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ON THE BITANGENTIAL INTERPOLATION PROBLEM FOR CONTRACTIVE VALUED FUNCTIONS IN THE POLYDISK

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ABSTRACT. We solve the bitangential interpolation problem with a finite number of interpolating points for the multivariable analogue of the Schur class introduced by J. Agler. The description of all solutions is parametrized by a Redheffer transform whose entries are given explicitly in terms of the interpolation data.

KEYWORDS: *Multidimensional unitary system, transfer or characteristic function, contour integral interpolation conditions, linear fractional parametrization.*

MSC (2000): Primary 47A57; Secondary 32B99.

1. INTRODUCTION

In this paper we study the bitangential interpolation problem for the class of contractive valued functions in the polydisk introduced by J. Agler in [2]. This class, which we denote by $\mathcal{S}_d^{p\times q}$ and call the *Schur class of the polydisk*, consists of all $\mathbb{C}^{p\times q}$-valued functions S analytic on the d-fold polydisk $\mathbb{D}^d$:

$$\mathbb{D}^d = \big\{z = (z_1, \ldots, z_d) \in \mathbb{C}^d : |z_k| < 1\big\}$$

and such that

$$\sup_{r<1} \|S(rT_1, \ldots, rT_d)\| \leqslant 1$$

for any $r < 1$ and for any d-tuple of commuting contractions $(T_1, \ldots, T_d)$. In the latter relation $S(rT_1, \ldots, rT_d)$ can be defined by the Cauchy integral formula

$$S(rT_1, \ldots, rT_d) = \frac{1}{(2\pi r\mathrm{i})^d} \int\limits_{r\mathbb{T}^d} S(z)(z_1 I - T_1)^{-1} \cdots (z_d I - T_d)^{-1}\,\mathrm{d}z_1 \cdots \mathrm{d}z_d.$$

It was shown in [7] (see [1] for the one-sided case) that S belongs to $\mathcal{S}_d^{p\times q}$ if and only if there exist d analytic operator-valued functions $H_k^1(z)$ on $\mathbb{D}^d$ with values equal to operators from an auxiliary Hilbert space $\mathcal{M}_k$ into $\mathbb{C}^p$ and d analytic operator-valued functions $H_k^2(z)$ on $\mathbb{D}^d$ ($k = 1, \dots, d$) with values in $\mathcal{L}(\mathcal{M}_k; \mathbb{C}^q)$ so that

$$\begin{aligned}\begin{pmatrix} I_p - S(z)S(\omega)^* & S(z) - S(\bar{\omega}) \\ S(\bar{z})^* - S(\omega)^* & I_q - S(\bar{z})^*S(\bar{\omega}) \end{pmatrix} \\ = \sum_{k=1}^d \begin{pmatrix} (1 - z_k\bar{\omega}_k)H_k^1(z)H_k^1(\omega)^* & (z_k - \bar{\omega}_k)H_k^1(z)H_k^2(\omega)^* \\ (z_k - \bar{\omega}_k)H_k^2(z)H_k^1(\omega)^* & (1 - z_k\bar{\omega}_k)H_k^2(z)H_k^2(\omega)^* \end{pmatrix} \\ = \sum_{k=1}^d \begin{pmatrix} 1 - z_k\bar{\omega}_k & z_k - \bar{\omega}_k \\ z_k - \bar{\omega}_k & 1 - z_k\bar{\omega}_k \end{pmatrix} \circ \begin{pmatrix} H_k^1(z) \\ H_k^2(z) \end{pmatrix} (H_k^1(\omega)^*, H_k^2(\omega)^*)\end{aligned} \tag{1.1}$$

where $\circ$ denotes the Schur entrywise matrix multiplication. The following alternative characterization of the class $\mathcal{S}_d^{p\times q}$ in terms of unitary d-variable colligations is given in [2] and [7].

Theorem 1.1. *The $\mathbb{C}^{p\times q}$-valued function S analytic in $\mathbb{D}^d$ belongs to $\mathcal{S}_d^{p\times q}$ if and only if there is an auxiliary Hilbert space $\mathcal{H}$ and a unitary operator*

$$\mathbf{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathbb{C}^q \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H} \\ \mathbb{C}^p \end{pmatrix}$$

and a d-fold orthogonal decomposition of $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d \tag{1.2}$$

such that

$$S(z) = D + C(I_{\mathcal{H}} - \mathbf{Z}(z)A)^{-1}\mathbf{Z}(z)B \tag{1.3}$$

where

$$\mathbf{Z} = z_1P_1 + \cdots + z_dP_d \tag{1.4}$$

and where P_k are orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}_k$. For S of the form (1.3) *it holds that*

$$\begin{pmatrix} I_p - S(z)S(\omega)^* & S(z) - S(\bar{\omega}) \\ S(\bar{z})^* - S(\omega)^* & I_q - S(\bar{z})^*S(\bar{\omega}) \end{pmatrix} = \begin{pmatrix} C(I - \mathbf{Z}(z)A)^{-1} \\ B^*(I - \mathbf{Z}(z)A^*)^{-1} \end{pmatrix}$$

$$\circ \begin{pmatrix} I - \mathbf{Z}(z)\mathbf{Z}(\omega)^* & \mathbf{Z}(z) - \mathbf{Z}(\omega)^* \\ \mathbf{Z}(z) - \mathbf{Z}(\omega)^* & I - \mathbf{Z}(z)\mathbf{Z}(\omega)^* \end{pmatrix} \circ ((I - A^*\mathbf{Z}(\omega)^*)^{-1}C^*, (I - A\mathbf{Z}(\omega)^*)^{-1}B),$$

or equivalently, that

$$I_p - S(z)S(\omega)^* = C(I - \mathbf{Z}(z)A)^{-1}(I - \mathbf{Z}(z)\mathbf{Z}(\omega)^*)(I - A^*\mathbf{Z}(\omega)^*)^{-1}C^* \tag{1.5}$$

$$S(z) - S(\bar{\omega}) = C(I - \mathbf{Z}(z)A)^{-1}(\mathbf{Z}(z) - \mathbf{Z}(\omega)^*)(I - A\mathbf{Z}(\omega)^*)^{-1}B \tag{1.6}$$

$$I_q - S(\bar{z})^*S(\bar{\omega}) = B^*(I - \mathbf{Z}(z)A^*)^{-1}(I - \mathbf{Z}(z)\mathbf{Z}(\omega)^*)(I - A\mathbf{Z}(\omega)^*)^{-1}B \tag{1.7}$$

and therefore, the representation (1.1) *is valid for*

$$H_k^1(z) = C(I - \mathbf{Z}(z)A)^{-1}P_k \quad \text{and} \quad H_k^2(z) = B^*(I - \mathbf{Z}(z)A^*)^{-1}P_k. \tag{1.8}$$

The representation (1.3) is called *the unitary realization* of $S \in \mathcal{S}_d^{p\times q}$.

We consider a bitangential interpolation problem in the class $\mathcal{S}_d^{p\times q}$ with interpolation conditions given in terms of contour integrals. For the one-variable case such a problem (which is called the *residue problem*) was introduced in [10] and studied for one-variable Schur functions in [4], [5], [6].

The set of interpolation points (*the spectra of the interpolation problem*) will be given by two sets

$$\mathbf{A} = \{A_1, \dots, A_d\} \text{ and } \mathbf{B} = \{B_1, \dots, B_d\}, \quad A_k \in \mathbb{C}^{n_L\times n_L}, B_k \in \mathbb{C}^{n_R\times n_R} \tag{1.9}$$

of commuting matrices with spectra inside the unit disk,

$$\begin{matrix} A_jA_k = A_kA_j, \quad B_jB_k = B_kB_j; \\ \operatorname{spec} A_k \subset \mathbb{D}, \quad \operatorname{spec} B_k \subset \mathbb{D} \end{matrix} \qquad (k, j = 1, \dots, d). \tag{1.10}$$

Let $\mathcal{H}^{p\times q}(\mathbb{D}^d)$ denote the space of all $\mathbb{C}^{p\times q}$-valued functions holomorphic on $\mathbb{D}^d$. We associate with the set $\mathbf{A}$ the Riesz operator $\mathbf{R_A} : \mathcal{H}^{n_R}(\mathbb{D}^d) \to \mathbb{C}^{n_L\times m}$ and $\mathbf{R_B} : \mathcal{H}^{n_R}(\mathbb{D}^d) \to \mathbb{C}^{n_R}$ defined by the rule

$$\begin{aligned} \mathbf{R_A} H &= \frac{1}{(2\pi \mathrm{i})^d} \int_{r\mathbb{T}^d} (z_1 I - A_1)^{-1} \cdots (z_d I - A_d)^{-1} H(z)\, \mathrm{d}z_1 \cdots \mathrm{d}z_d, \\ \mathbf{R_B} G &= \frac{1}{(2\pi \mathrm{i})^d} \int_{r\mathbb{T}^d} (z_1 I - B_1)^{-1} \cdots (z_d I - B_d)^{-1} G(z)\, \mathrm{d}z_1 \cdots \mathrm{d}z_d, \end{aligned} \tag{1.11}$$

where $r < 1$ is any number greater than the spectral radius of any matrix A_j. We shall also make frequent use of the operator

$$\mathbf{\Gamma_{A,B}}\{F(z)G(\omega)^*\} := \mathbf{R_A}(F(z))(\mathbf{R_B}(G(\omega)))^* \tag{1.12}$$

defined on $\mathcal{H}^{n_L\times m}(\mathbb{D}^d) \times \mathcal{H}^{n_R\times m}(\mathbb{D}^d)$.

We are given two sets $\mathbf{A}, \mathbf{B}$ of commuting matrices A_j, B_j satisfying (1.10), d matrices $\Lambda_1, \dots, \Lambda_d \in \mathbb{C}^{(n_L+n_R)\times(n_L+n_R)}$ partitioned by

$$\Lambda_k = \begin{pmatrix} \Lambda_k^L & \Lambda_k^{LR} \\ (\Lambda_k^{LR})^* & \Lambda_k^R \end{pmatrix}, \quad \Lambda_k^R \in \mathbb{C}^{n_R\times n_R},\ \Lambda_k^L \in \mathbb{C}^{n_L\times n_L},\ \Lambda_k^{LR} \in \mathbb{C}^{n_L\times n_R} \tag{1.13}$$

and matrices $X_L \in \mathbb{C}^{n_L\times p}$, $Y_L \in \mathbb{C}^{n_L\times q}$, $Y_R \in \mathbb{C}^{n_R\times q}$, $X_R \in \mathbb{C}^{n_R\times p}$. Given this data set

$$\mathcal{D} = \{\mathbf{A}, \mathbf{B}, \Lambda_1, \dots, \Lambda_d, X_L, Y_L, X_R, Y_R\}, \tag{1.14}$$

the associated bitangential interpolation problem is

Problem 1.2. Find all functions $S \in \mathcal{S}_d^{p\times q}$ satisfying the interpolation conditions

$$\mathbf{R_A}(X_L S(z)) = Y_L \tag{1.15}$$

$$\mathbf{R_B}(Y_R S(\bar{z})^*) = X_R \tag{1.16}$$

(where $\mathbf{R_A}$ and $\mathbf{R_B}$ are the operators defined via (1.11)) and such that for some choice of associated functions $H_k^1(z)$ and $H_k^2(z)$ in the representation (1.1), it holds that

$$\begin{pmatrix} \mathbf{R_A}(X_L H_k^1(z)) \\ \mathbf{R_B}(Y_R H_k^2(z)) \end{pmatrix} ((\mathbf{R_A}(X_L H_k^1(z)))^*, (\mathbf{R_B}(Y_R H_k^2(z)))^*) = \Lambda_k \tag{1.17}$$

for $k = 1, \dots, d$.

