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HIGHER DIMENSIONAL NEVANLINNA-PICK INTERPOLATION THEORY

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Abstract. We compute completely isometric representations of quotients of the operator algebra $\mathcal{S}_d$ generated by the d-shift introduced by Arveson. This gives rise to a higher dimensional generalization of Nevanlinna-Pick interpolation theory.

Quotients of $\mathcal{S}_d$ of dimension r admit a completely isometric representation by $r \times r$-matrices. There is an efficient criterion to decide whether an r-dimensional algebra of $r \times r$-matrices is a quotient of $\mathcal{S}_d$.

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1. INTRODUCTION

In [3], Arveson defines the *d-shift* $\mathbf{S} = (S_1, \ldots, S_d)$, acting on a certain Hilbert space H^2_d. The definitions of the d-shift and of H^2_d are recalled in Section 2 together with some properties we will need in the following. We write $\mathbb{B}(\mathcal{H})$ for the C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}$. Let

$$\mathbb{D}_d := \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid |z_1|^2 + \cdots + |z_d|^2 < 1\}$$

be the open Euclidean unit ball and let $\overline{\mathbb{D}}_d$ be its closure. We write $\mathcal{O}(\mathbb{D}_d)$ for the algebra of holomorphic functions on $\mathbb{D}_d$ and $\mathcal{O}(\overline{\mathbb{D}}_d)$ for the algebra of functions holomorphic in a neighborhood of $\overline{\mathbb{D}}_d$. Let $\mathcal{S}_d$ be the norm closed unital subalgebra of $\mathbb{B}(H^2_d)$ generated by $S_1, \ldots, S_d$. We call $f \in \mathcal{O}(\mathbb{D}_d)$ a *multiplier* on H^2_d iff $f \cdot g \in H^2_d$ for all $g \in H^2_d$ and write $\mathcal{M}_d$ for the algebra of multipliers. We have inclusions $\mathcal{M}_d \subset \mathbb{B}(H^2_d)$ and

$$\mathcal{O}(\overline{\mathbb{D}}_d) \subset \mathcal{S}_d \subset \mathcal{M}_d \subset H^2_d \subset \mathcal{O}(\mathbb{D}_d).$$

Assume given $z_1, \dots, z_m \in \mathbb{D}_d$ and $n \times n$-matrices $T_1, \dots, T_m \in \mathbb{M}_n$. There is $f \in \mathbb{M}_n(\mathcal{M}_d)$ with $\|f\| \leqslant 1$ and $f(z_j) = T_j$ for all j if and only if the block matrix with entries

$$\frac{1 - T_i T_j^*}{1 - \langle z_i, z_j \rangle} \in \mathbb{M}_n \tag{1.1}$$

is positive definite. Here $\langle z_i, z_j \rangle$ stands for the Hilbert space inner product on $\mathbb{C}^d$ with unit ball $\mathbb{D}_d$. There is a bounded $f \in \mathbb{M}_n(\mathcal{M}_d)$ with $f(z_j) = T_j$ for all j and positive real part in $\mathbb{B}(H^2_d)$ if the matrix with entries

$$\frac{T_i + T_j^*}{1 - \langle z_i, z_j \rangle} \in \mathbb{M}_n \tag{1.2}$$

is positive definite *and invertible*. If we allow for f to be an unbounded multiplier, it suffices to assume that the matrix in (1.2) is positive definite. In the scalar valued case $n = 1$, the solution f is unique if the matrix in (1.1) or (1.2) is positive and *not* invertible. The unique solution is a rational function with poles outside $\mathbb{D}_d$. There is an algorithm to compute it.

The same conditions (1.1) and (1.2) occur in classical Nevanlinna-Pick interpolation theory ([13], [11] and [14]), which is the special case $d = 1$. For $d = 1$, the 1-shift is the usual unilateral shift, $\mathcal{S}_1$ is the algebra $\overline{\mathcal{O}}(\overline{\mathbb{D}})$ of continuous functions on $\overline{\mathbb{D}}$ holomorphic on $\mathbb{D}$, and $\mathcal{M}_1 \cong H^\infty(\mathbb{D})$. Thus Nevanlinna-Pick interpolation theory is a special case of the assertions above.

After submitting the article, I learned that some of these interpolation results have been obtained independently also by Arias and Popescu([2]) and by Davidson and Pitts ([6]). These authors work with a non-commutative version of the d-shift and obtain interpolation results in that setting. Dividing out the commutator ideal then yields results about interpolation in $\mathbb{M}_n(\mathcal{M}_d)$. In this way, Theorem 4.1 and Theorem 7.3 become special cases of results in [2] and [6].

A map $\varphi : \mathcal{A} \to \mathcal{B}$ between operator algebras is called *completely contractive* iff the induced maps $\varphi_{(n)} : \mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_n(\mathcal{B})$ are contractive for all $n \in \mathbb{N}$ ([12]). Completely isometric maps and complete quotient maps are defined by requiring that $\varphi_{(n)}$ be isometric or a quotient map for all $n \in \mathbb{N}$, respectively.

The essential step in the proof of the interpolation results is to obtain a completely isometric representation of the quotient algebra $\mathcal{M}_d / I(z_1, \dots, z_m)$. Here $I(z_1, \dots, z_n)$ denotes the ideal of functions vanishing in the points $z_1, \dots, z_n$. As for the 1-dimensional case ([15]), it turns out that the compression of the standard representation $\mathcal{M}_d \to \mathbb{B}(H^2_d)$ to the subspace $H^2_d \ominus I(z_1, \dots, z_m)$ is completely isometric. Having this, one can solve the interpolation problem as in [15].

More generally, for suitable ideals $I \subset \mathcal{O}(\mathbb{D}_d)$, the representation of $\mathcal{M}_d/(I \cap \mathcal{M}_d)$ on $H^2_d \ominus (I \cap H^2_d)$ is completely isometric. This is proved by reduction to the case of finite codimensional ideals. For those, one can replace $\mathcal{M}_d/I$ by $\mathcal{Q} := \mathcal{S}_d/I$. The proof that the representation of $\mathcal{Q}$ on $H^2_d \ominus I$ is completely isometric is based on the universal property of the d-shift and the fact that $\mathcal{Q}$ can be represented completely isometrically at all, a consequence of [5].

We obtain a completely isometric representation of $\mathcal{S}_d/I$ for all ideals $I \subset \mathcal{S}_d$. In good cases, the canonical representation on $H^2_d \ominus I$ is completely isometric. We call such ideals *inner*. In general, one must add a representation coming from a *spherical operator*, that is, a d-tuple $\mathbf{Z} = (Z_1, \dots, Z_d)$ of commuting *normal* operators satisfying $Z_1 Z_1^* + \cdots + Z_d Z_d^* = 1$. It is quite remarkable that $\mathcal{S}_d/I$ has a completely isometric representation by $r \times r$-matrices if $\dim \mathcal{S}_d/I = r$.

The main difference between our methods and those of [2] and [6] is that we construct completely isometric representations of quotients of $\mathcal{S}_d$ and reduce questions about $\mathcal{M}_d$ to $\mathcal{S}_d$. In contrast, [2] and [6] work mostly with $\mathcal{M}_d$ and have little to say about quotients of $\mathcal{S}_d$.

In Section 8 we start with an r-dimensional commutative subalgebra $\mathcal{A} \subset \mathbb{M}_r$ and ask whether it can be written as a quotient of $\mathcal{S}_d$ by an inner ideal. There is an efficient algorithm to decide this question and obtain the quotient map $\mathcal{S}_d \to \mathcal{A}$. Having such a quotient map is very useful to check numerically whether a given representation of $\mathcal{A}$ is completely contractive.

A necessary condition for $\mathcal{A} \subset \mathbb{M}_r$ to be a quotient of $\mathcal{S}_d$ by an inner ideal is that $\mathcal{A} \cdot \mathcal{A}^* = \mathbb{M}_r$. In that case, we call $\mathcal{A}$ *expanding.* If $\mathcal{A}$ is expanding, a certain Hermitian sesquilinear form θ on $\mathcal{A}$ is defined. This form is diagonal in a suitable basis $X_1, \ldots, X_r$ of $\mathcal{A}$:

$$\theta\Big(\sum_{j=1}^{r} a_j X_j, \sum_{k=1}^{r} b_k X_k\Big) = \sum_{j=1}^{r} \varepsilon_j a_j \bar{b}_j$$

with certain $\varepsilon_j \in \{-1, 0, 1\}$ and always $\varepsilon_r = -1$. Let $\mathrm{p}(\mathcal{A})$, $\mathrm{o}(\mathcal{A})$, and $\mathrm{n}(\mathcal{A}) + 1$ be the numbers of **p**ositive, zer**o**, and **n**egative ε_j. These numbers are invariants of $\mathcal{A}$. The basis X_j can be ordered so that all the positive ε_j come first. $\mathcal{A}$ is a quotient of $\mathcal{S}_d$ iff $\mathrm{n}(\mathcal{A}) = 0$ and $\mathrm{p}(\mathcal{A}) \leqslant d$. The map sending $S_j \mapsto X_j$ for $j = 1, \ldots, \mathrm{p}(\mathcal{A})$ and $S_j \mapsto 0$ for $j > \mathrm{p}(\mathcal{A})$ is a complete quotient map. This criterion shows that the quotients of $\mathcal{S}_{r-1}$ form a closed subset with non-empty interior of the space of r-dimensional commutative subalgebras of $\mathbb{M}_r$.

2. PREPARATIONS: THE d-SHIFT

Arveson ([3]) defines a positive definite inner product on the space $\mathbb{C}[z_1, \ldots, z_d]$ of polynomials in d variables. The Hilbert space H_d^2 is the completion of $\mathbb{C}[z_1, \ldots, z_d]$ with respect to this inner product. There is a canonical continuous embedding $H_d^2 \subset \mathcal{O}(\mathbb{D}_d)$. The d-shift is the operator of multiplication by the coordinate functions, $(S_j f)(z) := z_j f(z)$ for all $f \in H_d^2$, $z \in \mathbb{D}_d$, $j = 1, \ldots, d$.

