J. OPERATOR THEORY
45(2001), 53–80

UNBOUNDED C^*-SEMINORMS AND UNBOUNDED C^*-SPECTRAL ALGEBRAS

SUBHASH J. BHATT, ATSUSHI INOUE and HIDEKAZU OGI

Communicated by Şerban Strătilă

ABSTRACT. Several $*$-algebras $\mathcal{A}$ carry with them unbounded C^*-seminorms in the sense that they are C^*-seminorms defined on $*$-subalgebras. Unbounded operator representations of $\mathcal{A}$ are constructed from such unbounded C^*-seminorms and they are investigated. The notions of spectrality and stability of unbounded C^*-seminorms are defined and studied.

KEYWORDS: *(Hereditary) spectral unbounded C^*-seminorms, unbounded $*$-representations, stable unbounded C^*-seminorms.*

MSC (2000): 47L60, 46H25, 46K10.

1. INTRODUCTION

Unbounded C^*-seminorms on $*$-algebras in the sense that they are C^*-seminorms defined on $*$-subalgebras have appeared in many mathematical and physical subjects (for example, locally convex $*$-algebras in [5]–[8] and [18], and the quantum field theory in [1], [14] and [32] etc.). But this systematical study has not yet done sufficiently. The main purpose of this paper is to do a systematical study of unbounded C^*-seminorms and to apply it to a study of unbounded $*$-representations and that of locally convex $*$-algebras.

The paper is organized as follows: In Section 2 we construct unbounded $*$-representations of a $*$-algebra from unbounded C^*-seminorms and investigate them. Let $\mathcal{A}$ be a $*$-algebra. Let p be a C^*-seminorm defined on $\mathcal{A}$. Every $*$-representation of the Hausdorff completion of $(\mathcal{A}, p)$ gives rise to a $*$-representation of $\mathcal{A}$ into bounded Hilbert space operators. However, there are a number of situations in which natural C^*-seminorms are defined on $*$-subalgebras of $\mathcal{A}$. Then they should lead to unbounded operator representations of $\mathcal{A}$. An *unbounded m^*-(respectively C^*-) seminorm* is a submultiplicative $*$-(respectively C^*-) seminorm p defined on a $*$-subalgebra $\mathcal{D}(p)$ of $\mathcal{A}$. Then $N_p := \{x \in \mathcal{D}(p) : p(x) = 0\}$ is a $*$-ideal of $\mathcal{D}(p)$ and $\mathfrak{N}_p := \{x \in \mathcal{D}(p) : ax \in \mathcal{D}(p), \forall a \in \mathcal{A}\}$ is a left ideal of $\mathcal{A}$.