Using the operator $\mathbf{\Gamma}$ defined by (1.12) and the block decompositions (1.13) of Λ_k one can rewrite (1.17) as

$$\mathbf{\Gamma}_{\mathbf{A},\mathbf{A}}\{X_L H_k^1(z) H_k^1(\omega)^* X_L^*\} = \Lambda_k^L \tag{1.18}$$

$$\mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\{X_L H_k^1(z) H_k^2(\omega)^* Y_R^*\} = \Lambda_k^{LR} \tag{1.19}$$

$$\mathbf{\Gamma}_{\mathbf{B},\mathbf{B}}\{Y_R H_k^2(z) H_k^2(\omega)^* Y_R^*\} = \Lambda_k^R. \tag{1.20}$$

In [3], the problem with the interpolation condition (1.15) was studied for matrix-valued functions with entries from the Hardy space $\mathbf{H}_2(\mathbb{D}^2)$ of the bidisk. In fact, a more general problem was considered in the framework of Hardy functions of two variables when the matrices A_1 and A_2 defining the spectra of interpolation, are not commuting. Nevertheless, the approach presented in [3] does not seem to extend to problems in more than two variables.

Example 1.3. (The two-sided Nevanlinna-Pick problem) Take $n+m$ points in the polydisk $z^{(j)} = (z_1^{(j)}, \dots, z_d^{(j)}) \in \mathbb{D}^d$ and $\omega^{(i)} = (\omega_1^{(i)}, \dots, \omega_d^{(i)}) \in \mathbb{D}^d$ and matrices

$$x_j \in \mathbb{C}^{\ell_j \times p}, \quad y_j \in \mathbb{C}^{\ell_j \times q}, \quad u_i \in \mathbb{C}^{r_i \times q}, \quad v_i \in \mathbb{C}^{r_i \times p}$$

$(j = 1, \dots, n;\ i = 1, \dots, m)$ and set

$$X_L = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad Y_L = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad Y_R = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad X_R = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$$

$$A_k = \begin{pmatrix} z_k^{(1)} I_{\ell_1} & & \\ & \ddots & \\ & & z_k^{(n)} I_{\ell_n} \end{pmatrix}, \quad B_k = \begin{pmatrix} \bar{\omega}_k^{(1)} I_{r_1} & & \\ & \ddots & \\ & & \bar{\omega}_k^{(n)} I_{r_n} \end{pmatrix}$$

$(k = 1, \dots, d)$. For such a choice of commuting matrices A_k, of commuting matrices B_k and of matrices X_L, Y_R, it holds for every function $S \in \mathcal{S}_d^{p \times q}$ that

$$\mathbf{R}_{\mathbf{A}}(X_L S(z)) = \begin{pmatrix} x_1 S(z^{(1)}) \\ \vdots \\ x_n S(z^{(n)}) \end{pmatrix}, \qquad \mathbf{R}_{\mathbf{B}}(Y_R S(\bar{z})^*) = \begin{pmatrix} u_1 S(\omega^{(1)})^* \\ \vdots \\ u_m S(\omega^{(m)})^* \end{pmatrix}$$

and thus, conditions (1.15) and (1.16) reduce respectively, to left-sided and right-sided Nevanlinna-Pick conditions

$$x_k S(z^{(k)}) = y_k \quad \text{and} \quad S(\omega^{(j)}) u_j^* = v_j^* \quad (k = 1, \dots, n;\ j = 1, \dots, m). \tag{1.21}$$

Similarly,

$$\mathbf{R}_{\mathbf{A}}(X_L H_k^1(z)) = \begin{pmatrix} x_1 H_k^1(z^{(1)}) \\ \vdots \\ x_n H_k^1(z^{(n)}) \end{pmatrix}, \quad \mathbf{R}_{\mathbf{B}}(Y_R H_k^2(z)) = \begin{pmatrix} u_1 H_k^2(\omega^{(1)}) \\ \vdots \\ u_m H_k^2(\omega^{(m)}) \end{pmatrix}$$

and conditions (1.17) reduce to

$$\begin{pmatrix} x_1 H_k^1(z^{(1)}) \\ \vdots \\ x_n H_k^1(z^{(n)}) \end{pmatrix} (H_k^1(z^{(1)})^* x_1^*, \dots, H_k^1(z^{(n)})^* x_n^*) = \Lambda_k^L$$

$$\begin{pmatrix} x_1 H_k^1(z^{(1)}) \\ \vdots \\ x_n H_k^1(z^{(n)}) \end{pmatrix} (H_k^2(\omega^{(1)})^* u_1^*, \dots, H_k^2(\omega^{(m)})^* u_m^*) = \Lambda_k^{LR}$$

$$\begin{pmatrix} u_1 H_k^2(\omega^{(1)}) \\ \vdots \\ u_m H_k(\omega^{(n)}) \end{pmatrix} (H_k^2(\omega^{(1)})^* u_1^*, \dots, H_k^2(\omega^{(m)})^* u_m^*) = \Lambda_k^R \quad (k = 1, \dots, d).$$

Such a problem was considered in [7] for the case where the $z^{(k)}$'s are disjoint from the $\omega^{(j)}$'s and the second type of interpolation condition (1.17) does not enter in.

Setting A_k to be general matrices (say, in Jordan form) one can deduce from (1.15) and (1.16) more general conditions than (1.21) involving partial derivatives of S of higher orders at different prescribed points in the polydisk $\mathbb{D}^d$.

In [7] a solution criterion and linear fractional parametrization for the set of all solutions of the problem with data set as in Example 1.3 was given (for the case where the $z^{(k)}$'s and $w^{(j)}$'s are distinct so no coupling interpolation conditions enter in). The main point of this paper is to recover these results for the more general Problem 1.2. The proof of the first part (existence criterion) follows the same idea as in [7]: solutions of the interpolation problem correspond to unitary colligation extensions of a partially defined isometric colligation constructed explicitly from the interpolation data. The parametrization of the set of all solutions follows an idea introduced by Arov and Grossman for the 1-variable case (see [9]). The block matrix function giving the linear fractional parametrization arises from a certain universal unitary colligation extension of the partially defined isometric colligation built from the interpolation data mentioned above. In [7] the verification of the parametrization was worked out via use of general principles of feedback connections of linear systems. Here we obtain a formula for the universal unitary colligation extension much more explicit than in [7], and verify the parametrization result via explicit computations using this formula rather than via general principles.

The paper [9] actually deals with a more abstract formulation (the Abstract Interpolation Problem or AIP) of the interpolation problem which is flexible enough to incorporate various more exotic types of interpolation. We plan to discuss a several variable version of the AIP in future work.

The paper is organized as follows. After the present Introduction, Section 2 derives the solvability condition, Section 3 introduces the partially defined d-variable isometric colligation associated with the interpolation data and an explicit formula for the universal unitary colligation extension required for the parametrization of the set of solutions, Section 4 obtains explicit formulas for the characteristic function of this universal unitary colligation extension, which generalize results for the one variable case obtained in [8], and finally, Section 5 verifies the linear fractional parametrization for the set of all solutions.

2. THE SOLVABILITY CRITERION

In this section we establish the solvability criterion of Problem 1.2. First we note some elementary properties of operators $\mathbf{R}$ and $\mathbf{\Gamma}$.

Lemma 2.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be collections of matrices as in* (1.9), (1.10) *and let* $\mathbf{R}$ *and* $\mathbf{\Gamma}$ *be operators defined via* (1.11) *and* (1.12), *respectively. Then*

(i) *For every constant matrix function* $W \in \mathbb{C}^{n_L \times n_R}$,

$$\mathbf{R}_{\mathbf{A}}(W) = \mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\{W\} = W. \tag{2.1}$$

(ii) *For every function* $F \in \mathcal{H}^{n_L \times m}(\mathbb{D}^d)$,

$$\mathbf{R}_{\mathbf{A}}((z_k I_{n_L} - A_k)F(z)) = 0 \quad (k = 1, \dots, d). \tag{2.2}$$

(iii) *For every choice of* $F \in \mathcal{H}^{n_L \times m}(\mathbb{D}^d)$ *and of* $G \in \mathcal{H}^{m \times \ell}(\mathbb{D}^d)$,

$$\mathbf{R}_{\mathbf{A}}(F(z)G(z)) = \mathbf{R}_{\mathbf{A}}(\widehat{F}G(z)) \tag{2.3}$$

where the (*constant*) *matrix* $\widehat{F} \in \mathbb{C}^{n_L \times m}$ *is defined by*

$$\widehat{F} = \mathbf{R}_{\mathbf{A}}(F(z)).$$

The first assertion of lemma follows from the spectral condition (1.10). Since $A_1, \dots, A_d$ are commuting matrices, it follows from (1.11) that the integrand in the right hand side of (2.2) is analytic with respect to z_k. Therefore, the integral which defines $\mathbf{R}_{\mathbf{A}}$ is equal to zero. The third assertion can be easily obtained by the residue calculus.

As a consequence of (2.2) we get

$$\mathbf{R}_{\mathbf{A}}(z_k F(z)) = A_k \mathbf{R}_{\mathbf{A}}(F(z)) \quad (k = 1, \dots, d) \tag{2.4}$$

and quite similarly,

$$\mathbf{R}_{\mathbf{B}}(z_k G(z)) = B_k \mathbf{R}_{\mathbf{B}}(G(z)) \quad (k = 1, \dots, d). \tag{2.5}$$

Therefore,

$$\mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\{z_k \bar{\omega}_j F(z)G(\omega)^*\} = A_k \mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\{F(z)G(\omega)^*\} B_j^*. \tag{2.6}$$

Theorem 2.2. *Problem* 1.2 *has a solution if and only if the matrices* Λ_k *are nonnegative*

$$\Lambda_k \geqslant 0 \quad (k = 1, \dots, d) \tag{2.7}$$

and satisfy the generalized Stein identity

$$\sum_{k=1}^{d} (M_k \Lambda_k M_k^* - N_k \Lambda_k N_k^*) = XX^* - YY^* \tag{2.8}$$

where

$$M_k = \begin{pmatrix} I_{n_L} & 0 \\ 0 & B_k \end{pmatrix}, \ N_k = \begin{pmatrix} A_k & 0 \\ 0 & I_{n_R} \end{pmatrix}, \ X = \begin{pmatrix} X_L \\ X_R \end{pmatrix}, \ Y = \begin{pmatrix} Y_L \\ Y_R \end{pmatrix}. \tag{2.9}$$

Proof. Here we check the necessity of conditions (2.7), (2.8). The proof of the sufficiency part is postponed up to Section 4 where it will be obtained as a consequence of stronger results. Let S be a solution of Problem 1.2, that is let relations (1.1), (1.15)-(1.17) be in force. The expression in the left hand side of (1.17) is necessarily nonnegative which gives (2.7). Substituting the partitionings (1.13) and (2.9) into (2.8) we conclude that (2.8) is equivalent to the following three equalities

$$\sum_{k=1}^{d}(\Lambda_k^L - A_k\Lambda_k^L A_k^*) = X_L X_L^* - Y_L Y_L^* \tag{2.10}$$

$$\sum_{k=1}^{d}(\Lambda_k^{LR} B_k^* - A_k\Lambda_k^{LR}) = X_L X_R^* - Y_L Y_R^* \tag{2.11}$$

$$\sum_{k=1}^{d}(\Lambda_k^R - B_k\Lambda_k^R B_k^*) = Y_R Y_R^* - X_R X_R^*. \tag{2.12}$$

Applying (2.6) to $F(z) = G(z) = X_L H_k^1(z)$ and using (1.18) we get

$$\begin{aligned}\mathbf{\Gamma}_{\mathbf{A},\mathbf{A}}\{z_k\bar{\omega}_k X_L H_k^1(z)H_k^1(\omega)^* X_L^*\} &= A_k\mathbf{\Gamma}_{\mathbf{A},\mathbf{A}}\{X_L H_k^1(z)H_k^1(\omega)^* X_L^*\}A_k^* \\ &= A_k\Lambda_k^L A_k^*.\end{aligned} \tag{2.13}$$

Summing up equalities (1.18) and (2.13) for all k and subtracting the second sum from the first one, we obtain

$$\mathbf{\Gamma}_{\mathbf{A},\mathbf{A}}\left\{X_L\sum_{k=1}^{d}(1 - z_k\bar{\omega}_k)H_k^1(z)H_k^1(\omega)^* X_L^*\right\} = \sum_{k=1}^{d}(\Lambda_k^L - A_k\Lambda_k^L A_k^*). \tag{2.14}$$

On the other hand the interpolation condition (1.15) implies

$$\begin{aligned}&\mathbf{\Gamma}_{\mathbf{A},\mathbf{A}}\big\{X_L(I_p - S(z)S(\omega)^*)X_L^*\big\} \\ &\quad = \mathbf{\Gamma}_{\mathbf{A},\mathbf{A}}\big\{X_L X_L^*\big\} - \mathbf{R}_{\mathbf{A}}(X_L S(z))(\mathbf{R}_{\mathbf{A}}(X_L S(\omega)))^* = X_L X_L^* - Y_L Y_L^*.\end{aligned} \tag{2.15}$$

By (1.1), the expressions in the left hand sides of (2.14) and (2.15) are equal. The equality of the right hand side expressions leads to (2.10).

