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A DEFORMATION QUANTIZATION PROCEDURE FOR C^*-ALGEBRAS

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Abstract. We discuss a framework for strict deformation quantization (in the sense of Rieffel) based on a GNS-type construction. These methods are applied then to prove that the quantum $\mathrm{SU}(N)$ groups produce a strict deformation quantization of the classical $\mathrm{SU}(N)$.

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The problem of the deformation for algebras (or rings) was brought to attention in the 60's, probably the most notable contribution being the one of Gerstenhaber ([5], [6]). His approach, based on algebras of formal power series, later was given the name of *formal deformation* (see also 1.30 in this paper).

Later, in the 80's, this theory was considerably revitalized by the work of Drinfeld (see [4] for the exposition and references) who introduced some important classes of deformations called *Quantum Groups*. To be a bit more specific, he considered formal deformations for two types of algebras:

(a) for $\mathrm{U}\mathfrak{g}$ — the universal enveloping algebra of a (simple) Lie algebra $\mathfrak{g}$;
(b) for $\mathrm{Fun}(G)$ (sometimes denoted $\mathrm{Pol}(G)$) — the algebra of polynomial functions on a (compact) Lie group.

When G is compact, the "quantum" $\mathrm{Fun}(G)$ becomes even a $*$-algebra. The algebras $\mathrm{U}\mathfrak{g}$ and $\mathrm{Fun}(G)$ (suppose $\mathfrak{g}$ is the Lie algebra of G) are deformed in such a way so they still become Hopf algebras satisfying the Tannaka-Krein duality. This is the reason why the "quantum" $\mathrm{Fun}(G)$ is regarded as the algebra of "coefficients' of representations" of the "quantum" $\mathrm{U}\mathfrak{g}$ (see [4], [12], [16], [17], [21], [23], [25]). It turns out that the "quantum" $\mathrm{Fun}(G)$ can be canonically completed to a

C^*-algebra called the "quantum" $C(G)$. The most important features of these C^*-algebras were abstracted by Woronowicz under the name of *Compact Matrix Pseudogroups*. This class of Hopf C^*-algebras (which besides the "quantum" $C(G)$'s, contains also C^*-algebras like $C^*_{\text{red}}(\Gamma)$ for Γ a finitely generated discrete group) was shown to allow generalizations of many notions and results from Harmonic Analysis. Among them, we note: the Haar measure, the Peter-Weyl Theorem, Orthogonality Relations and Tannaka-Krein Duality (see [23], [25]).

Motivated by the fact that the "quantum" $\text{Fun}(G)$'s give a formal deformation, a natural question arises in the C^*-algebraic context: *Do the "quantum" $C(G)$'s give a sort of deformation of C^*-algebras? More explicitly, do the "quantum" $C(G)$'s form a continuous field of C^*-algebras?*

Actually some very restrictive fields have to be considered. That is, those fields which contain the information on how the dense subalgebras of the "quantum" $\text{Fun}(G)$'s are "put together" by the formal deformation.

The right definition was introduced by Rieffel (see for example [14]) under the name of *(strict) deformation quantization.* If one is given a C^*-algebra A, by a (strict) deformation quantization of A one means a set of data $(\mathcal{A}, z, (\times_h, *_h, \|\cdot\|_h)_{h\in I})$ with

(a) I an interval which contains 0;
(b) $\mathcal{A}$ a dense $*$-subalgebra in A;
(c) $(\times_h, *_h, \|\cdot\|_h)$ a pre-C^*-algebra structure on $\mathcal{A}$, for all $h \in I$;
(d) $z : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ a bilinear map, such that:

(i) $(\times_0, *_0, \|\cdot\|_0)$ is exactly the pre-C^*-algebra structure on $\mathcal{A}$ "inherited" from A;

(ii) for any $a \in \mathcal{A}$, the map $h \mapsto \|a\|_h$ is continuous;

(iii) for any $h_0 \in I$ and any $a, b \in \mathcal{A}$ we have:

$$\lim_{h\to h_0} \|a \times_h b - a \times_{h_0} b\|_h = 0, \quad \lim_{h\to h_0} \|a^{*_h} - a^{*_{h_0}}\|_h = 0;$$

(iv) $\lim_{h\to 0} \left\|\frac{1}{h}(a \times_h b - a \times_0 b) - z(a, b)\right\|_h = 0.$

Note that if one takes A_h to be the completion of the pre-C^*-algebra $(\mathcal{A}, \times_h, *_h)$, one gets a continuous field of C^*-algebras $(A_h)_{h\in I}$ with $A_0 = A$. In its "standard" setting, the definition deals with the situation $A = C_0(M)$, the C^*-algebra of continuous functions which vanish at ∞ on a manifold M, $\mathcal{A} \subset C_0^\infty(M)$ and axiom (d) (iv) is replaced by

$$\lim_{h\to 0} \left\|\frac{1}{h}(a \times_h b - b \times_h a) - \{a, b\}\right\|_h = 0,$$

where $\{\cdot\, , \cdot\}$ is a Poisson bracket on $C_0^\infty(M)$ which leaves $\mathcal{A}$ invariant.

Since this definition was formulated, several examples have been constructed (Rieffel and Sheu; see [14], [15], [18], [19]).

Probably the most difficult step in checking a concrete example is the continuity property (d) (ii) in the above definition.

Motivated mainly by the "quantum" $\text{SU}(N)$ groups, we propose in this paper a framework for deformation quantization based on the existence of "continuous families" of faithful states.

Roughly speaking, to the definition of Rieffel we add the existence, for any $h \in I$, of a state $\varphi_h : A_h \to \mathbb{C}$ such that when we restrict it to the $*$-algebra $(\mathcal{A}, \times_h, *_h)$, we have:

(a) For any $a \in \mathcal{A}$ the map $h \mapsto \varphi_h(a)$ is continuous.

(b) For any $h \in I$, the GNS representation of $(\mathcal{A}, \times_h, *_h)$ associated to the state φ_h is isometric in the norm $\|\cdot\|_h$.

For this new set of data we propose the name *deformation algebras*. If a deformation algebra has a "smooth form" (see 1.27), then we get both formal deformations and deformation quantization (see 1.30 and 1.40).

It turns out that if the $*$-algebra structures are obtained by localizations of a $C(I)$-$*$-algebra (which is exactly $\mathcal{A} \otimes C(I)$ as a $C(I)$-module), the properties (d) (i), (d) (ii), (d) (iii) in Rieffel's definition automatically follow, provided that the norms $\|\cdot\|_h$ come from a "uniform representation theory" (see Theorem 2.5).

The material is organized as follows.

Section 1 is meant to formalize the notion of deformation algebras. First we examine the GNS construction "over $C(Q)$". Next we deal with the localizations associated to Hilbert modules and the GNS construction. Note that the "uniform" localization introduced in 1.12 appears also in [13], for exactly the same purposes (to get an upper semicontinuous family of seminorms). Then we give the definition of deformation algebras and we prove the result (Proposition 1.25) which relates them to deformation quantization. The section concludes with the constructions related to smooth forms. We took the opportunity to recall briefly the notion of formal deformation (Gerstenhaber) and to formulate some simple properties (which are essentially inspired from [26]).

Section 2 contains the main result (Theorem 2.5), which gives a sufficient condition for a GNS-$*$-algebra to be a deformation algebra. This condition is expressed in terms of:

(a) the continuity of the family of localized states,
(b) the uniform representability of the fiber algebras,
(c) faithfulness conditions in the fiber algebras.

The deformation procedure we propose here seems suitable for many families of C^*-algebras defined by means of generators and relations. Naively, Theorem 2.5 addresses the following problem. Suppose we have a family of C^*-algebras A_h, $h \in I$ defined by a same set of generators X but by a family M_h, $h \in I$ of sets of relations. (Here M_h are understood as subsets of $\mathbb{C}\{X\}$, the free $*$-algebra generated by X.) Then the question is: *When can the C^*-algebras A_h, $h \in I$ be put in a continuous field, for which each of the generators in X defines a continuous section?* If such a situation occurs, the next question is: *When does this field come from a deformation quantization?*

Section 3 deals with an example: the quantum $\mathrm{SU}(N)$. Here, using the "standard setting", one takes $M = \mathrm{SU}(N)$ and the Poisson bracket is the one used by Drinfeld ([4]), computed explicitly here for $N = 2, 3$. For $\mathcal{A}$ we use the algebra of polynomial functions. There are three important results which allow this example to work:

(a) The faithfulness of the Haar state. For $N = 2$ this fact already appears stated in [22], [24]. The general case is treated in [9].

(b) The freeness of $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N))$ as a $\mathbb{C}[q,q^{-1}]$-module. This uses the results of Koelink ([7]).

(c) The continuity of the family of Haar states. This appears to be previously unknown for the general case. The case $N=2$ is contained in [22], [24].

1. DEFORMATION ALGEBRAS

In this chapter we give the framework for deformation quantization. It is based on the generalization of the techniques related to $*$-algebras over $\mathbb{C}$. What is essential here is that instead of $*$-algebras over $\mathbb{C}$ we shall work with $*$-algebras over a commutative C^*-algebra. Then, instead of working with Hilbert spaces, one has to use Hilbert modules. It turns out that most of the techniques still work in this case.

Later, we shall use localizations to get $*$-algebras over $\mathbb{C}$ which will give the desired deformations. Throughout the entire section, Q will be a fixed compact Hausdorff space and $C(Q)$ will stand for the (commutative) C^*-algebra of complex valued continuous functions on Q. Here, for $f \in C(Q)$, the C^*-norm is

$$\|f\| = \sup\{|f(q)| : q \in Q\}.$$

1.1. DEFINITION. By a *$*$-algebra over $C(Q)$* (or shortly a *$C(Q)$-$*$-algebra*) we shall mean an algebra A over $C(Q)$ which is equipped with a map $a \mapsto a^*$ satisfying:

(i) $(a+b)^* = a^* + b^*$ for all $a, b \in A$;

(ii) $(ab)^* = b^* a^*$ for all $a, b \in A$;

(iii) $(fa)^* = \overline{f} a^*$ for all $a \in A$, $f \in C(Q)$;

(iv) $(a^*)^* = a$ for all $a \in A$.

We shall always suppose that $\mathbf{1} \cdot a = a$ for all $a \in A$, where $\mathbf{1}$ is the unit of $C(Q)$. In particular we also have $(\lambda a)^* = \overline{\lambda} a^*$ for all $a \in A$, $\lambda \in \mathbb{C}$.

A *homomorphism of $C(Q)$-$*$-algebras* $\theta : A \to B$ will be a homomorphism of $C(Q)$-algebras such that $\theta(a^*) = \theta(a)^*$ for all $a \in A$.

1.2. DEFINITION. Let A be a $C(Q)$-$*$-algebra. By a *$C(Q)$-positive functional on A* we mean a homomorphism of $C(Q)$-modules $\varphi : A \to C(Q)$ such that:

(i) $\varphi(a^*) = \overline{\varphi(a)}$ for all $a \in A$;

(ii) $\varphi(a^* a) \geqslant 0$ for all $a \in A$.

Exactly as in the case of algebras over $\mathbb{C}$, it is shown that in the unital case (i) follows from (ii).

1.3. Let A be a $C(Q)$-$*$-algebra and $\varphi : A \to C(Q)$ be a $C(Q)$-positive functional. As in the "classical" case we have the Cauchy-Schwartz inequality:

$$|\varphi(a^* b)|^2 \leqslant \varphi(a^* a) \cdot \varphi(b^* b), \qquad \text{for all } a, b \in A.$$

This inequality is, of course, in $C(Q)$. (The proof is simply by evaluating at any point $q \in Q$. Note that if we view A as a $*$-algebra, for any $q \in Q$ the map $A \ni a \mapsto \varphi(a)(q) \in \mathbb{C}$ is a positive functional.) In particular if we take $N_\varphi = \{a \in A : \varphi(a^* a) = 0\}$, this set will be also described as

$$N_\varphi = \{a \in A : \varphi(a^* b) = 0 \text{ for all } b \in B\}.$$

So N_φ is now a left ideal in A and also a $C(Q)$-submodule of A.

Take the quotient $C(Q)$-module A/N_φ. Define $\langle\cdot|\cdot\rangle_\varphi : A/N_\varphi \times A/N_\varphi \to C(Q)$ by

$$\langle a(\operatorname{mod} N_\varphi)|b(\operatorname{mod} N_\varphi)\rangle_\varphi = \varphi(a^*b).$$

Define also on A/N_φ a *norm* $\|\cdot\|_{2,\varphi}$ by

$$\|a(\operatorname{mod} N_\varphi)\|_{2,\varphi} = \|\langle a(\operatorname{mod} N_\varphi)|a(\operatorname{mod} N_\varphi)\rangle_\varphi\|^{1/2}.$$

Finally, take H_φ to be the Banach space obtained by the completion of A/N_φ with respect to the above norm. (Sometimes this space is denoted by $L^2_{C(Q)}(A,\varphi)$.)

1.4. Remarks. Take $f \in C(Q)$. Then for any $a \in A$ we have

$$\varphi((fa)^*(fa)) = \overline{f}f\varphi(a^*a).$$

This shows that

$$\|fa(\operatorname{mod} N_\varphi)\|_{2,\varphi} = \|\overline{f}f\varphi(a^*a)\|^{1/2} \leqslant \|f\|\cdot\|\varphi(a^*a)\|^{1/2} = \|f\|\cdot\|a(\operatorname{mod} N_\phi)\|_{2,\varphi}.$$

So, for any $f \in C(Q)$ the operator $M_f : A \to A$ defined by $M_f a = fa$ gives an operator $M_f : A/N_\varphi \to A/N_\varphi$, constructed similarly, but which is *continuous* with respect to the norm $\|\cdot\|_{2,\varphi}$. Hence M_f extends to the space H_φ. Now, if for $\xi \in H_\varphi$ and $f \in C(Q)$ we define $f\cdot\xi \overset{\text{def}}{=} M_f\xi$, it is easy to see that H_φ becomes a $C(Q)$-module. With respect to this structure all maps $A \to A/N_\varphi \hookrightarrow H_\varphi$ are homomorphisms of $C(Q)$-modules.

On the other hand, from the Cauchy-Schwartz inequality we get

$$\|\varphi(a^*b)\|^2 \leqslant \|\varphi(a^*a)\|\cdot\|\varphi(b^*b)\|,$$

which reads

$$\|\langle\xi|\eta\rangle_\varphi\| \leqslant \|\xi\|_{2,\varphi}\cdot\|\eta\|_{2,\varphi} \qquad \text{for all } \xi,\eta \in A/N_\varphi.$$

This shows that the map $\langle\cdot|\cdot\rangle_\varphi : A/N_\varphi \times A/N_\varphi \to C(Q)$ is continuous in the product topology, so it has a unique extension by continuity to a map $\langle\cdot|\cdot\rangle_\varphi : H_\varphi \times H_\varphi \to C(Q)$.

1.5. One can then easily prove that the Banach space H_φ is a Hilbert module over $C(Q)$, that is, it has the following properties:

(a) It is a $C(Q)$-module.

(b) It is equipped with a map $\langle\cdot|\cdot\rangle_\varphi : H_\varphi \times H_\varphi \to C(Q)$ satisfying:

(i) $\langle\xi+\eta|\zeta\rangle_\varphi = \langle\xi|\zeta\rangle_\varphi + \langle\eta|\zeta\rangle_\varphi$ for all $\xi,\eta,\zeta \in H_\varphi$;

(ii) $\langle\xi|\eta\rangle_\varphi = \overline{\langle\eta|\xi\rangle}_\varphi$ for all $\xi,\eta \in H_\varphi$;

(iii) $\langle f\xi|\eta\rangle_\varphi = \overline{f}\langle\xi|\eta\rangle_\varphi$ for all $\xi,\eta \in H_\varphi, f \in C(Q)$;

(iv) $\langle\xi|\xi\rangle_\varphi \geqslant 0$ for all $\xi \in H_\varphi$;

(v) if $\langle\xi|\xi\rangle_\varphi = 0$, then $\xi = 0$;

(vi) the norm on H_φ is given by

$$\|\xi\| = \|\langle\xi|\xi\rangle_\varphi\|^{1/2} \quad \text{for all } \xi \in H_\varphi.$$

1.6. For an arbitrary Hilbert module H over $C(Q)$ one defines $B(H)$ to be the set of all linear operators $T : H \to H$ for which there exists an operator $T^* : H \to H$ such that

$$\langle T\xi|\eta\rangle = \langle\xi|T^*\eta\rangle \quad \text{for all } \xi, \eta \in H.$$

The main properties of these operators are (see [1]):

(a) If $T \in B(H)$, then T is continuous.
(b) If $T \in B(H)$, then T^* is unique and $T^* \in B(H)$.
(c) Together with the operator norm, $B(H)$ becomes a C^*-algebra.
(d) For $T \in B(H)$ its norm is

$$\|T\| = \inf\{M \geqslant 0 : \langle T\xi|T\xi\rangle \leqslant M^2\langle\xi|\xi\rangle \text{ for all } \xi \in H\}.$$

(e) $B(H)$ is a $C(Q)$-module by $(fT)\xi = T(f\xi)$.
(f) If $T \in B(H)$ then $T : H \to H$ is a homomorphism of $C(Q)$-modules.

1.7. DEFINITION. By a *GNS-$*$-algebra over $C(Q)$* we shall mean a pair (A, φ) consisting of a $C(Q)$-$*$-algebra A and a $C(Q)$-positive functional $\varphi : A \to C(Q)$ such that for any $a \in A$, the operator $L_a : A/N_\varphi \to A/N_\varphi$ is continuous with respect to the norm $\|\cdot\|_{2,\varphi}$. Recall that $L_a(b \bmod N_\varphi)) = ab(\operatorname{mod} N_\varphi)$.

1.8. PROPOSITION. *Suppose A is a* GNS-$*$-*algebra over $C(Q)$. Then there exists a unique homomorphism of $C(Q)$-$*$-algebras $\pi_\varphi : A \to B(H)$ such that $\pi_\varphi(a)|A/N_\varphi = L_a$, for any $a \in A$.*

Proof. Define $\widetilde{L}_a$ to be the extension, by continuity, of the operator L_a from A/N_φ to H_φ. If $\xi, \eta \in A/N_\varphi$, say $\xi = b(\operatorname{mod} N_\varphi)$, $\eta = c(\operatorname{mod} N_\varphi)$ and $a \in A$, then

$$\begin{aligned}\langle\widetilde{L}_a\xi|\eta\rangle_\varphi &= \langle L_a\xi|\eta\rangle_\varphi = \langle ab(\operatorname{mod} N_\varphi)|c(\operatorname{mod} N_\varphi)\rangle_\varphi \\ &= \varphi((ab)^*c) = \varphi(b^*a^*c) = \langle\xi|\widetilde{L}_{a^*}\rangle_\varphi.\end{aligned}$$

Since the operators $\widetilde{L}_a$ and $\widetilde{L}_{a^*}$ are continuous, the equality

$$\langle\widetilde{L}_a\xi|\eta\rangle_\varphi = \langle\xi|\widetilde{L}_{a^*}\rangle_\varphi$$

holds for all $\xi, \eta \in H_\varphi$. This shows that $\widetilde{L}_a \in B(H_\varphi)$ and $(\widetilde{L}_a)^* = \widetilde{L}_{a^*}$. Then everything is clear if we take $\pi_\varphi(a) = \widetilde{L}_a$, for all $a \in A$. ∎

This homomorphism will be called the *GNS-homomorphism of A associated to φ*.

We leave GNS-$*$-algebras aside for a while, and start investigating the behavior of certain objects related to Hilbert modules (over $C(Q)$), when we "evaluate" them at points $q \in Q$.

1.9. Let H be a Hilbert module over $C(Q)$ and let $q \in Q$. Take $V_q = \{\xi \in H : \langle \xi | \xi \rangle (q) = 0\}$. Define $(\cdot | \cdot)_q : H/V_q \times H/V_q \to \mathbb{C}$ to be the map given as

$$(\xi (\operatorname{mod} V_q) | \eta (\operatorname{mod} V_q))_q = \langle \xi | \eta \rangle (q).$$

This map is correctly defined because

$$V_q = \{\xi \in H : \langle \xi | \eta \rangle (q) = 0 \ \text{ for all } \eta \in H\}$$

(use the Cauchy-Schwartz inequality: $|\langle \xi | \eta \rangle|^2 \leqslant \langle \xi | \xi \rangle \cdot \langle \eta | \eta \rangle$).