The inner product of H_d^2 can be characterized most easily by its reproducing kernel. Define $\langle z, x \rangle := z_1 \overline{x}_1 + \cdots + z_d \overline{x}_d$ for $z, x \in \mathbb{D}_d$ and $u_x(z) := (1 - \langle z, x \rangle)^{-1}$. We have $u_x \in H_d^2$ for all $x \in \mathbb{D}_d$ and

$$\langle f, u_x \rangle = f(x) \tag{2.1}$$

for all $f \in H_d^2$. Especially,

$$\langle u_x, u_y \rangle = (1 - \langle y, x \rangle)^{-1}. \tag{2.2}$$

Moreover, the vectors $\{u_x\}$ span a dense subset of H_d^2. Thus $(x, y) \mapsto u_y(x)$ is a reproducing kernel for the Hilbert space H_d^2. It also follows that

$$M_f^*(u_x) = \overline{f(x)} u_x, \tag{2.3}$$

where M_f^* is the adjoint of the operator M_f of multiplication by f. Thus u_x is a joint eigenvector for $\mathbf{S}^*$ with eigenvalue $\overline{x}$. There are no further joint eigenvectors for $\mathbf{S}^*$. The spectrum of the operator algebra $\mathcal{S}_d$ is homeomorphic to $\overline{\mathbb{D}}_d$. Thus functional calculus provides an inclusion $\mathcal{O}(\overline{\mathbb{D}}_d) \subset \mathcal{S}_d$.

A *d-contraction* is a d-tuple of commuting operators $\mathbf{T} = (T_1, \dots, T_d)$ on a Hilbert space $\mathcal{H}$ satisfying

$$\|T_1\xi_1 + \cdots + T_d\xi_d\|^2 \leqslant \|\xi_1\|^2 + \cdots + \|\xi_d\|^2 \tag{2.4}$$

for all $\xi_1, \dots, \xi_d \in \mathcal{H}$. An equivalent condition is that the $1 \times d$-matrix

$$\begin{pmatrix} T_1 & T_2 & \cdots & T_d \end{pmatrix}$$

be a contraction. The d-shift is the universal d-contraction in the following sense: If $\mathbf{T}$ is a d-contraction on $\mathcal{H}$, then there is a unique completely contractive representation $\varphi : \mathcal{S}_d \to \mathbb{B}(\mathcal{H})$ sending $S_j \mapsto T_j$. Conversely, if φ is a completely contractive representation of $\mathcal{S}_d$, then $\varphi(\mathbf{S}) = \big(\varphi(S_1), \dots, \varphi(S_d)\big)$ is a d-contraction. Arveson obtains much more detailed spatial information. We need some notation to formulate his results.

For $n \in \mathbb{N} \cup \{0, \infty\}$, let ℓ^2_n be the n-dimensional Hilbert space and let $n \cdot \mathbf{S}$ be the direct sum of n copies of the d-shift $\mathbf{S}$ acting on $\ell^2_n \otimes H^2_d$. A *spherical operator* is a d-tuple $\mathbf{Z} = (Z_1, \dots, Z_d)$ of commuting *normal* operators satisfying $Z_1Z_1^* + \cdots + Z_dZ_d^* = 1$. Let A be a set of operators on $\mathcal{H}$. A closed subspace $\mathcal{K}$ is called *co-invariant* for A iff its orthogonal complement is A-invariant. Equivalently, $\mathcal{K}$ is invariant for $A^* := \{T \in \mathbb{B}(\mathcal{H}) \mid T^* \in A\}$. Let $C^*(A)$ be the C^*-algebra generated by A. A closed subspace $\mathcal{K}$ is called *full* for A iff $C^*(A) \cdot \mathcal{K}$ is dense in $\mathcal{H}$.

THEOREM 2.1. (Arveson, [3]) *Let $d \in \mathbb{N}$ and let $\mathbf{T} = (T_1, \dots, T_d)$ be a d-contraction acting on some separable Hilbert space. Let $n \in \mathbb{N} \cup \{0, \infty\}$ be the rank of the operator $1 - T_1T_1^* - \cdots - T_dT_d^*$.*

Then there is a pair $(\mathbf{Z}, \mathcal{K})$ consisting of a spherical operator $\mathbf{Z}$ and a full co-invariant subspace $\mathcal{K}$ for the operator $n \cdot \mathbf{S} \oplus \mathbf{Z}$ such that $\mathbf{T}$ is unitarily equivalent to the compression of $n \cdot \mathbf{S} \oplus \mathbf{Z}$ to $\mathcal{K}$.

For $d = 1$, a spherical operator is a unitary operator and the dilation $n \cdot \mathbf{S} \oplus \mathbf{Z}$ occurring in Theorem 2.1 is the von Neumann-Wold decomposition of an isometry.

Up to a constant, the reproducing kernel of H^2_d is the $1/(d+1)$st power of the *Bergman kernel*

$$K_{\mathbb{D}_d}(x, y) = \frac{d!}{\pi^d}(1 - \langle x, y \rangle)^{-(d+1)}$$

of the domain $\mathbb{D}_d$ ([9]). Thus H^2_d is a "twisted Bergman space" in the terminology of [4]. These spaces are studied in harmonic analysis because they carry a natural projective representation of the automorphism group of the domain:

Theorem 2.2. *Let $h \in \mathrm{Aut}(\mathbb{D}_d)$ be an automorphism of $\mathbb{D}_d$, that is, a holomorphic map $\mathbb{D}_d \to \mathbb{D}_d$ with holomorphic inverse. Let*

$$\delta(z) := \big(\det Dh(z)\big)^{1/(d+1)},$$

where any holomorphic branch of the root is chosen, and let $(Tf)(z) := \delta(z) f\big(h(z)\big)$ for $f \in H^2_d$, $z \in \mathbb{D}_d$. Then T defines a unitary operator $H^2_d \to H^2_d$. This gives rise to a projective representation of $\mathrm{Aut}(\mathbb{D}_d)$ on H^2_d.

Moreover, $M_{f\circ h} \circ T = T \circ M_f$ for all $f \in \mathcal{S}_d$, so that $f \mapsto f \circ h$ is a completely isometric automorphism of $\mathcal{S}_d$.

The proof is based on the behavior of the Bergman kernel under biholomorphic mappings ([9], Proposition 6.1.7), which implies

$$\big(\det Dh(z)\big)^{\lambda}\overline{\big(\det Dh(w)\big)^{\lambda}} K_{\mathbb{D}_d}\big(h(z), h(w)\big)^{\lambda} = K_{\mathbb{D}_d}(z, w)^{\lambda}$$

for all $\lambda \in \mathbb{R}$, $z, w \in \mathbb{D}_d$, and $h \in \mathrm{Aut}(\mathbb{D}_d)$. See [1].

3. Completely isometric representations of quotients of $\mathcal{S}_d$

Let $I \subset \mathcal{S}_d$ be a closed ideal. Let $\widehat{I} \subset \mathrm{Spec}(\mathcal{S}_d)$ be the set of all maximal ideals containing I. Hence $\widehat{I} \cong \mathrm{Spec}(\mathcal{S}_d/I)$. Identify $\mathrm{Spec}(\mathcal{S}_d) \cong \overline{\mathbb{D}}_d$ and consider $\widehat{I} \subset \overline{\mathbb{D}}_d$. Let $\partial\widehat{I}$ be the smallest compact subset of $\widehat{I}$ such that $\widehat{I} = \overline{\widehat{I} \cap \mathbb{D}_d} \cup \partial\widehat{I}$. Thus

$$\partial\widehat{I} := \overline{\widehat{I} \setminus \overline{\widehat{I} \cap \mathbb{D}_d}}.$$

The ideal I is called *inner* iff $\partial\widehat{I} = \emptyset$ or equivalently $\widehat{I} \cap \mathbb{D}_d$ is dense in $\widehat{I}$. For example, the ideal $I := \{0\}$ is inner because $\widehat{I} = \overline{\mathbb{D}}_d$.

Let $P : H^2_d \to H^2_d \ominus I$ be the orthogonal projection onto the orthogonal complement of I. Let $\varphi_0 : \mathcal{S}_d \to \mathbb{B}(H^2_d \ominus I)$ be the compression $f \mapsto PfP$. The map φ_0 is a unital, completely contractive homomorphism because the closure $\overline{I}$ of I in H^2_d is $\mathcal{S}_d$-invariant. Moreover, $\ker \varphi_0 = \overline{I} \cap \mathcal{S}_d \supset I$. Consequently, φ_0 descends to a completely contractive representation of the quotient algebra $\mathcal{S}_d/I$. It sends $[S_j]$, the class of S_j in the quotient, to $S(I)_j := PS_jP$.

Let $\mathbf{N}(I)$ be a spherical operator with spectrum $\partial\widehat{I}$, acting on some Hilbert space $\mathcal{H}_\partial$. Compose the Gelfand transformation for the commutative Banach algebra $\mathcal{S}_d/I$ with the functional calculus for the normal multi-operator $\mathbf{N}(I)$ to get a completely contractive representation $\varphi_\partial : \mathcal{S}_d/I \to C(\widehat{I}) \to \mathbb{B}(\mathcal{H}_\partial)$ sending $[S_j] \mapsto \mathbf{N}(I)_j$. Thus

$$\psi := \varphi_0 \oplus \varphi_\partial : \mathcal{S}_d/I \to \mathbb{B}\big((H^2_d \ominus I) \oplus \mathcal{H}_\partial\big), \qquad \psi[S_j] := S(I)_j \oplus N(I)_j,$$

is a completely contractive representation of $\mathcal{S}_d/I$.

THEOREM 3.1. *The representation ψ is completely isometric. If I is inner, then the representation $\varphi_0 : \mathcal{S}_d/I \to \mathbb{B}(H^2_d \ominus I)$ is completely isometric.*

Proof. Any quotient of a unital operator algebra by a closed ideal is again a unital operator algebra ([5]). Thus $\mathcal{S}_d/I$ has a unital, completely isometric representation $\rho : \mathcal{S}_d/I \to \mathbb{B}(\mathcal{H})$. Let $\mathcal{A}$ be the closure of the range of ψ, that is, the unital operator algebra generated by the multi-operator $\mathbf{S}(I) \oplus \mathbf{N}(I)$ on $H^2_d \ominus I \oplus \mathcal{H}_\partial$. The theorem follows once the homomorphism $h : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ sending $\psi(S_j)$ to $\rho[S_j]$ is shown to be well defined and completely contractive. Then h is completely isometric.