In much the same way, applying (2.6) to $F(z) = G(z) = Y_R H_k^2(z)$ and using (1.20) and (1.16) we get (2.12). Finally, in view of (1.19),

$$\begin{aligned}&\mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\{(z_k - \bar{\omega}_k)X_L H_k^1(z)H_k^2(\omega)^* X_R^*\} \\ &\quad = A_k\mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\{X_L H_k^1(z)H_k^2(\omega)^* X_R^*\} - \mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\{X_L H_k^1(z)H_k^2(\omega)^* X_R^*\}B_k^* \\ &\quad = A_k\Lambda_k^{LR} - \Lambda_k^{LR}B_k^*\end{aligned}$$

which being summed up over all $k = 1, \ldots, d$, gives

$$\mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\left\{X_L\sum_{k=1}^{d}(z_k - \bar{\omega}_k)H_k^1(z)H_k^2(\omega)^* X_R^*\right\} = \sum_{k=1}^{d}(A_k\Lambda_k^{LR} - \Lambda_k^{LR}B_k^*). \tag{2.16}$$

On the other hand the interpolation conditions (1.15) and (1.16) imply

$$\begin{aligned}\mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\{X_L(S(z)-S(\bar{\omega}))Y_R^*\} &= \mathbf{R}_{\mathbf{A}}(X_LS(z))Y_R^* - X_L\mathbf{R}_{\mathbf{B}}(Y_RS(\bar{\omega})^*)^* \\ &= Y_LY_R^* - X_LX_R^*.\end{aligned} \tag{2.17}$$

By (1.1), the expressions in the left hand sides of (2.16) and (2.17) are equal. The equality of the right hand side expressions leads to (2.11). Thus, the equalities (2.10)–(2.12) hold and therefore (2.8) is in force. ∎

Note that conditions (2.7), (2.8) for the Nevanlinna-Pick interpolation problem were first obtained in [1].

3. THE UNIVERSAL UNITARY COLLIGATION ASSOCIATED WITH THE INTERPOLATION PROBLEM

We recall that a d-variable colligation is defined as a quadruple

$$\Omega = \left\{\mathcal{H} = \bigoplus_{k=1}^{d}\mathcal{H}_k, \mathcal{F}, \mathcal{G}, \mathbf{U}\right\}$$

consisting of three Hilbert spaces $\mathcal{H}$ (*the state space*) which is specified to have a fixed d-fold orthogonal decomposition, $\mathcal{F}$ (*the input space*) and $\mathcal{G}$ (*the output space*), together with a connecting operator

$$\mathbf{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{F} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{G} \end{pmatrix}. \tag{3.1}$$

The colligation is said to be *unitary* if the connecting operator $\mathbf{U}$ is unitary. A colligation

$$\widetilde{\Omega} = \left\{\widetilde{\mathcal{H}} = \bigoplus_{k=1}^{d}\widetilde{\mathcal{H}}_k, \mathcal{F}, \mathcal{G}, \widetilde{\mathbf{U}}\right\}$$

is said to be *unitarily equivalent* to the colligation Ω if there is a unitary operator $\alpha : \mathcal{H} \to \widetilde{\mathcal{H}}$ such that

$$\alpha P_k = \widetilde{P}_k\alpha \quad (k = 1, \dots, d) \quad \text{and} \quad \begin{pmatrix} \alpha & 0 \\ 0 & I_{\mathcal{G}} \end{pmatrix}\mathbf{U} = \widetilde{\mathbf{U}}\begin{pmatrix} \alpha & 0 \\ 0 & I_{\mathcal{F}} \end{pmatrix}$$

where P_k and $\widetilde{P}_k$ are orthogonal projections from $\mathcal{H}$ onto $\mathcal{H}_k$ and from $\widetilde{\mathcal{H}}$ onto $\widetilde{\mathcal{H}}_k$, respectively. The *characteristic function* of the colligation Ω is defined as

$$S_\Omega(z) = D + C\mathbf{Z}(z)(I_{\mathcal{H}} - A\mathbf{Z}(z))^{-1}B, \tag{3.2}$$

where $\mathbf{Z}$ is defined as in (3.26). Thus, Theorem 1.1 claims that a $\mathbb{C}^{p\times q}$-valued function S analytic in $\mathbb{D}^d$ belongs to the class $\mathcal{S}_d^{p\times q}$ if and only if it is the characteristic function of some d-variable unitary colligation

$$\Omega = \left\{\mathcal{H} = \bigoplus_{k=1}^{d}\mathcal{H}_k, \mathbb{C}^q, \mathbb{C}^p, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\}. \tag{3.3}$$

Remark 3.1. *Unitary equivalent colligations have the same characteristic function.*

In this section we associate certain *finite dimensional* (i.e. with finite dimensional state space, input space and output space) unitary colligation to Problem 1.2. It turns out that the characteristic function of this colligation (which is rational, according to (3.2)) is the transfer function of the Redheffer transform describing all solutions of Problem 1.2. We assume that necessary conditions (2.7) and (2.8) for Problem 1.2 to have a solution, are in force. Let M_k, N_k, X and Y be the matrices defined by (2.9) and let

$$W_1 = (M_1\Lambda_1^{1/2}, \dots, M_d\Lambda_d^{1/2}) \quad \text{and} \quad W_2 = (N_1\Lambda_1^{1/2}, \dots, N_d\Lambda_d^{1/2}). \tag{3.4}$$

The identity (2.8) provides the linear map

$$\mathbf{V} : \begin{pmatrix} W_1^* \\ Y^* \end{pmatrix} f \longrightarrow \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} f \tag{3.5}$$

to be an isometry from

$$\mathcal{D}_{\mathbf{V}} = \operatorname{Ran} \begin{pmatrix} W_1^* \\ Y^* \end{pmatrix} \subset \mathbb{C}^{nd+q} \quad \text{onto} \quad \operatorname{Ran} \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} \subset \mathbb{C}^{nd+p}$$

(here and in what follows we set $n := n_L + n_R$). The verification is straightforward: for every choice of $f, g \in \mathbb{C}^n$,

$$\begin{aligned} &\left\langle \begin{pmatrix} W_1^* \\ Y^* \end{pmatrix} f, \begin{pmatrix} W_1^* \\ Y^* \end{pmatrix} g \right\rangle - \left\langle \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} f, \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} g \right\rangle \\ &\qquad = g^*\Big(\sum_{k=1}^d M_k\Lambda_k M_k^* + YY^*\Big) f - g^*\Big(\sum_{k=1}^d N_k\Lambda_k N_k^* + XX^*\Big) f = 0. \end{aligned}$$

Let $\Delta(z)$ be the $\mathbb{C}^{n\times n}$-valued function defined by

$$\Delta(z) = \sum_{k=1}^d M_k\Lambda_k(M_k^* - z_k N_k^*) + YY^*. \tag{3.6}$$

Another representation of Δ,

$$\Delta(z) = \sum_{k=1}^d (N_k - z_k M_k)\Lambda_k N_k^* + XX^*, \tag{3.7}$$

follows from (2.8) and (3.6). Note that Δ takes the nonnegative value at the origin,

$$\Delta(0) = \sum_{k=1}^d M_k\Lambda_k M_k^* + YY^* = \sum_{k=1}^d N_k\Lambda_k N_k^* + XX^*, \tag{3.8}$$

which on account of (3.4) can be written as

$$\Delta(0) = W_1W_1^* + YY^* = W_2W_2^* + XX^*. \tag{3.9}$$

Let $\operatorname{rank}\Delta(0) = r \leqslant n$ and let $Q \in \mathbb{C}^{n\times r}$ be a matrix such that

$$\operatorname{rank} Q^*\Delta(0)Q = \operatorname{rank}\Delta(0) = r. \tag{3.10}$$

The latter relation implies in particular, that $Q^*\Delta(0)Q>0$, which allows to define the pseudoinverse matrix $\Delta(0)^{[-1]}$ as

$$\Delta(0)^{[-1]}=Q(Q^*\Delta(0)Q)^{-1}Q^*. \tag{3.11}$$

Note that (3.11) determines the Moore-Penrose pseudoinverse ([11]) if the columns of Q span the range $\operatorname{Ran}\Delta(0)$ of $\Delta(0)$.

Since $\mathbf{V}$ is an isometry, it follows from (3.9) that

$$\dim\mathcal{D}_{\mathbf{V}}=\dim\mathcal{R}_{\mathbf{V}}=\operatorname{rank}(W_1,Y)=\operatorname{rank}(W_2,Y)=r. \tag{3.12}$$

Introducing the orthogonal complements

$$\mathcal{D}_{\mathbf{V}}^{\perp}=\mathbb{C}^{nd+q}\ominus\mathcal{D}_{\mathbf{V}}\quad\text{and}\quad\mathcal{R}_{\mathbf{V}}^{\perp}=\mathbb{C}^{nd+p}\ominus\mathcal{R}_{\mathbf{V}}$$

we obtain as a corollary of (3.12) that

$$q':=\dim\mathcal{D}_{\mathbf{V}}^{\perp}=nd+q-r\quad\text{and}\quad p':=\dim\mathcal{R}_{\mathbf{V}}^{\perp}=nd+p-r. \tag{3.13}$$

Remark 3.2. *The orthogonal projection* $\mathbf{P}_{\mathcal{D}_{\mathbf{V}}^{\perp}}$ *of* $\mathbb{C}^{nd+q}$ *onto* $\mathcal{D}_{\mathbf{V}}^{\perp}$ *is given by the formula*

$$\mathbf{P}_{\mathcal{D}_{\mathbf{V}}^{\perp}}=I_{nd+q}-\begin{pmatrix}W_1^*\\ Y^*\end{pmatrix}\Delta(0)^{[-1]}(W_1,Y) \tag{3.14}$$

whereas the orthogonal projection $\mathbf{P}_{\mathcal{R}_{\mathbf{V}}^{\perp}}$ *of* $\mathbb{C}^{nd+p}$ *onto* $\mathcal{R}_{\mathbf{V}}^{\perp}$ *is given by the formula*

$$\mathbf{P}_{\mathcal{R}_{\mathbf{V}}^{\perp}}=I_{nd+p}-\begin{pmatrix}W_2^*\\ X^*\end{pmatrix}\Delta(0)^{[-1]}(W_2,X). \tag{3.15}$$

We let $T_1\in\mathbb{C}^{(nd+q)\times q'}$ and $T_2\in\mathbb{C}^{(nd+p)\times p'}$ be isometric matrices whose columns span $\mathcal{D}_{\mathbf{V}}^{\perp}$ and $\mathcal{R}_{\mathbf{V}}^{\perp}$ respectively. Then the projections $\mathbf{P}_{\mathcal{D}_{\mathbf{V}}^{\perp}}$ and $\mathbf{P}_{\mathcal{R}_{\mathbf{V}}^{\perp}}$ can be represented as

$$\mathbf{P}_{\mathcal{D}_{\mathbf{V}}^{\perp}}=T_1T_1^*,\quad\mathbf{P}_{\mathcal{R}_{\mathbf{V}}^{\perp}}=T_2T_2^*. \tag{3.16}$$

On the other hand, the following equalities

$$T_1^*T_1=I_{q'},\quad T_2^*T_2=I_{p'},\quad (W_1,Y)T_1=0,\quad (W_2,X)T_2=0 \tag{3.17}$$

hold by construction.