It is shown that any faithful nondegenerate $*$-representation $\Pi_p : \mathcal{A}_p \to \mathcal{B}(\mathcal{H})$ of the C^*-algebra $\mathcal{A}_p$ obtained by the Hausdorff completion of $(\mathcal{D}(p), p)$ leads to an unbounded $*$-representation π_p of $\mathcal{A}$ such that $\|\overline{\pi_p(x)}\| \leqslant p(x)$ for all $x \in \mathcal{D}(p)$. But, π_p is not necessarily nontrivial (that is, $\mathcal{H}_{\pi_p} \neq \{0\}$), and π_p is nontrivial if and only if $\mathfrak{N}_p \not\subset N_p$. We assume that an unbounded C^*-seminorm satisfies the condition $\mathfrak{N}_p \not\subset N_p$. Then π_p is always strongly nondegenerate. Here we say that a $*$-representation π is strongly nondegenerate if there exists a left ideal $\mathcal{I}$ of $\mathcal{A}$ contained in $\mathcal{A}_\flat^\pi := \{x \in \mathcal{A} : \pi(x) \text{ is bounded}\}$, such that $[\overline{\pi(\mathcal{I})}\mathcal{H}_\pi] = \mathcal{H}_\pi$, where $[\mathcal{K}]$ denotes the closed linear span of a subset $\mathcal{K}$ of a Hilbert space. We denote by $\mathrm{Rep}(\mathcal{A}, p)$ the set of all such $*$-representations π_p of $\mathcal{A}$. In order to investigate representations in $\mathrm{Rep}(\mathcal{A}, p)$ in details, we introduce the notions of nondegenerate, finite, uniformly semifinite, semifinite and weakly semifinite unbounded C^*-seminorms, and show that if p is weakly semifinite or semifinite, then there exists a strongly nondegenerate $*$-representation π_p in $\mathrm{Rep}(\mathcal{A}, p)$ such that $\|\overline{\pi_p(x)}\| = p(x)$ for all $x \in \mathcal{D}(p)$. Such a π_p is called *well-behaved*. In Section 3 we consider the converse direction of Section 2. We construct an unbounded C^*-seminorm r_π on $\mathcal{A}$ from a strongly nondegenerate $*$-representation π of $\mathcal{A}$ and a natural well-behaved representation $\pi_{r_\pi}^N$ of $\mathcal{A}$ constructed from r_π which is the restriction of the closure $\widetilde{\pi}$ of π. Further, it is shown that if p is a weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$ and π_p is any well-behaved $*$-representation, then r_{π_p} is a maximal extension of p. In Section 4 we define and characterize the notion of regular unbounded C^*-seminorms. An unbounded C^*-seminorm on a $*$-algebra $\mathcal{A}$ is *regular* if it is a restriction of the unbounded C^*-seminorm $\sup_\alpha p_\alpha$ defined by a family $\{p_\alpha\}$ of C^*-seminorms on $\mathcal{A}$. It is shown that given a semifinite unbounded C^*-seminorm p on $\mathcal{A}$, p is regular if and only if there exists a well-behaved $*$-representation π_p of $\mathcal{A}$ which is a restriction of the direct sum $\bigoplus_\alpha \pi_\alpha$ of bounded $*$-representations π_α of $\mathcal{A}$.

In Section 5 we construct the unbounded Gelfand-Naimark C^*-seminorm $|\cdot|_p$ on $\mathcal{A}$ from an unbounded m^*-seminorm p on $\mathcal{A}$. Yood ([33]) has investigated some aspects of bounded C^*-seminorms by re-examining the construction of Gelfand-Naimark pseudo-norm discussed in [9]. Here we extend some of Yood's results about C^*-seminorms to unbounded C^*-seminorms. In Section 6 we apply the results developed earlier to the study of spectral algebras. Following Palmer ([22]) a *spectral algebra* $\mathcal{A}$ is an algebra on which there is defined a submultiplicative seminorm p (called a *spectral seminorm*) such that $\{x \in \mathcal{A} : \ p(x) < 1\} \subset \mathcal{A}^{\mathrm{qr}}$(= the set of all quasi-regular elements of $\mathcal{A}$). The morale of [22] and [23] is that even though a spectral algebra need not be normable, it is rich enough to recapture the pure algebraic flavour of much of the spectral theory of Banach algebras. We call an unbounded m^*-seminorm p to be *spectral* (respectively *hereditary spectral*) if $\{x \in \mathcal{D}(p) : p(x) < 1\} \subset \mathcal{D}(p)^{\mathrm{qr}}$ (respectively $p\lceil\mathcal{B}$ is spectral for each $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$). An unbounded $*$-representation π of $\mathcal{A}$ is a *spectral $*$-representation* (respectively a *hereditary spectral $*$-representation*) if $\mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x) \subset \mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \bigcup \{0\}$ for all $x \in \mathcal{A}, C^*(\pi)$ being the C^*-algebra generated by $\overline{\pi(\mathcal{A}_\flat^\pi)}$ (respectively $\pi\lceil\mathcal{B}$ is spectral for each $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$). It is shown that there exists a strongly nondegenerate $*$-representation π of $\mathcal{A}$ such that $\pi_\flat := \pi\lceil\mathcal{A}_\flat^\pi$ is (hereditary) spectral if and only if there exists a maximal, weakly semifinite, (hereditary) spectral unbounded C^*-seminorm on $\mathcal{A}$. Further, we define the notion of stability of unbounded m^*- (or C^*-) seminorms and characterize it by spectral unbounded C^*-seminorms. An unbounded m^*-seminorm p on $\mathcal{A}$ is called *stable* if for any $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$, any $*$-representation π of $\mathcal{B}$ such that $\mathcal{B}\bigcap\mathcal{D}(p) \subset \mathcal{B}_\flat^\pi$ and $[\pi(\mathcal{B}\bigcap\mathcal{D}(p))\mathcal{D}(\pi)] = \mathcal{H}_\pi$ can be dilated to a $*$-representation ϱ of $\mathcal{A}$ such that $\mathcal{D}(p) \subset \mathcal{A}_\flat^\varrho$ and $[\varrho(\mathcal{D}(\varrho))\mathcal{D}(\varrho)] = \mathcal{H}_\varrho$. It is shown that a semifinite unbounded C^*-seminorm on $\mathcal{A}$ is hereditary spectral if and only if it is spectral and stable. In Section 7 we give some examples of (regular, spectral, weakly semifinite, semifinite) unbounded C^*-seminorms on special $*$-algebras (locally m-convex $*$-algebras, pro-C^*-algebras, M*-like (or C^*-like) locally convex $*$-algebras, Köthe sequence algebras, O*-algebras). Throughout this paper we assume that a $*$-algebra $\mathcal{A}$ has always an identity $\mathbb{1}$ to simplify the arguments. This assumption does not lose the generality.

