J. OPERATOR THEORY
45(2001), 3–18

HILBERT C^*-BIMODULES AND COUNTABLY GENERATED CUNTZ-KRIEGER ALGEBRAS

T. KAJIWARA, C. PINZARI and Y. WATATANI

Communicated by Şerban Strătilă

Abstract. Results by Cuntz and Kreiger on uniqueness, simplicity and the ideal structure of the algebras $\mathcal{O}_A$ associated with finite matrices with entries in $\{0,1\}$ are generalized to the case where A is an infinite matrix whose rows and columns are eventually zero, but not identically zero. Similar results have been recently obtained by Kumjian, Pask, Raeburn and Renault from the viewpoint of Renault's theory of groupoids. An alternative approach, based on the realization of $\mathcal{O}_A$ as an algebra generated by a Hilbert C^*-bimodule introduced by Pimsner, is proposed and compared.

Keywords: *Cuntz-Krieger algebra, Hilbert bimodule.*

MSC (2000): 46L35, 46L55.

1. INTRODUCTION

Kumjian, Pask, Raeburn and Renault generalized in [11] to Cuntz-Krieger algebras $\mathcal{O}_A$ associated with certain infinite matrices A with entries in $\{0,1\}$ results previously obtained by Cuntz and Krieger in the finite case about uniqueness, simplicity and the ideal structure ([5] and [4]). Their approach was based on the idea that $\mathcal{O}_A$ can be realized as the C^*-algebra associated with a groupoid, and therefore the description of the ideal theory of $\mathcal{O}_A$ can be reduced to Renault's analysis of the ideal theory of groupoid C^*-algebras ([17]). The main assumption the authors introduced to this aim was a graph-theoretic condition, called (K).

The purpose of the present paper is to present a self-contained proof, close to the original arguments of Cuntz and Krieger, which does not involve the theory of groupoids and to compare our results with the results in [11].

In a previous paper ([12]) we obtained some results concerning simplicity, uniqueness and ideal structure of the Cuntz-Krieger-Pimsner algebras $\mathcal{O}_X$ generated by a Hilbert bimodule X over a C^*-algebra A ([15]). However, to make our exposition as simple as possible, our assumption was that A is unital and X finite projective as a right A-module. When A is commutative and finite dimensional and X is finite projective as a right A-module, $\mathcal{O}_X$ reduces to a Cuntz-Krieger algebra over a finite matrix, as noted by Pimsner ([15]). Therefore we generalized to Hilbert C^*-bimodules the original conditions (I) and (II) introduced by Cuntz and Krieger on the defining matrix, that we called respectively (I) and (II)-*freeness*, the latter being stronger than the former. Our strategy of proof was suggested by arguments by Doplicher and Roberts for certain algebras $\mathcal{O}_\rho$ associated with an object ρ of a tensor C^*-category ([7], [8]). Indeed, if X is finite projective, $\mathcal{O}_X$ can be regarded as a Doplicher-Roberts algebra, considering X as an object of a semitensor C^*-category. In the general case, $\mathcal{O}_X$ is canonically contained in the Doplicher-Roberts algebra associated with X ([6]). We mention the works by Abadie, Eilers and Exel ([1]), Katayama ([9]) and Muhly and Solel ([13]), for more approaches to the algebra $\mathcal{O}_X$.

In the present paper we refine and develop the arguments in [12], for a particular class of countably generated Hilbert C^*-bimodules over σ-unital C^*-algebras. More specifically, we associate with any matrix $A = (a_{i,j})_{i,j\in\mathbb{N}}$ with entries in $\{0,1\}$, whose rows and columns are eventually zero (but no one of them is identically zero) a directed graph with the set of vertices denoted by Σ and the set of edges denoted by E, which is a subset of $\Sigma \times \Sigma$, i.e., a set-theoretic correspondence. As in [3], p. 526, $c_{00}(E)$ has a natural structure of pre-Hilbert bimodule over $c_0(\Sigma)$. Its completion X generates a C^*-algebra $\mathcal{O}_X$ isomorphic to $\mathcal{O}_A$.

We consider an approximated version of the notion of (I) and (II)-freeness suitable for the special case of Cuntz-Krieger algebras over infinite matrices, that we call conditions (I)$'$ and (II)$'$ respectively (actually these conditions can be formulated in a more general setting of certain countably generated Hilbert bimodules, as sketched in [16]). We use condition (I)$'$ to prove uniqueness and (II)$'$ to study the ideal structure. We show that the stronger condition (II)$'$ is equivalent to condition (K) considered in [11].

After the completion of the first draft of our work, we received a paper by Kumjian, Pask and Raeburn ([10]) where another graph-theoretic condition, called (L), was introduced to study the uniqueness problem. This induced us to review our paper once again. We show that condition (L) is equivalent to our condition (I)$'$.

In [14] Pask and Raeburn introduced a certain condition on the matrix A to compute the K-groups of countable Cuntz-Krieger algebras $\mathcal{O}_A$. Moreover, the authors proved in [10] that $\mathcal{O}_A$ is purely infinite under a further graph-theoretic condition, by reducing the proof to a result by Anantharaman-Delaroche ([2]). We give a direct proof by an argument essentially similar to that in [4] and [2]. Furthermore, we show how K-groups of $\mathcal{O}_A$ can be easily deduced from Pimsner's 6-term exact sequence ([15]) for the algebras $\mathcal{O}_X$.

2. BIMODULES GENERATING CUNTZ-KRIEGER ALGEBRAS

Let A be a C^*-algebra. A Hilbert A-module is a right Hilbert A-module X endowed with a non-degenerate isometric $*$-homomorphism ϕ of A into $\mathcal{L}_A(X_A)$, the algebra of adjointable right Hilbert A-module maps. Throughout the paper we will assume that A is a σ-unital C^*-algebra, that X is full and countably generated as a right Hilbert A-module and that $\phi(A)$ is contained in $\mathcal{K}_A(X_A)$, the algebra of compact operators on the right Hilbert module X_A. Let $F(X)$ be the Fock Hilbert A-bimodule $\bigoplus_{n=0}^{\infty} X^{\otimes n}$, where we set $X^{\otimes 0} = A$ for convenience. Let T_x be the creation operator on $F(X)$ by $x \in X$, and denote by $\mathcal{T}_X$ the C^*-algebra generated by $\{T_x\}_{x\in X}$. Denote by $J(X)$ the C^*-subalgebra of $\mathcal{L}_A(F(X)_A)$ generated by $\mathcal{L}_A\Big(\sum_{\text{finite}} X^{\otimes n}\Big)$, and let $M(J(X))$ be its multiplier algebra, that clearly contains $\mathcal{T}_X$. Define $\mathcal{O}_X$ to be the image of $\mathcal{T}_X$ in the corona algera $M(J(X))/J(X)$ under the quotient map. We denote by S_x the image of T_x. The C^*-algebra $\mathcal{O}_X$ was introduced by Pimsner in [15] and has a certain universality property with respect to the generating Hilbert C^*-bimodule X. We denote by $\mathcal{F}_{(s,r)}$ the norm closure of the linear span of $\{S_{x_1}\cdots S_{x_r}S^*_{y_s}\cdots S^*_{y_1}\}$. The condition $\phi(A)\subset\mathcal{K}_A(X_A)$ implies the following tower of inclusions:

$$\mathcal{F}_{(r,r+k)} \subset \mathcal{F}_{(r+1,r+k+1)} \subset \cdots .$$

We denote by $\mathcal{F}^{(k)}$ the norm closure of $\bigcup_{r=0}^{\infty}\mathcal{F}_{(r,r+k)}$. There is a short exact sequence

$$0 \to \mathcal{K}_A(F(X)_A) \to \mathcal{T}_X \to \mathcal{O}_X \to 0.$$

We summarize some lemmas for countably generated Hilbert C^*-bimodules over σ-unital C^*-algebras which were formally given in [12] for finitely generated projective modules over unital C^*-algebras. We denote by $\theta_{x,y}$ the "rank one" operator on X defined by $\theta_{x,y}(z) = x(y|z)_A$ for some $x,y,z\in X$.