Lemma 3.3. *The operator*

$$\mathbf{U}_0=\begin{pmatrix}U_{11}&U_{12}&U_{13}\\ U_{21}&U_{22}&U_{23}\\ U_{31}&U_{32}&0\end{pmatrix}:\begin{pmatrix}\mathbb{C}^{nd}\\ \mathbb{C}^{q}\\ \mathbb{C}^{p'}\end{pmatrix}\longrightarrow\begin{pmatrix}\mathbb{C}^{nd}\\ \mathbb{C}^{p}\\ \mathbb{C}^{q'}\end{pmatrix} \tag{3.18}$$

with entries specified by the rules

$$\begin{pmatrix}U_{11}&U_{12}\\ U_{21}&U_{22}\end{pmatrix}=\begin{pmatrix}W_2^*\\ X^*\end{pmatrix}\Delta(0)^{[-1]}(W_1,Y) \tag{3.19}$$

$$(U_{31},U_{32})=T_1^*,\qquad\begin{pmatrix}U_{13}\\ U_{23}\end{pmatrix}=T_2 \tag{3.20}$$

is a unitary extension of the isometry $\mathbf{V}$ *defined by* (3.5).

Proof. It follows from (3.9), (3.19) that

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}^* = \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} \Delta(0)^{[-1]}(W_2, X)$$

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}^* \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} W_1^* \\ Y^* \end{pmatrix} \Delta(0)^{[-1]}(W_1, Y)$$

which together with (3.14)–(3.16) and (3.20) imply

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}^* + \begin{pmatrix} U_{13} \\ U_{23} \end{pmatrix}(U_{13}^*, U_{23}^*) = I_{nd+q}$$

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}^* \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} + \begin{pmatrix} U_{31}^* \\ U_{32}^* \end{pmatrix}(U_{31}, U_{32}) = I_{nd+p}.$$

It follows immediately from the two latter equalities and (3.17) that $\mathbf{U}_0$ is unitary. To show that $\mathbf{U}_0$ is an extension of V, it suffices to check that

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} W_1^* \\ Y^* \end{pmatrix} x = \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} x \quad (\forall x \in \mathbb{C}^n). \tag{3.21}$$

Let $Q \in \mathbb{C}^{n \times r}$ be a matrix satisfying (3.10). Then every vector $x \in \mathbb{C}^n$ can be represented as

$$x = Qf + g \quad \text{for} \quad f \in \mathbb{C}^r \quad \text{and} \quad g \in \operatorname{Ker} \Delta(0). \tag{3.22}$$

By (3.9),

$$W_1^* g = 0, \quad W_2^* g = 0 \quad \text{and} \quad X^* g = 0 \quad (\forall g \in \operatorname{Ker} \Delta(0)). \tag{3.23}$$

Substituting (3.19) and (3.22) into the right hand side of (3.21) and taking into account (3.11) and (3.23) we get

$$\begin{aligned} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} W_1^* \\ Y^* \end{pmatrix} x &= \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} \Delta(0)^{[-1]}(W_1 W_1^* + Y Y^*) x \\ &= \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} Q(Q^* \Delta(0) Q)^{-1} Q^* \Delta(0)(Qf + g) \\ &= \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} Qf = \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} (Qf + g) = \begin{pmatrix} W_2^* \\ X^* \end{pmatrix} x \end{aligned}$$

and the lemma follows. ∎

The unitary operator $\mathbf{U}_0$ specified by (3.18)–(3.20) is the connecting operator of the finite dimensional unitary colligation

$$\Omega_0 = \left\{ \mathbb{C}^{nd} = \bigoplus_{k=1}^{d} \mathbb{C}^n, \begin{pmatrix} \mathbb{C}^q \\ \mathbb{C}^{p'} \end{pmatrix}, \begin{pmatrix} \mathbb{C}^p \\ \mathbb{C}^{q'} \end{pmatrix}, \mathbf{U}_0 \right\}. \tag{3.24}$$

According to (3.2), the characteristic function of this colligation is given by

$$\begin{aligned} \Sigma(z) &= \begin{pmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{pmatrix} \\ &= \begin{pmatrix} U_{22} & U_{23} \\ U_{32} & 0 \end{pmatrix} + \begin{pmatrix} U_{21} \\ U_{31} \end{pmatrix} (I - \mathbf{Z}(z) U_{11})^{-1} \mathbf{Z}(z)(U_{12}, U_{13}) \end{aligned} \tag{3.25}$$

where

$$\mathbf{Z}(z) = \begin{pmatrix} z_1 I_n & & \\ & \ddots & \\ & & z_d I_n \end{pmatrix}. \tag{3.26}$$

The formula (3.25) presents a unitary realization of the function $\Sigma \in \mathcal{S}_d^{(nd+q+p-r)\times(nd+q+p-r)}$. By Theorem 1.1,

$$I - \Sigma(z)\Sigma(\omega)^* = \begin{pmatrix} U_{21} \\ U_{31} \end{pmatrix} (I - \mathbf{Z}(z)U_{11})^{-1}(I - \mathbf{Z}(z)\mathbf{Z}(\omega)^*)(I - U_{11}^*\mathbf{Z}(\omega)^*)^{-1}(U_{21}^*, U_{31}^*)$$

$$\Sigma(z) - \Sigma(\bar{\omega}) = \begin{pmatrix} U_{21} \\ U_{31} \end{pmatrix} (I - \mathbf{Z}(z)U_{11})^{-1}(\mathbf{Z}(z) - \mathbf{Z}(\omega)^*)(I - U_{11}\mathbf{Z}(\omega)^*)^{-1}(U_{12}, U_{13})$$

$$I - \Sigma(\bar{z})^*\Sigma(\bar{\omega}) = \begin{pmatrix} U_{12}^* \\ U_{13}^* \end{pmatrix} (I - \mathbf{Z}(z)U_{11}^*)^{-1}(I - \mathbf{Z}(z)\mathbf{Z}(\omega)^*)(I - U_{11}\mathbf{Z}(\omega)^*)^{-1}(U_{12}, U_{13}).$$

Using Schur blockwise matrix multiplication, the latter three relations can be written as

$$\begin{pmatrix} I_p - \Sigma(z)\Sigma(\omega)^* & \Sigma(z) - \Sigma(\bar{\omega}) \\ \Sigma(\bar{z})^* - \Sigma(\omega)^* & I_q - \Sigma(\bar{z})^*\Sigma(\bar{\omega}) \end{pmatrix} = \sum_{k=1}^{d} \begin{pmatrix} 1 - z_k\bar{\omega}_k & z_k - \bar{\omega}_k \\ z_k - \bar{\omega}_k & 1 - z_k\bar{\omega}_k \end{pmatrix} \circ \begin{pmatrix} F_k^1(z) \\ F_k^2(z) \end{pmatrix} (F_k^1(\omega)^*, F_k^2(\omega)^*) \tag{3.27}$$

where

$$F_k^1(z) = \begin{pmatrix} U_{21} \\ U_{31} \end{pmatrix} (I - \mathbf{Z}(z)U_{11})^{-1} P_k \tag{3.28}$$

and

$$F_k^2(z) = \begin{pmatrix} U_{12}^* \\ U_{13}^* \end{pmatrix} (I - \mathbf{Z}(z)U_{11}^*)^{-1} P_k. \tag{3.29}$$

4. Explicit formulas for the characteristic function of the universal unitary colligation

In this section we give explicit formulas for the block entries Σ_{jk} of the characteristic function Σ defined via (3.25). Let $\Delta(z)$ be the function given by (3.6). Its value at zero $\Delta(0)$ has been already used for the construction of the colligation (3.24). It follows immediately from (3.8), that

$$\begin{aligned} \operatorname{Ker}\Delta(0) &= \{f \in \mathbb{C}^n : \Lambda_k M_k^* f = 0\,(\forall k),\ Y^* f = 0\} \\ &= \{f \in \mathbb{C}^n : \Lambda_k N_k^* f = 0\,(\forall k),\ X^* f = 0\} \\ &= \{f \in \mathbb{C}^n : \Lambda_k M_k^* f = 0,\ \Lambda_k N_k^* f = 0\,(\forall k),\ Y^* f = 0,\ X^* f = 0\}. \end{aligned} \tag{4.1}$$

Lemma 4.1. *For every $z \in \mathbb{D}^d$ it holds that*

$$\operatorname{Ker}\Delta(z) = \operatorname{Ker}\Delta(z)^* = \operatorname{Ker}\Delta(0). \tag{4.2}$$

Proof. Let $f \in \operatorname{Ker}\Delta(0)$. Then

$$\Lambda_k M_k^* f = 0, \quad \Lambda_k N_k^* f = 0 \quad (k = 1, \dots, d), \quad Y^* f = 0, \quad X^* f = 0 \tag{4.3}$$

and by (3.6), $\Delta(z)f = \Delta(z)^* f = 0$ at every point z. Therefore,

$$\operatorname{Ker}\Delta(0) \subseteq \operatorname{Ker}\Delta(z) \cap \operatorname{Ker}\Delta(z)^*. \tag{4.4}$$

Now suppose that $\Delta(z)f = 0$ for some choice of $f \in \mathbb{C}^n$ and $z \in \mathbb{D}^d$. Then using representations (3.7) and (3.6) for $\Delta(z)$ and $\Delta(z)^*$ respectively, we get

$$\begin{aligned} 0 = f^*(\Delta(z) + \Delta(z)^*)f &= f^*\Big(\sum_{k=1}^{d}(N_k - z_k M_k)\Lambda_k N_k^* + XX^* \\ &\quad + \sum_{k=1}^{d}(M_k - \bar{z}_k N_k)\Lambda_k M_k^* + YY^*\Big)f \\ &= f^*\Big(\sum_{k=1}^{d}(1 - |z_k|^2)M_k\Lambda_k M_k^* + \sum_{k=1}^{d}(z_k M_k - N_k)\Lambda_k(\bar{z}_k M_k^* - N_k^*) \\ &\quad + XX^* + YY^*\Big)f. \end{aligned}$$

Therefore, relations (4.3) hold and thus, $\Delta(0)f = 0$. Using the same chain of equalities we conclude that $\Delta(z)^* f = 0$ also implies $\Delta(0)f = 0$. Thus,

$$\operatorname{Ker}\Delta(z) \cup \operatorname{Ker}\Delta(z)^* \subseteq \operatorname{Ker}\Delta(0)$$

which together with (4.4) implies (4.2). ∎

Thus, we have shown that $\Delta(z)$ has a nonnegative real part in $\mathbb{D}^d$. Let $Q \in \mathbb{C}^{n\times r}$ be the matrix from the representation (3.11) of the pseudoinverse $\Delta^{[-1]}(0)$. According to (3.11) we introduce the function

$$\Delta(z)^{[-1]} = Q(Q^*\Delta(z)Q)^{-1}Q^*, \tag{4.5}$$

which is analytic in $\mathbb{D}^d$ on account of (4.2).