On H/V_q we then have a *norm* $\|\cdot\|_q$ given by

$$\|\xi (\operatorname{mod} V_q)\|_q = (\langle \xi | \xi \rangle (q))^{1/2} = \left[(\xi (\operatorname{mod} V_q) | \xi (\operatorname{mod} V_q))_q\right]^{1/2}.$$

Finally, define H_q to be the completion of H/V_q with respect to this norm. The space H_q becomes a Hilbert space (by extending the scalar product $(\cdot | \cdot)_q$ by continuity). This Hilbert space will be called the *GNS-localization of H at q*.

1.10. Proposition. *With the above notation, let $E_q : H \to H_q$ be the map given by $E_q(\xi) = \xi (\operatorname{mod} V_q)$.*

(i) *E_q is a continuous operator.*

(ii) *If we equip H_q with the $C(Q)$-module structure $f \cdot \eta = f(q)\eta$, then E_q is a homomorphism of $C(Q)$-modules.*

(iii) *For any $T \in B(H)$ there exists a unique operator $T_q \in \mathcal{B}(H_q)$ such that*

$$E_q T = T_q E_q.$$

(iv) *The correspondence $T \mapsto T_q$ gives a representation of the C^*-algebra $B(H)$ on the Hilbert space H_q.*

Proof. (i) is clear, since

$$\|E_q \xi\|_q^2 = \langle \xi | \xi \rangle (q) \leqslant \|\langle \xi | \xi \rangle\| = \|\xi\|^2.$$

(ii) This is again clear because if we take $\xi \in H$, $f \in C(Q)$, and define $g \in C(Q)$ by $g(p) = f(p) - f(q)$, $p \in Q$, then $f\xi = f(q)\xi + g\xi$. But now $g(q) = 0$, so

$$\langle g\xi | g\xi \rangle (q) = |g(q)|^2 \langle \xi | \xi \rangle (q) = 0.$$

This means $g\xi \in V_q$, so $E_q(f\xi) = E_q(f(q)\xi) = f(q) E_q \xi$.

(iii) Define, for $T \in B(H)$, the operator $T_q : H/V_q \to H/V_q$ by $T_q(\xi (\operatorname{mod} V_q)) = (T\xi)(\operatorname{mod} V_q)$. Using $\langle T\xi | T\xi \rangle \leqslant \|T\|^2 \langle \xi | \xi \rangle$, it follows that $TV_q \subset V_q$, so T_q is correctly defined. Moreover, from this inequality we also get

$$\langle T\xi | T\xi \rangle (q) \leqslant \|T\|^2 \langle \xi | \xi \rangle (q),$$

that is

$$\|T_q \xi (\operatorname{mod} V_q)\|_q \leqslant \|T\| \cdot \|\xi (\operatorname{mod} V_q)\|_q.$$

This allows us to extend (uniquely) T_q by continuity to the space H_q.

(iv) This is trivial. ∎

We shall call the above representation the *local representation of $B(H)$ at q*. We denote it by π_q.

1.11. LEMMA. *Let $T \in B(H)$. Then the map $Q \ni q \mapsto \|\pi_q(T)\| \in \mathbb{R}$ is lower semi-continuous, that is, for any $p \in Q$ we have*

$$\liminf_{q \to p} \|\pi_q(T)\| \geqslant \|\pi_p(T)\|.$$

Proof. Let $\xi \in H$. For any $q \in Q$ we have (with the notation from the previous proposition) $\|T_q\|^2 \cdot \|\xi (\operatorname{mod} V_q)\|_q^2 \geqslant \|T_q \xi (\operatorname{mod} V_q)\|_q^2$. This reads

$$\|\pi_q(T)\|^2 \cdot \langle \xi | \xi \rangle (q) \geqslant \langle T\xi | T\xi \rangle (q). \tag{1.1}$$

But for any $\eta \in H$, the map $q \mapsto \langle \eta | \eta \rangle (q)$ is continuous, so $\lim\limits_{q \to p} \langle \eta | \eta \rangle (q) = \langle \eta | \eta \rangle (p)$. But then, if we apply "$\liminf\limits_{q \to p}$" to both terms in (1.1), we get

$$\liminf_{q \to p} \|\pi_q(T)\|^2 \cdot \langle \xi | \xi \rangle (p) \geqslant \langle T\xi | T\xi \rangle (p).$$

This reads

$$\Big(\liminf_{q \to p} \|\pi_q(T)\| \Big) \cdot \|\zeta\|_p \geqslant \|\pi_p(T)\zeta\|_p$$

for any $\zeta \in H_p$ (actually only for $\zeta \in H/V_p$ but then for all ζ's by continuity). This last inequality gives us, of course,

$$\liminf_{q \to p} \|\pi_q(T)\| \geqslant \|\pi_p(T)\|. \quad \blacksquare$$

We turn our attention now to another type of localization. Take $q \in Q$ and denote by $C_q(Q)$ the space of all functions $f \in C(Q)$ with $f(q) = 0$. Let us remark that, since $B(H)$ is a $C(Q)$-module, for any $g \in C(Q)$ we can consider the operator $M_g \in B(H)$ defined as $M_g \xi = g\xi$. (One can easily check that $M_g^* = M_{\overline{g}}$; moreover $\|M_g\| \leqslant \|g\|$). Define (see also [13]) then J_q to be the smallest closed two-sided ideal of the C^*-algebra $B(H)$ that contains all the operators M_f, $f \in C_q(Q)$. In fact, J_q is the closed linear span of all operators of the form $M_f T$ with $f \in C_q(Q)$ and $T \in B(H)$. The canonical $*$-homomorphism, denoted

$$\pi_q^{\mathrm{unif}} : B(H) \to B(H)/J_q$$

will be called the *uniform localization of $B(H)$ at q.*

1.12. LEMMA. *Let $T \in B(H)$. Then the map $Q \ni q \mapsto \|\pi_q^{\mathrm{unif}}(T)\| \in \mathbb{R}$ is upper semi-continuous, that is, for any $p \in Q$ we have*

$$\limsup_{q \to p} \|\pi_q^{\mathrm{unif}}(T)\| \leqslant \|\pi_p^{\mathrm{unif}}(T)\|.$$

Proof. (See also [7], Chapter 2.) Let $\varepsilon > 0$ be fixed. Then

$$\|\pi_p^{\mathrm{unif}}(T)\| = \|T (\operatorname{mod} J_p)\| \stackrel{\mathrm{def}}{=} \inf\{\|T + S\| : S \in J_p\}.$$

In particular, there exists $S \in J_q$ such that

$$\|T + S\| \leqslant \|\pi_p^{\mathrm{unif}}(T)\| + \varepsilon. \tag{1.2}$$

Now, according to the description of J_p given above, there exist $f_1, \dots, f_n \in C_p(Q)$ and $X_1, \dots, X_n \in B(H)$ such that

$$\|M_{f_1}X_1 + \cdots + M_{f_n}X_n - S\| < \varepsilon. \tag{1.3}$$

For any $k = 1, 2, \dots, n$, using the continuity of the map f_k, we can find a neighborhood U_k of p such that

$$|f_k(q)| < \frac{\varepsilon}{n \cdot \|X_k\|}, \qquad \text{for all } q \in U_k. \tag{1.4}$$

By Urysohn's Lemma, for any such k we can find another neighborhood W_k of p, $W_k \subset U_k$ and a continuous function $\chi_k : Q \to [0,1]$ such that

$$\begin{aligned} \chi_k(q) &= 1 \quad \text{for all } q \in Q \setminus U_k \\ \chi_k(q) &= 0 \quad \text{for all } q \in W_k. \end{aligned}$$

Let $g_k = \chi_k \cdot f_k$. For $q \in Q$ we have

$$|f_k(q) - g_k(q)| = \begin{cases} 0 & \text{for all } q \in Q \setminus U_k, \\ (1 - \chi_k(q)) \cdot |f_k(q)| & \text{for all } q \in U_k, \end{cases}$$

which shows (use (1.4)) that

$$|f_k(q) - g_k(q)| < \frac{\varepsilon}{n \cdot \|X_k\|} \qquad \text{for all } q \in Q.$$

In particular

$$\|M_{f_k} - M_{g_k}\| = \|M_{f_k - g_k}\| \leqslant \|f_k - g_k\| < \frac{\varepsilon}{n \cdot \|X_k\|} \qquad \text{for all } k = 1, 2, \dots, n. \tag{1.5}$$

Take then $Y = M_{g_1}X_1 + \cdots + M_{g_n}X_n$. Note, on the one hand, that using (1.5), we have

$$\begin{aligned} \|M_{f_1}X_1 + \cdots + M_{f_n}X_n - Y\| &= \|(M_{f_1} - M_{g_1}) \cdot X_1 + \cdots + (M_{f_n} - M_{g_n}) \cdot X_n\| \\ &\leqslant \|M_{f_1} - M_{g_1}\| \cdot \|X_1\| + \cdots + \|M_{f_n} - M_{g_n}\| \cdot \|X_n\| < \varepsilon. \end{aligned}$$

Combine this inequality with (1.2) and (1.3). We obtain

$$\begin{aligned} \|T + Y\| &\leqslant \|T + S\| + \|M_{f_1}X_1 + \cdots + M_{f_n}X_n - S\| \\ &\quad + \|M_{f_1}X_1 + \cdots + M_{f_n}X_n - Y\| < \|\pi_p^{\text{unif}}(T)\| + 3\varepsilon. \end{aligned} \tag{1.6}$$

On the other hand, take $W = W_1 \cap \cdots \cap W_n$. This W is a neighborhood of p, and for any $q \in W$ we have $g_k(q) = 0$ for all $k = 1, 2, \dots, n$. In particular, $g_k \in C_q(Q)$ for all $q \in W$ and $k = 1, 2, \dots, n$. Hence $Y \in J_q$ for all $q \in W$. So, if $q \in W$,

$$\|T + Y\| \geqslant \inf\{\|T + Z\| : Z \in J_q\} \overset{\text{def}}{=} \|\pi_q^{\text{unif}}(T)\|.$$

Using (1.6), this inequality gives

$$\|\pi_q^{\text{unif}}(T)\| \leqslant \|\pi_p^{\text{unif}}(T)\| + 3\varepsilon \qquad \text{for all } q \in W.$$

In particular, we get $\limsup_{q \to p} \|\pi_q^{\text{unif}}(T)\| \leqslant \|\pi_p^{\text{unif}}(T)\| + 3\varepsilon$. But ε is arbitrary, so we get the desired result. ∎

1.13. LEMMA. *Let $T \in B(H)$.*

(i) *For any $q \in Q$ we have $\|T\| \geqslant \|\pi_q^{\mathrm{unif}}(T)\| \geqslant \|\pi_q(T)\|$.*

(ii) $\|T\| = \sup\{\|\pi_q(T)\| : q \in Q\}$.

(iii) *If for all $q \in Q$ we have $\|\pi_q^{\mathrm{unif}}(T)\| = \|\pi_q(T)\| \stackrel{\mathrm{def}}{=} p_q(T)$, then the map $Q \ni q \mapsto p_q(T) \in [0, \infty)$ is continuous.*

Proof. (i) The inequality $\|T\| \geqslant \|\pi_q^{\mathrm{unif}}(T)\|$ is trivial (simply because π_q^{unif} is a $*$-homomorphism). For the second inequality, since π_q is also a $*$-homomorphism, we have

$$\|\pi_q(T)\| = \inf\{\|T+S\| : S \in \operatorname{Ker} \pi_q\}.$$

So, in order to prove this second inequality it suffices to show that $J_q \subset \operatorname{Ker} \pi_q$. But if $f \in C_q(Q)$, $X \in B(H)$ and $\xi \in H$, we have

$$\begin{aligned}\|(M_f X)_q(\xi (\operatorname{mod} V_q)\|^2 &= \langle M_f X\xi | M_f X \xi\rangle(q) = \langle fX\xi | fX\xi\rangle(q)\\ &= |f(q)|^2 \langle X\xi | X\xi\rangle(q) = 0.\end{aligned}$$

That is, $(M_f X)_q = 0$ on H/V_q, so $\pi_q(M_f X) = 0$, i.e. $M_f X \in \operatorname{Ker} \pi_q$. Now everything is clear because $J_q = \overline{\operatorname{Span}}\{M_f X : f \in C_q(Q),\ X \in B(H)\}$.

(ii) By (i) we have $\|T\| \geqslant \sup\{\|\pi_q(T)\| : q \in Q\}$. Conversely, let $M = \sup\{\|\pi_q(T)\| : q \in Q\}$. Then, for any $q \in Q$, we have

$$\begin{aligned}\langle T\xi | T\xi\rangle(q) &= \big(T_q(\xi \operatorname{mod} V_q)) | T_q(\xi (\operatorname{mod} V_q))\big)_q\\ &= \|T_q \xi (\operatorname{mod} V_q)\|_q^2 \leqslant \|T_q\|^2 \cdot \|\xi (\operatorname{mod} V_q)\|_q^2\\ &= \|\pi_q(T)\|^2 \cdot \langle \xi | \xi\rangle(q) \leqslant M^2 \cdot \langle \xi|\xi\rangle(q), \qquad \text{for all } \xi \in H.\end{aligned}$$

This gives ($q \in Q$ was arbitrary) $\langle T\xi | T\xi\rangle \leqslant M^2 \langle \xi|\xi\rangle$. By Proposition 2.1.8 (d), this gives $\|T\| \leqslant M$.

(iii) Using Lemmas 1.11 and 1.12 we get, for all $q \in Q$,

$$p_q(T) \leqslant \liminf_{s \to q} p_s(T) \leqslant \limsup_{s \to q} p_s(T) \leqslant p_q(T),$$

so $\lim\limits_{s \to q} p_s(T) = p_q(T)$. ∎

1.14. REMARK. Lemma 1.13 (i) gives the existence of a $*$-homomorphism $\Phi_q : B(H)/J_q \to \mathcal{B}(H_q)$ such that $\pi_q = \Phi_q \circ \pi_q^{\mathrm{unif}}$.

We return now to GNS algebras and we investigate, in the same spirit as above, what happens when we "evaluate at q" a GNS-$*$-algebra.

1.15. DEFINITION. Suppose A is a $C(Q)$-$*$-algebra. For a fixed $q \in Q$ consider the map $e_q : C(Q) \mapsto \mathbb{C}$ given as $e_q(f) = f(q)$, $f \in C(Q)$. Clearly e_q is a $*$-algebra homomorphism which makes $\mathbb{C}$ to become a $C(Q)$-$*$-algebra. If we regard $\mathbb{C}$ only as a $C(Q)$ module, we can form $A_q = A \otimes_{C(Q)} \mathbb{C}$. Actually, since both A and $\mathbb{C}$ are $C(Q)$-algebras, A_q will become itself a $C(Q)$-algebra.

To distinguish the $C(Q)$-module structure on $\mathbb{C}$ we shall write $A_q = A \otimes_{e_q} \mathbb{C}$. In fact, A_q becomes a $C(Q)$-$*$-algebra. The involution is simply given by $(a \otimes \lambda)^* = a^* \otimes \overline{\lambda}$.

Because the $C(Q)$-module structure of A_q is defined only in terms of e_q and the $\mathbb{C}$-vector space structure (that is, for $x \in A_q$ and $f \in C(Q)$, $fx = f(q)x$), it will be reasonable to investigate A_q only as a $*$-*algebra* (over $\mathbb{C}$). We shall call A_q the *localization of* A *at* q.

Of course, the localization gives a functor from the category of $C(Q)$-$*$-algebras to the category of $*$-algebras.

If we consider the map $e_q : C(Q) \to \mathbb{C}$, then we can define another map

$$\mathrm{Id}_A \otimes_{C(Q)} e_q : A \otimes_{C(Q)} C(Q) \to A \otimes_{C(Q)} \mathbb{C}.$$

This map gives, of course, a homomorphism of $C(Q)$-$*$-algebras $\widetilde{e}_q : A \to A_q$.

Note that $\operatorname{Ker} e_q = C_q(Q)$, and if we denote by $\imath_q$ the inclusion $C_q(Q) \hookrightarrow C(Q)$, using the right exactness properties of the tensor product (over $C(Q)$) we get (see [8], Chapter V):

$$\operatorname{Ker}(\mathrm{Id}_A \otimes_{C(Q)} e_q) = \operatorname{Range}(\mathrm{Id}_A \otimes_{C(Q)} \imath_q).$$

But under the identification $A \otimes_{C(Q)} C(Q) \simeq A$, the $C(Q)$-submodule $\operatorname{Range}(\mathrm{Id}_A \otimes_{C(Q)} \imath_q)$ gets identified with the $C(Q)$-submodule $C_q(Q) \cdot A$ of A generated by the elements of the form $f \cdot a$ with $a \in A$ and $f \in C_q(Q)$. So we get an exact sequence of $C(Q)$-modules

$$0 \to C_q(Q) \cdot A \hookrightarrow A \overset{\widetilde{e_q}}{\to} A_q \to 0. \tag{1.7}$$

In fact, $C_q(Q) \cdot A$ is a two-sided ideal in A (as a $C(Q)$-algebra) and it is invariant under the involution. So (1.7) is an exact sequence of $C(Q)$-$*$-algebras.

1.16. PROPOSITION. *Let (A, φ) be a* GNS-$*$-*algebra over $C(Q)$ and let $q \in Q$. Then:*

(i) *There exists a unique positive functional $\varphi_q : A_q \to \mathbb{C}$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\widetilde{e_q}} & A_q \\ \varphi\big\downarrow & & \big\downarrow\varphi_q \\ C(Q) & \xrightarrow[e_q]{} & \mathbb{C} \end{array}$$

is commutative.

(ii) *If we take $\pi_\varphi : A \to B(H_\varphi)$ the* GNS-*homomorphism, there exists a unique representation $\Pi_q : A_q \to \mathcal{B}(H_{\varphi,q})$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\widetilde{e_q}} & A_q \\ \pi_\varphi\big\downarrow & & \big\downarrow\Pi_q \\ B(H_\varphi) & \xrightarrow[\pi_q]{} & \mathcal{B}(H_{\varphi,q}) \end{array}$$

is commutative. Here $H_{\varphi,q}$ is the GNS-*localization of the Hilbert module H_φ at q (cf.* 1.9.).

(iii) *There exists a unique $*$-homomorphism $\Sigma_q : A_q \to B(H_\varphi)/J_q$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\widetilde{e_q}} & A_q \\ \pi_\varphi\Big\downarrow & & \Big\downarrow\Sigma_q \\ B(H_\varphi) & \xrightarrow[\pi_q^{\mathrm{unif}}]{} & B(H_\varphi)/J_q \end{array}$$

is commutative.

(iv) *The $*$-homomorphisms Π_q and Σ_q are related by the commutative diagram*

$$\begin{array}{ccc} A_q & \xrightarrow{\Sigma_q} & B(H_\varphi)/J_q \\ \Pi_\varphi \searrow & & \nearrow \Phi_q \\ & \mathcal{B}(H_{\varphi,q}) & \end{array} .$$

(v) *(A_q, φ_q) is a* GNS-$*$-*algebra over $\mathbb{C}$ and the representation $\Pi_q : A_q \to \mathcal{B}(H_{\varphi,q})$ is unitary equivalent to its* GNS *representation.*

Proof. (i) Let $\psi_q = e_q \circ \varphi : A \to \mathbb{C}$. If $x \in C_q(Q) \cdot A$, say $x = f_1a_1 + \cdots + f_na_n$ with $f_1, \ldots, f_n \in C_q(Q)$ and $a_1, \ldots, a_n \in A$, then

$$\begin{aligned} \psi_q(x) = \varphi(x)(q) &= [f_1\varphi(a_1) + \cdots + f_n\varphi(a_n)](q) \\ &= f_1(q)\varphi(a_1)(q) + \cdots + f_n(q)\varphi(a_n)(q) = 0, \end{aligned}$$

because $f_1(q) = \cdots = f_n(q) = 0$. This computation shows that $\psi_q | C_q(Q) \cdot A = 0$ and so ψ_q factors through the canonical map $A \to A/\{C_q(Q) \cdot A\}$. But $A/\{C_q(Q) \cdot A\} \simeq A_q$ and so the desired φ_q exists. Its uniqueness is obvious.

In order to prove (iii) it suffices to show that $\pi_q^{\mathrm{unif}} \circ \pi_\varphi = 0$ on $C_q(Q) \cdot A$. But again if $x \in C_q(Q) \cdot A$, say $x = f_1a_1 + \cdots + f_na_n$ with $f_1, \ldots, f_n \in C_q(Q)$ and $a_1, \ldots, a_n \in A$, then

$$\pi_\varphi(x) = \pi_\varphi(f_1a_1 + \cdots + f_na_n) = M_{f_1}\pi_\varphi(a_1) + \cdots + M_{f_n}\pi_\varphi(a_n).$$

This shows exactly that $\pi_\varphi(x) \in J_q$, so $\pi_q^{\mathrm{unif}}(\pi_\varphi(x)) = 0$.