LEMMA 2.1. ([12]) *Let A be a C^*-algebra and X_A a right Hilbert A-module. For $x_1,\dots,x_n,y_1,\dots,y_n\in X$, we have*

$$\Big\|\sum_{i=1}^{n}\theta_{x_i,y_i}\Big\| = \big\|((x_i|x_j)_A)_{ij}^{1/2}((y_i|y_j)_A)_{ij}^{1/2}\big\|$$

where the right hand side norm is the C^*-norm of $M_n(A)$.

Using the above lemma, we deduce the following:

LEMMA 2.2. *Let A and D be C^*-algebras, and let X_A be a right Hilbert A-module. Let $\pi_A : A\to D$ be a $*$-homomorphism and let $\pi_X : X\to D$ be a contraction such that*

$$\pi_X(xa) = \pi_X(x)\pi_A(a), \quad \pi_A((x|y)_A) = \pi_X(x)^*\pi_X(y)$$

for $x,y\in X$ and $a\in A$. Then there exists a unique $*$-homomorphism $\pi_K : K = \mathcal{K}_A(X_A)\to D$ such that

$$\pi_K(\theta_{y,x}) = \pi_X(y)\pi_X(x)^* \quad \textit{and} \quad \pi_X(kx) = \pi_K(k)\pi_X(x)$$

for $x, y \in X$ and $k \in \mathcal{K}_A(X_A)$. Furthermore, if π_A is one-to-one, then π_K is faithful and π_X is an isometry.

Proof. By the proof of Lemma 2 in [12], we have for every n,

$$\Big\| \sum_{i=1}^{n} \pi_X(x_i)\pi_Y(y_i)^* \Big\| \leqslant \Big\| \sum_{i=1}^{n} \theta_{x_i, y_i} \Big\|.$$

Such elements are dense in $\mathcal{K}_A(X_A)$, which proves the lemma. ∎

Remark 2.3. Lemma 2.2 also holds for the spaces $\mathcal{K}_A(X_A, Y_A)$ of compact operators between different Hilbert A-modules X and Y.

A triple (π_X, π_A, D) satisfying properties stated in Lemma 2.2 and satisfying furthermore

$$\pi_X(\phi(a)x) = \pi_A(a)\pi_X(x), \quad a \in A,\, x \in X,$$

will be called a *representation* of the Hilbert A-bimodule X in D.

Of course, letting $\pi_A : A \to M(J(X))/J(X)$ denote the image in the quotient of the left action of A on the Fock space, $(S, \pi_A, \mathcal{O}_X)$ is a representation of X in $\mathcal{O}_X$, which furthermore satisfies

$$\pi_K(\phi(a)) = \pi_A(a) \quad \text{for } a \in \phi^{-1}(\mathcal{K}_A(X_A)) \cap A. \tag{2.1}$$

Since we assume that $\phi(A) \subset \mathcal{K}_A(X_A)$, equation (2.1) holds for $a \in A$. We will usually identify A with $\pi_A(A)$. Pimsner showed in [15] that $\mathcal{O}_X$ is the universal C^*-algebra generated by the image of a representation of X that satisfies (2.1).

Lemma 2.4. ([12], [15]) *Let X be a Hilbert A-bimodule and let $\mathcal{O}_X = C^*\{S_x \mid x \in X\}$. Then there exist isometries $\pi_m : X^{\otimes m} \to \mathcal{O}_X$ and $*$-isomorphisms*

$$\psi_{r,s} : \mathcal{K}_A(X_A^{\otimes r}, X_A^{\otimes s}) \to \mathcal{F}_{(r,s)}$$

for $m = 1, 2, 3, \ldots$, such that

$$\pi_m(x_1 \otimes \cdots \otimes x_m) = S_{x_1} \cdots S_{x_m}$$

and

$$\psi_{r,s}(\theta_{x_1 \otimes \cdots \otimes x_s, y_1 \otimes \cdots \otimes y_r}) = S_{x_1} \cdots S_{x_s} S_{y_r}^* \cdots S_{y_1}^*$$

for $x_i, y_j \in X$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant r$.

Proof. Apply Lemma 2.2 and the Remark 2.3 to $X^{\otimes r}$ in place of X and $X^{\otimes s}$ in place of Y. ∎

Set $\Sigma = \{1, 2, 3, \ldots\}$ and let $G = (G(i,j))_{i,j \in \Sigma}$ be an infinite matrix with entries in $\{0, 1\}$. We shall assume that no row and no column of G is identically zero. We associate to the matrix G the directed graph $\mathcal{G} = (\Sigma, E, s, r)$, where Σ is the set of vertices and $E = \{(i,j) \in \Sigma \times \Sigma \mid G(i,j) = 1\}$ is the set of edges. For an edge $\gamma = (i,j) \in E$, the source is given by $\mathrm{s}(\gamma) = i$ and the range is given by $\mathrm{r}(\gamma) = j$.

A graph $\mathcal{G}$ is *locally finite* if $\{i \in \Sigma \mid G(i,j) = 1\}$ and $\{j \in \Sigma \mid G(i,j) = 1\}$ are finite for any $i \in \Sigma$ and $j \in \Sigma$. We shall assume that $\mathcal{G}$ is locally finite throughout the paper.

Let $A = c_0(\Sigma)$ be the C^*-algebra of the functions on Σ vanishing at infinity and let $A_0 = c_{00}(\Sigma)$ be the dense $*$-subalgebra of functions with finite support. We denote by P_j the projection in A given by $P_j(i) = \delta_{ij}$.

Since the set E of edges is a subset of $\Sigma \times \Sigma$, we may regard E as a set-theoretic correspondence. The vector space $X_0 = c_{00}(E)$ of the function on E with finite support is an A-A-bimodule by

$$(a \cdot f \cdot b)(i,j) = a(i)f(i,j)b(j)$$

for $a, b \in A$, $f \in X_0$ and $(i,j) \in E$. We define an A-valued inner product on X_0 by

$$(f|g)_A(j) = \sum_i \overline{f(i,j)}g(i,j)$$

for $f, g \in X_0$. Then X_0 is a right pre-Hilbert A-module. We denote by X the completion of X_0. The left A-action on X_0 can be extended to $\phi : A \to \mathcal{L}_A(X_A)$ by continuity. Since G is a row finite matrix, $\phi(a) \subset \mathcal{K}_A(X_A)$. In particular, for $\gamma, \gamma' \in E$ and $j \in \Sigma$, we have

$$(\delta_\gamma|\delta_{\gamma'})_A = \delta_{\gamma,\gamma'}P_{r(\gamma)}, \quad \delta_\gamma P_j = \delta_{\mathrm{r}(\gamma),j}\delta_\gamma, \quad \phi(P_j)\delta_\gamma = \delta_{j,\mathrm{s}(\gamma)}\delta_\gamma.$$

Since no column of G is zero, the range map r is onto. Thus X_A is full. Let $\mathcal{O}_X = C^*\{S_x \mid x \in X\}$ be the C^*-algebra generated by the bimodule X introduced by Pimsner ([15]) as above.