Remark 4.2. *Let $\mathbf{P}_{\operatorname{Ker}\Delta(0)}$ denote the orthogonal projection from $\mathbb{C}^n$ onto $\operatorname{Ker}\Delta(0)$. Then*

$$I - \Delta(z)\Delta(z)^{[-1]} = \big(I - \Delta(z)\Delta(z)^{[-1]}\big)\mathbf{P}_{\operatorname{Ker}\Delta(0)}. \tag{4.6}$$

Proof. Take the representation (3.22) of any vector $x \in \mathbb{C}^n$. On account of (4.2), $\Delta(z)g = 0$ and using (4.5) we get

$$\big(I - \Delta(z)\Delta(z)^{[-1]}\big)\Delta(z)x = \big(I - \Delta(z)Q(Q^*\Delta(z)Q)^{-1}Q^*\big)\Delta(z)(Qf + g) = 0.$$

Since x is an arbitrary vector, the latter equality means that for every $z \in \mathbb{D}^d$,

$$\big(I - \Delta(z)\Delta(z)^{[-1]}\big)\mathbf{P}_{\operatorname{Ran}\Delta(z)} = 0, \tag{4.7}$$

where $\mathbf{P}_{\operatorname{Ran}\Delta(z)}$ denote the orthogonal projection from $\mathbb{C}^n$ onto $\operatorname{Ran}\Delta(z)$. In view of (4.2),

$$\mathbf{P}_{\operatorname{Ran}\Delta(z)} = I - \mathbf{P}_{\operatorname{Ker}\Delta(z)^*} = I - \mathbf{P}_{\operatorname{Ker}\Delta(0)}$$

which being substituted into (4.7), leads to (4.6). ∎

Lemma 4.3. *The following resolvent-like identity*

$$\Delta(z)^{[-1]} - \Delta(0)^{[-1]} = \Delta(0)^{[-1]} W_1 \mathbf{Z}(z) W_2^* \Delta(z)^{[-1]} \tag{4.8}$$

holds for every point $z = (z_1, \dots, z_d) \in \mathbb{D}^d$.

Proof. We begin with the equality

$$\Delta(0) - \Delta(z) = \sum_{k=1}^{d} z_k M_k \Lambda_k N_k^* = W_1 \mathbf{Z}(z) W_2^* \tag{4.9}$$

which follows directly from (3.6), (3.8) and (3.4). Using (3.11) and (4.5) we get

$$\Delta(z)^{[-1]} - \Delta(0)^{[-1]} = Q(Q^*\Delta(0)Q)^{-1}Q^*(\Delta(0) - \Delta(z))Q(Q^*\Delta(z)Q)^{-1}Q^*,$$

which together with (4.9) implies (4.8). ∎

To establish the explicit formula for $\Sigma(z)$ in terms of the interpolation data we apply the matrix equality

$$(I + BA^{-1}C)^{-1} = I - B(A + CB)^{-1}C \tag{4.10}$$

to the matrices

$$A = Q^*\Delta(0)Q, \quad B = -\mathbf{Z}(z)W_2^*Q, \quad C = Q^*W_1.$$

Taking into account (4.9) and (4.5) we get

$$\begin{aligned} (I - \mathbf{Z}(z)U_{11})^{-1} &= I + \mathbf{Z}(z)W_2^*Q(Q^*(\Delta(0) - W_1\mathbf{Z}(z)W_2^*)Q)^{-1}Q^*W_1 \\ &= I + \mathbf{Z}(z)W_2^*Q(Q^*\Delta(z)Q)^{-1}Q^*W_1 \\ &= I + \mathbf{Z}(z)W_2^*\Delta(z)^{[-1]}W_1. \end{aligned} \tag{4.11}$$

Furthermore, in view of (3.19), (4.8) and (4.11), it follows that

$$\begin{aligned} U_{21}(I - \mathbf{Z}(z)U_{11})^{-1} &= X^*\Delta(0)^{[-1]}W_1\big(I + \mathbf{Z}(z)W_2^*\Delta(z)^{[-1]}W_1\big) \\ &= X^*\big\{\Delta(0)^{[-1]} + \Delta(0)^{[-1]}W_1\mathbf{Z}(z)W_2^*\Delta(z)^{[-1]}\big\}W_1 \\ &= X^*\Delta(z)^{[-1]}W_1. \end{aligned} \tag{4.12}$$

Similarly,

$$\begin{aligned} (I - U_{11}\mathbf{Z}(z))^{-1}U_{12} &= \big\{I + W_2^*\Delta(z)^{[-1]}W_1\mathbf{Z}(z)\big\}W_2^*\Delta(0)^{[-1]}Y \\ &= W_2^*\big(\Delta(0)^{[-1]} + \Delta(z)^{[-1]}W_1\mathbf{Z}(z)W_2^*\Delta(0)^{[-1]}\big)Y \\ &= W_2^*\Delta(z)^{[-1]}Y \end{aligned} \tag{4.13}$$

and therefore,

$$(I - \mathbf{Z}(z)U_{11})^{-1}\mathbf{Z}(z)U_{12} = \mathbf{Z}(z)(I - U_{11}\mathbf{Z}(z))^{-1}U_{12} = \mathbf{Z}(z)W_2^*\Delta(z)^{[-1]}Y. \tag{4.14}$$

Taking adjoints in (4.13) and replacing z by $\bar{z}$ we obtain

$$U_{12}^*(I - \mathbf{Z}(z)U_{11}^*)^{-1} = Y^*(\Delta(\bar{z})^{[-1]})^*W_2. \tag{4.15}$$

Now we give explicit formulas for block entries Σ_{jk} of the function Σ: using (3.20), (3.25), (4.11), (4.12) and (4.14) we get

$$\begin{aligned}\Sigma_{11}(z) &= U_{22} + U_{21}(I - \mathbf{Z}(z)U_{11})^{-1}\mathbf{Z}(z)U_{12} \\ &= X^*\Delta(0)^{[-1]}Y + X^*\Delta(z)^{[-1]}W_1\mathbf{Z}(z)W_2^*\Delta(0)^{[-1]}Y \\ &= X^*\big(\Delta(0)^{[-1]} + \Delta(z)^{[-1]}W_1\mathbf{Z}(z)W_2^*\Delta(0)^{[-1]}\big)Y = X^*\Delta(z)^{[-1]}Y\end{aligned} \tag{4.16}$$

$$\begin{aligned}\Sigma_{12}(z) &= U_{23} + U_{21}(I - \mathbf{Z}(z)U_{11})^{-1}\mathbf{Z}(z)U_{13} \\ &= \big(U_{21}(I - \mathbf{Z}(z)U_{11})^{-1}\mathbf{Z}(z), I_p\big)\begin{pmatrix} U_{13} \\ U_{23}\end{pmatrix} \\ &= \big(X^*\Delta(z)^{[-1]}W_1\mathbf{Z}(z), I_p\big)T_2\end{aligned} \tag{4.17}$$

$$\begin{aligned}\Sigma_{21}(z) &= U_{32} + U_{31}(I - \mathbf{Z}(z)U_{11})^{-1}\mathbf{Z}(z)U_{12} \\ &= (U_{31}, U_{32})\begin{pmatrix} (I - \mathbf{Z}(z)U_{11})^{-1}\mathbf{Z}(z)U_{12} \\ I_q\end{pmatrix} \\ &= T_1^*\begin{pmatrix} \mathbf{Z}(z)W_2^*\Delta(z)^{[-1]}Y \\ I_q\end{pmatrix}\end{aligned} \tag{4.18}$$

$$\begin{aligned}\Sigma_{22}(z) &= U_{31}(I - \mathbf{Z}(z)U_{11})^{-1}\mathbf{Z}(z)U_{13} \\ &= T_1^*\begin{pmatrix} I_{nd} \\ 0\end{pmatrix}\big(I + \mathbf{Z}(z)W_2^*\Delta(z)^{[-1]}W_1\big)(I_{nd}, 0)\mathbf{Z}(z)T_2.\end{aligned} \tag{4.19}$$

5. DESCRIPTION OF THE SET OF ALL SOLUTIONS

In this section we describe the set of all solutions of Problem 1.2. The main result is:

Theorem 5.1. *Let Σ be the function decomposed as in* (3.25) *into four blocks Σ_{jk} specified by* (4.16)–(4.19). *Then S is a solution of Problem* 1.2 *if and only if S has a representation of the form*

$$S(z) = \Sigma_{11}(z) + \Sigma_{12}(z)\mathcal{E}(z)(I_{q'} - \Sigma_{22}(z)\mathcal{E}(z))^{-1}\Sigma_{21}(z) \tag{5.1}$$

where the free parameter $\mathcal{E}$ sweeps through the set $\mathcal{S}_d^{p'\times q'}$.

The transformation (5.1) acting on the free parameter $\mathcal{E}$ is sometimes called the Redheffer transformation with the transfer function Σ. The proof of Theorem 5.1 is divided in a number of steps and relies on the following auxiliary results.

Lemma 5.2. *Let* $\mathcal{E} \in \mathcal{S}_d^{p'\times q'}$, *let* $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \in \mathcal{S}_d^{(p+q')\times(q+p')}$ *and let* S *be of the form* (5.1). *Then* $S \in \mathcal{S}_d^{p\times q}$ *and moreover,*

$$I - S(z)S(\omega)^* = \Psi(z)(I - \mathcal{E}(z)\mathcal{E}(\omega)^*)\Psi(\omega)^* + (I, \Psi(z)\mathcal{E}(z))(I - \Sigma(z)\Sigma(\omega)^*)\begin{pmatrix} I \\ \mathcal{E}(\omega)^*\Psi(\omega)^* \end{pmatrix} \tag{5.2}$$

$$S(z) - S(\bar{\omega}) = \Psi(z)(\mathcal{E}(z) - \mathcal{E}(\bar{\omega}))\Phi(\bar{\omega}) + (I, \Psi(z)\mathcal{E}(z))(\Sigma(z) - \Sigma(\bar{\omega}))\begin{pmatrix} I \\ \mathcal{E}(\bar{\omega})\Phi(\bar{\omega}) \end{pmatrix} \tag{5.3}$$

$$I - S(\bar{z})^*S(\bar{\omega}) = \Phi(\bar{z})^*(I - \mathcal{E}(\bar{z})^*\mathcal{E}(\bar{\omega}))\Psi(\bar{\omega}) + (I, \Phi(\bar{z})^*\mathcal{E}(\bar{z})^*)(I - \Sigma(\bar{z})^*\Sigma(\bar{\omega}))\begin{pmatrix} I \\ \mathcal{E}(\bar{\omega})\Phi(\bar{\omega}) \end{pmatrix}, \tag{5.4}$$

where

$$\Psi(z) = \Sigma_{12}(z)(I - \mathcal{E}(z)\Sigma_{22}(z))^{-1} \quad \text{and} \quad \Phi(z) = (I - \Sigma_{22}(z)\mathcal{E}(z))^{-1}\Sigma_{21}(z). \tag{5.5}$$