(ii) and (iv) Define $\Pi_q = \Phi_q \circ \Sigma_q$, where Σ_q is given by (iii).

(v) Let $\mathcal{H}_q$ be the GNS space of A_q associated to φ_q. Take first $N_{\varphi_q} = \{x \in A_q : \varphi_q(x^*x) = 0\}$, then define on A_q/N_{φ_q} the scalar product

$$\big(x(\operatorname{mod} N_{\varphi_q}) | y(\operatorname{mod} N_{\varphi_q})\big)_{\varphi_q} = \varphi_q(x^*y),$$

and finally take $\mathcal{H}_q$ to be the completion of A_q/N_{φ_q} with respect to the corresponding norm. Recall that $H_{\varphi,q}$ is constructed in the following way. We take $V_q = \{\xi \in H_\varphi : \langle \xi | \xi \rangle_\varphi(q) = 0\}$ and we define on H_φ/V_q the scalar product

$$\big(\xi(\operatorname{mod} V_q) | \eta(\operatorname{mod} V_q)\big)_q = \langle \xi | \eta \rangle(q).$$

The Hilbert space $H_{\varphi,q}$ will be the completion of H_φ/V_q with respect to the corresponding norm. The space H_φ was defined itself also as a completion of A/N_φ, equipped with $\langle \cdot | \cdot \rangle_\varphi$. Denote the map $A \to A/N_\varphi \hookrightarrow H_\varphi$ by σ. So $\operatorname{Range} \sigma$ is a dense $C(Q)$-submodule of H_φ. Then, using the continuity of the operator

$E_q : H_q \to H_{\varphi,q}$ (see 1.10), $E_q(\operatorname{Range}\sigma)$ will be a dense subspace in $\operatorname{Range} E_q$. But, by construction, the space $\operatorname{Range} E_q$ is dense in $H_{\varphi,q}$, so $\operatorname{Range}(E_q \circ \sigma)$ will be dense in $H_{\varphi,q}$. Now define $U_0 : \operatorname{Range}(E_q \circ \sigma) \to \mathcal{H}_q$ by

$$U_0(E_q(\sigma(a))) = \widetilde{e}_q(a)(\operatorname{mod} N_{\varphi_q}) \quad \text{for all } a \in A.$$

Not only that shall we check that U_0 is correctly defined, but we shall also prove that U_0 is an isometry. Indeed, take $a, b \in A$ and let $\xi = E_q(\sigma(a))$, $\eta = E_q(\sigma(b))$. Then

$$\begin{aligned}
&(\widetilde{e}_q(a)(\operatorname{mod} N_{\varphi,q}) | \widetilde{e}_q(b)(\operatorname{mod} N_{\varphi,q}))_{\varphi_q} \\
&\quad = \varphi_q((\widetilde{e}_q(a))^* \widetilde{e}_q(b)) = \varphi_q(\widetilde{e}_q(a^*b)) = e_q(\varphi(a^*b)) = \varphi(a^*b)(q) \\
&\quad = \langle \sigma(a) | \sigma(b) \rangle (q) = \langle \sigma(a)(\operatorname{mod} V_q) | \sigma(b)(\operatorname{mod} V_q) \rangle_q = \langle E_q(\sigma(a)) | E_q(\sigma(b)) \rangle_q.
\end{aligned}$$

Now, $\operatorname{Range} U_0 = \{\widetilde{e}_q(a)(\operatorname{mod} N_{\varphi_q} : a \in A\}$. Since $\widetilde{e}_q(A) = A_q$, we get $\operatorname{Range} U_0 = A_q/N_{\varphi_q}$, that is $\operatorname{Range} U_0$ is dense in $\mathcal{H}_q$. Using the density of $\operatorname{Range}(E_q \circ \sigma)$ in $H_{\varphi,q}$, this will clearly give a unitary operator $U : H_{\varphi,q} \to \mathcal{H}_q$.

Take now $\xi \in A_q/N_{\varphi_q}$. Let $\eta \in \operatorname{Range}(E_q \circ \sigma)$ be such that $\xi = U_0\eta$, say $\eta = E_q(\sigma(b))$, $b \in A$. Then $\xi = \widetilde{e}_q(b)(\operatorname{mod} N_{\varphi_q})$. Take $x \in A_q$, say $x = \widetilde{e}_q(a)$, $a \in A$. Then

$$\begin{aligned}
U^{-1}(L_x\xi) &= U^{-1}\left[L_x(\widetilde{e}_q(b)(\operatorname{mod} N_{\varphi_q}))\right] = U^{-1}\left[x \cdot \widetilde{e}_q(b)(\operatorname{mod} N_{\varphi_q})\right] \\
&= U^{-1}\left[\widetilde{e}_q(a) \cdot (\widetilde{e}_q(b)(\operatorname{mod} N_{\varphi_q})\right] = U^{-1}\left[\widetilde{e}_q(ab)(\operatorname{mod} N_{\varphi_q})\right] \\
&= E_q(\sigma(ab)) = E_q(ab(\operatorname{mod} N_\varphi)) = E_q(\pi_\varphi(a) \cdot (b(\operatorname{mod} N_\varphi))) \\
&= E_q(\pi_\varphi(a)\sigma(b)).
\end{aligned} \tag{1.8}$$

Now, by Proposition 1.10, we have $E_qT = \pi_q(T)E_q$, for all $T \in B(H_\varphi)$. So the computation (1.8) can be continued to give

$$U^{-1}L_x\xi = \pi_q(\pi_\varphi(a)E_q(\sigma(b)) = (\pi_q \circ \pi_\varphi)(a)\eta. \tag{1.9}$$

But now, using part (ii),

$$(\pi_q \circ \pi_\varphi)(a) = \Pi_q(\widetilde{e}_q(a)) = \Pi_q(x).$$

So we can further continue (1.9) and conclude

$$U^{-1}L_x\xi = \Pi_q(x)\eta. \tag{1.10}$$

In particular, since U is unitary,

$$\|L_x\xi\| = \|U^{-1}L_x\xi\| = \|\Pi_q(x)\eta\| \leqslant \|\Pi_q(x)\| \cdot \|\eta\| = \|\Pi_q(x)\| \cdot \|U\eta\| = \|\Pi_q(x)\| \cdot \|\xi\|.$$

So L_x is continuous on A_q/N_{φ_q} and so (A_q, φ_q) is a GNS-$*$-algebra over $\mathbb{C}$. Extending (1.10) by continuity, we get that U intertwines Π_q with the GNS representation of (A_q, φ_q). ∎

If (A, φ) is a GNS-$*$-algebra over $C(Q)$, then the GNS-$*$-algebra (A_q, φ_q) given by the previous proposition will be called the *localization of* (A, φ) *at* q.

1.17. REMARK. Suppose ψ is another $C(Q)$-positive functional which makes again (A, ψ) a GNS-$*$-algebra over $C(Q)$. If, for some $q \in Q$ we have $\psi(a)(q) = \varphi(a)(q)$, for all $a \in A$, then $\psi_q = \varphi_q$. That is, the localization of (A, φ) at q depends only on the "evaluation of φ at q".

The above proposition associates to any GNS-$*$-algebra *over* $C(Q)$ a family (indexed by $q \in Q$) of GNS-$*$-algebras *over* $\mathbb{C}$. But if (B, ψ) is a GNS-$*$-algebra over $\mathbb{C}$, then B carries a C^*-seminorm p_ψ defined by $p_\psi(b) = \|\pi_\psi(b)\|$, where $\pi_\psi : B \to \mathcal{B}(L^2(B, \psi))$ is the GNS representation. (This seminorm can be also defined for GNS-$*$-algebras over $C(Q)$, but we will not use it in this generality.)

Proposition 1.16 has the following important

1.18. COROLLARY. *Let (A, φ) be a* GNS-$*$-*algebra over $C(Q)$. For any $q \in Q$, take p_{φ_q} to be the corresponding C^*-seminorm for the* GNS-$*$-*algebra (A_q, φ_q) (defined above). Then, for any $a \in A$, the map $Q \ni q \mapsto p_{\varphi_q}(\widetilde{e}_q(a)) \in [0, \infty)$ is lower semi-continuous.*

Proof. Use 1.16 (v) to get that $p_{\varphi_q}(\widetilde{e}_q(a)) = \|\Pi_q(\widetilde{e}_q(a)\|$. By 1.16 (ii) we get $p_{\varphi_q}(\widetilde{e}_q(a)) = \|\pi_q(\pi_\varphi(a))\|$. To conclude, simply apply Lemma 1.11. ∎

1.19. In exactly the same way (by Lemma 1.12), if we define on A_q another C^*-seminorm $p^{\text{unif}}_{\varphi,q}$ by $p^{\text{unif}}_{\varphi,q}(x) = \|\Sigma_q(x)\|$, we get that, for any $a \in A$, the map $Q \ni q \mapsto p^{\text{unif}}_{\varphi,q}(\widetilde{e}_q(a)) \in [0, \infty)$ is upper semi-continuous.

1.20. Using Lemma 1.13 we also obtain

(a) $p^{\text{unif}}_{\varphi,q}(x) \geqslant p_{\varphi_q}(x)$, for all $x \in A_q$;

(b) if $p^{\text{unif}}_{\varphi,q} = p_{\varphi_q}$ for all $q \in Q$, then for any $a \in A$ the map $Q \ni q \mapsto p_{\varphi_q}(\widetilde{e}_q(a)) \in [0, \infty)$ is continuous.

1.21. DEFINITION. Let (A, φ) be a GNS-$*$-algebra over $C(Q)$ and let $q \in Q$. We say that (A, φ) is *regular at* q if

(i) p_{φ_q} is a norm on A_q;

(ii) $p_{\varphi_q} = p^{\text{unif}}_{\varphi,q}$ on A_q.

In this case we simply denote $p_{\varphi_q}(\cdot)$ by $\|\cdot\|_{A_q}$.

1.22. REMARK. Suppose (A, φ) is regular at q for all $q \in Q$. Then, after completions, $(A_q)_{q \in Q}$ gives rise to a continuous field of C^*-algebras.

Indeed, let us take B_q to be the completion of A_q. As a "total" set of sections we take $\Upsilon_0 = \{\Gamma(a) : a \in A\}$, where the fields $\Gamma(a)$ are defined by

$$\Gamma(a) = (\widetilde{e}_q(a))_{q \in Q} \in \prod_{q \in Q} A_q \subset \prod_{q \in Q} B_q.$$

Clearly Υ_0 is invariant under the sum, product and involution. We then take Υ to be the "closure" of Υ_0. That is, Υ consists of all fields $(x_q)_{q \in Q} \in \prod_{q \in Q} B_q$ such that, for any $q_0 \in Q$ and any $\varepsilon > 0$, there exists a neighborhood U of q_0 and $a \in A$ with

$$\|x_q - \widetilde{e}_q(a)\|_{A_q} < \varepsilon, \quad \text{for all } q \in U.$$

By [3], Propositions 10.2.3 and 10.3.2 it follows that $\mathcal{E} = \big((B_q)_{q \in Q}, \Upsilon\big)$ is a continuous field of C^*-algebras.

Let us note that deformation quantizations (as defined in the introduction) give rise, after completions, to some restricted types of fields. These fields have the property that in each fiber one finds a faithful copy of the algebra $\mathcal{A}$ (with the corresponding deformed product, involution, and norm). This suggests that the regularity condition defined above may not be sufficient. It is then natural to introduce the following.

1.23. DEFINITION. Let (A, φ) be a GNS-$*$-algebra over $C(Q)$. We say that (A, φ) is a *deformation algebra over* $C(Q)$, if

(i) as a $C(Q)$-module, A is free;

(ii) (A, φ) is regular at q for all $q \in Q$.

The reason why we impose condition (i) is the following. Any free module A over $C(Q)$ has a $\mathbb{C}$-form. That is, if we regard A as a vector space over $\mathbb{C}$, there exists $\mathbb{C}$-linear subspace V such that the map

$$V \times C(Q) \ni (v, f) \mapsto fv \in A$$

gives an isomorphism of $C(Q)$-modules between $V \otimes_{\mathbb{C}} C(Q)$ and A.

Denote by Ψ_q the restriction of the map $\widetilde{e}_q$ to V. Then $\Psi_q : V \to A_q$ is an isomorphism of vector spaces.

1.24. NOTATION. Let (A, φ) be a deformation algebra over $C(Q)$ and let V be a $\mathbb{C}$-form for A. Define for each $q \in Q$ the GNS-$*$-algebra structure on V induced by Ψ_q. That is, we define for $v, w \in V$ their *q-product*

$$v \times_q w = {\Psi_q}^{-1} \left(\Psi_q(v) \Psi_q(w) \right).$$

For $v \in V$ we define its *q-adjoint* by

$$v^{*_q} = {\Psi_q}^{-1} \left(\Psi_q(v)^* \right).$$

For $v \in V$ we define its *q-norm*, by

$$\|v\|_q = \|\Psi_q(v)\|_{A_q}.$$

For $v \in V$, we define

$$\psi_q(v) = \varphi_q(\Psi_q(v)).$$

Denote $\sup_{q \in Q} \|v\|_q \overset{\text{def}}{=} \gamma(v)$.

So, as a GNS-$*$-algebra over $\mathbb{C}$, (A_q, φ_q) is identified with (V, ψ_q) equipped with the $*$-algebra structure given by the q-product and the q-involution. The C^*-norm $\| \cdot \|_q$ is the one that comes from the GNS representation associated with ψ_q.

1.25. PROPOSITION. *With the above notation, the following properties hold:*

(i) *For any $q \in Q$, $\| \cdot \|_q$ is a C^*-norm on the $*$-algebra $(V, \times_q, *_q)$.*

(ii) *For any $v \in V$, the map $Q \ni q \mapsto \|v\|_q \in [0, \infty)$ is continuous.*

(iii) *For any $q \in Q$ and any $v \in V$ we have*

$$\lim_{s \to q} \gamma(v^{*_s} - v^{*_q}) = 0.$$

(iv) *For any $q \in Q$ and any $v, w \in V$ we have*

$$\lim_{s \to q} \gamma(v \times_s w - v \times_q w) = 0.$$

(v) *For any $q \in Q$ and $v \in V$ we have*

$$\lim_{s\to q} \psi_s(v) = \psi_q(v).$$

Proof. (i) and (ii) follow right from the definition (see also 1.20 and 1.21). To prove the rest of the statements, we choose a basis $(v_i)_{i\in I}$ in V (as a vector space). Since V is a $\mathbb{C}$-form for A, $(v_i)_{i\in I}$ will be a basis for the free $C(Q)$-module A. For any $i, j \in I$ take $I_\times(i,j), I_*(i) \subset I$ to be finite sets and let the systems $\big(f_{ij}^k\big)_{k\in I_\times(i,j)}, \big(g_i^k\big)_{k\in I_*(i)} \subset C(Q)$ be such that

$$v_i v_j = \sum_{k\in I_\times(i,j)} f_{ij}^k v_k, \qquad v_i^* = \sum_{k\in I_*(i)} g_i^k v_k.$$

The above relations take place, of course, in A, which is a $C(Q)$-$*$-algebra.

Then, if we localize at q, we get

$$v_i \times_q v_j = \sum_{k\in I_\times(i,j)} f_{ij}^k(q) v_k, \qquad v_i^{*_q} = \sum_{k\in I_*(i)} g_i^k(q) v_k.$$

Because γ is obviously a norm on V, it suffices to prove (iii) for $v = v_i$ and (iv) for $v = v_i$, $w = v_j$. But, in this case everything is clear:

$$\begin{aligned}\lim_{s\to q} \gamma(v_i^{*_s} - v_i^{*_q}) &= \lim_{s\to q} \gamma\bigg(\sum_{k\in I_*(i)} (g_i^k(s) - g_i^k(q)) v_k\bigg)\\ &\leqslant \sum_{k\in I_*(i)} \lim_{s\to q} |g_i^k(s) - g_i^k(q)| \cdot \gamma(v_i^k) = 0.\end{aligned}$$

Similarly

$$\lim_{s\to q} \gamma(v_i \times_s v_j - v_i \times_q v_j) \leqslant \sum_{k\in I_\times(i,j)} \lim_{s\to q} |f_{ij}^k(s) - f_{ij}^k(q)| \cdot \gamma(v_k) = 0.$$

(v) is obvious since $\psi_s(v) = \varphi(v)(s)$ and $\varphi(v) \in C(Q)$. ∎

Using $\gamma(v) \geqslant \|v\|_s$ for all $s \in Q$, we get

1.26. COROLLARY. (i) $\lim_{s\to q} \|v_i^{*_s} - v_i^{*_q}\|_s = 0$ *for all $v \in V$;*
(ii) $\lim_{s\to q} \|v_i \times_s v_j - v_i \times_q v_j\|_s = 0$ *for all $v, w \in V$.*

So the system $\big(V, \times_q, *_q, \|\cdot\|_q\big)_{q\in Q}$ defines, after completions with respect to the C^*-norms $\|\cdot\|_q$, a continuous field of C^*-algebras over Q such that, if $a \in A$, then $\Gamma(a) = (\widetilde{e}_q(a))_{q\in Q} \in \prod_{q\in Q} A_q$ is a section in this continuous field (see 1.22). Moreover (use the notation from 1.16), the norm of such a section is $\|\Gamma(a)\| = \sup_{q\in Q} \|\widetilde{e}_q(a)\|_q$. But $\|\widetilde{e}_q(a)\|_q = p_{\varphi_q}(\widetilde{e}_q(a)) = \|\pi_{\varphi_q}(\widetilde{e}_q(a))\| = \|\Pi_q(\widetilde{e}_q(a))\| = \|\pi_q(\pi_\varphi(a))\|$, where $\pi_\varphi : A \to B(H_\varphi)$ is the GNS representation, while $\pi_q : B(H_\varphi) \to B(H_{\varphi,q})$ is the GNS localization. But then, using 1.13, it follows that $\|\Gamma(a)\| = \sup_{q\in Q} \|\pi_q(\pi_\varphi(a))\| = \|\pi_\varphi(a)\|$. Since the sections $\Gamma(a)$, $a \in A$

form a total set (in fact a dense $*$-algebra) in the C^*-algebra of sections in this field, it follows that the C^*-algebra of sections is exactly the completion of A in the norm $\|a\| = \|\pi_\varphi(a)\|$. This gives us the fact that the above continuous field is unique with the property that all the $\Gamma(a)$'s, $a \in A$ are continuous sections.

The next natural question that arises at this moment is: *When do we get a strict deformation quantization?* The missing ingredient is the bilinear map $z : V \times V \to V$. (See Introduction.) Through the remainder of this section Q will be supposed to be a compact C^∞-manifold.

1.27. For any $C(Q)$-module A and any subset $V \subset A$ we denote

$$V^\infty = \mathrm{Span}_{\mathbb{C}}\{fv : v \in V,\ f \in C^\infty(Q)\}.$$

DEFINITION. Let (A, φ) be a deformation algebra over $C(Q)$ and let V be a $\mathbb{C}$-form for A. We say that V is *smooth* if

(i) for any $v, w \in V$ we have $v \cdot w \in V^\infty$;
(ii) for any $v \in V$ we have $v^* \in V^\infty$.

1.28. PROPOSITION. *Let (A, φ) be a deformation algebra over $C(Q)$ and V be a smooth $\mathbb{C}$-form for A. Let $q \in Q$ be fixed and let $\sigma : (-\varepsilon, \varepsilon) \to Q$ be a smooth curve with $\sigma(0) = q$. Then*

(i) *There exists a unique sequence $(z^n_{\sigma,q})_{n=1}^\infty$ of bilinear maps $z^n_{\sigma,q} : V \times V \to V$ such that, for any $n \geqslant 1$ and any $v, w \in V$, we have*

$$\lim_{t\to 0} \frac{1}{t^n} \gamma(v \times_{\sigma(t)} w - v \times_q w - t z^1_{\sigma,q}(v, w) - \cdots - t^n z^n_{\sigma,q}(v, w)) = 0.$$

(ii) *There exists a unique sequence $(\lambda^n_{\sigma,q})_{n=1}^\infty$ of conjugate linear maps $\lambda^n_{\sigma,q} : V \to V$ such that, for any $n \geqslant 1$ and any $v \in V$, we have*

$$\lim_{t\to 0} \frac{1}{t^n} \gamma(v^{*_{\sigma(t)}} - v^{*_q} - t\lambda^1_{\sigma,q}(v) - \cdots - t^n \lambda^n_{\sigma,q}(v)) = 0.$$

(iii) *If $\sigma_1 : (-\varepsilon_1, \varepsilon_1) \to Q$ is another curve with $\sigma_1(0) = q$ and*

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\sigma_1\big|t = 0 = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\sigma\big|t = 0 \quad \textit{for } k = 1, 2, \ldots, n,$$

then $z^k_{\sigma_1,q} = z^k_{\sigma,q}$ and $\lambda^k_{\sigma_1,q} = \lambda^k_{\sigma,q}$, for all $k = 1, 2, \ldots, n$.