For $\alpha \in E$, S_{δ_α} will be denoted by S_α for short. Since $\{\delta_\alpha \mid \alpha \in E\}$ is an orthogonal basis for X, S_α ($\alpha \in E$) are partial isometries and their ranges are orthogonal. In fact, $S_\alpha^* S_\alpha = (\delta_\alpha|\delta_\alpha)_A = P_{\mathrm{r}(\alpha)}$. If $\alpha \neq \beta$, then

$$S_\alpha S_\alpha^* S_\beta S_\beta^* = S_\alpha(\delta_\alpha|\delta_\beta)_A S_\beta^* = S_\alpha \delta_{\alpha,\beta} P_{\mathrm{r}(\alpha)} S_\beta^* = 0.$$

Moreover, if $\mathrm{r}(\alpha) \neq \mathrm{r}(\beta)$, then the supports of S_α and S_β are orthogonal and $S_\alpha S_\beta^* = 0$. Let F be the edge matrix defined by $F(\alpha,\beta) = 1$ if $\mathrm{r}(\alpha) = \mathrm{s}(\beta)$ and $F(\alpha,\beta) = 0$ otherwise. Then the generators $\{S_\alpha \mid \alpha \in E\}$ satisfy

$$S_\alpha^* S_\alpha = \sum_\beta F(\alpha,\beta) S_\beta S_\beta^*.$$

Indeed, since $\phi(P_{\mathrm{r}(\alpha)}) = \sum_\beta \delta_{\mathrm{r}(\alpha),\mathrm{s}(\beta)} \theta_{\delta_\beta,\delta_\beta} \in \mathcal{K}_A(X_A)$,

$$S_\alpha^* S_\alpha = P_{\mathrm{r}(\alpha)} = \pi_K(\phi(P_{\mathrm{r}(\alpha)})) = \sum_\beta \delta_{\mathrm{r}(\alpha),\mathrm{s}(\beta)} S_\beta S_\beta^* = \sum_\beta F(\alpha,\beta) S_\beta S_\beta^*.$$

For $i \in \Sigma$, we may define $S_i = \sum_{\mathrm{s}(\alpha)=i} S_\alpha \in \mathcal{O}_X$, because no row of G is zero and the source map s is onto. If $\beta = (i,j) \in E$, then $S_\beta = S_i P_j$. In fact

$$S_i P_j = \sum_{\mathrm{s}(\alpha)=i} S_\alpha P_j = \sum_{\mathrm{s}(\alpha)=i} S_{\delta_\alpha P_j} = \sum_{\mathrm{s}(\alpha)=i} \delta_{\mathrm{r}(\alpha),j} S_\alpha = S_{(i,j)} = S_\beta.$$

We note that $S_i S_i^* = P_i$. In fact,

$$S_i S_i^* = \Bigg(\sum_{\mathrm{s}(\alpha)=i} S_\alpha\Bigg)\Bigg(\sum_{\mathrm{s}(\beta)=i} S_\beta\Bigg)^* = \sum_{\substack{\mathrm{s}(\alpha)=\mathrm{s}(\beta)=i \\ \mathrm{r}(\alpha)=\mathrm{r}(\beta)}} S_\alpha S_\beta^* = \sum_{\mathrm{s}(\beta)=i} S_\beta S_\beta^*.$$

Since no column of G is zero, the range map r is onto. We choose an edge ρ with $\mathrm{r}(\rho) = i$. Then

$$S_i S_i^* = \sum_\beta F(\rho, \beta) S_\beta S_\beta^* = P_{\mathrm{r}(\rho)} = P_i.$$

Hence, C^*-algebra $\mathcal{O}_X$ is also generated by $\{S_i \mid i \in \Sigma\}$ satisfying the relations

$$S_i^* S_i = \sum_j G(i,j) S_j S_j^*.$$

Moreover, for a finite path $(v_1, \ldots, v_m)$ consisting of m vertices, consider the corresponding $m-1$ edges $\alpha_i = (v_i, v_{i+1})$, $i = 1, \ldots, m-1$. It is useful to note that

$$S_{v_1} \cdots S_{v_{m-1}} P_{v_m} = S_{\alpha_1} \cdots S_{\alpha_{m-1}},$$

which will be used without further mention.

Let $v_i = \sum_{\mathrm{s}(\alpha)=i} \delta_\alpha$. Then $\{v_i \mid i \in \Sigma\}$ is a countable basis of X and $S_{v_i} = S_i$. We choose a numeration of the edge set $E = \{\gamma_i \mid i = 1, 2, 3, \ldots\}$ and put $u_i = \delta_{\gamma_i}$. Then $\{u_i \mid i = 1, 2, 3, \ldots\}$ is another countable basis of X. (In the discussion below we might as well make use of the basis $\{v_i \mid i \in \Sigma\}$ in place of $\{u_i \mid i = 1, 2, 3, \ldots\}$.) Let ${}^0\mathcal{O}_X$ be the $*$-algebra generated by $\{S_{u_i} \mid i = 1, 2, \ldots\}$. For $T \in \mathcal{O}_X$ we define $\sigma_k(T) = \sum_{i=1}^k S_{u_i} T S_{u_i}^*$.

Proposition 2.5. *For $T \in \mathcal{O}_X$, $\{\sigma_k(T)\}_k$ is a norm Cauchy sequence and converges to an element in $\mathcal{O}_X$.*

Proof. Let us choose, as first step, $T \in {}^0\mathcal{O}_X$. Then the sum $\sum_{i=1}^\infty S_{u_i} T S_{u_i}^*$ is in fact a finite sum, because D is column finite. Thus $\{\sigma_k(T)\}$ is a norm Cauchy sequence. Now, let T be a general element of $\mathcal{O}_X$. Then there exists a sequence $\{T_n\}_n$ in ${}^0\mathcal{O}_X$ with $T = \lim_{n\to\infty} T_n$. Since each σ_k is a completely positive map, we have $\|\sigma_k\| = \|\sigma_k(1)\| = \left\| \sum_{i=1}^k S_{u_i} S_{u_i}^* \right\| = 1$. Thus the sequence $\{\sigma_k(T_n)\}_n$ approximates $\sigma_k(T)$ uniformly in k. Therefore $\{\sigma_k(T)\}_k$ is also a norm Cauchy sequence. ∎

Definition 2.6. We denote by $\sigma(T)$ the norm limit $\lim_{k\to\infty} \sigma_k(T)$. The map σ is clearly completely positive.

Lemma 2.7. *For $T_1, T_2 \in A' \cap \mathcal{O}_X$,*

$$\sigma(T_1 T_2) = \sigma(T_1)\sigma(T_2).$$

Proof. For $N \in \mathbb{N}$,

$$\left(\sum_{i=1}^N S_{u_i} T_1 S_{u_i}^* \right) \left(\sum_{j=1}^N S_{u_j} T_2 S_{u_j}^* \right) = \sum_{i=1}^N S_{u_i} T_1 T_2 S_{u_i}^*.$$

Letting $N \to \infty$, we obtain the result. ∎

LEMMA 2.8. *If $T \in A' \cap \mathcal{O}_X$ and $\sigma(T) = 0$, then $T = 0$.*

Proof. The proof is analogous to that of Lemma 3.1 in [12]. ∎

The following lemma will be useful in the next section.

LEMMA 2.9. *For $T \in A' \cap \mathcal{O}_X$, and $x_1, \ldots, x_n \in X$,*

$$\sigma^m(T) S_{x_1} \cdots S_{x_m} = S_{x_1} \cdots S_{x_m} T.$$

Moreover, $\sigma^m(T) \in \mathcal{F}_{(m,m)}{}' \cap \mathcal{O}_X$. In particular, $\sigma(T)$ commutes with $A \subset \mathcal{F}_{(1,1)}$ and σ preserves $A' \cap \mathcal{O}_X$.