Proof. Using the functions Ψ and Φ from (5.5) and taking into account the identity

$$\mathcal{E}(z)(I_{q'} - \Sigma_{22}(z)\mathcal{E}(z))^{-1} = (I_{p'} - \mathcal{E}(z)\Sigma_{22}(z))^{-1}\mathcal{E}(z),$$

one can represent the function S of the form (5.1) as

$$S(z) = \Sigma_{11}(z) + \Psi(z)\mathcal{E}(z)\Sigma_{21}(z) = \Sigma_{11}(z) + \Sigma_{12}(z)\mathcal{E}(z)\Phi(z) \tag{5.6}$$

The following identities

$$\Sigma_{12}(z) + \Psi(z)\mathcal{E}(z)\Sigma_{22}(z) = \Psi(z), \quad \Sigma_{21}(z) + \Sigma_{22}(z)\mathcal{E}(z)\Phi(z) = \Phi(z) \tag{5.7}$$

follow immediately from (5.5) and imply together with (5.6), that

$$(I, \Psi(z)\mathcal{E}(z))\Sigma(z) = (I, \Psi(z)\mathcal{E}(z))\begin{pmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{pmatrix} = (S(z), \Psi(z)) \tag{5.8}$$

and

$$\Sigma(z)\begin{pmatrix} I \\ \mathcal{E}(z)\Phi(z) \end{pmatrix} = \begin{pmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{pmatrix}\begin{pmatrix} I \\ \mathcal{E}(z)\Phi(z) \end{pmatrix} = \begin{pmatrix} S(z) \\ \Phi(z) \end{pmatrix}. \tag{5.9}$$

Using (5.8) we get

$$(I, \Psi(z)\mathcal{E}(z))(I - \Sigma(z)\Sigma(\omega)^*)\begin{pmatrix} I \\ \mathcal{E}(\omega)^*\Psi(\omega)^* \end{pmatrix} = I + \Psi(z)\mathcal{E}(z)\mathcal{E}(\omega)^*\Psi(\omega)^* - S(z)S(\omega)^* - \Psi(z)\Psi(\omega)^*$$

which is equivalent to (5.2). Next, on account of (5.8) and (5.9),

$$\begin{aligned} &(I, \Psi(z)\mathcal{E}(z))(\Sigma(z) - \Sigma(\bar{\omega}))\begin{pmatrix} I \\ \mathcal{E}(\bar{\omega})\Phi(\bar{\omega}) \end{pmatrix} \\ &\quad = (S(z), \Psi(z))\begin{pmatrix} I \\ \mathcal{E}(\bar{\omega})\Phi(\bar{\omega}) \end{pmatrix} - (I, \Psi(z)\mathcal{E}(z))\begin{pmatrix} S(\bar{\omega}) \\ \Phi(\bar{\omega}) \end{pmatrix} \\ &\quad = S(z) - S(\bar{\omega}) - \Psi(z)(\mathcal{E}(z) - \mathcal{E}(\bar{\omega}))\Phi(\bar{\omega}) \end{aligned}$$

which is equivalent to (5.3). In much the same way one can check (5.4) with help of (5.9). Let $\mathcal{E} \in \mathcal{S}_d^{p' \times q'}$. Then

$$\begin{aligned} &\begin{pmatrix} I_{p'} - \mathcal{E}(z)\mathcal{E}(\omega)^* & \mathcal{E}(z) - \mathcal{E}(\bar{\omega}) \\ \mathcal{E}(\bar{z})^* - \mathcal{E}(\omega)^* & I_{q'} - \mathcal{E}(\bar{z})^*\mathcal{E}(\bar{\omega}) \end{pmatrix} \\ &\qquad = \sum_{k=1}^{d} \begin{pmatrix} 1 - z_k\bar{\omega}_k & z_k - \bar{\omega}_k \\ z_k - \bar{\omega}_k & 1 - z_k\bar{\omega}_k \end{pmatrix} \circ \begin{pmatrix} G_k^1(z) \\ G_k^2(z) \end{pmatrix} (G_k^1(\omega)^*, G_k^2(\omega)^*) \end{aligned} \tag{5.10}$$

for some operator-valued functions $G_k^1(z)$ and $G_k^2(z)$ analytic on $\mathbb{D}^d$. Substituting the latter representation together with (3.27) into (5.2)–(5.4) we get

$$\begin{aligned} &\begin{pmatrix} I_p - S(z)S(\omega)^* & S(z) - S(\bar{\omega}) \\ S(\bar{z})^* - S(\omega)^* & I_q - S(\bar{z})^*S(\bar{\omega}) \end{pmatrix} \\ &\qquad = \sum_{k=1}^{d} \begin{pmatrix} 1 - z_k\bar{\omega}_k & z_k - \bar{\omega}_k \\ z_k - \bar{\omega}_k & 1 - z_k\bar{\omega}_k \end{pmatrix} \circ \begin{pmatrix} H_k^1(z) \\ H_k^2(z) \end{pmatrix} (H_k^1(\omega)^*, H_k^2(\omega)^*) \end{aligned} \tag{5.11}$$

where

$$H_k^1(z) = (\Psi(z)G_k^1(z), (I, \Psi(z)\mathcal{E}(z))F_k^1(z)) \tag{5.12}$$

and

$$H_k^2(z) = (\Phi(\bar{z})^*G_k^2(z), (I, \Phi(\bar{z})^*\mathcal{E}(\bar{z})^*)F_k^2(z)). \tag{5.13}$$

Therefore, $S \in \mathcal{S}_d^{p \times q}$. ▮

Lemma 5.3. *Let Σ_{11} be the function given by* (4.16). *Then*

$$\mathbf{R}_{\mathbf{A}}(X_L\Sigma_{11}(z)) = Y_L \quad \textit{and} \quad \mathbf{R}_{\mathbf{B}}(Y_R\Sigma_{11}(\bar{z})^*) = X_R. \tag{5.14}$$

Proof. First we note that in view of (4.1), $\operatorname{Ker}\Delta(0) \subseteq \operatorname{Ker} Y^* \cap \operatorname{Ker} X^*$, and therefore,

$$\mathbf{P}_{\operatorname{Ker}\Delta(0)}Y = 0, \quad X^*\mathbf{P}_{\operatorname{Ker}\Delta(0)} = 0 \tag{5.15}$$

where $\mathbf{P}_{\operatorname{Ker}\Delta(0)}$ denotes the orthogonal projection from $\mathbb{C}^n$ onto $\operatorname{Ker}\Delta(0)$. Using (3.6) and taking into account (4.6) we get

$$\begin{aligned} XX^*\Delta(z)^{[-1]} &= \Delta(z)\Delta(z)^{[-1]} - \sum_{k=1}^{d}(z_kM_k - N_k)\Lambda_kN_k^*\Delta(z)^{[-1]} \\ &= I - \sum_{k=1}^{d}(z_kM_k - N_k)\Lambda_kN_k^*\Delta(z)^{[-1]} - \big(I - \Delta(z)\Delta(z)^{[-1]}\big)\mathbf{P}_{\operatorname{Ker}\Delta(0)}. \end{aligned} \tag{5.16}$$

In much the same way, it follows from (3.7) that

$$\begin{aligned} \Delta(z)^{[-1]}YY^* = I - \Delta(z)^{[-1]}\sum_{k=1}^{d}M_k\Lambda_k(M_k^* - z_kN_k^*) \\ - \mathbf{P}_{\operatorname{Ker}\Delta(0)}\big(I - \Delta(z)^{[-1]}\Delta(z)\big). \end{aligned} \tag{5.17}$$

Using (4.16), (5.16) and the first relation in (5.15), we get

$$X\Sigma_{11}(z) = XX^*\Delta(z)^{[-1]}Y = Y - \sum_{k=1}^{d}(z_kM_k - N_k)\Lambda_k N_k^*\Delta(z)^{[-1]}Y.$$

Substituting (1.13) and (2.9) into the latter equality we obtain

$$X_L\Sigma_{11}(z) = Y_L - \sum_{k=1}^{d}(z_kI_{n_L} - A_k)(\Lambda_k^L, \Lambda_k^{LR})N_k^*\Delta(z)^{[-1]}Y.$$

Applying the operator $\mathbf{R_A}$ to both parts of the latter equality, taking into account that $\Delta(z)^{[-1]}$ is analytic in $\mathbb{D}^d$ and using (2.1), (2.2), we get

$$\mathbf{R_A}(X_L\Sigma_{11}(z)) = \mathbf{R_A}(Y_L) - \sum_{k=1}^{d}\mathbf{R_A}\big((z_kI_{n_L} - A_k)(\Lambda_k^L, \Lambda_k^{LR})N_k^*\Delta(z)^{[-1]}\big)Y = Y_L$$

which proves the first relation in (5.14). The proof of the second assertion of lemma is quite similar: using (4.16), (5.17) and the second relation in (5.15) we get

$$\Sigma_{11}(z)Y^* = X^*\Delta(z)^{[-1]}YY^* = X^* - X^*\Delta(z)^{[-1]}\sum_{k=1}^{d}M_k\Lambda_k(M_k^* - z_kN_k^*).$$

Substituting partitions (1.13) and (2.9) into the latter equality and comparing the "right" blocks we obtain

$$\Sigma_{11}(z)Y_R^* = X_R^* - X^*\Delta(z)^{[-1]}\sum_{k=1}^{d}M_k\begin{pmatrix}\Lambda_k^{LR}\\ \Lambda_k^R\end{pmatrix}(B_k^* - z_kI_{n_R}).$$

Therefore,

$$Y_R\Sigma_{11}(\bar{z})^* = X_R + \sum_{k=1}^{d}(z_kI_{n_R} - B_k)\big((\Lambda_k^{LR})^*, \Lambda_k^R\big)M_k^*\big(\Delta(\bar{z})^{[-1]}\big)^*X$$

and applying the operator $\mathbf{R_B}$ to both parts of this latter equality we come to $\mathbf{R_B}(Y_R\Sigma_{11}(\bar{z})^*) = X_R$ which ends the proof of lemma. ∎

Lemma 5.4. *Let* Σ_{12} *and* Σ_{21} *be the functions defined by* (4.17) *and* (4.18), *respectively. Then*

$$\mathbf{R_A}(X_L\Sigma_{12}(z)) = 0 \quad \textit{and} \quad \mathbf{R_B}(Y_R\Sigma_{21}(\bar{z})^*) = 0. \tag{5.18}$$

Proof. We begin with equalities

$$(W_2, X)T_2 = 0 \quad \text{and} \quad (W_1, Y)T_1 = 0 \tag{5.19}$$

which hold by definition of T_1, T_2. Note also that in view of (4.1), $\operatorname{Ker}\Delta(0) \subseteq \operatorname{Ker}W_1^* \cap \operatorname{Ker}W_2^*$, and therefore,

$$\mathbf{P}_{\operatorname{Ker}\Delta(0)}W_2 = 0, \quad W_1^*\mathbf{P}_{\operatorname{Ker}\Delta(0)} = 0. \tag{5.20}$$

Using (4.17), (5.16) and taking into account the first relations from (5.19), (5.20) we get

$$
\begin{aligned}
X\Sigma_{12}(z) &= X(X^*\Delta(z)^{[-1]}W_1\mathbf{Z}(z), I_p)T_2 \\
&= \Big(\Big(I - \sum_{k=1}^{d}(z_kM_k - N_k)\Lambda_kN_k^*\Delta(z)^{[-1]}\Big)W_1\mathbf{Z}(z), X\Big)T_2 \\
&= \Big(W_1\mathbf{Z}(z) - W_2 - \sum_{k=1}^{d}(z_kM_k - N_k)\Lambda_kN_k^*\Delta(z)^{[-1]}W_1\mathbf{Z}(z), 0\Big)T_2.
\end{aligned}
$$