The multi-operator $\rho[\mathbf{S}] = \big(\rho[S_1], \dots, \rho[S_d]\big)$ is a d-contraction. We leave it to the reader to show that $\mathcal{H}$ can be chosen to be separable (or to extend Theorem 2.1 to the case of d-contractions on non-separable Hilbert spaces).

Apply Theorem 2.1 to $\rho[\mathbf{S}]$. This yields $n \in \mathbb{N} \cup \{0, \infty\}$, a spherical operator $\mathbf{Z}$ acting on some Hilbert space $\mathcal{H}_{\mathbf{Z}}$, and a full co-invariant subspace $\mathcal{K}$ of

$$\widehat{\mathcal{H}} := (\ell^2_n \otimes H^2_d) \oplus \mathcal{H}_{\mathbf{Z}}, \tag{3.1}$$

such that $\rho[\mathbf{S}]$ is unitarily equivalent to the compression of $n \cdot \mathbf{S} \oplus \mathbf{Z}$ to $\mathcal{K}$. Let $\widehat{\rho} : \mathcal{S}_d \to \mathbb{B}(\widehat{\mathcal{H}})$ be the representation defined by $\widehat{\rho}(S_j) := n \cdot S_j \oplus Z_j$. Since $\mathcal{K}$ is co-invariant for $n \cdot \mathbf{S} \oplus \mathbf{Z}$, its orthogonal complement $\mathcal{K}^\perp$ is $\widehat{\rho}(\mathcal{S}_d)$-invariant. Let

$$\mathcal{H}_2 := \widehat{\mathcal{H}} \ominus \widehat{\rho}(I) \cdot \widehat{\mathcal{H}} = \{\xi \in \widehat{\mathcal{H}} \mid \xi \perp \widehat{\rho}(f)\eta \text{ for all } f \in I,\ \eta \in \widehat{\mathcal{H}}\}.$$

We claim that $\mathcal{K} \subset \mathcal{H}_2$. Equivalently, $\widehat{\rho}(f)\eta \perp \mathcal{K}$ for all $f \in I$, $\eta \in \widehat{\mathcal{H}}$. This is evident for $\eta \in \mathcal{K}^\perp$ because $\mathcal{K}^\perp$ is $\widehat{\rho}(\mathcal{S}_d)$-invariant. Since the compression of $\widehat{\rho}(f)$ to $\mathcal{K}$ is $\rho[f] = \rho(0) = 0$, we also get $\widehat{\rho}(f)\eta \perp \mathcal{K}$ for $\eta \in \mathcal{K}$. Thus $\widehat{\rho}(f)\eta \perp \mathcal{K}$ for all $\eta \in \mathcal{K}^\perp \oplus \mathcal{K} = \widehat{\mathcal{H}}$ as desired. Equation (3.1) implies immediately that

$$\mathcal{H}_2 \cong \big(\ell^2_n \otimes (H^2_d \ominus I)\big) \oplus (\mathcal{H}_{\mathbf{Z}} \ominus \widehat{\rho}(I)\mathcal{H}_{\mathbf{Z}}).$$

Since $\mathbf{Z}$ is normal and $\widehat{\rho}(I)\mathcal{H}_{\mathbf{Z}}$ is $\mathbf{Z}$-invariant, the subspace $\widehat{\rho}(I)\mathcal{H}_{\mathbf{Z}}$ is also invariant for $\mathbf{Z}^*$, that is, a reducing subspace. Thus $C^*(n \cdot \mathbf{S} \oplus \mathbf{Z})$ maps $\widehat{\mathcal{H}} \ominus \widehat{\rho}(I)\mathcal{H}_{\mathbf{Z}} \supset \mathcal{K}$ into itself. Since $\mathcal{K}$ is full, it follows that $\widehat{\rho}(I) \cdot \mathcal{H}_{\mathbf{Z}} = \{0\}$. Therefore, $\operatorname{Spec}(\mathbf{Z}) \subset \widehat{I}$.

The homomorphism $\psi : \mathcal{S}_d \to \mathcal{A}$ gives rise to a continuous map $\psi^* : \operatorname{Spec}(\mathcal{A}) \to \operatorname{Spec}(\mathcal{S}_d) \cong \overline{\mathbb{D}}_d$. We claim that

$$\psi^*\big(\operatorname{Spec}(\mathcal{A})\big) = \widehat{I}. \tag{3.2}$$

This is of course a necessary condition for $\mathcal{S}_d/I \cong \mathcal{A}$. Since ψ annihilates I, it is clear that $\psi^*\big(\operatorname{Spec}(\mathcal{A})\big) \subset \widehat{I}$. Let $x \in \widehat{I} \cap \mathbb{D}_d$. Then $u_x \perp I$ by equation (2.1). Consequently,

$$f \mapsto \|u_x\|_2^{-2}\langle f u_x, u_x\rangle = \|u_x\|_2^{-2}\langle u_x, f^* u_x\rangle = f(x)$$

is a well defined and contractive linear functional on $\mathcal{A}$. Thus $x \in \operatorname{Spec}(\mathcal{A})$. Hence $\widehat{I} \cap \mathbb{D}_d \subset \operatorname{Spec}(\mathcal{A})$ and $\overline{\widehat{I} \cap \mathbb{D}_d} \subset \operatorname{Spec}(\mathcal{A})$ by compactness. The other points of $\widehat{I} \cap \partial\mathbb{D}_d$ are in $\partial\widehat{I} = \operatorname{Spec}\big(\mathbf{N}(I)\big) \subset \operatorname{Spec}(\mathcal{A})$. Equation (3.2) follows.

Since $\operatorname{Spec}(\mathcal{A}) \cong \widehat{I}$, the Gelfand transformation gives rise to a completely contractive homomorphism $g_1 : \mathcal{A} \to C(\widehat{I})$. Since $\mathbf{Z}$ is a normal operator with spectrum contained in $\widehat{I}$, functional calculus for $\mathbf{Z}$ gives rise to a $*$-representation $g_2 : C(\widehat{I}) \to \mathbb{B}(\mathcal{H}_{\mathbf{Z}})$. Let $f : \mathcal{A} \to \mathbb{B}(H^2_d \ominus I)$ be the compression to the first summand. Let $n \cdot f : \mathcal{A} \to \mathbb{B}\bigl(\ell^2_n \otimes (H^2_d \ominus I)\bigr)$ be the direct sum of n copies of f, that is, $n \cdot f : x \mapsto \mathrm{id}_{\ell^2_n} \otimes f(x)$. Clearly, $n \cdot f$ is a completely contractive representation.

Thus we get a completely contractive representation $n \cdot f \oplus (g_2 \circ g_1) : \mathcal{A} \to \mathbb{B}(\mathcal{H}_2)$ that maps $\psi(\mathbf{S})$ to the compression of $\widehat{\rho}(\mathbf{S})$ to $\mathcal{H}_2$. Compressing further to $\mathcal{H} \cong \mathcal{K} \subset \mathcal{H}_2$, we see that h is well defined and completely contractive, as desired. ∎

Suppose that I has finite codimension r, so that $\mathcal{Q} := \mathcal{S}_d/I$ is r-dimensional. Since the spectrum $\widehat{I}$ of $\mathcal{Q}$ is finite, we get $\partial\widehat{I} = \widehat{I} \cap \partial\mathbb{D}_d$. Let $\partial\widehat{I}$ have s elements $x_1, \dots, x_s$. We choose $\mathbf{N}(I)$ as a diagonal multi-operator on $\mathcal{H}_\partial := \mathbb{C}^s$. By Theorem 3.1, $\varphi_0 \oplus \varphi_\partial$ is a completely isometric representation of $\mathcal{Q}$. Let $\mathcal{Q}_0 := \varphi_0(\mathcal{Q})$.

CORALLARY 3.2. *The operator algebra $\mathcal{Q}$ is completely isometrically isomorphic to the orthogonal direct sum $\mathcal{Q}_0 \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, with s copies of $\mathbb{C}$ corresponding to $x_1, \dots, x_s \in \operatorname{Spec}(\mathcal{Q}) \cap \partial\mathbb{D}_d$. In addition, $\dim H^2_d \ominus I = r - s$.*

Thus $\mathcal{Q}$ has a completely isometric representation by $r \times r$-matrices.

Proof. We claim that $\operatorname{Spec}(\mathcal{Q}_0) \subset \mathbb{D}_d$. The orthogonal projection of $1 \in H^2_d$ to $H^2_d \ominus I$ is a cyclic vector for $\mathcal{Q}_0$. Hence $\dim \mathcal{Q}_0 = \dim H^2_d \ominus I$ is finite. Let $x \in \operatorname{Spec}(\mathcal{Q}_0)$ and let $\mathcal{Q}_0^* \subset \mathbb{B}(H^2_d \ominus I)$ be the algebra of adjoints of elements of $\mathcal{Q}_0$. The map $q \mapsto \overline{q(x)}$ is a character of $\mathcal{Q}_0^*$. Since $H^2_d \ominus I$ is finite dimensional, elementary linear algebra shows that there is an eigenvector $\eta \in H^2_d \ominus I$ with $\varphi_0(f)^*\eta = \overline{f(x)}\eta$ for all $f \in \mathcal{S}_d$. Since $H^2_d \ominus I$ is $\mathcal{S}_d^*$-invariant, $\eta \in H^2_d$ is also a joint eigenvector for $\mathbf{S}^*$. But we know all joint eigenvectors of $\mathbf{S}^*$: They are the vectors u_x with $x \in \mathbb{D}_d$. Therefore $x \in \mathbb{D}_d$.

Since $\operatorname{Spec}(\mathcal{Q}_0) \subset \mathbb{D}_d$, evaluation at $x_1, \dots, x_s$ provides s linearly independent linear functionals on the kernel of the quotient map $\mathcal{Q} \to \mathcal{Q}_0$. Therefore, $\dim \mathcal{Q}_0 \leqslant r - s$. The canonical map $\mathcal{Q} \to \mathcal{Q}_0 \oplus \mathbb{C}^s$ is completely isometric by Theorem 3.1. Dimension counting shows that it is a completely isometric *isomorphism* and that $\dim \mathcal{Q}_0 = r - s$. Hence also $\dim H^2_d = r - s$. ∎

The kernel of $\varphi_0 : \mathcal{S}_d \to \mathbb{B}(H^2_d \ominus I)$ is equal to the relative closure of I with respect to the H^2_d-norm. Thus Corollary 3.2 implies that a finite codimensional ideal $I \subset \mathcal{S}_d$ is inner if and only if it is relatively closed with respect to the H^2_d-norm.