Proof. We argue as in Lemma 3.2 from [12]. Consider the following sums, where the summation indices $i_1, \ldots, i_m$ run over a finite subset of $\mathbb{N}$:

$$\sum_{i_1,\ldots,i_m} S_{u_{i_1}} S_{u_{i_2}} \cdots S_{u_{i_m}} T S^*_{u_{i_m}} \cdots S^*_{u_{i_2}} S^*_{u_{i_1}} S_{x_1} \cdots S_{x_m}$$
$$= \sum_{i_1,\ldots,i_m} S_{u_{i_1}} S_{u_{i_2}} \cdots S_{u_{i_m}} S^*_{u_{i_m}} \cdots S^*_{u_{i_2}} S^*_{u_{i_1}} S_{x_1} \cdots S_{x_m} T.$$

Since $\{u_i \mid i = 1, 2, 3, \ldots\}$ is a basis of X, $\sum_{i=1}^{n} S_{u_i} S^*_{u_i} S_x = \sum_{i=1}^{n} S_{u_i (u_i|x)_A}$ converges to S_x in norm for $x \in X$. Taking the limits several times, we have $\sigma^m(T) S_{x_1} \cdots S_{x_m} = S_{x_1} \cdots S_{x_m} T$. ∎

Let $Y^m = \{(v_0, v_1, \ldots, v_m) \in \Sigma^{m+1} \mid G(v_j, v_{j+1}) = 1 \text{ for } j = 0, \ldots, n-1\}$ be the space of finite paths of length m consisting of $m+1$ vertices. Define an inclusion of C^*-algebras $b_m : C_0(Y^m) \to C_0(Y^{m+1})$ by

$$b_m(f)(v_0, v_1, \ldots, v_{m+1}) = f(v_0, v_1, \ldots, v_m)$$

for $f \in C_0(Y^m)$. Let $\mathcal{D}_A^m$ be the commutative C^*-subalgebra generated by A, $\sigma(A)$, $\ldots$ and $\sigma^m(A)$ in $A' \cap \mathcal{O}_X$. Let $a_m : \mathcal{D}_A^m \to \mathcal{D}_A^{m+1}$ be the canonical inclusion.

LEMMA 2.10. *There exist isomorphisms $\psi_m : \mathcal{D}_A^m \to C_0(Y^m)$, $m \in \mathbb{N}$, such that $\psi_{m+1} \circ a_m = b_m \circ \psi_m$ and $(\psi_m \circ \sigma \circ \psi_m^{-1}(f))(v_0, v_1, \ldots, v_{m+1}) = f(v_1, \ldots, v_{m+1})$ for $f \in C_0(Y^m)$.*

Proof. One checks that the map

$$\psi_m : P_{v_0} \sigma(P_{v_1}) \sigma^2(P_{v_2}) \cdots \sigma^m(P_{v_m}) \mapsto \delta_{(v_0, v_1, \ldots, v_m)}$$

extends to an $*$-isomorphism as in [5]. Then the rest is easily obtained. ∎

In particular, $(\psi_m(\sigma^j(P_k))(v_0, \ldots, v_{m+1}) = \delta_k(v_j)$ for j with $1 \leqslant j \leqslant m$. Taking the limit, we consider the commutative C^*-subalgebra $\mathcal{D}_A$ of $A' \cap \mathcal{O}_X$ generated by $\sigma^j(A)$, $j = 0, 1, 2, \ldots$. We identify its spectrum with the infinite path space

$$Y = \{(v_j)_{j \in \mathbb{N}} \mid G(v_j, v_{j+1}) = 1, \ j \in \mathbb{N}\}.$$

The cylinder set $Z(i_0, \ldots, i_m) = \{(v_j)_{j \in \mathbb{N}} \mid v_0 = i_0, \ldots, v_m = i_m\}$ corresponds to the projection $P_{i_0} \sigma(P_{i_1}) \cdots \sigma^m(P_{i_m})$. Moreover, the restriction of σ to $\mathcal{D}_A$ corresponds to the one-sided shift on the infinite path space Y.

3. (I)-FREENESS CONDITION

We shall consider a countable version of the condition (I) of [5]. A *loop* of the graph $\mathcal{G}$ is a finite path $(v_0, v_1, \dots, v_m)$ with $v_0 = v_m$. It is called a *simple loop* if the vertices $v_1, \dots, v_m$ are distinct. Let Σ_0 be the set of vertices $i \in \Sigma$ such that either there are at least two different loops based at i or there is an infinite path starting at i consisting of all different vertices.

Definition 3.1. We say that the matrix G (more precisely, the graph $\mathcal{G}$ corresponding to G) *satisfies condition* (I)$'$ if for every $v \in \Sigma$ there exists a path from v to some i in Σ_0. Equivalently, $\mathcal{G}$ satisfies (I)$'$ if for every vertex v either there is an infinite path starting at v consisting of all different vertices or there is a path from v to a vertex i such that there are at least two different loops based at i.

Following Kumjian, Pask and Raeburn ([10]), we say that a graph $\mathcal{G}$ satisfies condition (L) if all simple loops in $\mathcal{G}$ have exits, i.e., for any simple loop $(v_0, \dots, v_m)$ there exists a vertex v_k of the loop and a vertex u which does not belong to the loop such that (v_k, u) is an edge. It is shown in [10] that $\mathcal{G}$ satisfies (L) if and only if the associated groupoid is essentially free in the sense of [2], that is, the set of points of the unit space with the trivial isotropy group is dense in the unit space. Using this, they show a uniqueness theorem for the corresponding countably generated Cuntz-Krieger algebra.

Since we assume that no row and no column of D is identically zero, our condition (I)$'$ is equivalent to condition (L). In fact, suppose that (L) is not satisfied. Then there is a simple loop without an exit. Any vertex of the loop does not have a path to any vertex of Σ_0. Thus $\mathcal{G}$ does not satisfy (I)$'$. Conversely, suppose that (L) is satisfied. Since no row of D is identically zero, (L) implies that any vertex v has an edge (v, u) with $u \neq v$. Fix the vertex v. If there is an infinite path starting at v consisting of all different vertices, then there is nothing else to prove. Hence we may assume that there are no such paths. Using (L), we can choose a path from v to some vertex i such that there are at least two different loops based at i. Thus (I)$'$ is satisfied (see [10]).

We now show, following closely ideas of [5], how condition (I)$'$ implies that the Hilbert C^*-bimodule X satisfies a slightly modified version of (I)-freeness in [12]. This fact will enable us to get a proof of the uniqueness theorem that does not involve the theory of groupoids.

Lemma 3.2. *If $\mathcal{G}$ satisfies condition* (I)$'$*, then we have the following:*

(i) *For $i \in \Sigma$ and $k \in \mathbb{N}$ there exists an infinite path τ_i starting at i such that τ_i and its j-th translates are distinct, for $1 \leqslant j \leqslant k$.*

(ii) *Fix $n \in \mathbb{N}$. For vertices $i = 1, 2, \dots, n \in \Sigma$ and for $k \in \mathbb{N}$, there exist $r = r_{n,k} \in \mathbb{N}$ and finite paths τ_i^k starting at i $(i = 1, 2, \dots, n)$ with length r such that*

$$\psi_r^{-1}(t_i^k)\sigma^j(\psi_r^{-1}(t_i^k)) = 0 \quad \text{for } 1 \leqslant j \leqslant k,$$

where $t_i^k \in C_0(Y^r)$ is the characteristic function on one point set $\{\tau_i^k\}$.