Using the explicit formulas (3.4) for W_1, W_2 one can rewrite the latter equality as

$$
\begin{aligned}
X\Sigma_{12}(z) = \big((z_1M_1 - N_1)\Lambda_1^{1/2}, \dots, (z_dM_d - N_d)\Lambda_d^{1/2}, 0\big)T_2 \\
- \Big(\sum_{k=1}^{d}(z_kM_k - N_k)\Lambda_kN_k^*\Delta(z)^{[-1]}W_1\mathbf{Z}(z), 0\Big)T_2.
\end{aligned}
$$

Substituting partitions (1.13), (2.9) into the latter equality and comparing the upper blocks we obtain

$$
\begin{aligned}
X_L\Sigma_{12}(z) = \big((z_1I_{n_L} - A_1, 0)\Lambda_1^{1/2}, \dots, (z_dI_{n_L} - A_d, 0)\Lambda_d^{1/2}, 0\big)T_2 \\
- \Big(\sum_{k=1}^{d}(z_kI_{n_L} - A_k)(\Lambda_k^L, \Lambda_k^{LR})N_k^*\Delta(z)^{[-1]}W_1\mathbf{Z}(z), 0\Big)T_2.
\end{aligned}
$$

Applying the operator $\mathbf{R}_\mathbf{A}$ to both parts of the latter equality, taking into account that $\Delta(z)^{[-1]}$ is analytic in $\mathbb{D}^d$ and using (2.2), we get the first relation from (5.18). The second one is obtained in much the same way. ∎

Lemma 5.5. *$F_k^1(z)$ and $F_k^2(z)$ are the functions given by* (3.28) *and* (3.29) *and let $\mathbf{\Gamma}$ be the operator defined via* (1.12). *Then*

$$
\mathbf{\Gamma}_{\mathbf{A},\mathbf{A}}\left\{(X_L, 0)F_k^1(z)F_k^1(\omega)^*\begin{pmatrix} X_L^* \\ 0 \end{pmatrix}\right\} = \Lambda_k^L \tag{5.21}
$$

$$
\mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\left\{(X_L, 0)F_k^1(z)F_k^2(\omega)^*\begin{pmatrix} Y_R^* \\ 0 \end{pmatrix}\right\} = \Lambda_k^{LR} \tag{5.22}
$$

$$
\mathbf{\Gamma}_{\mathbf{B},\mathbf{B}}\left\{(Y_R, 0)F_k^2(z)F_k^2(\omega)^*\begin{pmatrix} Y_R^* \\ 0 \end{pmatrix}\right\} = \Lambda_k^R. \tag{5.23}
$$

Proof. Substituting (4.12) and (4.15) into (3.28) and (3.29), respectively, we obtain

$$
(I_p, 0)F_k^1(z) = X^*\Delta(z)^{[-1]}W_1P_k, \quad (I_q, 0)F_k^2(z) = Y^*(\Delta(\bar{z})^{[-1]})^*W_2P_k. \tag{5.24}
$$

Multiplying both sides of the first equality in (5.24) by X from the left an using (5.16) and the second relation in (5.20) we receive

$$
(X, 0)F_k^1(z) = XX^*\Delta(z)^{[-1]}W_1P_k = \Big(I - \sum_{k=1}^{d}(z_kM_k - N_k)\Lambda_kN_k^*\Delta(z)^{[-1]}\Big)W_1P_k
$$

and the comparison of the upper blocks leads to

$$(X_L,0)F_k^1(z) = \Big((I_{n_L},0) - \sum_{k=1}^d (z_kI_{n_L} - A_k)(\Lambda_k^L, \Lambda_k^{LR})N_k^*\Delta(z)^{[-1]}\Big)W_1P_k.$$

Applying the operator $\mathbf{R_A}$ to both parts of the latter equality, taking into account that $\Delta(z)^{[-1]}$ is analytic in $\mathbb{D}^d$ and using (2.1), (2.2), we get

$$\mathbf{R_A}((X_L,0)F_k^1(z)) = (I_{n_L},0)W_1P_k. \tag{5.25}$$

Similarly, using the second equality from (5.24) together with (5.17) and the first relation in (5.20) we get

$$\begin{aligned}(Y,0)F_k^2(z) &= YY^*(\Delta(\bar z)^{[-1]})^*W_2P_k \\ &= \Big(I - \sum_{k=1}^d (M_k - z_kN_k)\Lambda_kM_k^*\big(\Delta(\bar z)^{[-1]}\big)^*\Big)W_2P_k\end{aligned}$$

and the comparison of the lower blocks leads to

$$(Y_R,0)F_k^2(z) = \Big((0,I_{n_R}) + \sum_{k=1}^d (z_kI_{n_R} - B_k)\big((\Lambda_k^{LR})^*, \Lambda_k^R\big)M_k^*\big(\Delta(\bar z)^{[-1]}\big)^*\Big)W_2P_k.$$

Applying the operator $\mathbf{R_B}$ to both parts of the latter equality, we get

$$\mathbf{R_B}((Y_R,0)F_k^2(z)) = (0,I_{n_R})W_2P_k. \tag{5.26}$$

By definition (1.12) of $\mathbf{\Gamma}$ and in view of (5.25), (3.4) and (2.9),

$$\begin{aligned}\mathbf{\Gamma}_{\mathbf{A},\mathbf{A}}\Big\{(X_L,0)F_k^1(z)F_k^1(\omega)^*\begin{pmatrix} X_L^* \\ 0\end{pmatrix}\Big\} &= (I_{n_L},0)W_1P_kW_1^*\begin{pmatrix} I_{n_L} \\ 0\end{pmatrix} \\ &= (I_{n_L},0)M_k\Lambda_kM_k^*\begin{pmatrix} I_{n_L} \\ 0\end{pmatrix} = \Lambda_k^L, \\ \mathbf{\Gamma}_{\mathbf{A},\mathbf{B}}\Big\{(X_L,0)F_k^1(z)F_k^2(\omega)^*\begin{pmatrix} 0 \\ Y_R^*\end{pmatrix}\Big\} &= (I_{n_L},0)W_1P_kW_2^*\begin{pmatrix} 0 \\ I_{n_R}\end{pmatrix} \\ &= (I_{n_L},0)M_k\Lambda_kN_k^*\begin{pmatrix} 0 \\ I_{n_R}\end{pmatrix} = \Lambda_k^{LR},\end{aligned}$$

which prove (5.21) and (5.22). The equality (5.23) is verified in much the same way. ∎

Now we can prove the necessity part of Theorem 5.1.

Lemma 5.6. *Every function S of the form* (5.1) *satisfies the interpolation conditions* (1.15), (1.16) *and* (1.17).

Proof. Let Ψ and Φ be the functions defined in (5.5). Since the functions $\Psi(z)\mathcal{E}(z)$ and $\mathcal{E}(z)\Phi(z)$ are analytic in $\mathbb{D}^d$ it follows from (5.18) and by Lemma 2.1 that

$$\mathbf{R_A}(X_L\Sigma_{12}(z)\mathcal{E}(z)\Phi(z)) = 0$$

and

$$\mathbf{R_B}(Y_R\Sigma_{21}(\bar z)^*\mathcal{E}(\bar z)^*\Psi(\bar z)^*) = 0.$$

Now it follows immediately from (5.14) and representations (5.6) that S satisfies interpolation conditions (1.15) and (1.16).

It remains to show that the functions $H_k^1(z)$ and $H_k^2(z)$ from the representation (5.11) and defined by (5.12), (5.13), respectively, satisfy the interpolation conditions (1.18)–(1.20). It follows from (5.5) and (5.18) that

$$\mathbf{R_A}(X_L\Psi(z)) = 0 \quad \text{and} \quad \mathbf{R_A}(X_R\Phi(\bar{z})^*) = 0. \tag{5.27}$$

By Lemma 2.1, we get from (5.12), (5.13) and (5.27)

$$\mathbf{R_A}(X_L H_k^1(z)) = \mathbf{R_A}(0,(X_L,0)F_k^1(z)) \tag{5.28}$$

$$\mathbf{R_A}(Y_R H_k^2(z)) = \mathbf{R_A}(0,(Y_R,0)F_k^2(z)) \tag{5.29}$$

where $F_k^1(z)$ and $F_k^2(z)$ are the functions from the representation (3.27) given by formulas (3.28) and (3.29), respectively. By definition (1.12) of $\mathbf{\Gamma}$ and in view of (5.28), (5.29),

$$\mathbf{\Gamma_{A,A}}\{X_L H_k^1(z)H_k^1(\omega)^*X_L^*\} = \mathbf{\Gamma_{A,A}}\left\{(X_L,0)F_k^1(z)F_k^1(\omega)^*\begin{pmatrix} X_L^* \\ 0\end{pmatrix}\right\}$$

$$\mathbf{\Gamma_{A,B}}\{X_L H_k^1(z)H_k^2(\omega)^*Y_R^*\} = \mathbf{\Gamma_{A,B}}\left\{(X_L,0)F_k^1(z)F_k^2(\omega)^*\begin{pmatrix} Y_R^* \\ 0\end{pmatrix}\right\}$$

$$\mathbf{\Gamma_{B,B}}\{Y_R H_k^2(z)H_k^2(\omega)^*Y_R^*\} = \mathbf{\Gamma_{B,B}}\left\{(Y_R,0)F_k^2(z)F_k^2(\omega)^*\begin{pmatrix} Y_R^* \\ 0\end{pmatrix}\right\}$$

which together with (5.21)–(5.23) imply (1.18)–(1.20). ∎

Lemma 5.7. *Let S be a solution of Problem* 1.2. *Then it can be represented in the form* (5.1) *for some choice of the parameter* $\mathcal{E}\in\mathcal{S}_d^{p'\times q'}$.

Proof. By the construction, the coefficient matrix Σ of the transformation (5.1) is the characteristic function of the unitary colligation Ω_0 defined by (3.24). It is easily seen from (3.4) that the domain $\mathcal{D}_{\mathbf{V}} = \operatorname{Ran}\begin{pmatrix} W_1^* \\ Y^*\end{pmatrix}$ of the isometry $\mathbf{V}$ defined by (3.25), is the subspace of $\left(\bigoplus_{k=1}^{d}\operatorname{Ran}\Lambda_k^{1/2}\right)\oplus\operatorname{Ran}Y^*$.

It was shown in [7] that a function S is of the form (5.1) for some choice of the parameter $\mathcal{E}\in\mathcal{S}_d^{p'\times q'}$ if and only if S is the characteristic function of a unitary colligation

$$\widetilde{\Omega} = \left\{\widetilde{\mathcal{H}} = \bigoplus_{k=1}^{d}\widetilde{\mathcal{H}}_k, \mathbb{C}^q, \mathbb{C}^p, \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D}\end{pmatrix}\right\} \tag{5.30}$$

with the state space

$$\widetilde{\mathcal{H}} = \bigoplus_{k=1}^{d}\widetilde{\mathcal{H}}_k \quad \text{of the form} \quad \widetilde{\mathcal{H}}_k = \operatorname{Ran}\Lambda_k^{1/2}\oplus\mathcal{N}_k, \tag{5.31}$$

and such that

$$\left.\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D}\end{pmatrix}\right|_{\operatorname{Ran}\begin{pmatrix} W_1^* \\ Y^*\end{pmatrix}} = \mathbf{V}. \tag{5.32}$$

In other words, the colligation $\widetilde{\Omega}$ is the *coupling* of the colligation Ω_0 and some unitary colligation

$$\Omega' = \left\{ \mathcal{N} = \bigoplus_{k=1}^{d} \mathcal{N}_k, \mathbb{C}^{p'}, \mathbb{C}^{q'}, U' \right\}$$

where the dimensions p' and q' of the input and the output spaces are given by (3.13). At that, the parameter $\mathcal{E}$ in (5.1) is the characteristic function of Ω'.