2. REPRESENTATIONS INDUCED BY UNBOUNDED C^*-SEMINORMS

In this section we construct a family of $*$-representations of a $*$-algebra $\mathcal{A}$ induced by an unbounded C^*-seminorm on $\mathcal{A}$ and investigate the properties. We begin with the review of (unbounded) $*$-representations of $\mathcal{A}$. Throughout this section let $\mathcal{A}$ be a $*$-algebra with identity $\mathbb{1}$. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$ and let $\mathcal{L}^\dagger(\mathcal{D})$ denote the set of all linear operators X in $\mathcal{H}$ with the domain $\mathcal{D}$ for which $X\mathcal{D} \subset \mathcal{D}$, $\mathcal{D}(X^*) \supset \mathcal{D}$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra under the usual operations and the involution $X \to X^\dagger := X^*\lceil\mathcal{D}$. A $*$-subalgebra of the $*$-algebra $\mathcal{L}^\dagger(\mathcal{D})$ is said to be an O**-algebra on* $\mathcal{D}$ *in* $\mathcal{H}$. A $*$*-representation* π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with a domain $\mathcal{D}$ is a $*$-homomorphism of $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D})$ and $\pi(\mathbb{1}) = I$, and then we write $\mathcal{D}$ and $\mathcal{H}$ by $\mathcal{D}(\pi)$ and $\mathcal{H}_\pi$, respectively. Let π_1 and π_2 be $*$-representations of $\mathcal{A}$. If $\mathcal{H}_{\pi_1}$ is a closed subspace of $\mathcal{H}_{\pi_2}$ and $\pi_1(x) \subset \pi_2(x)$ for each $x \in \mathcal{A}$, then π_2 is said to be an *extension* of π_1 and denoted by $\pi_1 \subset \pi_2$. In particular, if $\pi_1 \subset \pi_2$ and $\mathcal{H}_{\pi_1} = \mathcal{H}_{\pi_2}$, then π_2 is said to be an *extension of* π_1 *in the same Hilbert space.* Let π be a $*$-representation of $\mathcal{A}$. If $\mathcal{D}(\pi)$ is complete with the graph topology t_π defined by the family of seminorms $\{\|\cdot\|_{\pi(x)} := \|\cdot\| + \|\pi(x)\cdot\| : x \in \mathcal{A}\}$, then π is said to be *closed*. It is well known that π is closed if and only if $\mathcal{D}(\pi) = \bigcap_{x\in\mathcal{A}} \mathcal{D}(\overline{\pi(x)})$. The *closure* $\widetilde{\pi}$ of π is defined by

$$\mathcal{D}(\widetilde{\pi}) = \bigcap_{x\in\mathcal{A}} \mathcal{D}(\overline{\pi(x)}) \quad \text{and} \quad \widetilde{\pi}(x)\xi = \overline{\pi(x)}\xi \quad \text{for } x \in \mathcal{A},\ \xi \in \mathcal{D}(\widetilde{\pi}).$$