Proof. (i) Recall the discussion in [5]. (ii) Just cut down the infinite path τ_i with a sufficient large length $r = r_{n,k}$ to get finite paths τ_i^k. ∎

Under above the situation, let $T_i^k = \psi_r^{-1}(t_i^k)$ and $Q_k = \sum_{i=1}^n T_i^k$. If $i \neq m$, then T_i^k and T_m^k are pairwise orthogonal. But we note there is the possibility that some T_i^k is equal to some $\sigma^j(T_m^k)$. Therefore we need to modify arguments of (I)-freeness in [12] a little bit to get over the trouble. We have the following modification of (I)-freeness in our case:

LEMMA 3.3. *Let $\mathcal{G}$ satisfy condition* (I)$'$. *Fix $n \in \mathbb{N}$. Then for $k \in \mathbb{N}$ the projections T_i^k $(i = 1, \dots, n)$ and Q_k defined as above satisfy the following properties:*

(i) *T_i^k is in $\mathcal{F}_{(r_{n,k}, r_{n,k})}$;*
(ii) *for $a \in A$ with $a(P_1 + \cdots + P_n) = a$, $a \mapsto aQ_k$ is completely isometric;*
(iii) *for $1 \leqslant j \leqslant k$, we have $T_i^k \sigma^j(T_i^k) = 0$.*

Proof. (i) is clear. (ii) follows from the fact that Q_k commutes with A and $P_i Q_k \neq 0$ for $i = 1, \dots, n$. (iii) is proved in Lemma 3.2. ∎

LEMMA 3.4. *Fix $n \in \mathbb{N}$ and $\ell \in \mathbb{N}$. Let T be a finite sum $\sum_i \theta_{x_i,y_i} \in \mathcal{K}_A(X_A^{\otimes \ell})$ for some $x_i, y_i \in (X^{\otimes \ell})(P_1 + \cdots + P_n)$. Then we have $\|T\sigma^\ell(Q_k)\| = \|T\|$.*

Proof. Use the formula

$$\left\| \left(\sum_{i=1} \theta_{x_i,y_i} \otimes 1_Y \right) \circ (1_X \otimes Q_k) \right\| = \left\| ((x_i|x_j)_A)_{ij}^{1/2} ((y_i|y_j)_A)_{ij}^{1/2} \operatorname{diag}(Q_k) \right\|$$

and Lemma 3.3 (ii) as in the proof of Theorem 4.3 of [12]. ∎

THEOREM 3.5. *Let D be a C^*-algebra, (V, ρ_A, D) a representation of the bimodule X in D such that $\rho_K(\phi(a)) = \rho_A(a)$, $a \in A$, where $\rho_K : K = \mathcal{K}_A(X_A) \to D$ is defined as in Lemma 2.2. Let $\varphi : \mathcal{O}_X \to D$ be the unique $*$-homomorphism such that $\varphi(S_x) = V_x$, $\varphi(\phi(a)) = \rho_A(a)$ and $\varphi(\pi_K(k)) = \rho_K(k)$ for $x \in X$, $a \in A$ and $k \in K = \mathcal{K}_A(X_A)$. Suppose that $a \to \rho_A(a) \in D$ is one-to-one. Then the restriction of φ to $\mathcal{F}^{(0)}$ is one-to-one. Moreover, if $\mathcal{G}$ satisfies* (I)$'$, *then φ is one-to-one.*

Proof. The first statement can be proved as in Theorem 4.3 (i) of [12].

Let us assume that $\mathcal{G}$ satisfies (I)$'$. Fix n and $\ell \in \mathbb{N}$. We first consider an element of the form:

$$B = \sum_{j=-k}^{k} B_j, \quad B_j \in \mathcal{K}_A\big(X_A^{\otimes \ell}, X_A^{\otimes(\ell+j)}\big)$$

where each B_j is a finite sum $\sum_i \theta_{x_i,y_i}$ with $x_i \in X^{\otimes(\ell+j)}$, $y_i \in X^{\otimes \ell}$, $x_i(P_1 + \cdots + P_n) = x_i$, $y_i(P_1 + \cdots + P_n) = y_i$. We choose projections $\{T_i^k\}_{i=1,\dots,n}$ and Q_k as in Lemmas 3.2 and 3.3. Let

$$B' = \sum_{i=1}^n \sigma^\ell(T_i^k) B \sigma^\ell(T_i^k) = \sum_{j=-k}^{k} B_j'.$$

Then

$$\varphi(B') = \varphi\Big(\sum_{i=1}^{n} \sigma^{\ell}(T_i^k)\Big(\sum_{j=-k}^{k} B_j\Big)\sigma^{\ell}(T_i^k)\Big) = \varphi\Big(\sum_{i=1}^{n} \sigma^{\ell}(T_i^k) B_0\Big)$$
$$= \varphi(\sigma^{\ell}(Q_k)B_0) = \varphi(B_0')$$

by Lemma 2.9 and Lemma 3.3 (iii). Now φ is faithful on the homogeneous subalgebra of $\mathcal{O}_X$, hence

$$\|\varphi(B_0)\| = \|B_0\| = \|\sigma^{\ell}(Q_k)B_0\| = \|B_0'\| = \|\varphi(B_0')\| = \|\varphi(B')\|.$$

Since $\{\sigma^{\ell}(T_i^k)\}_i$ is an orthogonal family, we have $\|\varphi(B')\| \leqslant \|\varphi(B)\|$. We have thus shown that $\|\varphi(B_0)\| \leqslant \|\varphi(B)\|$. Since $\mathcal{G}$ is locally finite and $X_0 = C_{00}(E)$ is dense in X, the set of elements B of this form, with n, k and ℓ ranging over $\mathbb{N}$, is norm dense in $\mathcal{O}_X$. Therefore we have the same inequality for any $B \in \mathcal{O}_X$. Thus there exists a conditional expectation from $\varphi(\mathcal{O}_X)$ to $\varphi(\mathcal{F}^{(0)})$. Using now well known arguments, we deduce that φ is faithful. ∎

The above theorem shows that the countably generated Cuntz-Krieger algebra $\mathcal{O}_X$ does not depend on the choice of the generators if $\mathcal{G}$ satisfies (I)$'$.

4. THE IDEAL STRUCTURE

We shall describe the ideal structure of $\mathcal{O}_X$ in terms of a certain class of ideals of A. A closed ideal J of A is called X-invariant if $(x|\phi(a)y)_A \in J$ for $x, y \in X$ and $a \in J$. For an X-invariant ideal, we define $J_X = \{a \in A; (x|\phi(a)y)_A \in J \text{ for } x, y \in X\}$, which is still a closed X-invariant ideal of A and contains J. We say that J is X-*saturated* if $J_X = J$.

Lemma 4.1. *Let $\mathcal{J}$ be a closed ideal of $\mathcal{O}_X$. Let $J = \mathcal{J} \cap A$. Then J is a closed X-invariant, X-saturated ideal of A.*

Proof. For $a \in J$, $x, y \in X$, $(x|ay)_A = x^*ay \in A \cap \mathcal{J} = J$. Assume that for every $x, y \in X$, $(x|ay)_A \in J$, that is, $S_{u_i}^* a S_{u_j} \in J$. Then $\sum_{i=1}^{N} S_{u_i} S_{u_i}^* a S_{u_j} \in \mathcal{J}$ for every j and $N \in \mathbb{N}$. Letting $N \to \infty$, the left hand side converges to $aS_{u_j} \in \mathcal{J}$ in the norm topology, since $\phi(a) \in \mathcal{K}_A(X_A)$. Thus $aS_{u_j} \in \mathcal{J}$. We have $a \in J$ by a similar argument. ∎

For each subset $S \subset \Sigma$, we put $J(S) = \{f \in A = c_0(\Sigma) \mid f|_{\Sigma \setminus S} = 0\}$. The map $S \to J(S)$ gives a bijective inclusion preserving correspondence between the lattice of subsets S of Σ and the lattice of closed two sided ideals of A. In particular, the notions of X-invariance and X-saturation have a clear interpretation, as stated in [16]. Recall from [4] and [11] that a subset S of Σ is called *hereditary* if all its vertices are connected only with vertices of S itself. An hereditary subset $S \subset \Sigma$ is called *saturated* if it contains any vertex with this property.

