore to identify ideals in $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$, we need to know about invariant ideals in induced C^*-algebras.

LEMMA 4.2. *Suppose H is a closed subgroup of a locally compact group G, and $\beta : H \to \operatorname{Aut} C$ is a strongly continuous action on a C^*-algebra C. If L is a closed $\operatorname{Ind}\beta$-invariant ideal in the induced C^*-algebra $\operatorname{Ind}_H^G(C,\beta)$, then $J = \{f(e) : f \in L\}$ is a β-invariant ideal of C, and*

$$L = \operatorname{Ind} J := \{f \in \operatorname{Ind}_H^G(C,\beta) : f(x) \in J \textit{ for all } x \in G\}.$$

Proof. If $j = f(e) \in J$ and $c \in C$, we can find g in $\operatorname{Ind}_H^G(C,\beta)$ such that $g(e) = c$ (Corollary 6.18 of [13]). Then $fg \in L$ and $jc = f(e)g(e)$ is in J, so J is an ideal of C. It is β-invariant because

$$\beta_h(f)(e) = f(eh^{-1}) = f(h^{-1}) = (\operatorname{Ind}\beta)_h(f)(e).$$

Next we prove that $L = \operatorname{Ind} J$. For $f \in L$ we have $f(x) = (\operatorname{Ind}\beta)_x^{-1}(f)(e)$, so the invariance of L implies that $L \subseteq \operatorname{Ind} J$. To see that $\operatorname{Ind} J \subseteq L$, we let $l \in \operatorname{Ind} J$, and find $f \in L$ such that $\|f - l\|$ is small; then l must be in L because L is closed. Fix $\varepsilon > 0$. For each $x \in G$, there exists $h_x \in L$ such that $\|h_x(y) - l(y)\| < \varepsilon$ for yH in a neighbourhood N_x of xH. Indeed, since $l(x) \in J$, there is $k \in L$ such that $l(x) = k(e)$, and we can take $h_x = (\operatorname{Ind}\beta)_x(k)$. For then $h_x(x) = (\operatorname{Ind}\beta)_x(k)(x) = k(e) = l(x)$, and hence $h_x|xH = l|xH$; the continuity of $xH \mapsto \|(h_x - l)(x)\|$ gives the required neighbourhood N_x. Since the map $sH \mapsto \|l(s)\|$ belongs to $C_0(G/H)$, there is a compact subset K of G/H such that $\|l(y)\| < \varepsilon$ for $yH \notin K$. By compactness, there are finite subcover $\{N_i\}_{1 \leqslant i \leqslant n}$ of K and $h_1, h_2, \ldots, h_n \in L$ such that $\|h_i(y) - l_y\| < \varepsilon$ for $yH \in N_i$ and $1 \leqslant i \leqslant n$. Now let $\{\rho_i\}$ be a partition of unity subordinate to $K^c \cup \{N_i\}$, and define $f(y) = \sum_{i=1}^{n} \rho_i(yH)h_i(y)$. ∎

LEMMA 4.3. *Suppose J is a nonzero closed ideal in $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$. Then there is a nonzero ideal I in Γ with positive cone $I^+ = \{x \in \Gamma^+ : i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_x) \in J\}$.*

Proof. Because J is nonzero, it follows from Corollary 2.7 of [3] that $E := \{x \in \Gamma^+ : i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_x) \in J\}$ is nonzero. We will prove that E is the positive cone in an order ideal I. For this we show that E is a subsemigroup of Γ^+ such that $e \leqslant y \leqslant x \in E$ implies $y \in E$; then $I := E - E$ has the required property.

Let $x, y \in E$. Then $i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_{x+y}) \in J$, because

$$\begin{aligned} i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_{x+y}) &= i_{B_{\Gamma^+,\infty}}((\mathbb{1}_e - \mathbb{1}_y) + (\mathbb{1}_y - \mathbb{1}_{x+y})) \\ &= i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_y) + i_{B_{\Gamma^+,\infty}}(\tau_y(\mathbb{1}_e - \mathbb{1}_x)) \\ &= i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_y) + i_{\Gamma^+}(y) i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_x) i_{\Gamma^+}(y)^*. \end{aligned}$$

Thus $x + y \in E$. Next suppose $x \in E$ and $e \leqslant z \leqslant x$. Then $0 \leqslant \mathbb{1}_e - \mathbb{1}_z \leqslant \mathbb{1}_e - \mathbb{1}_x$, so

$$0 \leqslant i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_z) \leqslant i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_x).$$

This implies that $i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_z) \in J$, because closed ideals of C^*- algebras are hereditary (Corollary 3.2.3 of [9]). Thus $z \in E$. ∎

Proof of Theorem 4.1. Let J be a nonzero $\widehat{\Gamma}$-invaraiant ideal of $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$. Then by Lemma 4.3 there is a nonzero order ideal I of Γ such that $I^+ = \{x \in \Gamma^+ : i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_x) \in J\}$. Since

$$i_{B_{\Gamma^+,\infty}}(\mathbb{1}_u - \mathbb{1}_v) = i_{B_{\Gamma^+,\infty}}(\tau_u(\mathbb{1}_e - \mathbb{1}_{v-u})) = i_{\Gamma^+}(u) i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_{v-u}) i_{\Gamma^+}(u)^*,$$

the $\widehat{\Gamma}$-invariant ideal $C_{I^+} \times_\tau \Gamma^+$ of $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$ is contained in J. We will prove that $C_{I^+} \times_\tau \Gamma^+ = J$.