Proof. Fix a basis $(v_i)_{i\in I}$ in V. Let, for $i, j \in I$, the sets $I_\times(i,j)$, $I_*(i)$, $(f^k_{ij})_{k\in I_\times(ij)}$, $(g^k_i)_{k\in I_*(i)}$ be given as in the proof of Proposition 1.25. That is

$$v_i v_j = \sum_{k\in I_\times(i,j)} f^k_{ij} v_k, \qquad v^*_i = \sum_{k\in I_*(i)} g^k_i v_k.$$

Since V is smooth, we have $f^k_{ij}, g^k_i \in C^\infty(Q)$.

Define

$$\alpha^k_{ij}(n) = (n!)^{-1} \cdot \frac{\mathrm{d}^n}{\mathrm{d}t^n}(t \mapsto f^k_{ij}(\sigma(t)))\big|t = 0,$$

and

$$\beta^k_i(n) = (n!)^{-1} \cdot \frac{\mathrm{d}^n}{\mathrm{d}t^n}(t \mapsto g^k_i(\sigma(t)))\big|t = 0.$$

Let $z^n_{\sigma,q} : V \times V \to V$ be the unique bilinear map defined, on the basis, by

$$z^n_{\sigma,q}(v_i, v_j) = \sum_{k \in I_\times(i,j)} \alpha^k_{ij}(n) v_k.$$

Similarly, define $\lambda^n_{\sigma,q} : V \to V$ to be the unique conjugate linear map given, on the basis, by

$$\lambda^n_{\sigma,q}(v_i) = \sum_{k \in I_*(i)} \beta^k_i(n) v_k.$$

In order to verify (i) and (ii) it suffices to show (use the fact that γ is a norm on V) that (i) holds for $v = v_i$, $w = v_j$ and (ii) holds for $v = v_i$.

But

$$\begin{aligned}
&\frac{1}{|t|^n} \cdot \gamma(v_i \times_{\sigma(t)} v_j - v_i \times_q v_j - t z^1_{\sigma,q}(v_i, v_j) - \cdots - t^n z^n_{\sigma,q}(v_i, v_j)) \\
&= \frac{1}{|t|^n \cdot \gamma\left(\sum_{k \in I_\times(i,j)} \left[f^k_{ij}(\sigma(t)) - f^k_{ij}(q) - t\alpha^k_{ij}(1) - \cdots - t^n \alpha^k_{ij}(n) \right] v_k \right)} \\
&\leqslant \sum_{k \in I_\times(i,j)} \left| \frac{1}{t^n} \cdot \left[f^k_{ij}(\sigma(t)) - f^k_{ij}(q) - t\alpha^k_{ij}(1) - \cdots - t^n \alpha^k_{ij}(n) \right] \right| \cdot \gamma(v_k).
\end{aligned}$$

Using Taylor expansion, for each $k \in I_\times(i,j)$ we have

$$\lim_{t \to 0} \frac{1}{t^n} \left[f^k_{ij}(\sigma(t)) - f^k_{ij}(q) - t\alpha^k_{ij}(1) - \cdots - t^n \alpha^k_{ij}(n) \right] = 0,$$

so we get (i).

(ii) is proved exactly in the same way. The uniqueness of the sequences $z^n_{\sigma,q}, \lambda^n_{\sigma,q}$ follows by induction. Indeed, let $\widetilde{z}^n_{\sigma,q}$ be another sequence which satisfies (i). Fix $v, w \in V$ and define

$$Z_n(t) = v \times_{\sigma(t)} w - v \times_q w - t z^1_{\sigma,q}(v, w) - \cdots - t^n z^n_{\sigma,q}(v, w)$$

and $\widetilde{Z}_n(t)$ in a similar way with $\widetilde{z}$'s instead of z's.

For $n = 0$ we have $Z_0(t) = \widetilde{Z}_0(t) = v \times_{\sigma(t)} w - v \times_q w$. Then if $Z_n(t) = \widetilde{Z}_n(t)$, we get

$$\begin{aligned}
\gamma(z^{n+1}_{\sigma,q}(v, w) - \widetilde{z}^{n+1}_{\sigma,q}) &= \gamma(t^{-(n+1)}(Z_{n+1}(t) - \widetilde{Z}_{n+1}(t))) \\
&\leqslant \frac{1}{|t|^{n+1}} \left[\gamma(Z_{n+1}(t)) + \gamma(\widetilde{Z}_{n+1}(t)) \right].
\end{aligned}$$

By assumption, we have $\lim_{t \to 0} \frac{1}{t^{n+1}} \cdot \gamma(Z_{n+1}(t)) = \lim_{t \to 0} \frac{1}{t^{n+1}} \cdot \gamma(\widetilde{Z}_{n+1}(t)) = 0$, so we get

$$\gamma(z^{n+1}_{\sigma,q}(v, w) - \widetilde{z}^{n+1}_{\sigma,q}(v, w)) = 0,$$

and since γ is a norm, we get $z^{n+1}_{\sigma,q}(v, w) = \widetilde{z}^{n+1}_{\sigma,q}(v, w)$. Similarly the uniqueness of the λ's follows.

Statement (iii) follows easily using formulas given for $z^n_{\sigma,q}$ and $\lambda^n_{\sigma,q}$ on the basis. ∎

1.29. COMMENTS. Suppose that $q \in Q$ is fixed and V and σ are also fixed as in Proposition 1.28. We denote the q-product simply by "$\cdot$" and the q-involution by "$*$". Let $z^n_{\sigma,q} \overset{\text{def}}{=} z_n$, $\lambda^n_{\sigma,q} \overset{\text{def}}{=} \lambda_n$ for $n \geqslant 1$. It is not difficult to prove (see [10] for this computation), that the maps z_n, λ_n, $n \geqslant 0$ satisfy the following identities:

(1.11.n) $\displaystyle\sum_{m=0}^{n} z_{n-m}(z_m(a,b),c) = \sum_{m=0}^{n} z_m(a, z_{n-m}(b,c))$ for all $n \geqslant 0$;

(1.12.n) $\displaystyle\sum_{m=0}^{n} \lambda_{n-m}(z_m(a,b)) = \sum_{m_1+m_2+m_3=n} z_{m_3}(\lambda_{m_1}(a), \lambda_{m_2}(b))$ for all $n \geqslant 0$;

(1.13.n) $\displaystyle\sum_{m=0}^{n} \lambda_{n-m}(\lambda_m(a)) = 0$ for all $n \geqslant 1$;

(1.13.0) $\lambda_0(\lambda_0(a)) = a$.

1.30. The above formulas allow us to make the following construction (q is fixed, as well as σ; use the above notation).

Take $V[[t]]$ to be the space of formal power series in t with coefficients in V. That is, every element in $V[[t]]$ is a power series

$$v = v_0 + v_1 t + v_2 t^2 + \cdots.$$

The vector space $V[[t]]$ becomes a $\mathbb{C}[[t]]$-module in the obvious way. If we regard the algebra $\mathbb{C}[[t]]$ as a $*$-algebra by $t^* = t$, we can define a $\mathbb{C}[[t]]$-$*$-algebra structure on $V[[t]]$ in the following way. It suffices to describe the product and involution for $v \in V$, i.e. for "constant" series. For these "generators", say $a, b \in V$, we define

$$a \bullet b = z_0(a,b) + z_1(a,b)t + z_2(a,b)t^2 + \cdots$$
$$a^\star = \lambda_0(a) + \lambda_1(a)t + \lambda_2(a)t^2 + \cdots.$$

Formulas (1.11) give the associativity of $\bullet$. Formulas (1.12) give

$$(v \bullet w)^\star = (v^\star) \bullet (w^\star), \quad \text{for all } v, w \in V[[t]].$$

Formulas (1.13) give

$$(v^\star)^\star = v, \quad \text{for all } v \in V[[t]].$$

Recall that $z_0(v,w) = v \cdot w$ and $\lambda_0(v) = v^*$. Having this in mind, our construction yields what is known in the literature as a *formal deformation of the $*$-algebra* $(V, \cdot, {}^*)$.

So, for any C^∞-curve $\sigma : (-\varepsilon, \varepsilon) \to Q$ with $\sigma(0) = q$, we have a formal deformation of the $*$-algebra $(V, \cdot, {}^*)$. As remarked in Proposition 1.28, this structure depends only on the derivatives of σ at 0.

1.31. We will focus now on the formulas (1.11)–(1.13) for $n = 1, 2$. We will rewrite (some of) them, using some operations which are used in homology. We recall (see [8]) the definition of the Hochschild complex $(C^n(D,D))_{n \geqslant 0}$ of an algebra D (over a commutative ring $\mathbf{k}$) with coefficients in itself:

$$C^n(D,D) \overset{\text{def}}{=} \{\omega : \underbrace{D \times \cdots \times D}_{n \text{ times}} \to D : \omega \text{ is } \mathbf{k}\text{-multilinear}\}.$$

For $n=0$, $C^0(D,D) \stackrel{\text{def}}{=} D$. Sometimes, to avoid the ambiguities, we shall use the notation $C^n_{\mathbf{k}}(D,D)$. Define $\delta : C^n(D,D) \to C^{n+1}(D,D)$ by

$$\begin{aligned}(\delta\omega)(d_1,\dots,d_{n+1}) \stackrel{\text{def}}{=} d_1\cdot\omega(d_2,\dots,d_{n+1}) - \omega(d_1d_2,d_3,\dots,d_{n+1}) + \cdots \\ \cdots + (-1)^n\omega(d_1,d_2,\dots,d_n,d_{n+1}) + (-1)^{n+1}\omega(d_1,\dots,d_n)\cdot d_{n+1}.\end{aligned}$$

If $\omega \in C^0(D,D) = D$,

$$(\delta\omega)(d) \stackrel{\text{def}}{=} d\cdot\omega - \omega\cdot d.$$

The space of n-cocycles is

$$Z^n(D,D) \stackrel{\text{def}}{=} \{\omega \in C^n(D,D) : \delta\omega = 0\},$$

and the space of n-coboundaries is

$$B^n(D,D) \stackrel{\text{def}}{=} \delta C^{n-1}(D,D), \quad B^0(D,D) \stackrel{\text{def}}{=} \{0\}.$$

For $\omega \in C^n(D,D)$, $\nu \in C^m(D,D)$, $n,m \geqslant 1$, one defines their *Gerstenhaber composition product* (see [5]), $\omega\circ\nu \in C^{n+m-1}(D,D)$ by

$$\begin{aligned}&(\omega\circ\nu)(d_1,\dots,d_{n+m-1}) \\ &= \sum_{j=0}^{n-m}(-1)^{j(m-1)}\omega(d_1,\dots,d_j,\nu(d_{j+1},\dots,d_{j+m}),d_{j+m+1},\dots,d_{n+m-1}).\end{aligned}$$

In general we have

$$Z^n(D,D)\circ Z^m(D,D) \not\subset Z^{n+m-1}(D,D).$$

Instead, it is true that if we take the composition commutator

$$[\omega,\nu]_\circ = \omega\circ\nu - (-1)^{(n-1)(m-1)}\nu\circ\omega,$$

then

$$[Z^n(D,D), Z^m(D,D)]_\circ \subset Z^{n+m-1}(D,D).$$

But if we take $\omega \in Z^{2k}(D,D)$, then $[\omega,\omega]_\circ = 2\omega\circ\omega$. So if $\omega \in Z^{2k}(D,D)$, then $\omega\circ\omega \in Z^{4k-1}(D,D)$. (Assume that 2 is invertible in $\mathbf{k}$.)

Note that if $\omega,\nu \in C^1(D,D)$, then $\omega\circ\nu$ is exactly the composition of ω and ν as maps $: D \to D$.

If moreover D is a $*$-algebra, then on each $C^n(D,D)$ we have an involution defined by

$$\omega^*(d_1,\dots,d_n) \stackrel{\text{def}}{=} \omega(d_1^*,\dots,d_n^*)^*.$$

An easy computation shows that, for any $\omega \in C^n(D,D)$ we have

$$\delta(\omega^*) = (-1)^{n+1}(\delta\omega)^*.$$

We now turn our attention to the formulas (1.11)–(1.13) for $n \leqslant 2$. Let us denote by $\rho_n : V \to V$ the map $\rho_n(a) = \lambda_0(\lambda_n(a))$, $n \geqslant 0$. The maps ρ_n are linear.

As before, the point $q \in Q$ and the curve σ are supposed now to be fixed. Regard V as the $*$-algebra A_q on which the product and involution are simply denoted by "$\cdot$" and "$*$". Also denote, in A_q, the commutator $a\cdot b - b\cdot a$ by

$[a, b]$. We regard $z_n \in C^2(A_q, A_q)$ and $\rho_n \in C^1(A_q, A_q)$. Of course $z_0(a, b) = a \cdot b$, $\lambda_0(a) = a^*$ and $\lambda_n(a) = \rho_n(a)^*$. The relations (1.11.1), (1.11.2) are

$$z_1(a \cdot b, c) + z_1(a, b) \cdot c = a \cdot z_1(b, c) + z_1(a, b \cdot c)$$

$$z_2(a \cdot b, c) + z_1(z_1(a, b), c) + z_2(a, b) \cdot c = a \cdot z_2(b, c) + z_1(a, z_1(b, c)) + z_2(a, b \cdot c).$$

They are equivalent, using the above notations, with

(1.11.1)′ $\qquad \delta z_1 = 0, \quad$ that is, z_1 is a 2-cocycle.

(1.11.2)′ $\qquad \delta z_2 = z_1 \circ z_1.$

Relation (1.12.1) is

$$\lambda_1(a \cdot b) + z_1(a, b)^* = z_1(a^*, b^*) + \lambda_1(b) \cdot a^* + b^* \cdot \lambda_1(a),$$

which reads

(1.12.1)′ $$z_1 - z_1^* = \delta \rho_1.$$

Relation (1.13.1) is $\lambda_1(a^*) + \lambda_1(a)^* = 0$, which is the same as

(1.13.1)′ $$\rho_1^* = -\rho_1.$$

Let us recall that the 2-cocycle z_1 (on A_q) depends only on the first derivative $\dot{\sigma}(0)$ of σ. This means that it corresponds actually to a tangent vector at q for Q. That is why we shall denote z_1 by $z_{X,q}$ where $X = \dot{\sigma}(0)$. The same is true for ρ_1 which will be denoted by $\rho_{X,q}$.

What we have obtained is the following.

1.32. PROPOSITION. *Let (A, φ) be a deformation algebra over $C(Q)$. Let V be a smooth $\mathbb{C}$-form for A. Then*

(a) *For any $q \in Q$ and any $X \in T_q Q$ there exist unique maps*

$$z_{X,q} \in C^2(A_q, A_q), \quad \rho_{X,q} \in C^1(A_q, A_q)$$

such that for any curve σ with $\sigma(0) = q$ and $\dot{\sigma}(0) = X$, we have

(i) $z^1_{\sigma,q} = z_{X,q}$,

(ii) $\lambda^1_{\sigma,q}(a) = \rho_{X,q}(a)^{*_q}$ *for all* $a \in A_q$.

(b) *The maps $z_{X,q}$ and $\rho_{X,q}$ satisfy*

(i) $z_{X,q} \in Z^2(A_q, A_q)$,

(ii) $z_{X,q} \circ z_{X,q} \in B^3(A_q, A_q)$,

(iii) $\rho^*_{X,q} = -\rho_{X,q}$,

(iv) $z_{X,q} - z^*_{X,q} = \delta \rho_{X,q}$.

1.33. COMMENTS. (I) In the case when Q is a manifold with boundary, the definition of the maps $z^n_{\sigma,q}, \lambda^n_{\sigma,q}$ given by Proposition 1.28 makes sense if we allow curves σ which have as domain either $[0, \varepsilon)$ or $(-\varepsilon, 0]$. (This is relevant only for points $q \in \partial Q$.)

(II) If Y is a vector field on Q (the manifold Q may have boundary), then all the 2-cocycles $z_{Y_q,q}, q \in Q$ give rise to a 2-cocycle z_Y on the $C(Q)$-algebra A. Similarly, all the $\rho_{Y_q,q}$'s give a map $\rho_Y \in C^1_{C(Q)}(A, A)$. These facts rely on the well-known results about currents on manifolds (or, equivalently, the dependence on parameters for solutions of differential equations).

Take a cover of Q with open sets $D_1, \dots, D_n$ such that for each $k = 1, \dots, n$ we have an interval I_k either of the form $I_k = [0, \varepsilon_k)$ or of the form $I_k = (-\varepsilon_k, 0]$, and smooth maps $\Phi_k : I_k \times D_k \to Q$ such that

$$\Phi_k(0, q) = q, \quad \frac{\partial \Phi_k}{\partial t}(t, q) = Y_{\Phi_x(t,q)}, \qquad \text{for all } q \in D_k,\ t \in I_k,\ k = 1, \dots, n.$$

For each k, $V \otimes C^\infty(D_k) \stackrel{\text{def}}{=} B_k$ becomes a $*$-algebra over $C^\infty(D_k)$. Taking the second derivatives $\frac{\partial^2 \Phi_k}{\partial^2 t}(0, q)$ we obtain elements in $z_k^2 \in C^2_{C^\infty(D_k)}(B_k, B_k)$ such that $z_Y \circ z_Y = \delta z_k^2$. Here we view z_Y as a 2-cocycle on B_k. Using suitable extensions we can suppose the existence of maps $z_k^2 \in C^2_{C(Q)}(A, A)$ such that $z_Y \circ z_Y = \delta z_k^2$ "on D_k". Finally, this enables us, using a partition of unity, to find $z_2 \in C^2_{C(Q)}(A, A)$ such that $z_Y \circ z_Y = \delta z_2$, that is $z_Y \in B^3(A, A)$.

(III) The above properties can be expressed using the following.

DEFINITION. (compare to [26]) By a *strict Poisson algebra* (over a commutative ring $\mathbf{k}$) we mean a pair (D, z) with

(i) $z \in Z^2(D, D)$;

(ii) $z \circ z \in B^3(D, D)$.

If, moreover, $\mathbf{k}$ is a $*$-algebra and D is a $*$-algebra over $\mathbf{k}$, the system (D, z) will be called a *strict Poisson $*$-algebra* if

(iii) $z - z^* \in B^2(D, D)$.

(IV) So, the above considerations bring us to the conclusion that (A, z_Y) is a strict Poisson $*$-algebra over $C(Q)$. Of course, by localization at $q \in Q$ we get the pair $(A_q, z_{Y_q, q})$ which is a strict Poisson $*$-algebra over $\mathbb{C}$.

(V) Note that (1.11.1), (1.11.2) make exactly the definition of the strict Poisson algebra. It turns out that (1.12.1), (1.13.1) follow from the definition of the strict Poisson $*$-algebra. Indeed, if (D, z) is a strict Poisson $*$-algebra, let $\rho_0 \in C^1(D, D)$ be such that $z^* - z = \delta \rho_0$. If we put $\rho = \frac{1}{2}(\rho_0 - \rho_0^*)$ we obtain $\rho = -\rho^*$ and

$$\delta \rho = \frac{1}{2}(\delta \rho_0 - \delta(\rho_0^*)) = \frac{1}{2}(\delta \rho_0 - (\delta \rho_0)^*) = z - z^*.$$

One can easily derive, by computation, the following

1.34. PROPOSITION. (Compare to [26]; see [10] for the proof) *Let (D, z) be a strict Poisson algebra. Denote $z(a, b) - z(b, a)$ by $\{a, b\}$. Let $z_2 \in C^2(D, D)$ be such that $z \circ z = \delta z_2$ and let $z_2(a, b) - z_2(b, a) \stackrel{\text{def}}{=} \pi_2(a, b)$. Then*

$$\{a, b \cdot c\} - \{a, b\} \cdot c - b \cdot \{a, c\} = z([a, b], c) + z(b, [a, c]) - [a, z(b, c)], \tag{1.14.1}$$

$$[\{a, b\}, c] + [\{b, c\}, a] + [\{c, a\}, b] + \{[a, b], c\} + \{[b, c], a\} + \{[c, a], b\} = 0, \tag{1.15.1}$$

$$\begin{aligned} &\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} + \pi_2([a, b], c) \\ &\quad + \pi_2([b, c], a) + \pi_2([c, a], b) + [\pi_2(a, b), c] + [\pi_2(b, c), a] \\ &\quad + [\pi_2(c, a), b] = 0, \qquad \textit{for all } a, b, c \in D. \end{aligned} \tag{1.15.2}$$

If, moreover, (D, z) is a strict Poisson $*$-algebra and $\rho \in C^1(D, D)$ is such that

$$z - z^* = \delta\rho; \tag{1.12.1$''$}$$

$$\rho^* = -\rho, \tag{1.13.1$''$}$$

then, for all $a, b \in D$, we have

$$\{a, b\} - \{b^*, a^*\}^* = [a, \rho(b)] + [\rho(a), b] - \rho([a, b]). \tag{1.16.1}$$

1.35. COROLLARY. (See also [26]) *Suppose (D, z) is a strict Poisson algebra. Let $Z(D)$ be the center of D, that is, $Z(D) = \{a \in D : [a, b] = 0$ for all $b \in D\}$. Then*

(a) *If $a, b \in Z(D)$ we have $\{a, b\} \in Z(D)$.*

(b) *For all $a, b, c \in Z(D)$ we have*

(i) $\{a, b \cdot c\} = \{a, b\} \cdot c + \{a, c\} \cdot b$,

(ii) $\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0$.