Then $\widetilde{\pi}$ is the smallest closed extension of π. The *weak commutant* $\pi(\mathcal{A})'_{\mathrm{w}}$ of π is defined by

$$\pi(\mathcal{A})'_{\mathrm{w}} = \{C \in \mathcal{B}(\mathcal{H}_\pi) : C\pi(x)\xi = \pi(x^*)^*C\xi,\ \forall x \in \mathcal{A},\ \forall \xi \in \mathcal{D}(\pi)\},$$

where $\mathcal{B}(\mathcal{H}_\pi)$ is the set of all bounded linear operators on $\mathcal{H}_\pi$, and it is a weakly closed $*$-invariant subspace of $\mathcal{B}(\mathcal{H}_\pi)$, but it is not necessarily an algebra. It is known that $\pi(\mathcal{A})'_{\mathrm{w}}\mathcal{D}(\pi) \subset \mathcal{D}(\pi)$ if and only if $\pi(\mathcal{A})'_{\mathrm{w}}$ is a von Neumann algebra and $\overline{\pi(x)}$ is affiliated with the von Neumann algebra $(\pi(\mathcal{A})'_{\mathrm{w}})'$ for each $x \in \mathcal{A}$. For more details we refer to [16], [19], [26] and [29].

DEFINITION 2.1. A mapping p of a subspace $\mathcal{D}(p)$ of $\mathcal{A}$ into $\mathbb{R}^+ = [0, \infty)$ is said to be an *unbounded (semi)norm* on $\mathcal{A}$ if it is a (semi)norm on $\mathcal{D}(p)$, and p is said to be an *unbounded m^*- (respectively C^*-) (semi)norm* on $\mathcal{A}$ if $\mathcal{D}(p)$ is a $*$-subalgebra of $\mathcal{A}$ and p is a submultiplicative $*$- (respectively C^*-) (semi)norm on $\mathcal{D}(p)$.

By [31], if a seminorm p on a $*$-algebra $\mathcal{A}$ is a C^*-seminorm, that is, it satisfies the C^*-property $p(x^*x) = p(x)^2$, $\forall x \in \mathcal{A}$, then it is a m^*-seminorm on $\mathcal{A}$, that is, $p(x^*) = p(x)$ and $p(xy) \leqslant p(x)p(y)$ for $\forall x, y \in \mathcal{A}$.

Let p be an unbounded C^*-seminorm on $\mathcal{A}$. We put

$$N_p = \{x \in \mathcal{D}(p) : p(x) = 0\} \quad \text{and} \quad \mathfrak{N}_p = \{x \in \mathcal{D}(p) : ax \in \mathcal{D}(p),\ \forall a \in \mathcal{A}\}.$$

Then N_p is a $*$-ideal of $\mathcal{D}(p)$ and $\mathfrak{N}_p$ is a left ideal of $\mathcal{A}$, and the quotient $*$-algebra $\mathcal{D}(p)/N_p$ is a normed $*$-algebra with the C^*-norm $\|x + N_p\|_p := p(x)$ $(x \in \mathcal{D}(p))$. We denote by $\mathcal{A}_p$ the C^*-algebra obtained by the completion of $\mathcal{D}(p)/N_p$, and denote by $\mathrm{Rep}(\mathcal{A}_p)$ the set of all faithful nondegenerate $*$-representations Π_p of the C^*-algebra $\mathcal{A}_p$ on Hilbert spaces $\mathcal{H}_{\Pi_p}$. It is well known that $\mathrm{Rep}(\mathcal{A}_p) \neq \emptyset$. For each $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$ we can define a bounded $*$-representation π_p^0 of $\mathcal{D}(p)$ on the Hilbert space $\mathcal{H}_{\Pi_p}$ by

$$\pi_p^0(x) = \Pi_p(x + N_p), \quad x \in \mathcal{D}(p).$$

The natural question arises: Can we extend the bounded $*$-representation π_p^0 of the $*$-algebra $\mathcal{D}(p)$ to a (generally unbounded) $*$-representation of the $*$-algebra $\mathcal{A}$? We show that this question has affirmative answer.