4. QUOTIENTS OF $\mathcal{M}_d$

We are going to use Theorem 3.1 to compute completely isometric representations of quotients $\mathcal{M}_d/(I \cap \mathcal{M}_d)$ for suitable ideals $I \subset \mathcal{O}(\mathbb{D}_d)$. To explain when an ideal $I \subset \mathcal{O}(\mathbb{D}_d)$ is suitable, we need some notation. For $x \in \mathbb{D}_d$, let $\mathcal{O}_x$ be the ring of germs of holomorphic functions near x. There is a canonical map $\pi_x : \mathcal{O}(\mathbb{D}_d) \to \mathcal{O}_x$. Let $I_x := \pi_x(I) \cdot \mathcal{O}_x$ be the ideal in $\mathcal{O}_x$ generated by the image of I. We call I *local* iff $\pi_x(f) \in I_x$ for all $x \in \mathbb{D}_d$ already implies $f \in I$. Equivalently, I is the ring of global sections of a coherent sheaf on $\mathbb{D}_d$.

It is clear that a local ideal is closed in the topology of locally uniform convergence. Conversely, if the ideal I is closed in this topology, it is local. This deep theorem is due to Henri Cartan ([7], p. 181).

Let $\mathcal{D}_x \subset \mathcal{O}'_x$ be the space of functionals of the form $l(f) = P(\partial z_1, \dots, \partial z_k)$ $f(x)$, where P is some polynomial in differentiation operators. Thus $\mathcal{D}_x$ is the dual space of the ring of formal power series at x with its canonical product topology. We view $\mathcal{D}_x \subset \mathcal{O}(\mathbb{D}_d)'$ in the obvious way and let $\mathcal{D}$ be the linear span of the subspaces $\mathcal{D}_x$.

The ideal structure of the ring $\mathcal{O}_x$ is quite well understood. It turns out that an ideal $J \subset \mathcal{O}_x$ is automatically of the form $N^\perp := \{f \in \mathcal{O}_x \mid f \perp N\}$ for some subspace $N \subset \mathcal{D}_x$, where $f \perp N$ means $l(f) = 0$ for all $l \in N$. Hence a local ideal $I \subset \mathcal{O}(\mathbb{D}_d)$ satisfies $I = (I^\perp \cap \mathcal{D})^\perp$.

We claim that any local ideal $I \subset \mathcal{O}(\mathbb{D}_d)$ can be described as the intersection of a net of finite codimensional local ideals (I_j). We will use this to reduce assertions about arbitrary local ideals to the finite codimensional case. Actually, one can construct a decreasing *sequence* of ideals whose intersection is I. Since the proof of this stronger statement is rather unpleasant, we prefer to work with nets.

To prove the claim, we consider a finite subset $j := \{l_1, \dots, l_m\} \subset I^\perp \cap \mathcal{D}$. By definition of $\mathcal{D}$, these differentiation operators involve only derivatives up to some finite order s at finitely many points of $\mathbb{D}_d$. Let $I'_j \subset \mathcal{O}(\mathbb{D}_d)$ be the ideal of functions vanishing in these finitely many points up to order s and let $I_j := I + I'_j \subset \mathcal{O}(\mathbb{D}_d)$. It is not hard to see that I_j is a local ideal of finite codimension that is annihilated by the functionals $l_1, \dots, l_m$. If we let j run through all finite subsets of $I^\perp \cap \mathcal{D}$, we obtain a net of ideals (I_j) with $\bigcap I_j \supset I$ and $\bigcap I_j \subset (I^\perp \cap \mathcal{D})^\perp = I$.

If $I \subset \mathcal{O}(\mathbb{D}_d)$ is a local ideal, we write $H^2_d \ominus I$ for $H^2_d \ominus (I \cap H^2_d)$, $\mathcal{M}_d/I$ for $\mathcal{M}_d/(I \cap \mathcal{M}_d)$, and $\mathcal{S}_d/I$ for $\mathcal{S}_d/(I \cap \mathcal{S}_d)$. The subspaces $I \cap \mathcal{M}_d$ and $I \cap H^2_d$ are closed with respect to the norms of $\mathcal{M}_d$ and H^2_d, respectively. It can happen easily that $I \cap \mathcal{M}_d = 0$. In this case, most assertions in the following are rather empty.

If $I \subset \mathcal{O}(\mathbb{D}_d)$ is a finite codimensional local ideal, then each element of $\mathcal{O}(\mathbb{D}_d)/I$ can be represented by a polynomial. Thus $\mathcal{S}_d/I \cong \mathcal{M}_d/I \cong H^2_d \ominus I \cong \mathcal{O}(\mathbb{D}_d)/I$. Furthermore, $I \cap \mathcal{S}_d$ is relatively closed with respect to the H^2_d-norm and thus inner.

Theorem 4.1. *Let $I \subset \mathcal{O}(\mathbb{D}_d)$ be a local ideal. The subspace $H^2_d \ominus I \subset H^2_d$ is co-invariant for $\mathcal{M}_d$. The compression $\varphi : \mathcal{M}_d \to \mathbb{B}(H^2_d \ominus I)$ of the standard representation of $\mathcal{M}_d$ to $H^2_d \ominus I$ descends to a completely isometric representation of $\mathcal{M}_d/I$. Its image $\varphi(\mathcal{M}_d)$ is equal to the weak closure of $\varphi(\mathcal{S}_d)$ and equal to the commutant of $\varphi(\mathbf{S})$, that is, the set of operators commuting with $\varphi(\mathbf{S})$.*

Let $T \in \mathbb{M}_n\big(\mathbb{B}(H^2_d \ominus I)\big)$ commute with $1_n \otimes \varphi(\mathbf{S})$. If $\|T\| \leqslant 1$, then $T = \varphi_{(n)}(\widehat{T})$ for some $\widehat{T} \in \mathbb{M}_n(\mathcal{M}_d)$ with $\|\widehat{T}\| \leqslant 1$.

Proof. The subspace $H^2_d \ominus I$ is the orthogonal complement of the closed $\mathcal{M}_d$-invariant subspace $I \cap H^2_d$ and thus co-invariant for $\mathcal{M}_d$. Hence φ is a completely contractive representation with kernel $I \cap H^2_d \cap \mathcal{M}_d = I \cap \mathcal{M}_d$.

We will show the following: if $T \in \mathbb{M}_n\big(\mathbb{B}(H^2_d \ominus I)\big)$ commutes with $1_n \otimes \varphi(\mathbf{S})$ and $\|T\| \leqslant 1$, then there is a net $(\widehat{T}_j)$ in $\mathbb{M}_n(\mathcal{S}_d)$ with $\|\widehat{T}_j\| \leqslant 1$ for all j such that $\varphi_{(n)}(\widehat{T}_j)$ converges towards T in the weak operator topology. Thus the commutant of $1_n \otimes \mathbf{S}$ and the weak closure of $\mathbb{M}_n\big(\varphi(\mathcal{S}_d)\big)$ are equal.

The unit ball of $\mathbb{M}_n(\mathcal{M}_d)$ is weakly compact. Hence a subnet of $(\widehat{T}_j)$ converges weakly towards some $\widehat{T} \in \mathbb{M}_n(\mathcal{M}_d)$. Necessarily, $\varphi_{(n)}(\widehat{T}) = T$ because φ is continuous with respect to the weak operator topology. The theorem follows.

It remains to construct the net $(\widehat{T}_j)$. We assume that $n = 1$ to simplify notation. The argument is the same in the matrix valued case.

Let (I_j) be a net of finite codimensional local ideals in $\mathcal{O}(\mathbb{D}_d)$ with $I = \bigcap I_j$. Let $P : H^2_d \to H^2_d \ominus I$ and $P_j : H^2_d \to H^2_d \ominus I_j$ be the orthogonal projections. Let $\varphi_j : \mathcal{S}_d \to \mathbb{B}(H^2_d \ominus I_j)$ be the compression of the standard representation.

$P1 = P(1) \in H^2_d \ominus I$ is a cyclic vector for $\varphi(\mathcal{S}_d)$. Since T and $\varphi(\mathcal{S}_d)$ commute, $T(f) = P(T(P1) \cdot f)$ for all $f \in H^2_d \ominus I$. Thus $P_j T P_j(f) = P_j T(f) = P_j(T(P1) \cdot f)$ for all $f \in H^2_d \ominus I_j$. Since $\mathcal{S}_d/I_j \cong \mathcal{O}(\mathbb{D}_d)/I_j$, the operator $P_j T P_j \in \mathbb{B}(H^2_d \ominus I_j)$ must be in the range of φ_j. The homomorphism φ_j is a complete quotient map by Theorem 3.1. Hence there is $\widehat{T}_j \in \mathcal{S}_d$ with $\|\widehat{T}_j\| \leqslant 1$ and $\varphi_j(\widehat{T}_j) = (1 - 1/j) P_j T P_j$. Thus $\langle \varphi(\widehat{T}_j)\xi, \eta\rangle = (1 - 1/j)\langle T\xi, \eta\rangle$ for all $\xi, \eta \in H^2_d \ominus I_j$. Hence $\lim \langle \varphi(\widehat{T}_j)\xi, \eta\rangle = \langle T\xi, \eta\rangle$, whenever $\xi, \eta \in \Sigma := \bigcup\limits_{j=1}^{\infty} H^2_d \ominus I_j$.

Since Σ is dense in $H^2_d \ominus I$ and $\{\varphi(\widehat{T}_j)\}$ is uniformly bounded, $\varphi(\widehat{T}_j)$ converges in the weak operator topology towards T. By the way, $\big(\varphi(\widehat{T}_j)\big)$ converges even in the $*$-strong operator topology. That is, $\widehat{T}_j \xi \to T\xi$ and $\widehat{T}_j^* \xi \to T^* \xi$ for all ξ. ∎

For $I = \{0\}$, Theorem 4.1 asserts that $\mathcal{S}_d$ is weakly dense in $\mathcal{M}_d$.