If (D, z) is a strict Poisson $*$-algebra, then, for all $a, b \in Z(D)$, we have $\{a, b\} = \{b^*, a^*\}^*$.

1.36. COMMENTS. Usually, for a commutative algebra D, by a *Poisson bracket on D* one means a Lie algebra structure $\{\cdot, \cdot\}$ on D such that, for any $a, b, c \in D$ we have

$$\{a, b \cdot c\} = \{a, b\} \cdot c + \{a, c\} \cdot b. \tag{1.17}$$

So Corollary 1.35 says that if (D, z) is a strict Poisson algebra, then $\{\cdot, \cdot\}$, as defined in Proposition 1.33, is a Poisson bracket on $Z(D)$.

However, at present it is not clear whether all Poisson brackets on commutative algebras come from strict Poisson algebra structures. (Of course, any Poisson bracket on a commutative algebra is itself a 2-cocycle and we could try with $z = \frac{1}{2}\{\cdot, \cdot\}$. But it seems hopeless to check that condition (ii) in the definition holds for z.)

1.37. REMARK. Suppose D is commutative and $(x_i)_{i \in I}$ is a set of generators for D as an algebra. Then, using (1.17), a Poisson bracket on D is uniquely determined by its values "on generators", i.e. by the system $\{x_i, x_j\}_{i,j \in I}$.

In the conclusion of this section we shall recall Rieffel's definition (see [14], [15]) of deformation quantization. We will be able to conclude that smooth $\mathbb{C}$-forms of deformation algebras give deformation quantizations when restricted along curves in Q.

1.38. DEFINITION. Let B be a C^*-algebra. By a *deformation quantization of B* one means a system $(V, z, (\times_h, *_h, \|\cdot\|_h)_{h \in I})$ with

(i) I an interval of one of the forms $I = (-\varepsilon, \varepsilon')$ or $I = [0, \varepsilon)$ or $I = (-\varepsilon, 0]$ with $\varepsilon, \varepsilon' > 0$;

(ii) $(V, \times_h, *_h)$ is a $*$-algebra for every $h \in I$;

(iii) V is a dense $*$-subalgebra in B and $a \times_0 b = ab, a^{*_0} = a^*$ for all $a, b \in V$;

(iv) for every $h \in I$, $\|\cdot\|_h$ is a C^*-norm on the $*$-algebra $(V, \times_h, *_h)$;

(v) $\|a\|_0 = \|a\|$ for all $a \in V$ ($\|\cdot\|$ is the norm of B);

(vi) for any $a \in V$ the map $h \mapsto \|a\|_h$ is continuous on I;

(vii) for any $h_0 \in I$ and any $a, b \in V$ we have

$$\lim_{h \to h_0} \|a \times_h b - a \times_{h_0} b\|_h = 0; \qquad \lim_{h \to h_0} \|a^{*_h} - a^{*_{h_0}}\|_h = 0;$$

(viii) $z : V \times V \to V$ is a map such that

$$\lim_{h \to 0} \left\| \frac{a \times_h b - ab}{h} - z(a,b) \right\|_h = 0, \qquad \text{for all } a, b \in V.$$

In this situation we call the deformation *in the z-direction.*

1.39. COMMENTS. In concrete examples one works with some variants of the above definition. For instance, when B is commutative, axiom (viii) is replaced by

$$\lim_{h \to 0} \left\| \frac{a \times_h b - b \times_h a}{h} - \{a, b\} \right\|_h = 0,$$

where $\{\cdot\,,\cdot\} : V \times V \to V$ is a Poisson bracket.

In the non-commutative case it is natural (according to the above discussion) to require that (V, z) be a strict Poisson algebra.

The results from this section give easily the following.

1.40. PROPOSITION. *Let (A, φ) be a deformation algebra over $C(Q)$. Suppose Q is a C^∞-manifold (possibly with boundary), and suppose V is a smooth $\mathbb{C}$-form for A. Fix $q \in Q$ and a smooth curve $\sigma : I \to Q$ (with I an interval of the form described at the Definition* 1.38*) such that $\sigma(0) = q$.*

Let B be the completion of the $*$-algebra $A_q = (V, \times_q, *_q)$ with respect to the C^*-norm $\|\cdot\|_q$. For any $h \in I$, let $\times_h = \times_{\sigma(h)}, *_h = *_{\sigma(h)}$ and $\|\cdot\|_h = \|\cdot\|_{\sigma(h)}$. Take $z = z^1_{\sigma,q} : V \times V \to V$ to be the map given by Proposition 1.28.

Then (V, z) is a strict Poisson algebra and $(V, z, (\times_h, *_h, \|\cdot\|_h)_{h \in I})$ is a deformation quantization of B in the z-direction. Moreover, if A_q is commutative, then $\{a, b\} = z(a, b) - z(b, a)$ defines a Poisson bracket on $A_q = V$ and

$$\lim_{h \to 0} \left\| \frac{a \times_h b - b \times_h a}{h} - \{a, b\} \right\|_h = 0, \qquad \textit{for all } a, b \in V.$$

2. THE DEFORMATION QUANTIZATION PROCEDURE

In this section we shall give a sufficient condition which ensures that certain families of C^*-algebras give rise, in a natural way, to deformation algebras. The types of C^*-algebras we will deal with are defined by means of generators, relations and bound conditions. We begin by introducing some terminology, which will make the exposition easier.

2.1. DEFINITIONS. Let A be a $*$-algebra over $\mathbb{C}$.

(i) By a *bounded representation theory* for A we shall mean a full subcategory $\mathcal{R}$ of the category of all $*$-representations of A (on Hilbert spaces), such that for any $a \in A$ the quantity $p_{\mathcal{R}}(a) \stackrel{\text{def}}{=} \sup\{\|\pi(a)\| : \pi \in \mathcal{R}\}$ is finite.

(ii) Suppose $\mathcal{R}$ is a bounded representation theory for the $*$-algebra A. Note that the map $p_{\mathcal{R}}(a) : A \to [0, \infty)$ defined above is a C^*-seminorm. We can then form a C^*-algebra denoted by $C^*_{\mathcal{R}}(A)$, defined as the separate-completion of A with respect to the C^*-seminorm $p_{\mathcal{R}}$. If the seminorm $p_{\mathcal{R}}$ is already a norm, the representation theory $\mathcal{R}$ is said to be *faithful*.

2.2. REMARKS. (i) An extreme case is the one in which a $*$-algebra A has the property that the category of *all* $*$-representations of A is bounded. In this case the $*$-algebra A will be called a *maximally bounded $*$-algebra*. In this case, using in the above construction $\mathcal{R}$ to be the category of all $*$-representations, we get a C^*-algebra denoted by $C^*_{\max}(A)$.

(ii) On the other extreme, one can consider the category $\mathcal{R}$ to consist of a single representation. For example, consider a GNS-$*$-algebra (A, φ) (over $\mathbb{C}$). Taking $\mathcal{R} = \{\pi_\varphi\}$ and apply the above construction we get a C^*-algebra, denoted simply by $\Gamma(A, \varphi)$.

(iii) Here is now a canonical method of producing bounded representation theories. Suppose A is a $*$-algebra over $\mathbb{C}$. By a *bound condition on A* we mean a pair (X, β) consisting of a set X of generators for A (as a $*$-algebra), and a map $\beta : X \to [0, \infty)$. In the presence of a bound condition (X, β) we can define $\mathcal{R}(X, \beta)$ to be the category of all $*$-representations $\pi : A \to B(H_\pi)$ having the property that $\|\pi(x)\| \leqslant \beta(x)$, for all $x \in X$. It is obvious that $\mathcal{R}(X, \beta)$ is a bounded representation theory for A. If the representation theory $\mathcal{R}(X, \beta)$ is faithful, then the bound condition (X, β) will also be called *faithful*.

(iv) Let X be an arbitrary set and $\beta : X \to [0, \infty)$ be any map. Consider $\mathbb{C}\{X\}$ to be the free $*$-algebra generated by X. For a subset $M \subset \mathbb{C}\{X\}$, we denote by $\mathcal{R}_M(X, \beta)$ the category of all representations $\pi \in \mathcal{R}(X, \beta)$ with the property that $\pi(m) = 0$ for all $m \in M$. Clearly we get a bounded representation theory for $\mathbb{C}\{X\}$. The C^*-algebra $C^*_{\mathcal{R}_M(X,\beta)}(\mathbb{C}\{X\})$ will simply be denoted by $C^*_M(X, \beta)$ and will be called the *universal C^*-algebra generated by X, subject to relations $m = 0$, $m \in M$ and bound condition $\|x\| \leqslant \beta(x)$, $x \in X$.* An alternative construction for this C^*-algebra is the following. Consider $\mathcal{I}(M)$ to be the two-sided $*$-ideal of $\mathbb{C}\{X\}$ generated by M. Denote the quotient $\mathbb{C}\{X\}/\mathcal{I}(M)$ simply by $\mathbb{C}\{X|M\}$ and write $\mathbb{C}\{X\} \ni a \mapsto \widehat{a} \in \mathbb{C}\{X|M\}$ for the quotient $*$-homomorphism. Then, if we consider the set $\widehat{X} = \{\widehat{x} : x \in X\} \subset \mathbb{C}\{X|M\}$, and if we define $\widehat{\beta} : \widehat{X} \to [0, \infty)$ by $\widehat{\beta}(\widehat{x}) = \inf\{\beta(y) : y \in X,\ \widehat{y} = \widehat{x}\}$, then $C^*_M(X, \beta)$ is canonically isomorphic to $C^*_{\mathcal{R}(\widehat{X},\widehat{\beta})}(\mathbb{C}\{X|M\})$. If the representation theory $\mathcal{R}(\widehat{X}, \widehat{\beta})$ (for $\mathbb{C}\{X|M\}$) is faithful we call the system (X, M, β) a *faithful presentation.* (Note that this condition has the following interpretation: We consider $\mathcal{I}_\beta(M) \subset \mathbb{C}\{X\}$ to be the intersection of all kernels of the $*$-representations in $\mathcal{R}_M(X, \beta)$. We always have $\mathcal{I}(M) \subset \mathcal{I}_\beta(M)$. The system (X, M, β) is a faithful presentation if and only if $\mathcal{I}(M) = \mathcal{I}_\beta(M)$.)

2.3. DEFINITIONS. Let A be $*$-algebra over $\mathbb{C}$, $\mathcal{R}$ be a bounded representation theory for A and $\varphi : A \to \mathbb{C}$ be a positive linear functional.

(i) φ is said to be *$\mathcal{R}$-regular* if the following conditions hold:

(a) (A, φ) is a GNS-$*$-algebra over $\mathbb{C}$,
(b) $p_\varphi = p_{\mathcal{R}}$.

(Here $p_\varphi : A \to [0, \infty)$ is defined by $p_\varphi(a) = \|\pi_\varphi(a)\|$, where $\pi_\varphi : A \to \mathcal{B}(L^2(A, \varphi))$ is the GNS representation.) Note that if φ is $\mathcal{R}$-regular, then there exists a (unique) $*$-isomorphism $\Phi : \Gamma(A, \varphi) \to C^*_{\mathcal{R}}(A)$ such that $\Phi \circ \iota_\varphi = \iota_{\mathcal{R}}$, where $\iota_\varphi : A \to \Gamma(A, \varphi)$ and $\iota_{\mathcal{R}} : A \to C^*_{\mathcal{R}}(A)$ are the canonical $*$-homomorphisms.

(ii) φ is said to be $\mathcal{R}$-*continuous* if it is continuous in the topology given by the seminorm $p_{\mathcal{R}}$, that is, there exists a constant $C \geqslant 0$ such that $|\varphi(a)| \leqslant C \cdot p_{\mathcal{R}}(a)$ for all $a \in A$. It is worth mentioning that regularity does not imply, in general, continuity. (In fact, if (A, φ) is a GNS-$*$-algebra over $\mathbb{C}$, we may even have $p_\varphi(a) = 0$ for all $a \in A$, but still φ not being the null functional.) If φ is $\mathcal{R}$-continuous, then there exists a (necessarily unique) positive linear functional $\widetilde{\varphi} : C^*_{\mathcal{R}}(A) \to \mathbb{C}$ such that $\varphi = \widetilde{\varphi} \circ \iota_{\mathcal{R}}$, where $\iota_{\mathcal{R}} : A \to C^*_{\mathcal{R}}(A)$ denotes the canonical $*$-homomorphism.

If φ is $\mathcal{R}$-continuous, then a sufficient condition for $\mathcal{R}$-regularity is the faithfulness of the functional $\widetilde{\varphi} : C^*_{\mathcal{R}}(A) \to \mathbb{C}$. (This means that, whenever $x \in C^*_{\mathcal{R}}(A)$ is such that $\widetilde{\varphi}(x^*x) = 0$, it follows that $x = 0$.) If this is the case, then φ is said to be $\mathcal{R}$-*faithful*.

With these preparations, we are ready to state the main result in this section. We begin by fixing the following:

2.4. NOTATION. Let W be a vector space (over $\mathbb{C}$) and let A be the free $C(Q)$-module $A = W \otimes_{\mathbb{C}} C(Q)$. We identify W with $A_q = A \otimes_{e_q} \mathbb{C}$ by the map $\Theta_q : w \mapsto (w \otimes_{\mathbb{C}} 1) \otimes_{e_q} 1$. For any linear map $\psi : W \to C(Q)$ let $\psi_q : A_q \to \mathbb{C}$ be defined as $\psi_q = e_q \circ \psi \circ \Theta_q^{-1}$. For any subset $X \subset W$ we denote $\Theta_q(X)$ by X_q, and for any map $\beta : X \to [0, \infty)$ denote by β_q the map $\beta \circ \Theta_q^{-1} : X_q \to [0, \infty)$.

2.5. THEOREM. *Suppose the $C(Q)$-module A is equipped with a $C(Q)$-$*$-algebra structure.*

(i) *Fix $X \subset W$ a subset such that $Y = \{x \otimes 1 : x \in X\}$ generates A as a $C(Q)$-$*$-algebra. (This implies that for any $q \in Q$, the set $X_q \subset A_q$ generates A_q as a $*$-algebra.)*

(ii) *Take $\beta : X \to [0, \infty)$ to be any map with the property that for any $q \in Q$ the bound condition (X_q, β_q) is faithful. For each $q \in Q$, let $\mathcal{R}_q = \mathcal{R}(X_q, \beta_q)$ be the bounded representation theory for the $*$-algebra A_q (over $\mathbb{C}$), associated with the bound condition (X_q, β_q).*

(iii) *Let $\psi : W \to C(Q)$ be a linear map such that, for each $q \in Q$, the functional ψ_q is positive and $\mathcal{R}_q$-regular. Let $\varphi : A = W \otimes_{\mathbb{C}} C(Q) \to C(Q)$ be the map defined by $\varphi(w \otimes f) = f \cdot \psi(w)$, $w \in W$, $f \in C(Q)$.*

Then (A, φ) is a deformation algebra.

Proof. STEP I. We show that φ is $C(Q)$-positive.

First, let us remark that, for $w \in W$, we have $\Theta_q(w) = \widetilde{e_q}(w \otimes 1)$, so

$$\psi_q(\Theta_q(w)) = e_q(\psi(w)) = e_q(\varphi(w \otimes 1)) = \varphi_q(\widetilde{e}_q(w \otimes 1)) = \varphi_q(\Theta_q(w)).$$

That is, ψ_q coincides with the functional φ_q obtained by localizing the pair (A, φ) at q. Note that, by construction, $\varphi : A \to C(Q)$ is $C(Q)$-linear. Here we used the notation from 1.16. Note also that $\widetilde{e}_q : A \to A_q$ is a homomorphism of $*$-algebras, and $A_q = \{\widetilde{e}_q(w \otimes 1) : w \in W\}$. Take now $a \in A$. Then

$$\varphi(a^*a)(q) = \varphi_q(\widetilde{e}_q(a)^*\widetilde{e}_q(a)) = \psi_q(\widetilde{e}_q(a)^*\widetilde{e}_q(a)) \geqslant 0,$$

because ψ_q is positive on A_q. So $\varphi(a^*a) \geqslant 0$ in $C(Q)$ for all $a \in A$, that is, φ is positive.

STEP II. We prove now that (A, φ) is a GNS-$*$-algebra over $C(Q)$.

This means that, on A/N_φ, all operators L_a, $a \in A$ are continuous. Remark that if L_a and L_b are continuous then $L_{a+b} = L_a + L_b$ and $L_{ab} = L_aL_b$ will also be continuous. Also, if L_a is continuous and $f \in C(Q)$, since M_f is continuous, $L_{fa} = M_fL_a$ will also be continuous. So, according to these remarks, since Y generates A as a $C(Q)$-$*$-algebra, it suffices to show that for any $y \in Y$, both L_y and L_{y^*} are continuous. Take $y \in Y$ and $x \in X$ with $y = x \otimes 1$. Let a be one of the elements y or y^*. For any $b \in A$ we have

$$\varphi(b^*a^*ab)(q) = \varphi_q(\widetilde{e}_q(b)^*\widetilde{e}_q(a)^*\widetilde{e}_q(a)\widetilde{e}_q(b)). \tag{2.1}$$

But on A_q, the positive functional φ_q is $\mathcal{R}_q$-regular. In particular, (A_q, φ_q) is a GNS-$*$-algebra and $p_{\varphi_q} = p_{\mathcal{R}_q}$. Here p_{φ_q} is the C^*-seminorm associated with (A_q, φ_q). So (2.1) yields

$$\varphi(b^*a^*ab)(q) \leqslant p_{\varphi_q}(\widetilde{e}_q(a))^2 \cdot \varphi_q(\widetilde{e}_q(b)^*\widetilde{e}_q(b)) \leqslant p_{\mathcal{R}_q}(\widetilde{e}_q(a))^2 \cdot \varphi(b^*b)(q). \tag{2.2}$$

But $p_{\mathcal{R}_q}$ is a C^*-seminorm and $\widetilde{e}_q$ is a $*$-homomorphism, so $p_{\mathcal{R}_q}(\widetilde{e}_q(a)) = p_{\mathcal{R}_q}(\widetilde{e}_q(a)^*)$. So, in the case $a = y^*$ we get $p_{\mathcal{R}_q}(\widetilde{e}_q(a)) = p_{\mathcal{R}_q}(\widetilde{e}_q(y))$. That is, in any of the two cases ($a = y$ or $a = y^*$) (2.2) gives

$$\varphi(b^*a^*ab)(q) \leqslant p_{\mathcal{R}_q}(\widetilde{e}_q(y))^2 \cdot \varphi(b^*b)(q). \tag{2.3}$$

But $p_{\mathcal{R}_q}(\widetilde{e}_q(y)) = p_{\mathcal{R}_q}(\widetilde{e}_q(x \otimes 1)) = p_{\mathcal{R}_q}(\Theta_q(x))$. Note that $\Theta_q(x) \in X_q$, so by the construction of $\mathcal{R}_q$ as $\mathcal{R}(X_q, \beta_q)$, we obtain $p_{\mathcal{R}_q}(\Theta_q(x)) \leqslant \beta_q(\Theta_q(x)) = \beta(x)$. With this evaluation, (2.3) yields $\varphi(b^*a^*ab)(q) \leqslant \beta(x)^2 \cdot \varphi(b^*b)(q)$, for all $b \in A$, $q \in Q$. This reads

$$\varphi(b^*a^*ab) \leqslant \beta(x)^2 \cdot \varphi(b^*b), \qquad \text{for all } b \in A.$$

Consequently, the operator L_a is bounded and $\|L_a\| \leqslant \beta(x)$. This concludes Step II.

STEP III. We show that for any $q \in Q$ we have $p^{\mathrm{unif}}_{\varphi,q} = p_{\varphi_q}$.