LEMMA 4.2. *The map $S \to J(S)$ above defined restricts to a bijective inclusion preserving correspondence from hereditary saturated subsets of Σ onto closed, X-invariant, X-saturated ideals of A.*

Let $J = J(S)$ be a closed X-invariant, X-saturated ideal of A. As in [12], we put

$$X_J = \{x \in X \mid (x|x)_A \in J\}.$$

Then by Proposition 10 in [12], X_J is a submodule of X and X/X_J is a Hilbert A/J bimodule. This bimodule is associated to a graph obtained from the original one by removing vertices in S and edges whose ranges are contained S. The matrix corresponding to this graph is the submatrix of the original matrix G whose rows and columns do not belong to S. It still has the property that no row or column is identically zero. ∎

PROPOSITION 4.3. *Let J be a closed X-invariant, X-saturated ideal of A. Let $\mathcal{J}$ be the closed ideal of $\mathcal{O}_X$ generated by J. Then we have $\mathcal{J} \cap A = J$.*

Proof. Clearly we have $J \subset \mathcal{J} \cap A$. $\mathcal{J}$ is the norm closure of the linear combinations of the set of products of $j \in J$ and S_x, $S_{x'}^*$'s. For $j \in J$, we have

$$jS_x = \lim_{N\to\infty} \sum_{i=1}^{N} S_{u_i} S_{u_i}^* j S_x.$$

Since J is X-invariant, jS_x is a limit in norm of elements of the form $\sum_{i=1}^{N'} S_{u_i} j_i$ where $j_i \in J$. This argument shows that $\mathcal{J}$ is the norm closure of the linear combinations of the elements $\{S_{x_1} \cdots S_{x_r} j S_{y_s}^* \cdots S_{y_1}^*\}$.

We have $\mathcal{J} \cap A \subset \mathcal{J}^{(0)} \cap A$, where $\mathcal{J}^{(0)}$ is gauge invariant subalgebra in $\mathcal{J}$. Let $T \in \mathcal{J}^{(0)}$. Since T is gauge invariant, T is contained in the norm closure of the linear span of elements of the form $\sum_{i=1}^{N} S_{x_1^i} \cdots S_{x_{r_i}^i} j^i S_{y_{r_i}^i}^* \cdots S_{y_1^i}^*$. Thus we have

$$\mathcal{J}^{(0)} = \text{ closure of } \bigcup_{r=1}^{\infty} X^{\otimes r} J (X^{\otimes r})^*$$

where $I_r := X^{\otimes r} J (X^{\otimes r})^*$ is the closure in norm of linear combinations of the form $\left\{ \sum_{\text{finite}} S_{x_1} \cdots S_{x_r} j S_{y_r}^* \cdots S_{y_1}^* \right\}$. Here we note that I_r is a closed ideal in $\mathcal{F}_{(r,r)}$ and also that $I_r \subset I_{r+1}$, since J is X-invariant.

We show that $\mathcal{J}^{(0)} \cap A = J$. Let π_r be the composition of the map $A \to \mathcal{F}_{(r,r)}$ with $\mathcal{F}_{(r,r)} \to \mathcal{F}_{(r,r)}/I_r$. Let π_∞ be the composition of the map $A \to \mathcal{F}^{(0)}$ with $\mathcal{F}^{(0)} \to \mathcal{F}^{(0)}/\mathcal{J}^{(0)}$. Then $\ker \pi_r = A \cap I_r$ and $\ker \pi_\infty = A \cap \mathcal{J}^{(0)}$. Therefore there are obvious monomorphisms:

$$A/A \cap I_r \to \mathcal{F}_{(r,r)}/I_r, \quad A/A \cap \mathcal{J}^{(0)} \to \mathcal{F}^{(0)}/\mathcal{J}^{(0)}.$$

In general, if J is a closed ideal of a C^*-algebra B and $a \in B$, we denote by $[a]_{B/J}$ the quotient image of a in B/J.

For $a \in A$,

$$\|[a]_{A/A\cap\mathcal{J}^{(0)}}\| = \|[a]_{\mathcal{F}^{(0)}/\mathcal{J}^{(0)}}\| = \operatorname{dist}(a, \mathcal{J}^{(0)}) = \lim_{r\to\infty} \operatorname{dist}(a, I_r)$$
$$= \lim_{r\to\infty} \|[a]_{\mathcal{F}_{(r,r)}/I_r}\| = \lim_{r\to\infty} \|[a]_{A/A\cap I_r}\|.$$

We now show that for all $r \in \mathbb{N}$, $A \cap I_r = J$. For $b \in X^{\otimes r} J (X^{\otimes r})^*$, we may write b in the form

$$b = \lim_{N\to\infty} \sum_{i=1}^{N} S_{x_1^i} \cdots S_{x_r^i} j^i S^*_{y_r^i} \cdots S^*_{y_1^i}$$

where $j^i \in J$ for any i. Since J is norm closed and X-invariant we have that $x'^* b y' \in J$ for any pair of elements x' and y' in $X^{\otimes r}$. Then for any pair of elements x'' and y'' in $X^{\otimes r-1}$, x_1, y_1 in X, we have that $x_1^*(x''^* b y'')y_1 \in J$. Since J is X-saturated, $x''^* b y'' \in J_X = J$. Repeating the argument a finite number of steps, we deduce that $b \in J$.

It follows that

$$\|[a]_{A/A\cap\mathcal{J}^{(0)}}\| = \|[a]_{A/J}\|,$$

therefore $J = A \cap \mathcal{J}^{(0)}$. ∎

This proposition shows that the map $\mathcal{J} \mapsto \mathcal{J} \cap A$ is a surjection from the set of ideals in $\mathcal{O}_X$ to the set of X-invariant, X-saturated ideals in A.

Theorem 4.4. *If $\mathcal{G}$ satisfies* (I)$'$, *$\mathcal{O}_X$ is simple if and only if there are no proper hereditary, saturated subsets in Σ.*

Proof. If there exist no proper X-invariant, X-saturated ideal in A, then by Theorem 3.5 every non-trivial $*$-representation of $\mathcal{O}_X$ is faithful. If there exists a proper X-invariant, X-saturated ideal J in A, then by Proposition 4.3 the closed two sided ideal $\mathcal{J}$ generated by J is not equal to $\mathcal{O}_X$. ∎

In [11], the authors prove a theorem of this type under a condition, called (K), stronger than (I)$'$. We remark as in [16] that there is no proper hereditary, saturated subset in Σ if and only if the graph $\mathcal{G}$ is cofinal in the sense of [11], that is, for every vertex v and every infinite path $(u_0, u_1, \ldots)$, there exists a finite path $(t_0, \ldots, t_k)$ such that $t_0 = v$ and $t_k = u_r$.

In order to classfy the closed ideals in $\mathcal{O}_X$, we need an analogue of condition (II) in [4] which enables us to apply (and extend) an argument of our previous paper ([12]). For a subset S of the vertex set Σ of $\mathcal{G}$, we denote by $\mathcal{G} \setminus S$ the graph obtained from $\mathcal{G}$ by removing S and edges whose sources or ranges are contained in S.

Definition 4.5. $\mathcal{G}$ satisfies *condition* (II)$'$ if for every hereditary, saturated subset S of Σ, the graph $\mathcal{G} \setminus S$ satisfies (I)$'$.

Definition 4.6. ([11]) $\mathcal{G}$ satisfies *condition* (K) if there exist no vertices v for which there is precisely one loop based at v.

We show that condition (K) is equivalent to our condition (II)$'$. In fact, assume that (K) is satisfied. Let S be a hereditary, saturated subset of Σ. Fix a vertex $v \in \Sigma \setminus S$. Consider first the case that there exist at least two different loops in $\mathcal{G}$ based at v. Since S is hereditary, these two loops are in $\mathcal{G} \setminus S$. And nothing else needs to be proven. Hence by (K) we may assume that there exists no loop based at v. Since no row of D is zero and S is saturated, there is $v_1 \in \Sigma \setminus S$ such that $v_1 \neq v$ and (v, v_1) is an edge. If there exist two different loops based at v_1, nothing else needs to be proven. Otherwise we can similarly choose another $v_2 \in \Sigma \setminus S$ different from v and v_1 such that (v, v_1, v_2) is a path in $\mathcal{G}$. Iterating this argument, we deduce that in $\mathcal{G} \setminus S$ either there is a path from v to some v_n satisfying that there are two different loops based at v_n, or there is an infinite path $(v, v_1, v_2, \ldots)$ consisting of all different vertices. Hence (II)$'$ is satisfied.