5. THE FANTAPPIÈ TRANSFORM

Composing the adjoint of the inclusion $H_d^2 \to \mathcal{O}(\mathbb{D}_d)$ with the canonical conjugate linear isomorphism $(H_d^2)' \to H_d^2$, we obtain a continuous, conjugate linear map $\mathcal{F} : \mathcal{O}(\mathbb{D}_d)' \to H_d^2$. This map is characterized by $l(f) = \langle f, \mathcal{F}(l)\rangle$ for all $l \in \mathcal{O}(\mathbb{D}_d)'$ and $f \in H_d^2$. Define $\delta_x \in \mathcal{O}(\mathbb{D}_d)'$ by $\delta_x(f) := f(x)$ for all $f \in \mathcal{O}(\mathbb{D}_d)$. Equation (2.1) asserts that $\mathcal{F}(\delta_x) = u_x$, so that $\mathcal{F}(\delta_x)(y) = u_x(y) = (1-\langle y, x\rangle)^{-1} = \overline{\delta_x(u_y)}$. Since the functionals δ_x span a weak-$*$-dense subspace of $\mathcal{O}(\mathbb{D}_d)'$, we conclude that

$$\mathcal{F}(l)(y) = \overline{l(u_y)} \quad \text{for all } l \in \mathcal{O}(\mathbb{D}_d)',\ y \in \mathbb{D}_d. \tag{5.1}$$

We might call $\mathcal{F}$ the *Fantappiè transform* because (5.1) without conjugations is the definition of the Fantappiè transform for the domain $\mathbb{D}_d$ ([8]). The main theorem about the Fantappiè transform in [8] asserts in our special case that $\mathcal{F}$ is a homeomorphism from $\mathcal{O}(\mathbb{D}_d)'$ onto $\mathcal{O}(\overline{\mathbb{D}}_d)$.

PROPOSITION 5.1. *Let $I \subset \mathcal{O}(\mathbb{D}_d)$ be a local ideal. Define $N := I^\perp \cap \mathcal{D} \subset \mathcal{O}(\mathbb{D}_d)'$.*

The subspace $\mathcal{F}(N) \subset H_d^2 \ominus I$ is dense. Every function in $\mathcal{F}(N)$ is a rational function. Thus rational functions are dense in $H_d^2 \ominus I$. In particular, if I has finite codimension, then $H_d^2 \ominus I$ contains only rational functions.

Proof. Since I is local, $I = N^\perp$ and therefore $I \cap H_d^2 = \mathcal{F}(N)^\perp$. Thus $\mathcal{F}(N)$ is dense in $H_d^2 \ominus I$. If $l(f) = P(\partial z_1, \ldots, \partial z_d)f(x_0)$ for a certain polynomial P of degree k, then $\mathcal{F}(l)(y) = \overline{l(u_y)} = p(y)(1 - \langle y, x_0\rangle)^{-k-1}$ for another polynomial p. Thus $\mathcal{F}(N)$ contains only rational functions. ∎

The commutative algebra $\mathcal{O}(\mathbb{D}_d)$ acts on its dual space $\mathcal{O}(\mathbb{D}_d)'$ by $f \cdot l(h) := l(f \cdot h)$ for all $f, h \in \mathcal{O}(\mathbb{D}_d)$, $l \in \mathcal{O}(\mathbb{D}_d)'$. Using the Fantappiè transform, we get a corresponding action on $\mathcal{O}(\overline{\mathbb{D}}_d)$ by $f \cdot \mathcal{F}(l) := \mathcal{F}(f \cdot l)$. We have

$$\langle g, \mathcal{F}(f \cdot l)\rangle = f \cdot l(g) = l(f \cdot g) = \langle f \cdot g, \mathcal{F}(l)\rangle = \langle M_f g, \mathcal{F}(l)\rangle = \langle g, M_f^* \mathcal{F}(l)\rangle \tag{5.2}$$

for all $f \in \mathcal{M}_d$, $l \in \mathcal{O}(\mathbb{D}_d)'$, $g \in H_d^2$. Hence

$$M_f^*\big(\mathcal{F}(l)\big) = \mathcal{F}(f \cdot l) \tag{5.3}$$

for all $f \in \mathcal{M}_d$, $l \in \mathcal{O}(\mathbb{D}_d)'$. If $f \in \mathcal{O}(\mathbb{D}_d)$ is not necessarily a multiplier, we still define a bounded linear map $M_f^* : \mathcal{O}(\overline{\mathbb{D}}_d) \to \mathcal{O}(\overline{\mathbb{D}}_d)$ by (5.3) and view M_f^* as a densely defined unbounded operator on H_d^2. However, already for $d = 1$ the adjoint of M_f^* need not be densely defined. Thus M_f^* may fail to be contained in the adjoint of another unbounded operator.

Let $f \in \mathcal{O}(\mathbb{D}_d)$. We say that M_f^* has *positive real part* and write $\operatorname{Re} M_f^* \geqslant 0$ iff the $\mathbb{R}$-bilinear form $\xi, \eta \mapsto \operatorname{Re}\langle M_f^*\xi, \eta\rangle$ on $\mathcal{O}(\overline{\mathbb{D}}_d)$ is positive definite. Observe that the map $f \mapsto \operatorname{Re}\langle M_f^*\xi, \eta\rangle$ is a continuous functional on $\mathcal{O}(\mathbb{D}_d)$ for fixed $\xi, \eta \in \mathcal{O}(\overline{\mathbb{D}}_d)$. Therefore, the set of functions $f \in \mathcal{O}(\mathbb{D}_d)$ with $\operatorname{Re} M_f^* \geqslant 0$ is closed in the topology of locally uniform convergence.

Let $I \subset \mathcal{O}(\mathbb{D}_d)$ be a local ideal, $I^\perp \subset \mathcal{O}(\mathbb{D}_d)'$ its annihilator, and $f \in \mathcal{O}(\mathbb{D}_d)/I$. Then $l \mapsto f \cdot l$ well defines a bounded linear map $I^\perp \to I^\perp$ because I is an ideal. Using (5.3), we define a bounded operator $M_f^* : \mathcal{F}(I^\perp) \to \mathcal{F}(I^\perp)$ and view M_f^* as an unbounded operator on $H_d^2 \ominus I \supset \mathcal{F}(I^\perp)$. This operator is densely defined by Proposition 5.1. We say that M_f^* has positive real part iff the $\mathbb{R}$-bilinear form $a, b \mapsto \operatorname{Re}\langle M_f^* \mathcal{F}(a), \mathcal{F}(b)\rangle$ on $I^\perp$ is positive definite. It should be evident how to carry these definitions over to matrix valued holomorphic functions.

Theorem 5.2. *Let $T \in \mathbb{M}_n(\mathcal{O}(\mathbb{D}_d)/I)$. Then there is $\widehat{T} \in \mathbb{M}_n\big(\mathcal{O}(\mathbb{D}_d)\big)$ with $T = [\widehat{T}]$ such that $M_{\widehat{T}}^*$ has positive real part if and only if M_T^* has positive real part.*

If M_T^* is bounded and $\operatorname{Re} M_T^*$ is positive and invertible, then $\widehat{T}$ can be chosen such that $M_{\widehat{T}}^* \in \mathbb{M}_n(\mathcal{M}_d)$ is bounded and $\operatorname{Re} M_{\widehat{T}}^*$ is positive and invertible.

Proof. The second half is easier because it does not involve unbounded operators. In [10] the *positive cone* of an operator algebra $\mathcal{A}$ is defined to be the set of all $x \in \mathcal{A}$ for which $x + x^*$ is positive and invertible. Functional calculus with $\mathcal{C}(z) := (1-z)/(1+z)$ is a bijection between the positive cone and the open unit ball of $\mathcal{A}$.

Assume that M_T is bounded and that $2 \operatorname{Re} M_T := M_T + M_T^*$ is invertible. The operator $\mathcal{C}(M_T)$ has norm strictly less than 1, hence can be lifted to an operator $\widehat{\mathcal{C}(M_T)} \in \mathbb{M}_n(\mathcal{M}_d)$ of norm strictly less than 1 by Theorem 4.1. Then $\mathcal{C}\big(\widehat{\mathcal{C}(M_T)}\big) \in \mathbb{M}_n(\mathcal{M}_d)$ is the desired lifting of T.

In the unbounded situation, the assertion is proved by compressing to the complements of finite codimensional ideals. Let (I_j) be a net of finite codimensional local ideals in $\mathcal{O}(\mathbb{D}_d)$ with $I = \bigcap I_j$. Let $P_j : H_d^2 \to H_d^2 \ominus I_j$ be the orthogonal projection. Proposition 5.1 implies that $P_j M_T^* P_j$ is defined on all of $H_d^2 \ominus I_j$. We have $\operatorname{Re} P_j M_T^* P_j \geqslant 0$, so that $1/j + P_j M_T^* P_j$ has positive and invertible real part.

Hence there is $\widehat{T}_j \in \mathbb{M}_n(\mathcal{M}_d)$ with positive and invertible real part whose compression to $H_d^2 \ominus I_j$ is $1/j + P_j M_T^* P_j$. View $\widehat{T}_j$ as a holomorphic function $\mathbb{D}_d \to \mathbb{M}_n$, then $\operatorname{Re} \widehat{T}_j(x) \geqslant 0$ for all $x \in \mathbb{D}_d$. Thus $(\widehat{T}_j)$ is a normal family. A subnet of $(\widehat{T}_j)$ converges locally uniformly towards a holomorphic function $\widehat{T} : \mathbb{D}_d \to \mathbb{M}_n$. Since $\operatorname{Re} M_{\widehat{T}_j} \geqslant 0$ for all j, it follows that $\operatorname{Re} M_{\widehat{T}} \geqslant 0$. Furthermore, $[\widehat{T}] = [T]$ in $\mathbb{M}_n(\mathcal{O}(\mathbb{D}_d)/I_j)$ for all j and thus $[\widehat{T}] = T$ in $\mathbb{M}_n(\mathcal{O}(\mathbb{D}_d)/I)$. ∎

6. Uniqueness and construction of solutions

For scalar valued interpolation, we show that Sarason's criterion (see [15]) for the uniqueness of solutions generalizes to our situation. Let $I \subset \mathcal{O}(\mathbb{D}_d)$ be a local ideal and $\xi \in H_d^2 \ominus I$ with $\|\xi\| = 1$. Let $T \in \mathcal{M}_d/I$ satisfy $\|T\| = 1$. We call ξ a *maximal vector* for T iff $\|T\xi\| = 1$. Let $T \in \mathcal{O}(\mathbb{D}_d)/I$ satisfy $\operatorname{Re} M_T^* \geqslant 0$. We call ξ a *zero vector* for $\operatorname{Re} M_T^*$ iff $\xi \in \mathcal{O}(\overline{\mathbb{D}}_d) \subset H_d^2$ and $\operatorname{Re}\langle \xi, M_T^* \xi\rangle = 0$. If M_T^* is bounded, this is equivalent to $\xi \in \ker \operatorname{Re} M_T^*$, that is, $M_T^* \xi = -T\xi$.