Suppose $C_{I^+} \times_\tau \Gamma^+ \neq J$. Then $J/(C_{I^+} \times_\tau \Gamma^+)$ is nonzero, and $\Psi \circ \Theta \circ \Phi(J)$ is a nonzero $\widehat{\Gamma}$-invariant ideal of the induced algebra $\operatorname{Ind}_{I^\perp}^{\widehat{\Gamma}} B_{(\Gamma/I)^+,\infty} \times_\tau (\Gamma/I)^+$. By Lemma 4.2, there is a nonzero invariant ideal E of $B_{(\Gamma/I)^+,\infty} \times_\tau (\Gamma/I)^+$ such that

$$\Psi \circ \Theta \circ \Phi(J) = \operatorname{Ind} E := \{f \in \operatorname{Ind}_{I^\perp}^{\widehat{\Gamma}} B_{(\Gamma/I)^+,\infty} \times_\tau (\Gamma/I)^+ : f(\gamma) \in E \text{ for all } \gamma \in \widehat{\Gamma}\}.$$

Now Corollary 2.7 of [3] says there is a nonzero element $[u]$ of $(\Gamma/I)^+$ such that

$$1 - j_{(\Gamma/I)^+}([u]) j_{(\Gamma/I)^+}([u])^* = j_{B_{(\Gamma/I)^+,\infty}}(\mathbb{1}_{[e]} - \mathbb{1}_{[u]}) \in E,$$

where $j_{B_{(\Gamma/I)^+,\infty}} : B_{(\Gamma/I)^+,\infty} \to B_{(\Gamma/I)^+,\infty} \times_\tau (\Gamma/I)^+$ is the canonical embedding. But from the characterisations of $Q : B_{(\Gamma/I)^+,\infty} \times_{\tau\circ q} \Gamma^+ \to B_{(\Gamma/I)^+,\infty} \times_\tau (\Gamma/I)^+$ and Ψ we see that

$$\begin{aligned} j_{B_{(\Gamma/I)^+,\infty}}(\mathbb{1}_{[e]} - \mathbb{1}_{[u]}) &= Q(i_{B_{(\Gamma/I)^+,\infty}}(\mathbb{1}_{[e]} - \mathbb{1}_{[u]})) \\ &= Q(\widehat{\tau \circ q}_\gamma^{-1}(i_{B_{(\Gamma/I)^+,\infty}}(\mathbb{1}_{[e]} - \mathbb{1}_{[u]}))) \quad \text{for all } \gamma \in \widehat{\Gamma} \\ &= \Psi(i_{B_{(\Gamma/I)^+,\infty}}(\mathbb{1}_{[e]} - \mathbb{1}_{[u]}))(\gamma) \quad \text{for all } \gamma \in \widehat{\Gamma}. \end{aligned}$$

So $\Psi(i_{B_{(\Gamma/I)^+,\infty}}(\mathbb{1}_{[e]} - \mathbb{1}_{[u]})) \in \operatorname{Ind} E = \Psi \circ \Theta \circ \Phi(J)$. Hence $i_{B_{(\Gamma/I)^+,\infty}}(\mathbb{1}_{[e]} - \mathbb{1}_{[u]}) \in \Theta \circ \Phi(J)$, and $i_{B_{(\Gamma/I)^+,\infty}}(\mathbb{1}_{[e]} - \mathbb{1}_{[u]}) = \Theta \circ \Phi(k)$ for some $k \in J$. Thus

$$i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_u) - k \in \ker \Theta \circ \Phi = C_{I^+} \times_\tau \Gamma^+ \subset J,$$

and therefore $i_{B_{\Gamma^+,\infty}}(\mathbb{1}_e - \mathbb{1}_u)$ belongs to J. But this means $u \in I^+$, and $[u] = [e]$ is the zero element of Γ/I, which is a contradiction. So $C_{I^+} \times_\tau \Gamma^+ = J$.