We know that, in general, we have $p^{\mathrm{unif}}_{\varphi,q} \geqslant p_{\varphi_q}$ (see 1.20). So it suffices to prove $p_{\varphi_q} \geqslant p^{\mathrm{unif}}_{\varphi,q}$. According to the $\mathcal{R}_q$-regularity of φ_q, we have $p_{\varphi_q} = p_{\mathcal{R}_q}$. So what needs to be shown is $p_{\mathcal{R}_q} \geqslant p^{\mathrm{unif}}_{\varphi,q}$. Recall (see Proposition 1.16) that, for any $z \in A_q$, we have $p^{\mathrm{unif}}_{\varphi,q}(z) = \|\Sigma_q(z)\|$, where $\Sigma_q : A_q \to B(H_\varphi)/J_q$ is the $*$-homomorphism given by $\Sigma_q \circ \widetilde{e}_q = \pi^{\mathrm{unif}}_q \circ \pi_\varphi$. Here H_φ is the GNS space of A over $C(Q)$, $\pi_\varphi : A \to B(H_\varphi)$ is the GNS homomorphism and $\pi^{\mathrm{unif}}_q : B(H_\varphi) \to B(H_\varphi)/J_q$ is the quotient map. Fix $q \in Q$ and take ρ a faithful $*$-representation of the C^*-algebra $B(H_\varphi)/J_q$ (on some Hilbert space). Consider the representation $\rho \circ \Sigma_q$ of A_q. Let $z \in X_q$, say $z = \Theta_q(x)$, $x \in X$. Then

$$\begin{aligned} \|(\rho \circ \Sigma_q)(z)\| &\leqslant \|\Sigma_q(z)\| = \|(\Sigma_q \circ \widetilde{e}_q)(x \otimes 1)\| \\ &= \|(\pi^{\mathrm{unif}}_q \circ \pi_\varphi)(x \otimes 1)\| \leqslant \|\pi_\varphi(x \otimes 1)\|. \end{aligned} \tag{2.4}$$

But, according to the proof of Step II, we have $\|L_{x\otimes 1}\| \leqslant \beta(x)$, that is $\|\pi_\varphi(x\otimes 1)\| \leqslant \beta(x)$. Then (2.4) gives

$$\|(\rho\circ\Sigma_q)(z)\| \leqslant \beta(x) = \beta_q(\Theta_q(x)) = \beta_q(z).$$

So the representation $\rho\circ\Sigma_q$ satisfies

$$\|(\rho\circ\Sigma_q)(z)\| \leqslant \beta_q(z), \quad \text{for all } z\in X_q.$$

But this shows exactly that $\rho\circ\Sigma_q \in \mathcal{R}(X_q,\beta_q) \stackrel{\text{def}}{=} \mathcal{R}_q$. In particular

$$p_{\mathcal{R}_q}(u) \geqslant \|(\rho\circ\Sigma_q)(u)\| = \|\Sigma_q(u)\| = p^{\text{unif}}_{\varphi,q}(u), \quad \text{for all } u\in A_q,$$

and we are done.

STEP IV. To conclude the proof, the only thing that remains to be shown is that, for any $q\in Q$, p_{φ_q} is a norm on A_q. But this is clear since $p_{\varphi_q} = p_{\mathcal{R}_q}$ and $\mathcal{R}_q$ is assumed to be faithful, that is, $p_{\mathcal{R}_q}$ is a norm. ∎

The above result would be applied in the following framework. Suppose we are given a compact manifold Q together with a fixed set X and a map $\beta : X\to [0,\infty)$. Suppose for each $q\in Q$ we are given a subset $M_q\subset \mathbb{C}\{X\}$. Construct the C^*-algebras $A_q \stackrel{\text{def}}{=} C^*_{M_q}(X,\beta)$, described in 2.2 (iv). Then the above theorem gives us a criterion for the family of C^*-algebras $\{A_q\}_{q\in Q}$ to be assembled into a continuous field, for which all the elements $x\in X$ define continuous sections. First, let us denote the $*$-algebras $\mathbb{C}\{X|M_q\}$ simply by V_q, and the representation theory $\mathcal{R}(\widehat{X},\widehat{\beta})$ by $\mathcal{R}_q$. (We have used the notation from 2.2 (iv). Recall that A_q will then be naturally isomorphic to $C^*_{\mathcal{R}_q}(V_q)$.)

Assume for every $q\in Q$ we are given a positive functional $\varphi_q : V_q\to\mathbb{C}$. We now outline the steps needed to check that all the conditions in the theorem are satisfied.

STEP I. Find an index set I and for each q find a system $(v_i^q)_{i\in I}\subset V_q$ such that:

(I)(a) For each $q\in Q$, the system $(v_i^q)_{i\in I}$ is a basis for in V_q (as a linear space over $\mathbb{C}$).

(I)(b) For all $i,j\in I$ there exists a finite set of indices $I_\times(i,j)$ such that

$$v_i^q\cdot v_j^q \in \operatorname{Span}\{v_k^q : k\in I_\times(i,j)\}$$

for all $q\in Q$.

(I)(c) For all $i\in I$ there exists a finite set of indices $I_*(i)$ such that

$$(v_i^q)^* \in \operatorname{Span}\{v_k^q : k\in I_*(i)\}$$

for all $q\in Q$.

(I)(d) For all triplets (i,j,k) with $k\in I_\times(i,j)$, the functions $f_{ij}^k : Q\to\mathbb{C}$ defined by

$$v_i^q\cdot v_j^q = \sum_{k\in I_\times(i,j)} f_{ij}^k(q) v_k^q$$

are smooth.

(I)(e) For all pairs (i,k) with $k \in I_*(i)$, the functions $g_i^k : Q \to \mathbb{C}$ defined by

$$(v_i^q)^* = \sum_{k \in I_*(i)} g_i^k(q) v_k^q$$

are smooth.

(I)(f) For any $i \in I$ the map $q \mapsto \varphi_q(v_i^q)$ is continuous.

STEP II. For any $q \in Q$, check that:
(II)(a) The representation theory $\mathcal{R}_q$ is faithful for V_q.
(II)(b) The positive functional φ_q is $\mathcal{R}_q$-regular.

STEP III. (Conclusion) Fix $q \in Q$ and a smooth curve $\sigma : I \to Q$ with $\sigma(0) = q$. Take V a vector space with a fixed basis $(v_i)_{i \in I}$. Define for any $p \in Q$ the isomorphism $e_p : V \to V_p$ by $e_p(v_i) = v_i^p$. Put for any $h \in I$ the $*$-algebra structure $(\times_h \cdot *_h)$ on V defined by

$$v \times_h w = e_{\sigma(h)}^{-1}(\underbrace{e_{\sigma(h)}(v) \cdot e_{\sigma(h)}(w)}_{\text{in } B_{\sigma(h)}}), \qquad v^{*_h} = e_{\sigma(h)}^{-1}(\underbrace{e_{\sigma(h)}(v)^*}_{\text{in } B_{\sigma(h)}}).$$

Define also the norms $\|\cdot\|_h$ by

$$\|v\|_h = \|e_{\sigma(h)}(v)\| \quad \text{in } B_{\sigma(h)}.$$

Compute $z = z_{\sigma,q}^1$ as in Proposition 1.28.

Then, using the identification $e_q : V \xrightarrow{\sim} V_q \subset A_q$, the system $(V, z, (\times_h, *_h, \|\cdot\|_h)_{h \in I})$ is a deformation quantization for A_q in the z-direction.

3. APPLICATION TO QUANTUM SU(N) GROUPS

In this section we shall describe an example to which the framework described in Section 2 can be applied. Other examples are discussed in [10] and [11].

Let us introduce the quantum SU(N) groups. For the moment we do not specify the space Q of parameters. The generic value for q will be $q \in (-1,1]$, $q \neq 0$. We shall follow the descriptions given in [23], [25], [12], [7].

3.1. DEFINITIONS. We fix $N \in \mathbb{N}$, $N \geqslant 2$. We take $\mathrm{Fun}_q(\mathrm{U}(N))$ to be the unital algebra over $\mathbb{C}$ generated by N^2+1 elements, labeled $(t_{ij})_{i,j=1,N}$ and d subject to the following relations

$$t_{ij} \cdot t_{il} = q t_{il} \cdot t_{ij} \qquad \text{for } j < l, \tag{3.1}$$

$$t_{ij} \cdot t_{kj} = q t_{kj} \cdot t_{ij} \qquad \text{for } i < k, \tag{3.2}$$

$$t_{ij} \cdot t_{kl} = t_{kl} \cdot t_{ij} \qquad \text{for } i > k, j < l, \tag{3.3}$$

$$t_{ij} \cdot t_{kl} - t_{kl} \cdot t_{ij} = (q - q^{-1}) t_{il} \cdot t_{kj} \qquad \text{for } i < k, j < l, \tag{3.4}$$

$$d \cdot t_{ij} = t_{ij} \cdot d \qquad \text{for all } i,j, \tag{3.5}$$

$$d \cdot D = D \cdot d = 1, \tag{3.6}$$

where $D = \det_q(t_{ij})_{i,j} \stackrel{\text{def}}{=} \sum_{\sigma\in\mathfrak{S}_N} (-q)^{I(\sigma)} t_{1\sigma(1)} \cdots t_{N\sigma(N)}$. Here, for any permutation $\sigma \in \mathfrak{S}_N$, we denote by $I(\sigma)$ the number of inversions of σ, that is

$$I(\sigma) = \mathrm{Card}\{(i,j) : i < j,\ \sigma(i) > \sigma(j)\}.$$

With this notation, for any $i, j = \overline{1, N}$ one takes the (i,j)-*minor of* D to be

$$D^{ij} \stackrel{\text{def}}{=} \sum_{\substack{\alpha:\{1,\ldots,i-1,i+1,\ldots,N\}\to \\ \to\{1,\ldots,j-1,j+1,\ldots,N\} \\ \alpha\ \text{bijective}}} (-q)^{I(\alpha)} t_{1\alpha(1)} \cdots t_{i-1,\alpha(i-1)} \cdot t_{i+1,\alpha(i+1)} \cdots t_{N\alpha(N)}.$$

Then $\mathrm{Fun}_q(\mathrm{U}(N))$ becomes a $*$-algebra if we set

$$t_{ij}^* = (-q)^{j-i} D^{ij} d \quad d^* = D.$$

We follow [7] and recall some important properties which will be used later for the checking of Step I. We need to introduce some notation:

(i) for a system of positive integers $m = (m_{ij})_{\substack{i,j=1,N \\ i+j\leqslant N}}$ we denote

$$t_{11}^{m_{11}} \cdots t_{1,N-1}^{m_{1,N-1}} \cdot t_{21}^{m_{21}} \cdots t_{2,N-2}^{m_{2,N-2}} \cdots t_{N-1,1}^{m_{N-1,1}} \stackrel{\text{def}}{=} t_+^m;$$

(ii) similarly, for a system of positive integers $n = (n_{ij})_{\substack{i,j=1,N \\ i+j\geqslant N+2}}$ we denote

$$t_{2N}^{n_{2N}} \cdot t_{3,N-1}^{n_{3,N-1}} \cdot t_{3N}^{n_{3N}} \cdots t_{N2}^{n_{N2}} \cdots t_{NN}^{n_{NN}} \stackrel{\text{def}}{=} t_-^n;$$

(iii) for $p = (p_1, \ldots, p_N) \in \mathbb{N}^N$ denote

$$t_{N1}^{p_1} \cdot t_{N-1,2}^{p_2} \cdots t_{1N}^{p_N} \stackrel{\text{def}}{=} t_0^p;$$

(iv) for such p denote $\min\{p_1, \ldots, p_N\}$ by $p_{\min}$;

(v) for m, n, p as above and $l \in \mathbb{N}$, let

$$X(m,n,p,l) \stackrel{\text{def}}{=} t_-^n \cdot t_0^p \cdot t_+^m \cdot d^l.$$

With these notations one has the following.

3.2. THEOREM. ([7]) (i) $\{X(m,n,p,l) : \min(p_{\min}, l) = 0\}$ *is a basis for* $\mathrm{Fun}_q(\mathrm{U}(N))$.

(ii) *Let m, n, p, l be arbitrary, with $p_{\min}$, $l \geqslant 1$. Then*

$$X(m,n,p,l) \in \mathrm{Span}\{X(m',n',p',l') : p'_{\min} < p_{\min}\}.$$

COMMENTS. Statement (i) is contained in [7], Theorem 3.4. Statement (ii) is an auxiliary result used for the proof of (i) (see [7], Formulas (3.3), (3.4) and page 206).

If we expand, using (ii), an element $X(m,n,p,l)$ as a linear combination of elements $X(m',n',p',l')$ with $p'_{\min} < p_{\min}$, using [7], Formulas (3.3) and (3.4), one can see that the coefficients which would appear are polynomials in q and q^{-1}.

This allows us to draw the following conclusion. If we denote by $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{U}(N))$ the algebra over $\mathbb{C}[q,q^{-1}]$ generated by the t_{ij}'s and d subject to relations (3.1)–(3.6), then $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{U}(N))$ is a free $\mathbb{C}[q,q^{-1}]$-module with basis $\{X(m,n,p,l) : \min(p_{\min}, l) = 0\}$ and part (ii) of the theorem holds with "$\mathrm{Span}_{\mathbb{C}[q,q^{-1}]}$".

If we make $\mathbb{C}[q,q^{-1}]$ a $*$-algebra by $q^* = q$, $(q^{-1})^* = q^{-1}$, then $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{U}(N))$ becomes a $*$-algebra over $\mathbb{C}[q,q^{-1}]$.

3.3. DEFINITION. For q as before ($q \in (-1,1]$, $q \neq 0$) we define

$$\mathrm{Fun}_q(\mathrm{SU}(N)) \stackrel{\text{def}}{=} \mathrm{Fun}_q(\mathrm{U}(N))/J_q$$

where J_q is the two-sided ideal generated by $d-1$.

Note that d is in the center of $\mathrm{Fun}_q(\mathrm{U}(N))$, so $J_q = \mathrm{Fun}_q(\mathrm{U}(N)) \cdot (d-1)$.

We can also define

$$\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N)) \stackrel{\text{def}}{=} \mathrm{Fun}_{[q,q^{-1}]}(\mathrm{U}(N))/J_{[q,q^{-1}]}$$

where $J_{[q,q^{-1}]} = \mathrm{Fun}_{[q,q^{-1}]}(\mathrm{U}(N)) \cdot (d-1)$. Then $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N))$ will be an algebra over $\mathbb{C}[q,q^{-1}]$.

Note that $d^* - 1 = D - 1 = D(1-d)$ so $d^* - 1 \in J_q$. Thus J_q is invariant under the involution. Hence $\mathrm{Fun}_q(\mathrm{SU}(N))$ is a $*$-algebra. Similarly $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N))$ is a $*$-algebra over $\mathbb{C}[q,q^{-1}]$.

Let $\pi : \mathrm{Fun}_q(\mathrm{U}(N)) \to \mathrm{Fun}_q(\mathrm{SU}(N))$ be the canonical surjection. The same notation will be used for the map $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{U}(N)) \to \mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N))$. We denote $\pi(t_{ij})$ by u_{ij}.

3.4. LEMMA. (i) *There exists a unique unital $*$-homomorphism*

$$\Lambda : \mathrm{Fun}_q(\mathrm{SU}(N)) \to \mathrm{Fun}_q(\mathrm{U}(N))$$

such that

$$\begin{aligned} \Lambda(u_{ij}) &= t_{ij} && \textit{for all } i > 1 \textit{ and all } j, \\ \Lambda(u_{1j}) &= t_{1j} \cdot d && \textit{for all } j. \end{aligned}$$

(ii) *For this $*$-homomorphism we have $\pi \circ \Lambda = \mathrm{Id}$.*

Proof. (i) We define $\Lambda_0 : \mathrm{Fun}_q(\mathrm{U}(N)) \to \mathrm{Fun}_q(\mathrm{U}(N))$ on the generators by

$$\begin{aligned} \Lambda_0(t_{ij}) &= t_{ij} && \text{for } i > 1 \text{ and all } j, \\ \Lambda_0(t_{1j}) &= t_{1j} \cdot d && \text{for all } j, \\ \Lambda_0(d) &= 1. \end{aligned}$$

Since d is in the center of $\mathrm{Fun}_q(\mathrm{U}(N))$, all the relations (3.1)–(3.5) are verified by the $\Lambda_0(t_{ij})$'s. Again it is clear that

$$\Lambda_0(d) \cdot \det_q(\Lambda_0(t_{ij}))_{i,j} = \det_q(\Lambda_0(t_{ij}))_{i,j} = d \cdot \det_q(t_{ij})_{i,j} = 1,$$

and similarly $\det_q(\Lambda_0(t_{ij})) \cdot \Lambda_0(d) = 1$. Hence Λ_0 makes sense as a homomorphism of algebras. Again it is easy to show that Λ_0 is a $*$-homomorphism.

But, clearly, $\Lambda_0(J_q) = \{0\}$, hence Λ_0 gives the desired map Λ.

(ii) Since $\pi(d) = 1$, we get $(\pi \circ \Lambda)(u_{ij}) = \pi(t_{ij}) = u_{ij}$ for all i, j. ∎

We shall use now similar notation to that introduced in 3.1.

Denote $\pi(t_+^m)$ by u_+^m, $\pi(t_-^p)$ by u_-^n and $\pi(t_0^p)$ by u_0^p. Denote $u_-^n \cdot u_0^p \cdot u_+^m$ by $Y(m,n,p)$. From Koelink's theorem we get:

3.5. COROLLARY. *$\{Y(m,n,p) : p_{\min} = 0\}$ is a basis for* $\mathrm{Fun}_q(\mathrm{SU}(N))$.

Proof. For m and p systems as in 3.1, denote $m_{11}+m_{12}+\cdots+m_{1,N-1}+p_N \overset{\text{not}}{=} l(m,p)$. Then we obtain $\Lambda(Y(m,n,p)) = X(m,n,p,l(m,p))$. But if $p_{\min} = 0$, then $\min(p_{\min}, l(m,p)) = 0$. This shows, using Koelink's theorem, that the set $\{\Lambda(Y(m,n,p)) : p_{\min} = 0\}$ is linearly independent. Hence the $Y(m,n,p)$'s are themselves linearly independent.

Let m,n,p be arbitrary now, with $p_{\min} \geqslant 1$. Using part (ii) of Koelink's theorem, by induction, we can find $l \in \mathbb{N}$ large enough, such that

$$X(m,n,p,l) \in \mathrm{Span}\{X(m',n',p',l') : p'_{\min} = 0\}.$$

This gives

$$\pi(X(m,n,p,l)) \in \mathrm{Span}\{\pi(X(m',n',p',l') : p'_{\min} = 0\}.$$

But

$$\pi(X(m,n,p,l)) = Y(m,n,p) = \pi(X(m,n,p,l'')) \quad \text{for all } l'' \in \mathbb{N}.$$

So we obtain

$$\pi(X(m,n,p,l'') \in \mathrm{Span}\{Y(m',n',p') : p'_{\min} = 0\}.$$

But, using part (i) of Koelink's theorem,

$$\mathrm{Span}\{\pi(X(m,n,p,l''))\} = \pi(\mathrm{Fun}_q(\mathrm{U}(N))) = \mathrm{Fun}_q(\mathrm{SU}(N)),$$

which gives

$$\mathrm{Fun}_q(\mathrm{SU}(N)) = \mathrm{Span}\{Y(m',n',p') : p'_{\min} = 0\}. \quad \blacksquare$$

REMARK. The same conclusion is obtained if we work over $\mathbb{C}[q,q^{-1}]$. This means that $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N))$ is a free $\mathbb{C}[q,q^{-1}]$-module with basis $\{u_-^n \cdot u_0^p \cdot u_+^m : p_{\min} = 0\}$.

NOTATION. Let r be a positive integer, and $k : \{1,\ldots,r\} \to \{1,\ldots,N\} \times \times \{1,\ldots,N\}$ an arbitrary map. We denote by $u_{[k]}$ the element $u_{k_1} \cdots u_{k_r} \in \mathrm{Fun}_q(\mathrm{SU}(N))$. Elements of this form will be called *monomials*.

3.6. REMARK. Following the proof of Koelink's theorem (see [7]), one can show the following fact: *For any k as above, there exists a set $I(k)$ of triplets (m,n,p) as in Corollary* 3.5 (i.e. $p_{\min} = 0$) *and unique numbers $f_k^{(m,n,p)}(q)$, $(m,n,p) \in I(k)$, such that*

$$u_{[k]} \quad = \sum_{(m,n,p)\in I(k)} f_k^{(m,n,p)}(q) Y(m,n,p).$$

This is clear, but all the coefficients $f_k^{(m,n,p)}(q)$ are polynomials in q and q^{-1}, and the set $I(k)$ can be chosen big enough that it does not depend on q. In particular, this gives the same type of expansion $u_{[k]} = \sum_{(m,n,p)\in I(k)} f_k^{(m,n,p)}(q) Y(m,n,p)$ in $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N))$.