Conversely, assume that (K) is not satisfied. Then there is a vertex v for which there is precisely one loop ℓ based at v. If the loop ℓ has no exits, then (L), which is equivalent to (I)$'$, is not satisfied. Thus neither is (II)$'$. Hence we may assume that the loop ℓ has an exit. Let $Z = \{e_1, \ldots, e_l\} \subset \Sigma$ be the set of exits from the loop ℓ. Let T be the set of $u \in \Sigma$ such that there is a path from some $e_i \in Z$ to u and u is not on the loop ℓ. Then T is hereditary. Let S be the saturation of T. Then S is a hereditary, saturated subset of Σ that consists of those $i \in \Sigma$ for which there is $n \in \mathbb{N}$ such that every path with source i and length n has range in T. One can easily show that the loop ℓ and S are still disjoint. Therefore ℓ is a loop in $\mathcal{G} \setminus S$ with no exit. Hence $\mathcal{G} \setminus S$ does not satisfy (I)$'$. Thus (II)$'$ is not satisfied.

Proposition 4.7. *Assume that $\mathcal{G}$ satisfies* (II)$'$. *Then every ideal $\mathcal{J}$ of $\mathcal{O}_X$ is gauge invariant and generated by the intersection $J = \mathcal{J} \cap A$.*

Proof. By Lemma 4.1, J is an X-invariant and X-saturated ideal of A. We define X_J as before. Then there exists a representation of the Hilbert A/J-bimodule X/X_J in $\mathcal{O}_X/\mathcal{J}$ satisfying (2.1), and the representation of A/J is faithful. Note that X/X_J satisfies condition (I)$'$, so this representation extends to a $*$-isomorphim from $\mathcal{O}_{X/X_J}$ to $\mathcal{O}_X/\mathcal{J}$ by Theorem 3.5.

In particular, the gauge action $\beta : \mathbb{T} \to \operatorname{Aut} \mathcal{O}_{X/X_J}$ induces an action $\alpha : \mathbb{T} \to \operatorname{Aut} \mathcal{O}_X/\mathcal{J}$ such that for $z \in \mathbb{T}$ and $x \in X$, $\alpha_z([S_x]_{\mathcal{O}_X/\mathcal{J}}) = z[(S_x)]_{\mathcal{O}_X/\mathcal{J}}$. Therefore, if γ denote the gauge action of $\mathbb{T}$ on $\mathcal{O}_X$, for $T \in \mathcal{O}_X$, $\alpha_z([T]_{\mathcal{O}_X/\mathcal{J}}) = [\gamma_z(T)]_{\mathcal{O}_X/\mathcal{J}}$. This shows that $\mathcal{J}$ is gauge invariant, and therefore generated by the subspaces $\mathcal{J} \cap \mathcal{F}_{(r,s)}$.

We show now that $\mathcal{J} \cap \mathcal{F}_{(s,r)} = X^{\otimes r} J X^{\otimes s *}$. Let $T \in \mathcal{J} \cap \mathcal{F}_{(s,r)}$. Then

$$\begin{aligned} T &= \lim \Bigg(\sum_{i_1, \ldots, i_r} S_{u_{i_1}} \cdots S_{u_{i_r}} S_{u_{i_r}}^* \cdots S_{u_{i_1}}^* \Bigg) T \Bigg(\sum_{j_1, \ldots, j_s} S_{u_{j_1}} \cdots S_{u_{j_s}} S_{u_{j_s}}^* \cdots S_{u_{j_1}}^* \Bigg) \\ &= \lim \sum_{\substack{i_1, \ldots, i_r \\ j_1, \ldots, j_s}} S_{u_{i_1}} \cdots S_{u_{i_r}} T^{i_1, \ldots, i_r, j_1, \ldots, j_s} S_{u_{j_s}}^* \cdots S_{u_{j_1}}^*, \end{aligned}$$

where all limits are taken with respect to the norm topology. We have

$$T^{i_1, \ldots, x_r, j_1, \ldots, j_s} = S_{u_{i_r}}^* \cdots S_{u_{i_1}}^* T S_{u_{j_1}} \cdots S_{u_{j_s}} \in A \cap \mathcal{J} = J.$$

Then we have $T \in X^{\otimes r} J (T^{\otimes s})^*$. The reverse inclusion $X^{\otimes s} J X^{\otimes r *} \subseteq \mathcal{J} \cap \mathcal{F}_{(r,s)}$ is obvious. ∎

We have thus proved the following result:

Theorem 4.8. *Assume that $\mathcal{G}$ satisfies* (II)$'$. *Then the lattice of closed ideals of $\mathcal{O}_X$ is isomorphic to the lattice of X-invariant, X-saturated ideals of A. And it is also isomorphic to the lattice of hereditary, saturated subsets of Σ.*

5. PURE INFINITENESS

Kumjian, Pask and Raeburn obtained a nice condition on an infinite locally finite graph $\mathcal{G}$ in order that the associated Cuntz-Krieger algebra be purely infinte. Their strategy is based on the realization of the algebra as a groupoid C^*-algebra and on the observation that their graph theoretic condition makes it possible to reduce to a result in [2].

In this section we shall give an alternative proof of the same result which does not involve the theory of groupoid C^*-algebras. Instead, we shall use an argument essentially due to [4] and [2]. Recall that a C^*-algebra is called *purely infinite* if every non-zero hereditary C^*-subalgebra contains an infinite projection.

Theorem 5.1. ([10]) *If a graph $\mathcal{G}$ satisfies condition* (L) *and every vertex of $\mathcal{G}$ connects to a loop, then $\mathcal{O}_X$ is purely infinite.*

Proof. Let $E : \mathcal{O}_X \to \mathcal{F}^{(0)}$ be the canonical conditional expectation onto the fixed point algebra under the gauge action. For every nonzero hereditary C^*-subalgebra $\mathcal{H}$ of $\mathcal{O}_X$, choose a non-zero positive element $T \in \mathcal{H}$. We may suppose that $\|E(T)\| = 1$. Then there exist $n \in \mathbb{N}$, $\ell \in \mathbb{N}$, $k \in \mathbb{N}$ and a positive $B \in {}^0\mathcal{O}_X$ satisfying

$$B = \sum_{j=-k}^{k} B_j, \quad B_j \in \mathcal{K}_A(X_A^{\otimes \ell}, X_A^{\otimes(\ell+j)})$$

and $\|T - B\| < \frac{1}{4}$, where each B_j is a finite sum $\sum_i \theta_{x_i,y_i}$ with $x_i \in X^{\otimes(\ell+j)}$, $y_i \in X^{\otimes \ell}$, $x_i(P_1 + \cdots + P_n) = x_i$, $y_i(P_1 + \cdots + P_n) = y_i$. Put $B_0 = E(B)$; then $\|B_0\| \geqslant \frac{3}{4}$. Since B_0 is in a finite dimensional C^*-subalgebra of $\mathcal{K}_A(X^{\otimes \ell})$, there exists a spectral projection $e \in \mathcal{K}_A(X^{\otimes \ell})$ of B_0 corresponding to the maximum eigenvalue of B_0. Then $eB_0e \geqslant \frac{3}{4}e$. Choose T_i^k and Q_k as in Lemmas 3.2 and 3.3. Since $\|e\sigma^\ell(Q_k)\| = \|e\|$ by Lemma 3.4, there is some i with $e\sigma^\ell(T_i^k) \neq 0$. By Lemma 2.9, e commutes with $R = \sigma^\ell(T_i^k)$. Hence $f = eR$ is a non-zero projecion in $\mathcal{K}_A(X_A^{\otimes m})$ for some m. Since $RBR = RB_0R$, we have $fBf = fB_0f$. We also have $fB_0f \geqslant \frac{3}{4}f$. Since for any vertex i there is a path $(v_0, v_1, \ldots, v_r)$ starting at $i = v_0$ and connecting to a loop $(v_r, v_{r+1}, \ldots, v_s, v_r)$ with an exit, there is a non-zero partial isometry V_i with $V_iV_i^* \lneq V_i^*V_i \leqslant P_i$. In fact, put $V_i = S_{v_0} \cdots S_{v_{r-1}}(S_{v_r} \cdots S_{v_s}P_{v_r})S_{v_{r-1}}^* \cdots S_{v_0}^*$. Since $\{S_\alpha S_\beta^* \mid \alpha, \beta$ finite paths with edge-lengths $m\}$ is a system of matrix units of $\mathcal{K}_A(X^{\otimes m})$, every projection in $\mathcal{K}_A(X^{\otimes m})$ is equivalent to a finite sum of $S_\alpha S_\alpha^*$. And the projection $S_\alpha S_\alpha^*$ is equivalent to $S_\alpha^* S_\alpha \in A$, which is a finite sum of $\{P_i \mid i \in \Sigma\}$. Therefore we can find a projection p and a partial isometry W with $W^*W = p$, $WW^* \lneq p$, and