Next suppose I_1 and I_2 are distinct order ideals of Γ. Then one has to contain the other; say I_1 is strictly contained in I_2. We want to show that $C_{I_1^+} \times_\tau \Gamma^+$

is strictly contained in $C_{I_2^+} \times_\tau \Gamma^+$. Let $y \in I_2^+ \setminus I_1^+$. Then $i_{B_{\Gamma^+,\infty}}(1\!\!1_e - 1\!\!1_y)$ is in $C_{I_2^+} \times_\tau \Gamma^+$, and we claim that it is not in $C_{I_1^+} \times_\tau \Gamma^+$. If $i_{B_{\Gamma^+,\infty}}(1\!\!1_e - 1\!\!1_y) \in C_{I_1^+} \times_\tau \Gamma^+$, then $\Theta \circ \Phi\big(i_{B_{\Gamma^+,\infty}}(1\!\!1_e - 1\!\!1_y)\big) = i_{B_{(\Gamma/I)^+,\infty}}(1\!\!1_{[e]} - 1\!\!1_{[y]})$ vanishes in $B_{(\Gamma/I_1)^+,\infty} \times_\tau \Gamma^+$. Hence $\Psi\big(i_{B_{(\Gamma/I_1)^+,\infty}}(1\!\!1_{[e]} - 1\!\!1_{[y]})\big) = 0$ in the induced algebra $\operatorname{Ind} B_{(\Gamma/I_1)^+,\infty} \times_\tau (\Gamma/I_1)^+$, and $i_{B_{(\Gamma/I_1)^+,\infty}}(1\!\!1_{[e]} - 1\!\!1_{[y]}) = 0$ in $B_{(\Gamma/I_1)^+,\infty} \times_\tau (\Gamma/I_1)^+$. But this implies that $1\!\!1_{[e]} - 1\!\!1_{[y]} = 0$, so $y \in I_1^+$, which contradicts the assumption that $y \notin I_1^+$. So $C_{I_1^+} \times_\tau \Gamma^+$ must be strictly contained in $C_{I_2^+} \times_\tau \Gamma^+$. This completes the proof of Theorem 4.1. ∎

COROLLARY 4.4. *Every $\widehat{\Gamma}$-invariant ideal in the commutator ideal $\mathcal{C}$ of the Toeplitz algebra $\mathcal{T}(\Gamma)$ is the ideal $\mathcal{C}_I$ generated by $\{T_u T_u^* - T_v T_v^* : v - u \in I^+\}$ for some order ideal I of Γ.*

Proof. The isomorphism of $B_{\Gamma^+} \times_\tau \Gamma^+$ onto $\mathcal{T}(\Gamma)$ is equivariant, and takes $B_{\Gamma^+,\infty} \times_\tau \Gamma^+$ to $\mathcal{C}$ and $C_{I^+} \times_\tau \Gamma^+$ to $\mathcal{C}_I$. Thus the result follows immediately from the Theorem 4.1. ∎

Acknowledgements. These results are from the author's PhD thesis ([1]). She wishes to thank her supervisor Iain Raeburn for his helpful comments and suggestions. She greatly appreciate all facilities provided by the Centre of Mathematics ITB for writing this paper.

REFERENCES

1. S. ADJI, Crossed products of C^*-algebras by semigroups of endomorphisms, Ph.D. Dissertation, University of Newcastle, Australia, 1995.
2. S. ADJI, Invariant ideals of crossed products by semigroups of endomorphisms, in *Functional Analysis and Global Analysis*, Proc. of the Conference at Manila in October 1996, T. Sunada, P.W. Sy(Eds), Springer-Verlag, Singapore 1997, pp. 1–8.
3. S. ADJI, M. LACA, M. NILSEN, I. RAEBURN, Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups, *Proc. Amer. Math. Soc.* **122**(1994), 1133– 1141.
4. A.H. CLIFFORD, Note on Hahn's theorem on ordered abelian groups, *Proc. Amer. Math. Soc.* **5**(1954), 860–863.
5. R.G. DOUGLAS, On the C^*-algebra of a one-parameter semigroup of isometries, *Acta Math.* **128**(1972), 143–152.
6. S. ECHTERHOFF, On induced covariant systems, *Proc. Amer. Math. Soc.* **108**(1990), 703–706.
7. M. LACA, I. RAEBURN, Extending multipliers from semigroups, *Proc. Amer. Math. Soc.* **123**(1995), 355–363.
8. G.J. MURPHY, Ordered groups and Toeplitz algebas, *J. Operator Theory* **18**(1987), 303–326.
9. G.J. MURPHY, *C^*-Algebras and Operator Theory*, Academic Press, San Diego 1990.
10. G.J. MURPHY, Toeplitz operators and algebras, *Math. Z.* **208**(1991), 355–362.
11. D. OLSEN, G.K. PEDERSEN, Partially inner C^*-dynamical systems, *J. Funct. Anal.* **66**(1986), 262–281.

12. J.C. QUIGG, I. RAEBURN, Induced C^*-algebras and Landstad duality for twisted coactions, *Trans. Amer. Math. Soc.* **347**(1995), 2885–2915.
13. I. RAEBURN, D.P. WILLIAMS, Morita equivalence and continuous-trace C^*-algebras, *Math. Surveys Monogr.*, vol. 60, Amer. Math. Soc., Providence, RI 1998.

SRIWULAN ADJI
Department of Mathematics
Bandung Institute of Technology
Bandung 40132
INDONESIA

Received June 6, 1998; revised January 30, 1999.