PROPOSITION 2.2. *Let p be an unbounded C^*-seminorm on $\mathcal{A}$. For any $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$, there exists a $*$-representation π_p of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\pi_p}$ such that $\|\overline{\pi_p(b)}\| \leqslant p(b)$ for each $b \in \mathcal{D}(p)$ and $\|\overline{\pi_p(x)}\| = p(x)$ for each $x \in \mathfrak{N}_p$.*

Proof. Let $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$. We put

$$\begin{aligned} &\mathcal{D}(\pi_p) = \text{ linear span of } \{\Pi_p(x + N_p)\xi : x \in \mathfrak{N}_p, \xi \in \mathcal{H}_{\Pi_p}\}, \\ &\pi_p(a)\Big(\sum_k \Pi_p(x_k + N_p)\xi_k\Big) = \sum_k \Pi_p(ax_k + N_p)\xi_k \quad \text{(finite sums)} \end{aligned}$$

for $a \in \mathcal{A}$, $\{x_k\} \subset \mathfrak{N}_p$ and $\{\xi_k\} \subset \mathcal{H}_{\Pi_p}$. Since

$$\begin{aligned} (\Pi_p(ax + N_p)\xi | \Pi_p(y + N_p)\eta) &= (\xi | \Pi_p((ax + N_p)^*(y + N_p))\eta) \\ &= (\xi | \Pi_p(x^*a^*y + N_p)\eta) \\ &= (\xi | \Pi_p(x^* + N_p)\Pi_p(a^*y + N_p)\eta) \\ &= (\Pi_p(x + N_p)\xi | \Pi_p(a^*y + N_p)\eta) \end{aligned}$$

for each $a \in \mathcal{A}, x, y \in \mathfrak{N}_p$ and $\xi, \eta \in \mathcal{H}_{\Pi_p}$, it follows that $\pi_p(a)$ is a well-defined linear operator on $\mathcal{D}(\pi_p)$ for each $a \in \mathcal{A}$, so that it is easily shown that π_p is a $*$-representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_{\pi_p} := [\mathcal{D}(\pi_p)] = \overline{\mathcal{D}(\pi_p)}^{\|\cdot\|}$ (the closure of $\mathcal{D}(\pi_p)$ in $\mathcal{H}_{\Pi_p}$) with domain $\mathcal{D}(\pi_p)$. Take an arbitrary $b \in \mathcal{D}(p)$. By the definition of π_p we have $\pi_p(b) = \pi_p^0(b)\lceil\mathcal{D}(\pi_p)$, and hence

$$\|\overline{\pi_p(b)}\| \leqslant \|\Pi_p(b + N_p)\| \leqslant \|b + N_p\|_p = p(b).$$

Suppose $x \in \mathfrak{N}_p$. It is sufficient to show that $\|\overline{\pi_p(x)}\| \geqslant p(x)$. If $p(x) = 0$, then it is obvious. Suppose $p(x) \neq 0$. We put $y = x/p(x) \in \mathfrak{N}_p$. For each $\xi \in \mathcal{H}_{\Pi_p}$ with $\|\xi\| \leqslant 1$, we have

$$\|\Pi_p(y + N_p)\xi\| \leqslant \|\Pi_p(y + N_p)\| \, \|\xi\| = p(y)\|\xi\| \leqslant 1,$$

and so

$$\begin{aligned}\|\overline{\pi_p(y)}\| = \|\overline{\pi_p(y^*)}\| &\geqslant \sup\left\{\|\pi_p(y^*)\Pi_p(y + N_p)\xi\| : \xi \in \mathcal{H}_{\Pi_p} \text{ such that } \|\xi\| \leqslant 1\right\} \\ &= \sup\left\{\|\Pi_p(y^*y + N_p)\xi\| : \xi \in \mathcal{H}_{\Pi_p} \text{ such that } \|\xi\| \leqslant 1\right\} \\ &= \|\Pi_p(y^*y + N_p)\| = p(y^*y) = p(y)^2 = 1.\end{aligned}$$