If I has finite codimension, then a maximal vector for T exists and can be computed explicitly whenever $\|T\| = 1$. A zero vector for $\operatorname{Re} M_T^*$ exists and can be computed explicitly whenever $\operatorname{Re} M_T^* \geqslant 0$ and $\operatorname{Re} M_T^*$ is *not* invertible.

Theorem 6.1. *Let $I \subset \mathcal{O}(\mathbb{D}_d)$ be a local ideal.*

Let $T \in \mathcal{M}_d/I$ with $\|T\| = 1$. If $\xi \in H^2_d \ominus I$ is a maximal vector for T, then there is a unique $f \in \mathcal{M}_d$ with $\|f\| \leqslant 1$ and $[f] = T$. Namely, $f(z) = (T\xi)(z)/\xi(z)$ for all $z \in \mathbb{D}_d$ with $\xi(z) \neq 0$.

Let $T \in \mathcal{O}(\mathbb{D}_d)/I$ with $\operatorname{Re} M_T^* \geqslant 0$. If $\xi \in H^2_d \ominus I$ is a zero vector for T, then there is a unique $f \in \mathcal{O}(\mathbb{D}_d)$ with $\operatorname{Re} M_f^* \geqslant 0$ and $[f] = T$. Namely, $f(z) = -(M_T^*\xi)(z)/\xi(z)$ for all $z \in \mathbb{D}_d$ with $\xi(z) \neq 0$. If $T \in \mathcal{M}_d/I$, then $-M_T^*\xi = T\xi$.

If I is finite codimensional, then the solution is a rational function in both cases.

Proof. We consider first the case $\|T\| \leqslant 1$. By Theorem 4.1, there is $f \in \mathcal{M}_d$ with $\varphi(f) = T$ and $\|f\| = 1$. Since $\|f\xi\| = 1$ and $\|P_I^\perp(f\xi)\| = \|T\xi\| = 1$, it follows that $f \cdot \xi = T\xi$. View f, ξ, and $T\xi$ as holomorphic functions on $\mathbb{D}_d$. Since $\xi \neq 0$, the set of those $z \in \mathbb{D}_d$ with $\xi(z) \neq 0$ is dense. On this set, $f(z) = (T\xi)(z)/\xi(z)$. This determines f uniquely on all of $\mathbb{D}_d$.

If I is finite codimensional, then $H^2_d \ominus I$ only contains rational functions by Proposition 5.1. Thus f is a rational function as a quotient of two rational functions.

In the case $\operatorname{Re} M_T^* \geqslant 0$ we use Theorem 5.2 to obtain a lift $f \in \mathcal{O}(\mathbb{D}_d)$ with $\operatorname{Re} M_f^* \geqslant 0$. The remaining assertions follow as above if we show that $f \cdot \xi = -M_T^*\xi$.

Since f lifts T, we have $M_f^*\xi = M_T^*\xi$. Let $l \in \mathcal{O}(\mathbb{D}_d)'$ and $\eta := \mathcal{F}(l)$. Using $\langle M_f^*\xi, \eta\rangle = l(M_T^*\xi)$, $\langle \xi, M_f^*\eta\rangle = \overline{l(f \cdot \xi)}$, and $\operatorname{Re}\langle \xi, M_f^*\xi\rangle = 0$, we compute

$$\begin{aligned}\operatorname{Re}\langle \xi + \lambda\eta, M_f^*(\xi + \lambda\eta)\rangle &= \operatorname{Re}\bigl[\overline{\lambda}(\langle \xi, M_f^*\eta\rangle + \langle M_T^*\xi, \eta\rangle) + |\lambda|^2\langle \eta, M_f^*\eta\rangle\bigr] \\ &= \operatorname{Re}\bigl[\overline{\lambda} l(f \cdot \xi + M_T^*\xi) + |\lambda|^2\langle \eta, M_f^*\eta\rangle\bigr]\end{aligned}$$

for all $\lambda \in \mathbb{C}$. Since $\operatorname{Re} M_f^* \geqslant 0$, this expression is non-negative for all $\lambda \in \mathbb{C}$. Hence $l(f \cdot \xi + M_T^*\xi) = 0$ for all $l \in \mathcal{O}(\mathbb{D}_d)'$. Therefore, $f \cdot \xi = -M_T^*\xi$. ∎

Theorem 6.2. *Let $I \subset \mathcal{O}(\mathbb{D}_d)$ be a local ideal. Then $\mathcal{M}_d \cap I$ is dense in $H^2_d \cap I$ with respect to the H^2_d-norm. In particular, if $H^2_d \cap I \neq \{0\}$, then $\mathcal{M}_d \cap I \neq \{0\}$.*

Proof. Let P_I be the orthogonal projection $H^2_d \to H^2_d \cap I$. Since the closed linear span of the functions u_x, $x \in \mathbb{D}_d$, is norm dense in H^2_d, the closed linear span of the functions $P_I(u_x)$ is norm dense in $H^2_d \cap I$. Thus it suffices to show that $\eta := P_I(u_x)$ lies in $\mathcal{M}_d \cap I$ for all $x \in \mathbb{D}_d$. By definition,

$$\eta = u_x - P_I^\perp(u_x) \in \mathbb{C} \cdot u_x + (H^2_d \ominus I) = H^2_d \ominus I_x,$$

where $I_x := \{f \in I \mid f(x) = 0\}$. If $f \in \mathcal{O}(\mathbb{D}_d)$, then $f \cdot \eta - f(x)\eta \in I_x$. Thus the projection of $f \cdot \eta$ to $H^2_d \ominus I_x$ equals $f(x)\eta$. Consequently, the rank one operator $T := |\eta\rangle\langle u_x|$ is the compression of M_η to $H^2_d \ominus I_x$. Evidently, $u_x/\|u_x\|_{H^2_d}$ is a maximal vector for $T/\|T\|$. Thus Theorem 6.1 implies that $g(z) := u_x(x)\eta(z)/u_x(z)$ is the unique lifting of T to a multiplier with minimal norm $\|T\|$. In particular, $g \in \mathcal{M}_d$. Since $u_x \in \mathcal{O}(\overline{\mathbb{D}}_d) \subset \mathcal{M}_d$, it follows that $\eta \in \mathcal{M}_d$ as desired. ∎

7. INTERPOLATION IN FINITELY MANY POINTS

Let $z_1, \dots, z_m \in \mathbb{D}_d$, $T_1, \dots, T_m \in \mathbb{M}_n$. All $f \in \mathbb{M}_n(\mathcal{M}_d)$ with $f(z_j) = T_j$ have the same class $[T]$ in $\mathbb{M}_n\big(\mathcal{M}_d/I(z_1, \dots, z_m)\big)$. Represent $\mathcal{M}_d/I(z_1, \dots, z_m)$ completely isometrically on $\mathcal{H} := H^2_d \ominus I(z_1, \dots, z_m)$. By (2.1), the vectors $e_j := u_{z_j}$, $j = 1, \dots, m$, are a basis of $\mathcal{H}$. If $f \in \mathbb{M}_n(\mathcal{M}_d)$, then

$$M_f^*(u_x \otimes \xi) = u_x \otimes (f(x)^* \xi) \tag{7.1}$$

for all $x \in \mathbb{D}_d$, $\xi \in \mathbb{C}^n$ by (2.3). Consequently,

$$\langle M_f^*(u_x \otimes \xi), u_y \otimes \eta\rangle = \langle u_x, u_y\rangle \cdot \langle f(x)^*\xi, \eta\rangle = (1 - \langle y, x\rangle)^{-1} \cdot \langle f(x)^*\xi, \eta\rangle. \tag{7.2}$$

Thus $[T]^*$ can be computed explicitly with respect to the basis $\{e_j\}$.

This basis is not orthonormal but a linearly independent frame. Recall that a set of vectors $(\xi_j)_{j\in J}$ in a Hilbert space $\mathcal{H}$ is called a *frame* iff there are numbers $A, B \in (0, \infty)$ such that, for all $\eta \in \mathcal{H}$,

$$A\|\eta\|^2 \leqslant \sum_{j\in J} |\langle \xi_j, \eta\rangle|^2 \leqslant B\|\eta\|^2.$$

PROPOSITION 7.1. *Let $\mathcal{H}$ be a Hilbert space, $(\xi_j)_{j\in J}$ a frame, and $T \in \mathbb{B}(\mathcal{H})$. Then T has positive real part iff the matrix with entries*

$$\widetilde{T}_{ij} := \langle T\xi_j, \xi_i\rangle + \langle \xi_j, T\xi_i\rangle \tag{7.3}$$

is positive definite.

Proof. With every frame one can associate a bounded linear map $\sigma : \mathcal{H} \to \ell^2(J)$ mapping η to $(\langle \eta, \xi_j\rangle)_{j\in J}$. Moreover, $\sigma^*\sigma$ is invertible. If T has positive real part, then so has $\sigma T\sigma^* \in \mathbb{B}\big(\ell^2(J)\big)$ because

$$\operatorname{Re}(\sigma T \sigma^*) = \frac{1}{2}(\sigma T\sigma^* + \sigma T^*\sigma^*) = \sigma \operatorname{Re}(T)\sigma^*. \tag{7.4}$$

Conversely, if $\sigma T\sigma^*$ has positive real part then so has $\sigma^*\sigma T\sigma^*\sigma \in \mathbb{B}(\mathcal{H})$ and hence T because $\sigma^*\sigma$ is invertible. Let $\{\delta_j\}_{j\in J}$ be the canonical basis of $\ell^2(J)$. Then $\sigma^*\delta_j = \xi_j$ and thus

$$\langle 2\operatorname{Re}(\sigma T\sigma^*)\delta_j, \delta_i\rangle = \langle 2\sigma \operatorname{Re}(T)\sigma^*\delta_j, \delta_i\rangle = \langle 2\operatorname{Re}(T)\xi_j, \xi_i\rangle = \langle T\xi_j, \xi_i\rangle + \langle \xi_j, T\xi_i\rangle.$$

Hence the matrix $\widetilde{T}$ belongs to the operator $2\operatorname{Re}(\sigma T\sigma^*)$. ∎

If σ is invertible (as in our finite dimensional situation) then the matrix described in (7.3) is invertible iff T has invertible real part by (7.4).