Let us pause for a moment to recapitulate what we have defined and to accommodate these structures with the notation suggested at the end of Section 2.

Suppose $N \geqslant 2$ is fixed. We will take X to be a set indexed by N^2+1 elements, denoted 1 and x_{ij}, $1 \leqslant i,j \leqslant N$. We will denote by M_q the subsets in $\mathbb{C}\{X\}$ which define the relations for $\mathrm{Fun}_q(\mathrm{SU}(N))$. So $V_q \overset{\text{def}}{=} \mathbb{C}\{X|M_q\}$ is exactly the algebra $\mathrm{Fun}_q(\mathrm{SU}(N))$. We now define the map $\beta : X \to [0,\infty)$ simply by $\beta(1) = \beta(x_{ij}) = 1$ for all i,j. Let $\mathcal{R}_q$ be the representation theory for $V_q = \mathrm{Fun}_q(\mathrm{SU}(N))$ consisting of all representations $\pi : \mathrm{Fun}_q(\mathrm{SU}(N)) \to B(H_\pi)$ with the property that $\|\pi(1)\| \leqslant 1$ and $\|\pi(u_{ij})\| \leqslant 1$ for all i,j. (Using the notation from 2.2 (iv), this is exactly the representation theory $\mathcal{R}(\widehat{X},\widehat{\beta})$ for $\mathrm{Fun}_q(\mathrm{SU}(N))$, since the canonical $*$-homomorphism : $\mathbb{C}\{X\} \to \mathrm{Fun}_q(\mathrm{SU}(N))$ acts on the generators as $1 \mapsto 1$ and $x_{ij} \mapsto u_{ij}$.) When there is need to make distinctions for different values of q, the elements $u_{ij} \in \mathrm{Fun}_q(\mathrm{SU}(N))$ will be denoted by u^q_{ij}. If we need to make a distinction for different values of N, we will use the notation $u(N)^q_{ij}$ instead of u^q_{ij} and V^N_q instead of V_q.

3.7. Remarks. (i) One can show that if we take $u \in \mathrm{Mat}_N(\mathrm{Fun}_q(\mathrm{SU}(N)))$ to be the matrix $u = (u_{ij})_{i,j=1,N}$, then u is *unitary*. This means

$$\sum_{k=1}^{N} u_{ik} \cdot u^*_{jk} = \sum_{k=1}^{N} u^*_{ki} \cdot u_{kj} = \delta_{ij} \cdot 1. \tag{3.7}$$

(ii) In particular if $\mathcal{H}_0$ is a pre-Hilbert space and $\pi_0 : \mathrm{Fun}_q(\mathrm{SU}(N)) \to L(\mathcal{H}_0)$ is a homomorphism of algebras (here $L(\mathcal{H}_0)$ is the algebra of linear, but not necessarily continuous operators) such that

$$\langle \pi_0(x)\xi|\eta\rangle = \langle \xi|\pi_0(x^*)\eta\rangle \quad \text{for all } x \in \mathrm{Fun}_q(\mathrm{SU}(N)),\ \xi,\eta \in \mathcal{H}_0,$$

then by (3.7) we get

$$\langle \pi_0(u_{ij})\xi|\pi_0(u_{ij})\xi\rangle \leqslant \langle \xi|\xi\rangle \quad \text{and} \quad \langle \pi_0(u^*_{ij})\xi|\pi_0(u^*_{ij})\xi\rangle \leqslant \langle \xi|\xi\rangle \quad \text{for all } \xi \in \mathcal{H}_0.$$

This shows that all the operators $\pi_0(u_{ij}), \pi_0(u^*_{ij})$ are continuous on $\mathcal{H}_0$, and π_0 will give a $*$-representation π of $\mathrm{Fun}_q(\mathrm{SU}(N))$ on the completion of $\mathcal{H}_0$. For such a representation it follows that $\|\pi(u_{ij})\| \leqslant 1$, for all i,j. Since any representation $\pi \in \mathrm{Rep}(\mathrm{Fun}_q(\mathrm{SU}(N)))$ can be constructed in this way, we get that $\mathrm{Fun}_q(\mathrm{SU}(N))$ is a maximally bounded $*$-algebra (see 2.2 (i)). We apply the construction in 2.2 (i) to $\mathrm{Fun}_q(\mathrm{SU}(N))$ and obtain a C^*-algebra, which is denoted by $C(\mathrm{SU}_q(N))$ and called *the algebra of "continuous functions" on quantum* $\mathrm{SU}(N)$ *at* q.

(iii) In fact, using (i), any $*$-representation of $\mathrm{Fun}_q(\mathrm{SU}(N))$ belongs to the representation theory $\mathcal{R}_q$. Hence, on $\mathrm{Fun}_q(\mathrm{SU}(N))$ the C^*-seminorms $p_{\max}$ and $p_{\mathcal{R}_q}$ coincide.

(iv) Using (ii) we can see that if a positive functional $\varphi : \mathrm{Fun}_q(\mathrm{SU}(N)) \to \mathbb{C}$ is given, then $(\mathrm{Fun}_q(\mathrm{SU}(N)), \varphi)$ is a GNS-$*$-algebra. Moreover, any such functional is $\mathcal{R}_q$-continuous, since $\mathrm{Fun}_q(\mathrm{SU}(N))$ is unital. Hence, it extends to a positive functional on $C(\mathrm{SU}_q(N))$, still denoted by φ.

A first set of properties we shall use is contained in the following.

3.8. Proposition. (a) *There exists a unique unital $*$-homomorphism* $\Delta_N : \mathrm{Fun}_q(\mathrm{SU}(N)) \to \mathrm{Fun}_q(\mathrm{SU}(N)) \otimes \mathrm{Fun}_q(\mathrm{SU}(N))$ *such that*

$$\Delta_N(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj} \quad \textit{for all } i,j = 1,N.$$

(b) *There exists a unique unital* $*$*-homomorphism* $\varepsilon_N : \mathrm{Fun}_q(\mathrm{SU}(N)) \to \mathbb{C}$ *such that*

$$\varepsilon_N(u_{ij}) = \delta_{ij} \quad \textit{for all } i, j.$$

(c) *There exists a unique unital* $\mathbb{C}$*-linear anti-automorphism* S_N *of* $\mathrm{Fun}_q(\mathrm{SU}(N))$ *such that*

$$S_N(u_{ij}) = u_{ij}^* \quad \textit{for all } i, j.$$

(d) *The triple* $(\Delta_N, \varepsilon_N, S_N)$ *determines a Hopf* $*$*-algebra structure on* $\mathrm{Fun}_q(\mathrm{SU}(N))$*,* i.e.

(i) $(\Delta_N \otimes \mathrm{Id}) \circ \Delta_N = (\mathrm{Id} \otimes \Delta_N) \circ \Delta_N$ *as maps* : $\mathrm{Fun}_q(\mathrm{SU}(N)) \to \mathrm{Fun}_q(\mathrm{SU}(N)) \otimes \mathrm{Fun}_q(\mathrm{SU}(N)) \otimes \mathrm{Fun}_q(\mathrm{SU}(N))$*;*

(ii) $(\varepsilon_N \otimes \mathrm{Id})(\Delta_N(x)) = (\mathrm{Id} \otimes \varepsilon_N)(\Delta(x)) = x$ *for all* $x \in \mathrm{Fun}_q(\mathrm{SU}(N))$*;*

(iii) $S_N(S_N(x^*)^*) = x$*, for all* $x \in \mathrm{Fun}_q(\mathrm{SU}(N))$*;*

(iv) *If we take the* $m_N : \mathrm{Fun}_q(\mathrm{SU}(N)) \otimes \mathrm{Fun}_q(\mathrm{SU}(N)) \to \mathrm{Fun}_q(\mathrm{SU}(N))$ *to be the map* $m_N(x \otimes y) = xy$*, then*

$$m_N \circ (\mathrm{Id} \otimes S_N)(\Delta_N(x)) = m_N \circ (S_N \otimes \mathrm{Id})(\Delta_N(x)) = \varepsilon_N(x) \cdot 1$$

for all $x \in \mathrm{Fun}_q(\mathrm{SU}(N))$.

(e) *The map* Δ_N *extends to a unique unital* $*$*-homomorphism*

$$\Delta_N : C(\mathrm{SU}_q(N)) \to C(\mathrm{SU}_q(N)) \otimes C(\mathrm{SU}_q(N)).$$

(It is known (*see* [2]) *that the* C^**-algebras* $C(\mathrm{SU}_q(N))$ *are nuclear. So the* C^**-tensor product involved in the definition of* $C(\mathrm{SU}_q(N)) \otimes C(\mathrm{SU}_q(N))$ *is unambiguously defined.) The map* ε_N *extends to a unique unital* $*$*-homomorphism* $\varepsilon_N : C(\mathrm{SU}_q(N)) \to \mathbb{C}$*. The formulas* (i), (ii) *hold for these extensions also.*

(f) *On the* C^**-algebra* $C(\mathrm{SU}_q(N))$ *there exists a unique state* τ_q^N *such that*

$$(\mathrm{Id} \otimes \tau_q^N)(\Delta_N(x)) = (\tau_q^N \otimes \mathrm{Id})(\Delta_N(x)) = \tau_q^N(x) \cdot 1 \quad \textit{for all } x \in C(\mathrm{SU}_q(N)).$$

(Both $(\tau \otimes \mathrm{Id}) \circ \Delta$ *and* $(\mathrm{Id} \otimes \tau) \circ \Delta$ *are viewed as maps* : $C(\mathrm{SU}_q(N)) \to C(\mathrm{SU}_q(N))$*.)*

Proof. See [25], [23], [12]. ∎

When $q = 1$, $C(\mathrm{SU}_1(N))$ is isomorphic to the commutative C^*-algebra of continuous functions on $\mathrm{SU}(N)$. In analogy with the case $q = 1$, τ_q^N is called the *Haar state*. The formula for the Haar state on quantum $\mathrm{SU}(N)$ has been found in Sheu's paper ([20]).

Going back to the framework suggested at the end of Section 2, we will choose the positive functionals $\varphi_q : V_q = \mathrm{Fun}_q(\mathrm{SU}(N)) \to \mathbb{C}$ to be exactly the restrictions of the Haar states.

Let us recall now a key result from [9].

3.9. THEOREM. *The Haar states* $\tau : C(\mathrm{SU}_q(N)) \to \mathbb{C}$ *are faithful.*

As a consequence, we get that the GNS-representation of $C(\mathrm{SU}_q(N))$ associated with τ is isometric. Hence, the GNS-representation of $\mathrm{Fun}_q(\mathrm{SU}(N))$ associated with φ_q will implement the defining C^*-seminorm on $\mathrm{Fun}_q(\mathrm{SU}(N))$, that is, we have $p_{\varphi_q} = p_{\max} = p_{\mathcal{R}_q}$. But we also know (cf. [24] and [25]) that $p_{\max}$ is a norm on $\mathrm{Fun}_q(\mathrm{SU}(N))$. So we have:

3.10. *The conditions* (II)(a) *and* (II)(b) *in the remark after the proof of Theorem* 2.5 *are satisfied.*

We now proceed with the verifications for the conditions in Step I in the remark after the proof of Theorem 2.5. First, we need to choose a "common basis" in all the algebras $V_q^N = \mathrm{Fun}_q(\mathrm{SU}(N))$. (For the moment q is "generic", i.e. $q \in (-1, 1]$, $q \neq 0$.) For a fixed $N \geqslant 2$ we take I^N to be the set of all triples (m, n, p) with

(i) $m = (m_{ij})_{\substack{i,j=1,N \\ i+j<N+1}} \subset \mathbb{N}$,

(ii) $n = (n_{ij})_{\substack{i,j=1,N \\ i+j>N+1}} \subset \mathbb{N}$,

(iii) $p = (p_1, \ldots, p_N) \subset \mathbb{N}$ with $p_{\min} \overset{\text{def}}{=} \min(p_1, \ldots, p_N) = 0$.

For any $i = (m, n, p) \in I^N$ we take

$$v_i^q = Y(m,n,p) = u_{2N}^{n_{2N}} \cdots u_{N2}^{n_{N2}} \cdot \ldots \cdot u_{NN}^{n_{NN}} \cdot u_{N1}^{p_1} \cdots u_{1N}^{p_N} \cdot u_{11}^{m_{11}} \cdots u_{N-1,1}^{m_{N-1,1}},$$

where $u_{ij} = u(N)_{ij}^q$ are the canonical generators for $\mathrm{Fun}_q(\mathrm{SU}(N))$ (see 3.5 and 3.6).

To distinguish the different N's we shall write $v(N)_i^q$ instead of v_i^q.

Using 3.6, it easily follows that

3.11. *The properties* (I)(a), (I)(b), (I)(c), (I)(d), (I)(e) *in the remark after the proof of Theorem* 2.5 *are satisfied.*

So the only thing we are left to prove is Condition (I)(f) in the remark after the proof of Theorem 2.5. This will be done by induction on N. Here are some technical results we shall use for this purpose.

3.12. PROPOSITION. *Assume $N \geqslant 2$ and q "generic".*

(i) *There exists a unique unital $*$-homomorphism* $\Gamma_N : \mathrm{Fun}_q(\mathrm{SU}(N+1)) \to \mathrm{Fun}_q(\mathrm{SU}(N))$ *such that*

$$\begin{aligned} \Gamma_N(u(N+1)_{ij}) &= u(N)_{ij} && \text{if } i, j \leqslant N, \\ \Gamma_N(u(N+1)_{N+1,N+1}) &= 1, \\ \Gamma_N(u(N+1)_{j,N+1}) &= \Gamma_N(u(N)_{N+1,j}) = 0 && \text{if } j \leqslant N. \end{aligned}$$

(ii) *The $*$-homomorphism Γ_N extends to the completions.*

Proof. Statement (i) is clear (easy computations).

By the universality property of "$C^*_{\max}$", clearly Γ_N extends as a $*$-homomorphism

$$\Gamma_N : C(\mathrm{SU}_q(N+1)) \to C(\mathrm{SU}_q(N)). \quad \blacksquare$$

COMMENTS. In the case $q = 1$ the map Γ_N is given by the inclusion $\mathrm{SU}(N) \hookrightarrow \mathrm{SU}(N+1)$ defined by $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. So we can interpret the above fact as a "quantum" analogue of this situation.

Consider the map $F : C(\mathrm{SU}_q(N+1)) \to C(\mathrm{SU}_q(N+1))$ defined by

$$F = \{(\tau^N \circ \Gamma_N) \otimes \mathrm{Id}_{C(\mathrm{SU}_q(N+1))} \otimes (\tau^N \circ \Gamma_N)\} \circ \Delta^2$$

where $\tau^N : C(\mathrm{SU}_q(N)) \to \mathbb{C}$ is the Haar state for $\mathrm{SU}_q(N)$ and $\Delta^2 = (\Delta\otimes\mathrm{Id})\circ\Delta = (\mathrm{Id}\otimes\Delta)\circ\Delta$ is the iterated comultiplication. (We identify $\mathbb{C}\otimes C(\mathrm{SU}_q(N+1))\otimes\mathbb{C}$ with $C(\mathrm{SU}_q(N+1))$.)

The next result collects the some facts proved in [9]:

3.13. THEOREM. (i) *The Haar state* $\tau^{N+1} : C(\mathrm{SU}_q(N+1)) \to \mathbb{C}$ *factors through* F *as* $\tau^{N+1} = \tau^{N+1}\circ F$.

(ii) *The map* F *is a conditional expectation with the range equal to the unital* C^**-subalgebra of* $C(\mathrm{SU}_q(N+1))$ *generated by* $u(N+1)^q_{N+1,N+1}$.

(iii) *When we regard* $\mathrm{Fun}_q(\mathrm{SU}(N+1))$ *as a* $*$*-subalgebra of* $C(\mathrm{SU}_q(N+1))$*, the subspace* $F(\mathrm{Fun}_q(\mathrm{SU}(N+1))$ *is precisely the unital* $*$*-subalgebra generated by* $u(N+1)^q_{N+1,N+1}$.

The above result tells us that the Haar state τ^{N+1} is completely determined by the knowledge of the Haar state on $C(\mathrm{SU}_q(N))$ and the knowledge of how the Haar state τ^{N+1} acts on the unital $*$-subalgebra of $\mathrm{Fun}_q(\mathrm{SU}(N+1))$ generated by $u(N+1)^q_{N+1,N+1}$. In fact we can say a bit more than that.

Let us denote the unital $*$-subalgebra of $\mathrm{Fun}_q(\mathrm{SU}(N+1))$ generated by $u(N+1)^q_{N+1,N+1}$ simply by C_q^{N+1}. Also denote the element $1-u(N+1)^q_{N+1,N+1}\cdot (u(N+1)^q_{N+1,N+1})^*$ simply by K^q_{N+1}. Finally, for each pair of integers (r,s) with $s\geqslant 0$, define the element $a(N+1)^q_{r,s}\in C_q^{N+1}$ as

$$a(N+1)^q_{r,s} = \begin{cases} (u(N+1)^q_{N+1,N+1})^r\cdot (K_q^{N+1})^s & \text{if } r\geqslant 0,\\ (u(N+1)^q_{N+1,N+1})^{*-r}\cdot (K_q^{N+1})^s & \text{if } r<0.\end{cases}$$

With these notations, another result from [9] states:

3.14. PROPOSITION. (i) *The family* $\{a(N+1)^q_{r,s}\in C_q^{N+1} : r,s\in\mathbb{Z},\ s\geqslant 0\}$ *is a basis for* C^{N+1} *(as a linear space over* $\mathbb{C}$*).*

(ii) *If* (r,s) *is a pair of integers with* $s\geqslant 0$ *but* $r\neq 0$*, then* $\tau_q^{N+1}(a(N+1)^q_{r,s} = 0$.

3.15. REMARK. Let $i=(m,n,p)$ be an index in the set I^{N+1} (see the notation following 3.10). Define $|i| = \sum m_{ab}+\sum n_{cd}+\sum p_e$. The results from [17], Theorem 3.1. give the fact that $F(v(N+1)^q_i)$ not only belongs to the subalgebra C_q^{N+1} but in fact belongs to $\mathrm{Span}\{a(N+1)^q_{rs} \mid r,s\in\mathbb{Z},\ s\geqslant 0,\ |r|+s\leqslant |i|\}$.

These results say that, besides the knowledge of the Haar state on $C(\mathrm{SU}_q(N))$, in order to know how τ^{N+1} acts, it suffices to know how τ^{N+1} acts on the $*$-subalgebra generated by K_q^{N+1}. Since anyway we have $\|u(N+1)^q_{N+1,N+1}\|\leqslant 1$ if follows that the element $K_q^{N+1}\in C(\mathrm{SU}_q(N+1))$ is positive and has norm $\leqslant 1$. Consider then, by functional calculus, the unique unital $*$-homomorphism $\Phi_q^{N+1} : C[0,1]\to C(\mathrm{SU}_q(N+1))$ with the property that $\Phi_q^{N+1}(\mathrm{Id}_{[0,1]}) = K_q^{N+1}$. We then can find a unique Borel probability measure on $[0,1]$ with the property that $\tau_q^{N+1}(\{K_q^{N+1}\}^s) = \int_0^1 t^s\,\mathrm{d}\mu_q^{N+1}(t)$ for any $s\geqslant 0$. With this notation, the measure μ_q^{N+1} and the state $\tau_q^N : C(\mathrm{SU}_q(N))\to\mathbb{C}$ completely determine the state $\tau_q^{N+1} : C(\mathrm{SU}_q(N+1))\to\mathbb{C}$. This fact will be used in our inductive proof of (I)(f).

3.16. LEMMA. *Assume Condition* (I)(f) *in the remark after the proof of Theorem 2.5 is satisfied for N. Then for any $i \in I_{N+1}$ there exists a system of functions $(h^i_{r,s})_{|r|+s\leqslant|i|}$ all of which are continuous functions on $(-1,1]\setminus\{0\}$, such that*

$$F(v(N+1)^q_i) = \sum_{|r|+s\leqslant|i|} h^i_{rs}(q)a(N+1)^q_{rs},$$

for all for all $q \in (-1,1]\setminus\{0\}$.