$p \leqslant f$. At this point the rest of the proof continues exactly as in Proposition 2.4 of [2]. In fact we have

$$W^*BW = W^*fB_0fW \geqslant \frac{3}{4}W^*fW \geqslant \frac{3}{4}p$$

and

$$W^*TW \geqslant W^*BW - \frac{1}{4}p \geqslant \frac{p}{2}.$$

It follows that W^*TW is invertible in $p\mathcal{O}_Xp$. We denote by C its inverse and we put $U = C^{1/2}W^*T^{1/2}$. We have $UU^* = p$ and $U^*U \leqslant \|C\|T$. Therefore U^*U is an infinite projection in $\mathcal{H}$. ∎

6. COMPUTATION OF K-GROUPS

We can apply directly the following 6-term exact sequence due to Pimsner ([15]) for the computation of the K-theory of $\mathcal{O}_X$. In this way we avoid a long proof in [14]. We denote by $[X]$ the element in $\mathrm{KK}(A, A)$ determined by X. As in [15], $[X]$ gives a canonical endomorphism of $\mathrm{K}_*(A)$, where $* = 0, 1$.

PROPOSITION 6.1. ([15]) *With the notation as above, we have the following* 6*-term exact sequence.*

$$\begin{array}{ccccc}
\mathrm{K}_0(A) & \stackrel{i-[X]}{\longrightarrow} & \mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(\mathcal{O}_X) \\
\uparrow & & & & \downarrow \\
\mathrm{K}_1(\mathcal{O}_X) & \longleftarrow & \mathrm{K}_1(A) & \stackrel{i-[X]}{\longleftarrow} & \mathrm{K}_1(A)
\end{array}$$

The 6-term exact sequence for countably generated Cuntz-Krieger algebras had been given in [14] under some condition called (J) using a different technique. We simply apply Pimsner's formula as above and get the following:

$$\begin{array}{ccccc}
\bigoplus_{i=1}^{\infty}\mathbb{Z} & \stackrel{I-G^{\mathrm{t}}}{\longrightarrow} & \bigoplus_{i=1}^{\infty}\mathbb{Z} & \longrightarrow & \mathrm{K}_0(\mathcal{O}_X) \\
\uparrow & & & & \downarrow \\
\mathrm{K}_1(\mathcal{O}_X) & \longleftarrow & \{0\} & \longleftarrow & \{0\}
\end{array}$$

In fact, $[X] : \mathrm{K}_0(A) \ni [AP_i] \mapsto [AP_i \bigotimes_A X] = \bigoplus_{\{j \mid G(i,j)=1\}} [AP_j]$.

Therefore we immediately get the K-theory of countably generated Cuntz-Krieger algebras.

PROPOSITION 6.2.

$$\mathrm{K}_1(\mathcal{O}_X) \simeq \mathrm{Ker}(I - G^{\mathrm{t}}), \quad \mathrm{K}_0(\mathcal{O}_X) \simeq \bigoplus_{i=1}^{\infty}\mathbb{Z}/\,\mathrm{Im}(I - G^{\mathrm{t}}).$$

The first and third authors are supported by the Grant-in aid for Scientific Research of JSPS and the second author is supported by the EU and NATO-CNR.

REFERENCES

1. B. Abadie, S. Eilers, R. Exel, Morita equivalence for crossed products by Hilbert C^*-bimodules, *Trans. Amer. Math. Soc.* **350**(1998), 3043–3054.
2. C. Anantharaman-Delaroche, Purely infinite C^*-algebras arising from dynamical systems, *Bull. Soc. Math. France* **125**(1997), 199–225.
3. A. Connes, *Noncommutative Differential Geometry*, Academic Press, 1994.
4. J. Cuntz, A class of C^*-algebras and topological Markov chains. II: Reducible chains and the Ext-functor for C^*-algebras, *Invent. Math.* **63**(1981), 25–40.
5. J. Cuntz, W. Krieger, A class of C^*-algebras and topological Markov chains, *Invent. Math.* **36**(1980), 251–268.
6. S. Doplicher, C. Pinzari, R. Zuccante, The C^*-algebra of a Hilbert bimodule, *Boll. Un. Mat. Ital. A (7)* **8**(1988), 263–281.
7. S. Doplicher, J.E. Roberts, Duals of compact Lie groups realized in the Cuntz algebras and their actions on C^*-algebras, *J. Funct. Anal.* **74**(1987), 96–120.
8. S. Doplicher, J.E. Roberts, A new duality theory for compact groups, *Invent. Math.* **98**(1989), 157–218.
9. Y. Katayama, Generalized Cuntz algebra $\mathcal{O}_N^M$, *R.I.M.S. Kokyuroku* **858**(1994), 87–90.
10. A. Kumjian, D. Pask, I. Raeburn, Cuntz-Krieger algebras of directed graphs, *Pacific J. Math.* **184**(1998), 161–174.
11. A. Kumjian, D. Pask, I. Raeburn, J. Renault, Graphs, groupoids and Cuntz-Krieger algebras, *J. Funct. Anal.* **144**(1997), 505–541.
12. T. Kajiwara, C. Pinzari, Y. Watatani, Ideal structure and simplicity of the C^*-algebras generated by Hilbert bimodules, *J. Funct. Anal.* **159**(1998), 295–322.
13. P.S. Muhly, B. Solel, On the simplicity of some Cuntz-Pimsner algebras, *Math. Scand.* **83**(1998), 53–73.
14. D. Pask, I. Raeburn, K-group of countable Cuntz-Krieger algebras, *Publ. Res. Inst. Math. Sci.* **32**(1996), 415–443.
15. M. Pimsner, A class of C^*-algebras generating both Cuntz-Krieger algebras and crossed products by **Z**, *Free Probability Theory*, Fields Inst. Comm. **12** (1997), 189–212.
16. C. Pinzari, The ideal structure of Cuntz-Krieger-Pimsner algebras and Cuntz-Krieger algebras over infinite matrices, in *Operator Algebras and Quantum Field Theory*, International Press, 1997, pp. 136–150.
17. J. Renault, *A Groupoid Approach to C^*-Algebras*, Lecture Notes Math., vol. 793, Springer-Verlag, 1980.

TSUYOSHI KAJIWARA
Department of Environmental and Mathematical Sciences
Okayama University
Tsushima, Okayama 700
JAPAN

E-mail: kajiwara@math.ems.okayama-u.ac.jp

CLAUDIA PINZARI
Dipartimento di Matematica
Università di Roma "Tor Vergata"
00133 Roma
ITALY

E-mail: pinzari@axp.mat.uniroma2.it

YASUO WATATANI
Graduate School of Mathematics
Kyushu University
Ropponmatsu, Fukuoka 810
JAPAN

E-mail: watatani@math.kyushu-u.ac.jp

Received March 3, 1998.