THEOREM 7.2. *Let $z_1, \dots, z_m \in \mathbb{D}_d$ and let $T_1, \dots, T_m \in \mathbb{M}_n$.*

There is $F \in \mathcal{O}(\mathbb{D}_d, \mathbb{M}_n)$ with $F(z_j) = T_j$ for all j and $\operatorname{Re} M_F^* \geqslant 0$ if and only if the block matrix $A \in \mathbb{M}_m \otimes \mathbb{M}_n$ with entries

$$\frac{T_i + T_j^*}{1 - \langle z_i, z_j\rangle} \in \mathbb{M}_n$$

is positive definite.

Proof. Let $v_1, \dots, v_n$ be the standard basis of $\mathbb{C}^n$. Then the vectors $e_i \otimes v_j$ form a frame for $\mathcal{H} \otimes \ell^2_n$. Equation (7.2) implies $\langle [T]^*(e_j \otimes v_\mu), e_i \otimes v_\nu\rangle + \langle e_j \otimes v_\mu, [T]^*(e_i \otimes v_\nu)\rangle = \langle (T_j^* + T_i)v_\mu, v_\nu\rangle(1 - \langle z_i, z_j\rangle)^{-1}$. By Proposition 7.1, $[T]^*$ has positive real part iff the matrix A is positive definite. The assertion now follows from Theorem 5.2. ∎

THEOREM 7.3. *Let $z_1, \dots, z_m \in \mathbb{D}_d$ and let $T_1, \dots, T_m \in \mathbb{M}_n$.*

There is $F \in \mathbb{M}_n(\mathcal{M}_d)$ with $\|F\| \leqslant 1$ and $F(z_j) = T_j$ for all $j = 1, \dots, m$ iff the block matrix $A' \in \mathbb{M}_m \otimes \mathbb{M}_n$ with entries

$$\frac{1 - T_i T_j^*}{1 - \langle z_i, z_j \rangle} \in \mathbb{M}_n$$

is positive definite.

Proof. Let $B := (\beta_{i,j})$ be the matrix whose entries are the inner products

$$\beta_{i,j} := \langle e_j, e_i \rangle = (1 - \langle z_i, z_j \rangle)^{-1}$$

by (2.2). Since the inner product in H_d^2 is positive definite, the matrix B is positive and invertible. Thus we can form the vectors

$$\widetilde{e}_j := B^{-1/2} e_j = \sum_{k=1}^{m} (B^{-1/2})_{kj} e_k.$$

A straightforward computation shows that $\{\widetilde{e}_j\}$ is an orthonormal basis. Moreover, the operator $B : e_j \to \sum_k \beta_{kj} e_k$ still has the matrix (β_{ij}) in the basis $(\widetilde{e}_j)$ because B and $B^{-1/2}$ commute. Some linear algebra and (7.2) yield

$$\begin{aligned}
[T]^* \widetilde{e}_j \otimes \xi &= \sum_{k=1}^{m} [T]^* (B^{-1/2})_{kj} e_k \otimes \xi = \sum_{k=1}^{m} (B^{-1/2})_{kj} e_k \otimes T_k^* \xi \\
&= \sum_{k,l=1}^{m} \widetilde{e}_l (B^{1/2})_{lk} (B^{-1/2})_{kj} \otimes T_k^* \xi \\
&= (B^{1/2} \otimes \mathrm{id}_{\mathbb{C}^n}) \circ \operatorname{diag}(T_1^*, \dots, T_m^*) \circ (B^{-1/2} \otimes \mathrm{id}_{\mathbb{C}^n})(\widetilde{e}_j \otimes \xi).
\end{aligned}$$

Hence with respect to the orthonormal basis $\widetilde{e}_j$, the operator $[T]$ is given by

$$[T] = (B \otimes \mathrm{id})^{-1/2} \operatorname{diag}(T_1, \dots, T_n)(B \otimes \mathrm{id})^{1/2}$$

because $B \otimes \mathrm{id}$ is self-adjoint. This operator has norm at most 1 iff

$$1 - [T] \circ [T]^* = 1 - (B \otimes \mathrm{id})^{-1/2} \operatorname{diag}(T_i)(B \otimes \mathrm{id}) \operatorname{diag}(T_j^*)(B \otimes \mathrm{id})^{-1/2}$$

is positive. Since $B \otimes \mathrm{id}$ is invertible, this is equivalent to the positivity of $B \otimes \mathrm{id} - \operatorname{diag}(T_i)(B \otimes \mathrm{id}) \operatorname{diag}(T_j^*)$. This is the matrix A'. ∎

8. ALGEBRAS OF MATRICES

Fix $r \in \mathbb{N}$. Let E be the set of all commutative unital subalgebras of $\mathbb{M}_r$ with $\dim \mathcal{A} = r$. For example, if $T \in \mathbb{M}_r$ is a single operator with r different eigenvalues, then the unital subalgebra generated by T is of this form. We consider E as a subset of the Grassmannian manifold of r-dimensional vector subspaces of $\mathbb{M}_r$. This yields a natural topology on E.

An algebra $\mathcal{A} \in E$ is called *expanding* iff $\mathcal{A} \cdot \mathcal{A}^* = \mathbb{M}_r$. Dimension counting shows that this is equivalent to the bijectivity of the linear map $m : \mathcal{A} \otimes \mathcal{A}^* \to \mathbb{M}_r$, $x \otimes y^* \mapsto x \cdot y^*$. Let $x_1, \dots, x_r$ be a basis of $\mathcal{A}$. By definition, $\mathcal{A}$ is expanding iff the matrices $x_i x_j^*$ are linearly independent. This in turn is equivalent to the non-vanishing of a certain determinant. Thus the set of expanding algebras is dense in E. We are going to define invariants $\mathrm{p}(\mathcal{A})$ and $\mathrm{n}(\mathcal{A})$ for expanding algebras $\mathcal{A}$ that determine whether $\mathcal{A}$ is a quotient of $\mathcal{S}_d$.

We call $\xi \in \mathbb{C}^n$ a *co-eigenvector* for $\mathcal{A}$ iff ξ is a joint eigenvector for $\mathcal{A}^*$.

An expanding algebra $\mathcal{A}$ has a cyclic vector. In fact, any co-eigenvector ξ is cyclic because $\mathcal{A}^* \cdot \xi = \mathbb{C}\xi$ implies $\mathbb{C}^r = \mathbb{M}_r \cdot \xi = \mathcal{A} \cdot \mathcal{A}^* \xi = \mathcal{A} \cdot \xi$. If ξ is cyclic, then the map $\mathcal{A} \to \mathbb{C}^r$, $x \mapsto x \cdot \xi$, is bijective. Thus if ξ_1 and ξ_2 are two cyclic vectors, then there are $T_1, T_2 \in \mathcal{A}$ with $\xi_2 = T_1 \xi_1$ and $\xi_1 = T_2 \xi_2$. $T_1 T_2$ is the unique element of $\mathcal{A}$ with $T_1 T_2 \xi_2 = \xi_2$, so that $T_1 T_2 = 1$. Similarly, $T_2 T_1 = 1$, so that T_1 and T_2 are inverses of each other.

For $x \in \mathbb{M}_r$, we define a sesquilinear form $\theta_x : \mathcal{A}' \times \mathcal{A}' \to \mathbb{C}$ on the dual $\mathcal{A}'$ by

$$\theta_x(l_1, l_2) := (l_1 \otimes l_2^*)\big(m^{-1}(x)\big)$$

for $l_1, l_2 \in \mathcal{A}'$, where $l_2^*(y^*) := \overline{l_2(y)}$ for all $y \in \mathcal{A}^*$. This form is associated to $m^{-1}(x) \in \mathcal{A} \otimes \mathcal{A}^*$ in a *natural* way. This is the reason for working on the dual of $\mathcal{A}$. If x is self-adjoint, then θ_x is Hermitian, that is, $\overline{\theta_x(l_1, l_2)} = \theta_x(l_2, l_1)$.

Let ξ be a cyclic vector and let $x := -P_\xi$ be the negative of the orthogonal projection onto $\mathbb{C}\xi$. The above construction applied to $-P_\xi$ yields a Hermitian sesquilinear form θ_ξ. Let $\mathrm{p}(\mathcal{A})$, $\mathrm{o}(\mathcal{A})$, and $\mathrm{n}(\mathcal{A}) + 1$ be the number of **p**ositive, zer**o**, and **n**egative eigenvalues of θ_ξ. By definition, $\mathrm{p}(\mathcal{A}) + \mathrm{o}(\mathcal{A}) + \mathrm{n}(\mathcal{A}) = r - 1$.

These numbers do not depend on the choice of the cyclic vector ξ. For if η is another cyclic vector, then $\eta = T\xi$ for an invertible operator $T \in \mathcal{A}$. Thus $P_\eta = cTP_\xi T^*$ for some constant $c > 0$. Therefore, $m^{-1}(P_\eta) = cT \cdot m^{-1}(P_\xi) \cdot T^*$. Multiplication by T is an invertible transformation on $\mathcal{A}$. Let $T' : \mathcal{A}' \to \mathcal{A}'$ be the transpose of it. Then $\theta_\eta(l_1, l_2) = c\theta_\xi(T' l_1, T' l_2)$ for all $l_1, l_2 \in \mathcal{A}'$. Thus the forms θ_η and θ_ξ have the same numbers of positive, negative, and zero eigenvalues. Hence the numbers $\mathrm{p}(\mathcal{A})$, $\mathrm{o}(\mathcal{A})$, $\mathrm{n}(\mathcal{A})$ are well defined invariants of the algebra $\mathcal{A}$.