The mentioned result is a multivariable analogue of the corresponding 1-variable result of D. Arov and L. Grossman. We refer also to [9] for application of these ideas to the one-variable abstract interpolation problem.

By Theorem 1.1, S is the characteristic function of some unitary colligation Ω of the form (3.3). In other words, S admits a unitary realization (1.3) with the state space $\mathcal{H}$ decomposed into a d-fold orthogonal sum (1.2), and the equality (1.1) holds for functions H_k^1 and H_k^2 defined via (1.8). The functions H_k^1 and H_k^2 are analytic and take respectively $\mathcal{L}(\mathcal{H}_k; \mathbb{C}^p)$ and $\mathcal{L}(\mathcal{H}_k; \mathbb{C}^q)$ values in $\mathbb{D}^d$. Then the function H_1 and H_2 defined as

$$H^1(z) = H_1^1(z)P_1 + \cdots + H_d^1(z)P_d = C(I_{\mathcal{H}} - \mathbf{Z}(z)A)^{-1}, \tag{5.33}$$

$$H^2(z) = H_1^2(z)P_1 + \cdots + H_d^2(z)P_d = B^*(I_{\mathcal{H}} - \mathbf{Z}(z)A^*)^{-1}, \tag{5.34}$$

are analytic and respectively $\mathcal{L}(\mathcal{H}; \mathbb{C}^p)$- and $\mathcal{L}(\mathcal{H}; \mathbb{C}^q)$-valued in $\mathbb{D}^d$. With respect to the decomposition (1.2), H^j has the following block-operator form

$$H^j(z) = (H_1^j(z), \ldots, H_d^j(z)) \quad (j = 1, 2). \tag{5.35}$$

The interpolation conditions (1.15), (1.16) and (1.17) which are assumed to be satisfied by S, force certain restrictions on the connecting operator $\mathbf{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Substituting (1.3) into (1.15) and (1.16) we get

$$\mathbf{R_A}(X_L(D + C(I_{\mathcal{H}} - \mathbf{Z}(z)A)^{-1}\mathbf{Z}(z)B)) = Y_L$$

and

$$\mathbf{R_B}(Y_R(D^* + B^*\mathbf{Z}(z)(I_{\mathcal{H}} - A^*\mathbf{Z}(z))^{-1}C^*)) = X_R$$

which are equivalent on account of (5.33) and (5.34) to

$$X_L D + \mathbf{R_A}(X_L H^1(z)\mathbf{Z}(z))B = Y_L \tag{5.36}$$

and

$$Y_R D^* + \mathbf{R_B}(Y_R H^2(z)\mathbf{Z}(z))C^* = X_R, \tag{5.37}$$

respectively. It also follows from (5.33) and (5.34) that

$$C + H^1(z)\mathbf{Z}(z)A = H^1(z), \quad B^* + H^2(z)\mathbf{Z}(z)A^* = H^2(z)$$

and therefore, that

$$X_L C + \mathbf{R_A}(X_L H^1(z)\mathbf{Z}(z))A = \mathbf{R_A}(X_L H^1(z)) \tag{5.38}$$

and

$$Y_R B^* + \mathbf{R_B}(Y_R H^2(z)\mathbf{Z}(z))A^* = \mathbf{R_B}(Y_R H^2(z)). \tag{5.39}$$

The equalities (5.36) and (5.38) can be written in matrix form as

$$(\mathbf{R_A}(X_L H^1(z)\mathbf{Z}(z)), X_L)\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (\mathbf{R_A}(X_L H^1(z)), Y_L), \tag{5.40}$$

whereas the equalities (5.37) and (5.39) are equivalent to

$$(\mathbf{R_B}(Y_R H^2(z)\mathbf{Z}(z)), Y_R)\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = (\mathbf{R_B}(Y_R H^2(z)), X_R). \tag{5.41}$$

Since the functions H_k^j are analytic in $\mathbb{D}^d$, the following operators

$$T_k^1 := \mathbf{R_A}(X_L H_k^1(z)), \quad T_k^2 := \mathbf{R_B}(Y_R H_k^2(z)) \tag{5.42}$$

are bounded and act from $\mathcal{H}_k$ into $\mathbb{C}^{n_L}$ and into $\mathbb{C}^{n_R}$, respectively. It follows from (5.35) and (5.42) that

$$\mathbf{R_A}(X_L H^1(z)) = (T_1^1, \ldots, T_d^1) \quad \text{and} \quad \mathbf{R_B}(Y_R H^2(z)) = (T_1^2, \ldots, T_d^2). \tag{5.43}$$

Using (5.35) and (1.4) we get $H^j(z)\mathbf{Z}(z) = (z_1 H_1^j(z), \ldots, z_d H_d^j(z))$ $(j = 1, 2)$ and therefore,

$$\mathbf{R_A}(X_L H^1(z)\mathbf{Z}(z)) = (A_1 T_1^1, \ldots, A_d T_d^1) \tag{5.44}$$

$$\mathbf{R_B}(Y_R H^2(z)\mathbf{Z}(z)) = (B_1 T_1^2, \ldots, B_d T_d^2). \tag{5.45}$$

Substituting (5.43)–(5.45) into (5.40) and (5.41) we obtain

$$(A_1 T_1^1, \ldots, A_d T_d^1, X_L)\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (T_1^1, \ldots, T_d^1, Y_L)$$

and

$$(B_1 T_1^2, \ldots, B_d T_d^2, Y_R)\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = (T_1^2, \ldots, T_d^2, X_R).$$

Since the operator $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is unitary, we conclude from these two relations that for every choice of $f \in \mathbb{C}^n$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} T_1^{1*} & T_1^{2*}B_1^* \\ \vdots & \\ T_d^{1*} & T_d^{2*}B_d^* \\ Y_L^* & Y_R^* \end{pmatrix} f = \begin{pmatrix} T_1^{1*}A_1^* & T_1^{2*} \\ \vdots & \\ T_d^{1*}A_1^* & T_d^{2*} \\ X_L^* & X_R^* \end{pmatrix} f. \tag{5.46}$$

Now we use the interpolation conditions (1.17): substituting (5.42) into (1.17) we obtain the following factorizations

$$L_k L_k^* = \Lambda_k \quad (k = 1, \ldots, d),$$

where $L_k = \begin{pmatrix} T_k^1 \\ T_k^2 \end{pmatrix}$ and where T_k^1 and T_k^2 are defined in (5.42). Therefore, the linear transformations U_k defined by the rule

$$U_k : L_k^* f \to \Lambda_k^{1/2} f \quad (f \in \mathbb{C}^n) \tag{5.47}$$

is the unitary map from $\operatorname{Ran} L_k^*$ onto $\operatorname{Ran} \Lambda_k^{1/2}$. Setting

$$\mathcal{N}_k := \mathcal{H}_k \ominus \operatorname{Ran} L_k^* \quad \text{and} \quad \widetilde{\mathcal{H}}_k := \operatorname{Ran} \Lambda_k^{1/2} \oplus (\mathcal{H}_k \ominus \operatorname{Ran} L_k^*) = \operatorname{Ran} \Lambda_k^{1/2} \oplus \mathcal{N}_k,$$

let us define the unitary map $\widetilde{U}_k : \mathcal{H}_k \to \widetilde{\mathcal{H}}_k$ by the rule

$$\widetilde{U}_k g = \begin{cases} U_k g & \text{for } g \in \operatorname{Ran} L_k^*, \\ g & \text{for } g \in \mathcal{N}_k. \end{cases} \tag{5.48}$$

The operator

$$\widetilde{U} := \bigoplus_{k=1}^{d} \widetilde{U}_k : \mathcal{H} \to \widetilde{\mathcal{H}} := \bigoplus_{k=1}^{d} \widetilde{\mathcal{H}}_k \tag{5.49}$$

is unitary and satisfies

$$\widetilde{U} P_k = \widetilde{P}_k \widetilde{U} \quad (k = 1, \ldots, d)$$

where P_k and $\widetilde{P}_k$ are orthogonal projections from $\mathcal{H}$ onto $\mathcal{H}_k$ and from $\widetilde{\mathcal{H}}$ onto $\widetilde{\mathcal{H}}_k$, respectively. Introducing the operators

$$\widetilde{A} = \widetilde{U} A \widetilde{U}^*, \quad \widetilde{B} = \widetilde{U} B, \quad \widetilde{C} = C \widetilde{U}^*, \quad \widetilde{D} = D \tag{5.50}$$

we construct the colligation $\widetilde{\Omega}$ via (5.31). By definition, $\widetilde{\Omega}$ is unitarily equivalent to the initial colligation Ω defined in (3.3). By Remark 3.1, $\widetilde{\Omega}$ has the same characteristic function as Ω, that is, $S(z)$. To apply the general result from [7], it suffices to check (5.32). But this equality easily follows from (5.47)–(5.50). ∎

Theorem 5.1 is a consequence of Lemmas 5.6 and 5.7.

REFERENCES

1. J. AGLER, Some interpolation theorems of Nevanlinna-Pick type, preprint.
2. J. AGLER, On the representation of certain holomorphic functions defined on a polydisk, in *Topics in Operator Theory: Ernst D. Hellinger Memorial Volume*, (Ed. L. de Branges, I. Gohberg and J. Rovnyak), vol. 48, Oper. Theory Adv. Appl., Birkhäuser Verlag, Basel 1990, pp. 47–66.
3. D. ALPAY, V. BOLOTNIKOV, Interpolation in the Hardy space of the bidisk, *Proc. Amer. Math. Soc.* **127**(1999), 1789–1799.
4. J. BALL, I. GOHBERG, L. RODMAN, *Interpolation of Rational Matrix Functions*, Birkhäuser Verlag, Basel 1990.
5. J. BALL, I. GOHBERG, L. RODMAN, Tangential interpolation problems for rational matrix functions, *Proc. Sympos. Pure Math.*, vol. 40, Amer. Math. Soc., Providence 1990, pp. 59–86.
6. J. BALL, I. GOHBERG, L. RODMAN, Two-sided Lagrange-Sylvester interpolation problems for rational matrix functions, *Proc. Sympos. Pure Math.*, vol. 51, Amer. Math. Soc., Providence 1990, pp. 17–83.
7. J. BALL, T. TRENT, Unitary colligations, reproducing kernel Hilbert spaces and Nevanlinna-Pick interpolation in several variables, *J. Funct. Anal.* **157**(1998), 1–61.
8. V. BOLOTNIKOV, H. DYM, On degenerate interpolation, entropy and extremal problems for matrix Schur functions, *Integral Equations Operator Theory* **32**(1998), 367–435.

9. V. Katsnelson, A. Kheifets, P. Yuditskii, An abstract interpolation problem and the extension theory of isometric operators, in *Topics in Interpolation Theory*, (Ed. H. Dym, B. Fritzsche, V. Katsnelson, and B. Kirstein), Oper. Theory Adv. Appl., vol. 95, Birkhäuser Verlag, Basel 1997, pp. 283–297. Translated from: Operators in function spaces and problems in function theory, pp. 83–96 (Naukova-Dumka, Kiev, 1987. Edited by V.A. Marchenko).
10. A. Nudelman, On a generalization of classical interpolation problems, *Sov. Math. Dokl.* **23**(1997), 125–128.
11. R.A. Penrose, A generalized inverse for matrices. *Proceedings Cambridge Philos. Soc.* **51**(1955), 406–413.

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