Proof. Due to 3.15, we already know that the desired system of functions exists (and is unique). So, the only thing we need to check is the continuity. First, using the hypothesis, we get the existence of a finite set of indices J and a system of continuous functions $f_j : (-1,1]\setminus\{0\} \to \mathbb{C}$, $j \in J$ such that

$$F(v(N+1)^q_i) = \sum_{j\in J} f_j(q)v(N+1)^q_j,$$

for all $q \in (-1,1]\setminus\{0\}$. This follows from the fact that the map $(\Gamma_N \otimes \mathrm{Id} \otimes \Gamma_N)\circ \Delta^2 : \mathrm{Fun}_q(\mathrm{SU}(N+1)) \to \mathrm{Fun}_q(\mathrm{SU}(N)) \otimes \mathrm{Fun}_q(\mathrm{SU}(N+1)) \otimes \mathrm{Fun}_q(\mathrm{SU}(N))$ gives rise to a $\mathbb{C}[q,q^{-1}]$-linear map $(\Gamma_N \otimes \mathrm{Id} \otimes \Gamma_N)\circ \Delta^2 : \mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N+1)) \to \mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N)) \otimes_{\mathbb{C}[q,q^{-1}]} \mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N+1)) \otimes_{\mathbb{C}[q,q^{-1}]} \mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N))$.

So, we get the identities

$$\sum_{|r|+s\leqslant|i|} h^i_{rs}(q)a(N+1)^q_{rs} = \sum_{j\in J} f_j(q)v(N+1)^q_j, \tag{3.8}$$

for all $q \in (-1,1]\setminus\{0\}$.

Let us expand each $a(N+1)^q_{rs}$, with $|r|+s \leqslant |i|$ in the basis $v(N+1)^q_i$, $i \in I^{N+1}$. We find a "big" finite set of indices $L(i) \subset I^{N+1}$ and continuous functions $H^l_{rs} : (-1,1]\setminus\{0\} \to \mathbb{C}$ such that

$$a(N+1)^q_{rs} = \sum_{l\in L(i)} H^l_{rs}(q)v(N+1)^q_l,$$

for all r,s with $|r|+s \leqslant |i|$ and all $q \in (-1,1]\setminus\{0\}$. We can do this because the a's are monomials (see 3.6).

Fix now $q_0 \in (-1,1]\setminus\{0\}$. Because the a's are linearly independent there exists a finite subset indexed by all pairs (r,s) with $|r|+s \leqslant |i|$, denoted $L_0 = \{l_{rs} : r,s \in \mathbb{Z},\ s \geqslant 0,\ |r|+s \leqslant |i|\}$, such that the matrix $(H^{l_{rs}}_{r's'}(q))_{(r,s),(r',s')}$ is invertible for all q in a neighborhood of q_0. Using this matrix we can "exchange" the a's with the v_l's with $l \in L_0$ using continuous coefficients. This means that we can find a two systems G^l_{rs}, $Z^l_{l'}$, $|r|+s \leqslant |i|$, $l \in L_0$, $l' \in L(i)\setminus L_0$ consisting of continuous functions defined on a neighborhood U of q_0, such that

$$v(N+1)^q_l = \sum_{|r|+s\leqslant|i|} G^l_{rs}(q)a(N+1)^q_{rs} + \sum_{l'\in L(i)-L_0} Z^l_{l'}(q)v(N+1)^q_{l'}, \tag{3.9}$$

for all $l \in L_0$ and all $q \in U$. (The G's are exactly the coefficients of the inverse of the matrix formed with the H's.)

Using the fact that the set $\{a(N+1)^q_{rs} : |r|+s \leqslant |i|\} \cup \{v(N+1)^q_{l'} : l' \in L(i)\setminus L_0\}$ is linearly independent, we see that if we make (for those $j \in L_0$) the "exchanges" given by (3.9) in (3.8) we obtain the continuity of the h's on the neighborhood U. Since this can be done for any q_0, the lemma is proved. ∎

3.17. THEOREM. *For any $N \geqslant 2$ and any $i \in I^N$, the map $(-1,1] \ni q \mapsto \tau_q^N(v(N)_i^q) \in \mathbb{C}$ is continuous.*

Proof. We will prove the theorem by induction on N. Let us start with the case $N = 2$. For this, we shall use the explicit formulas from [22], Appendix.

In this case

$$I^2 = \{(m, n, p_1, p_2) : m, n, p_1, p_2 \in \mathbb{N}, \min(p_1, p_2) = 0\}.$$

For $i = (m, n, p_1, p_2)$, the basis vector is $v(2)_i^q = (u(2)_{22}^q)^n \cdot (u(2)_{21}^q)^{p_1} \cdot (u(2)_{12}^q)^{p_2} \cdot (u(2)_{11}^q)^m$. Using (cf. [24]) the notation $u(2)_{11}^q \overset{\text{def}}{=} \alpha_q$, $u(2)_{21}^q \overset{\text{def}}{=} \gamma_q$, it is easy to see that $u(2)_{22}^q = \alpha_q^*$ and $u(2)_{12}^q = -q\gamma_q^*$.

So the basis is

$$\{\alpha_q^{*n}\gamma_q^p\alpha_q^m : m, n, p \in \mathbb{N}\} \cup \{(-q)^p\alpha_q^{*n}\gamma_q^{*p}\alpha_q^m : m, n, p \in \mathbb{N},\ p \geqslant 1\}.$$

The formulas from [22], Appendix are:

(i) if $m \neq n$ or $p \geqslant 1$, then

$$\tau_q^2(\alpha_q^{*n}\gamma_q^p\alpha_q^m) = \tau_q^2(\alpha_q^{*n}\gamma_q^{*p}\alpha_q^m) = 0;$$

(ii) in the remaining case

$$\begin{aligned}\tau_q^2(\alpha_q^{*m}\alpha_q^m) &= \tau_q^2((1 - \gamma_q^*\gamma_q) \cdots (1 - q^{2(1-m)}\gamma_q^*\gamma_q)) \\ &= \int_0^1 (1 - t)(1 - q^{-2}t) \cdots (1 - q^{2(1-m)}t)\, \mathrm{d}_{q^2}t\end{aligned}$$

where $d_{q^2}t$ is the measure that gives the q^2-integral. That is, for $q^2 \neq 1$,

$$\int_0^1 f(t)\, \mathrm{d}_{q^2}t = (1 - q^{-2}) \cdot \sum_{n \geqslant 0} q^{2n} f(q^{2n}),$$

and $d_1 t$ is the Lebesgue integral. These formulas clearly show the desired continuity.

Next we prove the inductive step. Assume the theorem holds for N. Using Lemma 3.16, for each $i \in I^{N+1}$ we have

$$F(v(N+1)_i^q) = \sum_{|s|+p \leqslant |i|} h_{(s,p)}^i(q) \cdot a(N+1)_{(s,p)}^q$$

with $h_{(s,p)}^i$ continuous functions.

But then, by the 3.14 (ii) and 3.15 we have

$$\tau_q^{N+1}(v(N+1)_i^q) = \sum_{p \leqslant |i|} h_{(0,p)}^i(q) \cdot \int_0^1 t^p\, \mathrm{d}\mu_q^{N+1}(t).$$

So, the only thing to be proved is that the probability measures μ_q^{N+1}, $q \in (-1,1] \setminus \{0\}$ form a continuous family (in the weak topology).

Fix $q_0 \in (-1,1]\setminus\{0\}$. Using a compactness argument, in order to show that $\lim\limits_{q\to q_0}\mu_q^{N+1} = \mu_{q_0}^{N+1}$ (weakly), it suffices to show the following: If $q_n \to q_0$ and $\mu_{q_n}^{N+1} \to \mu$, then $\mu = \mu_{q_0}^{N+1}$. But this is clear if we define

$$\widetilde{\tau}_{q_0}^{N+1}(v(N+1)_i^{q_0}) = \sum_{p\leqslant|i|} h_{(0,p)}^i(q_0)\cdot \int_0^1 t^p\,\mathrm{d}\mu(t),$$

for then we get

$$\widetilde{\tau}_{q_0}^{N+1}(v(N+1)_i^{q_0}) = \lim_{n\to\infty}\sum_{p\leqslant|i|} h_{(0,p)}^i(q_n)\cdot\int_0^1 t^p\,\mathrm{d}\mu_{q_n}^{N+1}(t) = \lim_{n\to\infty}\tau_{q_n}^{N+1}(v(N+1)_i^{q_n}).$$

Using the fact that $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N+1))$ is actually a Hopf algebra over $\mathbb{C}[q,q^{-1}]$, one can easily see that $\widetilde{\tau}_{q_0}^{N+1}$ is a functional on $\mathrm{Fun}_{q_0}(\mathrm{SU}(N+1))$ which satisfies $(\widetilde{\tau}_q^{N+1}\otimes\mathrm{Id})\circ\Delta = (\mathrm{Id}\otimes\widetilde{\tau}_q^{N+1})\circ\Delta$ as linear maps : $\mathrm{Fun}_{q_0}(\mathrm{SU}(N+1)) \to \mathrm{Fun}_{q_0}(\mathrm{SU}(N+1))$, and $\widetilde{\tau}_q^{N+1}(1) = 1$. But it is known (see [24]) that such a functional is unique and it must coincide with the restriction of the Haar state to $\mathrm{Fun}_{q_0}(\mathrm{SU}(N+1))$. This means that $\widetilde{\tau}_{q_0}^{N+1} = \tau_{q_0}^{N+1}$ (on $\mathrm{Fun}_{q_0}(\mathrm{SU}(N+1))$).

But then, using the fact that the conditional expectation $F : C(\mathrm{SU}_{q_0}(N+1)) \to C_{q_0}^{N+1}$ acts as the identity on $C_{q_0}^{N+1}$, we have

$$\int_0^1 t^p\,\mathrm{d}\mu(t) = \tau_{q_0}^{N+1}(a(N+1)_{(0,p)}^{q_0}) = \int_0^1 t^p\,\mathrm{d}\mu_{q_0}^{N+1}(t)$$

for all p, which gives $\mu = \mu_{q_0}^{N+1}$. So the case $N+1$ is proved. ∎

The above result say that condition (I)(f) in the remark after the proof of Theorem 2.5 also holds.

3.18. CONCLUSION. Let us take now $Q = [\varepsilon,1]$, for some fixed $\varepsilon \in (0,1)$, and $\sigma : [0,1-\varepsilon] \to Q$ to be the curve given by $\sigma(h) = 1-h$. Having (see Section 2) the conditions (I)(a)–(I)(f) and (II)(a), (II)(b) satisfied, we get a system $(\mathrm{Fun}_1(\mathrm{SU}(N))z, (\times_h, *_h, \|\cdot\|_h))_{h\in\mathrm{I}})$ which is a deformation quantization for $C(\mathrm{SU}(N))$ in the z-direction (see below for a discussion on z). Moreover:

(i) The $\mathbb{C}$-linear map $\mathrm{Fun}_1(\mathrm{SU}(N)) \to \mathrm{Fun}_{1-h}(\mathrm{SU}(N))$ defined on the basis by $v(N)_i^1 \mapsto v(N)_i^{1-h}$, $i \in I^N$, establishes an isometric $*$-isomorphism between the normed $*$-algebra $(\mathrm{Fun}_1(\mathrm{SU}(N))\times_h, *_h, \|\cdot\|_h)$ and the normed $*$-algebra $(\mathrm{Fun}_{1-h}(\mathrm{SU}(N)), \|\cdot\|_{C^*_{\max}})$.

(ii) After completions, one gets a continuous field $(C(\mathrm{SU}_q(N))_{q\in[\varepsilon,1]}$ of C^*-algebras in which the systems $(u(N)_{ij}^q)_{q\in[\varepsilon,1]}$, $1 \leqslant i,j \leqslant N$ all define continuous sections.

The algebra $\mathrm{Fun}_1(\mathrm{SU}(N))$ is the algebra of functions on $\mathrm{SU}(N)$ which are polynomials in the coordinates $(u_{ij})_{i,j=1,N}$.

The 2-cocycle z can be computed using the "multiplication table" in the basis $(v_i)_{i\in I^N}$. This leads to complicated formulas. Instead, we shall find the corresponding Poisson bracket. This only requires to find the values of the 2-cocycle

z on the generators, that is, to find $z(u_{ab}, u_{a'b'})$. For this, we need to describe $u_{ab} \times_{1-h} u_{a'b'}$ in terms of the canonical basis $(v_i)_{i \in I^N}$. That is, we need to compute the products $u^q_{ab} \cdot u^q_{a'b'}$ in $\mathrm{Fun}_q(\mathrm{SU}(N))$ and express the result in the basis v^q_i, $i \in I^N$.

Recall that

$$I^N = \{(m,n,p) : m = (m_{ab})_{a+b<N+1},\ n = (n_{ab})_{a+b>N+1},\ p = (p_1, \ldots, p_N) \\ \text{with } \min(p_1, \ldots, p_N) = 0\}$$

and for $i = (m,n,p) \in I^N$

$$v_i = u_{2N}^{n_{2N}} \cdots u_{N2}^{n_{N2}} \cdots u_{NN}^{n_{NN}} \cdot u_{N1}^{p_1} \cdots u_{1N}^{p_N} \cdot u_{11}^{m_{11}} \cdots u_{1,N-1}^{m_{1,N-1}} \cdots u_{N-1,1}^{m_{N-1,1}}$$

(in all the algebras $\mathrm{Fun}_q(\mathrm{SU}(N))$).

This gives a total ordering on the set $\{1, \ldots, N\} \times \{1, \ldots, N\}$ defined by

$$(2,N) < \cdots < (k, N-k+2) < \cdots < (k,N) < \cdots < (N,2) < \cdots < (N,N) \\ < (N,1) < (N-1,2) < \cdots < (1,N) \\ < (1,1) < \cdots < (1,N-1) < \cdots < (k,1) < \cdots < (k,N-k) < \cdots < (N-1,1).$$

Suppose $N \geqslant 3$. If $(a,b) \leqslant (a',b')$, then clearly $u_{ab} \times_q u_{a'b'}$ will be a basis vector, so $z(u_{ab}, u_{a'b'}) = 0$.

If (a,b) and (a',b') are in the "wrong" order, i.e. $(a,b) > (a',b')$, then we use the relations in $\mathrm{Fun}_q(\mathrm{SU}(N))$ (see formulas (3.1)–(3.6)). So:

(i) If $a = a'$ and $b < b'$ then $u_{ab} \times_q u_{a'b'} = q u_{a'b'} \times_q u_{ab}$, now with $u_{a'b'} \times_q u_{ab}$ a basis vector. So, if we take into account $q = 1 - h$, we will get

$$z(u_{ab}, u_{a'b'}) = -u_{a'b'} u_{ab}.$$

(ii) If $a = a'$ and $b > b'$, then $u_{ab} \times_q u_{a'b'} = q^{-1} u_{a'b'} \times_q u_{ab}$. So we get

$$z(u_{ab}, u_{a'b'}) = u_{a'b'} u_{ab}.$$

(iii) If $a > a'$ and $b = b'$ then $u_{ab} \times_q u_{a'b'} = q^{-1} u_{a'b'} \times_q u_{ab}$, which gives

$$z(u_{ab}, u_{a'b'}) = u_{a'b'} u_{ab}.$$

(iv) If $a < a'$ and $b = b'$ then $u_{ab} \times_q u_{a'b'} = q u_{a'b'} \times_q u_{ab}$, which gives

$$z(u_{ab}, u_{a'b'}) = -u_{a'b'} u_{ab}.$$

(v) If $a < a'$ and $b > b'$, then $u_{ab} \times_q u_{a'b'} = u_{a'b'} \times_q u_{ab}$, so

$$z(u_{ab}, u_{a'b'}) = 0.$$

(iv) If $a < a'$ and $b < b'$ then $u_{ab} \times_q u_{a'b'} = u_{a'b'} \times_q u_{ab} + (q - q^{-1}) u_{ab'} \times_q u_{a'b}$. Note that now $u_{ab'} \times_q u_{a'b} = u_{a'b} \times_q u_{ab'}$, so this is anyway a basis vector. This gives

$$z(u_{ab}, u_{a'b'}) = -2 u_{ab'} \times_q u_{a'b}.$$

(vii) If $a > a'$ and $b > b'$, then $u_{ab} \times_q u_{a'b'} = u_{a'b'} \times_q u_{ab} - (q - q^{-1}) u_{a'b} \times_q u_{ab'}$. As before we get

$$z(u_{ab}, u_{a'b'}) = 2 u_{ab'} \times_q u_{a'b}.$$

(viii) If $a > a'$ and $b < b'$ exactly as in (v) we get

$$z(u_{ab}, u_{a'b'}) = 0.$$

The above formulas can be used to describe the Poisson bracket on $\mathrm{Fun}_1(\mathrm{SU}(N))$ defined by z.

EXAMPLE. Suppose $N = 3$. The corresponding Poisson bracket is given by the following formulas

(i) If $(ab; a'b')$ is of one of the forms

$$\begin{array}{cccccc} (12;11) & (13;11) & (21;11) & (31;11) & (13;12) & (22;12) \\ (32;12) & (23;13) & (33;13) & (22;21) & (23;21) & (31;21) \\ (23;22) & (32;22) & (33;23) & (32;31) & (33;31) & (33;32) \end{array}$$

then $\{u_{ab}, u_{a'b'}\} = u_{ab}u_{a'b'}$.

(ii) If $(ab; a'b')$ is of one of the forms

$$\begin{array}{ccc} (22;11) & (23;11) & (32;11) \\ (33;11) & (23;12) & (33;12) \\ (32;21) & (33;21) & (33;22) \end{array}$$

then $\{u_{ab}, u_{a'b'}\} = 2u_{ab'}u_{a'b}$.

(iii) If $(ab; a'b')$ is of one of the forms

$$\begin{array}{ccc} (21;12) & (31;12) & (21;13) \\ (22;13) & (31;13) & (32;13) \\ (31;22) & (31;23) & (32;23) \end{array}$$

then $\{u_{ab}, u_{a'b'}\} = 0$.

For $N = 2$ the arguments are similar. Recall that the basis vectors are here

$$v_{(m,n,p_1,p_2)} = u_{22}^n u_{21}^{p_1} u_{12}^{p_2} u_1^m$$

with $\min(p_1, p_2) = 0$. We compute the values $z(u_{ab}, u_{a'b'})$ in a similar way. But everything works as in the case $N \geqslant 3$ except for the element $u_{12} \times_q u_{21} = u_{21} \times_q u_{12}$ which does not belong to the basis.

For $N = 2$ the "multiplication table" for the u_{ab}'s looks like

$$\begin{array}{ll} u_{11} \times_q u_{11} = v_{(0,2,0,0)} & u_{12} \times_q u_{11} = v_{(0,1,0,1)} \\ u_{11} \times_q u_{12} = qv_{(0,1,0,1)} & u_{12} \times_q u_{12} = v_{(0,0,0,2)} \\ u_{11} \times_q u_{21} = qv_{(0,1,1,0)} & u_{12} \times_q u_{21} = qv_{(1,1,0,0)} - qv_{(0,0,0,0)} \\ u_{11} \times_q u_{22} = v_{(0,0,0,0)} - q^2 v_{(0,2,0,0)} & u_{12} \times_q u_{22} = v_{(1,0,0,1)} \\ \cdots \quad \text{etc.} & \end{array}$$

This leads, for the Poisson bracket, to the following formulas

(i) If $(ab; a'b')$ is of one of the forms

$$(12;11) \quad (21;11) \quad (22;12) \quad (22;21),$$

then $\{u_{ab}, u_{a'b'}\} = u_{ab}u_{a'b'}$.

(ii) $\{u_{22}, u_{11}\} = -2u_{22}u_{11}$.

(iii) $\{u_{21}, u_{12}\} = 0$.

3.19. FINAL REMARKS. (i) It can be shown that the Poisson bracket on $\mathrm{Fun}_1(\mathrm{SU}(N))$ can be extended to a Poisson bracket on $C^\infty(\mathrm{SU}(N))$ (see [4]).

(ii) Note that the $*$-algebra $\mathrm{Fun}_{[q,q^{-1}]}(\mathrm{SU}(N))$, over $\mathbb{C}[q, q^{-1}]$, carries an additional structure of a Hopf algebra over $\mathbb{C}[q, q^{-1}]$. This enables us to conclude that for any $f \in V_1 = \mathrm{Fun}_1(\mathrm{SU}(N))$ if we take $\Delta_q(f) \in \mathrm{Fun}_q(\mathrm{SU}(N)) \otimes \mathrm{Fun}_q(\mathrm{SU}(N))$, $S_q(f) \in \mathrm{Fun}_q(\mathrm{SU}(N))$ and $\varepsilon_q(f) \in \mathbb{C}$, then

$$\lim_{q\to q_0} \|\Delta_q(f) - \Delta_{q_0}(f)\|_q = 0, \tag{3.10}$$

$$\lim_{q\to q_0} \|S_q(f) - S_{q_0}(f)\|_q = 0, \tag{3.11}$$

$$\lim_{q\to q_0} \varepsilon_q(f) = \varepsilon_{q_0}(f). \tag{3.12}$$

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