Let ξ be even a co-eigenvector, not just cyclic. Let $I_\xi := \{T \in \mathcal{A} \mid T^*\xi = 0\}$ be the corresponding maximal ideal. Choose any basis $X_1, \dots, X_{r-1}$ of I_ξ and let $X_r := 1$. Express $m^{-1}(-P_\xi)$ in this basis:

$$-P_\xi = \sum_{j,k=1}^{r} c_{jk} X_j X_k^*. \tag{8.1}$$

Since $X_k^* \xi = 0$ for all $k \leqslant r - 1$, we get $-\xi = -P_\xi(\xi) = \sum_{j=1}^{r} c_{jr} X_j \xi$. Since also $-\xi = -X_r \xi$, it follows that $c_{jr} = 0$ for $j \leqslant r - 1$ and $c_{rr} = -1$. Since $-P_\xi$ is self-adjoint, $\overline{c_{jk}} = c_{kj}$ and thus $c_{rj} = 0$ for $j \leqslant -1$. Diagonalizing the form (c_{jk}), we obtain another basis T_j of I_ξ such that

$$-P_\xi = \varepsilon_1 T_1 T_1^* + \cdots + \varepsilon_{r-1} T_{r-1} T_{r-1}^* - 1$$

with certain $\varepsilon_j \in \{-1, 0, +1\}$. We may assume the T_j ordered so that all the positive ε_j come first. The sesquilinear form θ_ξ on $\mathcal{A}'$ is equal to

$$\theta_\xi(l_1, l_2) = \sum_{j=1}^{r-1} \varepsilon_j l_1(T_j)\overline{l_2(T_j)} - l_1(1) l_2(1).$$

Thus $\mathrm{p}(\mathcal{A})$, $\mathrm{o}(\mathcal{A})$, and $\mathrm{n}(\mathcal{A})$ are equal to the number of positive, zero, and negative ε_j. It follows that $\mathrm{n}(\mathcal{A}) \geqslant 0$.

As a result, in order to check that $\mathcal{A}$ is expanding, compute the invariants $\mathrm{p}(\mathcal{A})$, etc., and the operators T_j, we have to do the following: Find a joint eigenvector ξ for $\mathcal{A}^*$. Compute the coefficients c_{jk} in equation (8.1) (this amounts to inverting the linear map m). Diagonalize the Hermitian matrix (c_{jk}). There are efficient numerical algorithms for performing these computations.

THEOREM 8.1. *Assume that* $\mathrm{n}(\mathcal{A}) = 0$ *and that* $\mathrm{p}(\mathcal{A}) \leqslant d$. *Then* $\mathcal{A}$ *is completely isometric to a quotient of* $\mathcal{S}_d$ *by some inner ideal. Let* ξ *be a co-eigenvector and*

$$1 - P_\xi = T_1 T_1^* + \cdots + T_p T_p^*$$

for certain $T_1, \dots, T_p \in I_\xi$. *Then* $\mathrm{n}(\mathcal{A}) = 0$ *and* $\mathrm{p}(\mathcal{A}) = p$, *and* $(T_1, \dots, T_p, 0, \dots, 0)$ *is a* d*-contraction. The homomorphism* $\mathcal{S}_d \to \mathcal{A}$ *defined by this* d*-contraction is a complete quotient map.*

Conversely, assume that $I \subset \mathcal{S}_d$ *is an inner ideal of codimension* r. *Then* $\mathcal{S}_d/I \subset \mathbb{B}(H_d^2 \ominus I) \cong \mathbb{M}_r$ *is expanding and* $\mathrm{n}(\mathcal{A}) = 0$, $\mathrm{p}(\mathcal{A}) \leqslant d$.

We mention without proof that if $\mathcal{A} \subset \mathbb{M}_r$ is completely isometric to a quotient of $\mathcal{S}_d$ by an inner ideal, then it is unitarily equivalent to such a quotient. Thus $\mathcal{A}$ is expanding and satisfies $\mathrm{n}(\mathcal{A}) = 0$, $\mathrm{p}(\mathcal{A}) \leqslant d$.

Proof. The constructions above the theorem show that we can write $1 - p_\xi = T_1 T_1^* + \cdots + T_p T_p^*$ if and only if $\mathrm{n}(\mathcal{A}) = 0$ and $\mathrm{p}(\mathcal{A}) = p$. By assumption, the operator $\Delta := 1 - T_1 T_1^* - \cdots - T_p T_p^*$ is a rank one projection. In particular, $\mathbf{T} = (T_1, \dots, T_p, 0, \dots, 0)$ is a d-contraction. Let $\varphi : \mathcal{S}_d \to \mathcal{A}$ be the corresponding completely contractive homomorphism sending $\mathbf{S}$ to $\mathbf{T}$.

Let $\Omega := \mathrm{Spec}(\mathbf{T})$. This is a finite subset of $\overline{\mathbb{D}}_d$. We claim that $\Omega \subset \mathbb{D}_d$. Otherwise, there is $x \in \mathrm{Spec}(\mathbf{T})$ with $\|x\|_2 = 1$. Let $X := \sum_{j=1}^{d} x_j T_j$. Since $\mathbf{T}$ is a d-contraction, X is a contraction. By construction, 1 is an eigenvalue of X. Let $0 \neq V \subset \mathbb{C}^r$ be the corresponding eigenspace with $X|V = \mathrm{id}$. Since also $\|X\| \leqslant 1$, it follows that $X(\mathbb{C}^r \ominus V) \perp V$. Thus V is a reducing subspace for X. Some linear algebra shows that the orthogonal projection P_V can be written as a polynomial in X and therefore lies in $\mathcal{A}$. Thus P_V commutes with $\mathcal{A}$ and thus also with $\mathcal{A}^*$. Since $\mathcal{A}$ is expanding, P_V commutes with $\mathbb{M}_r = \mathcal{A} \cdot \mathcal{A}^*$. Thus $P_V = \mathrm{id}$, that is, $X = 1$. This contradicts $T_1, \dots, T_d \in I_\xi \not\ni 1$.

By Theorem 2.1, the operator $\mathbf{T}$ can be written as the compression of $n \cdot \mathbf{S} \oplus \mathbf{Z}$ to a full co-invariant subspace $\mathcal{K}$. In the proof of Theorem 3.1, it is shown that

$\mathrm{Spec}(\mathbf{Z}) \subset \widehat{I}$ if $\mathbf{Z}$ is the spherical part of the dilation of a completely contractive representation of $\mathcal{S}_d/I$. Since $\mathrm{Spec}(\mathcal{A}) \cap \partial\mathbb{D}_d = \emptyset$, there can be no non-trivial spherical part $\mathbf{Z}$.

The number n is the rank of the defect operator Δ by Theorem 2.1, thus $n=1$. Consequently, $\mathbf{T}$ is the compression of the d-shift $\mathbf{S}$ to a full co-invariant subspace. By Theorem 3.1 and Corollary 3.2, φ is a complete isometry $\mathcal{S}_d/\ker\varphi \cong \mathcal{A}$.

Conversely, let $I \subset \mathcal{S}_d$ be an inner ideal of finite codimension r and let $\mathcal{A} := \mathcal{S}_d/I$. Theorem 3.1 asserts that the standard representation of $\mathcal{A}$ on $H^2_d \ominus I$ is completely isometric. It is shown in [3] that the closed linear span of $\mathcal{S}_d \cdot \mathcal{S}_d^*$ contains the algebra $\mathbb{K}(H^2_d)$ of compact operators on $\mathbb{B}(H^2_d)$. Thus $\mathcal{A}$ is expanding:

$$\mathcal{A} \cdot \mathcal{A}^* = P_I^\perp \cdot \mathcal{S}_d \cdot P_I^\perp \cdot \mathcal{S}_d^* \cdot P_I^\perp = P_I^\perp \cdot \mathcal{S}_d \cdot \mathcal{S}_d^* \cdot P_I^\perp \supset P_I^\perp \cdot \mathbb{K}(H^2_d) \cdot P_I^\perp = \mathbb{M}_r.$$

By Theorem 2.2, the automorphism group of $\mathbb{D}_d$ operates completely isometrically on $\mathcal{S}_d$. Since it operates transitively on $\mathbb{D}_d \subset \mathrm{Spec}(\mathcal{S}_d)$, we may assume without loss of generality that $0 \in \mathrm{Spec}(\mathcal{S}_d/I)$. Thus $1 = u_0 \in H^2_d \ominus I$. In fact, 1 is a co-eigenvector for $\mathcal{A}$. Arveson computes that $1 - S_1S_1^* - \cdots - S_dS_d^*$ is the rank one projection onto $\mathbb{C} \cdot 1$. Let $S(I)_j := P_I^\perp S_j P_I^\perp = P_I^\perp S_j$. We conclude that $P_I^\perp - S(I)_1 S(I)_1^* - \cdots - S(I)_d S(I)_d^*$ is still the rank one projection onto $\mathbb{C} \cdot 1$. Consequently, $\mathrm{n}(\mathcal{A}) = 0$ and $\mathrm{p}(\mathcal{A}) \leqslant d$. ∎

The sesquilinear form θ depends continuously on $\mathcal{A} \in E$ in a suitable sense. Thus an eigenvalue of θ cannot change its sign without becoming zero in between. Hence if $\mathrm{o}(\mathcal{A}) = 0$, then p and n are constant in a neighborhood of $\mathcal{A}$. Thus the set of r-dimensional quotients of $\mathcal{S}_{r-1}$ with $\mathrm{p}(\mathcal{A}) = r - 1$ is an open subset of E. The boundary of this set consists of the r-dimensional quotients of $\mathcal{S}_{r-2}$. The set of expanding algebras with $\mathrm{n}(\mathcal{A}) > 0$, $\mathrm{o}(\mathcal{A}) = 0$ is an open subset of algebras that are not quotients of any $\mathcal{S}_d$.

Let $\mathcal{A} \cong \mathcal{S}_d/I$ and let $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be a representation. Then ρ is completely contractive iff $\rho[\mathbf{S}]$ is a d-contraction. This is quite an efficient criterion to check whether a representation is completely contractive. Furthermore, quotients of $\mathcal{A}$ can be computed explicitly. These are reasons why it is desirable to write an algebra as a quotient of $\mathcal{S}_d$.

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