Hence, we have $\|\overline{\pi_p(x)}\| \geqslant p(x)$. This completes the proof. ∎

We simply sketch the method of the construction of the $*$-representation π_p:

REMARK 2.3. Let p be an unbounded C^*-seminorm on $\mathcal{A}$. As above, we can construct a set $\{\pi_p\}$ of $*$-representations of $\mathcal{A}$ from any $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$, but π_p is not necessarily nontrivial, that is, the case $\mathcal{H}_{\pi_p} = \{0\}$ may arise (Example 7.1, (2)). It is clear that $\mathcal{H}_{\pi_p} \neq \{0\}$ if and only if $\mathfrak{N}_p \not\subset N_p$. Hereafter we shall assume that unbounded C^*-seminorms satisfy always this condition: $\mathfrak{N}_p \not\subset N_p$.

Let p be an unbounded C^*-seminorm on $\mathcal{A}$. We denote by $\mathrm{Rep}(\mathcal{A}, p)$ the set of all $*$-representations of $\mathcal{A}$ constructed as above by $(\mathcal{A}, p)$, that is,

$$\mathrm{Rep}(\mathcal{A}, p) = \{\pi_p : \Pi_p \in \mathrm{Rep}(\mathcal{A}_p)\}.$$

DEFINITION 2.4. An unbounded m^*-seminorm q on $\mathcal{A}$ is said to be *nondegenerate* if $\mathcal{D}(q)^2$ is total in $\mathcal{D}(q)$ with respect to the seminorm q. An unbounded m^*-seminorm q on $\mathcal{A}$ is said to be *finite* if $\mathcal{D}(q) = \mathfrak{N}_q$; and q is said to be *uniformly semifinite* if there exists a net $\{u_\alpha\}$ in $\mathfrak{N}_q$ such that $u_\alpha^* = u_\alpha$ and $q(u_\alpha) \leqslant 1$ for each α and $\lim_\alpha q(xu_\alpha - x) = 0$ for each $x \in \mathcal{D}(q)$; and q is said to be *semifinite* if $\mathfrak{N}_q$ is dense in $\mathcal{D}(q)$ with respect to the seminorm q. An unbounded C^*-seminorm p on $\mathcal{A}$ is said to be *weakly semifinite* if $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p) := \{\pi_p \in \mathrm{Rep}(\mathcal{A}, p) : \mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi_p}\} \neq \emptyset$. An element π_p of $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ is said to be a *well-behaved* $*$-representation of $\mathcal{A}$ in $\mathrm{Rep}(\mathcal{A}, p)$.

Definition 2.5. A $*$-representation π of $\mathcal{A}$ is said to be *strongly nondegenerate* if there exists a left ideal $\mathcal{I}$ of $\mathcal{A}$ contained in the bounded part $\mathcal{A}_\flat^\pi := \{x \in \mathcal{A} : \overline{\pi(x)} \in \mathcal{B}(\mathcal{H}_\pi)\}$ of π such that $[\overline{\pi(\mathcal{I})}\mathcal{H}_\pi] = \mathcal{H}_\pi$.

Proposition 2.6. *Let p be an unbounded C^*-seminorm on $\mathcal{A}$ and $\pi_p \in \operatorname{Rep}(\mathcal{A}, p)$. Then the following statements hold:*

(1) $[\overline{\pi_p(\mathfrak{N}_p)}\mathcal{H}_{\pi_p}] = \mathcal{H}_{\pi_p}$*, and so π_p is strongly nondegenerate.*

(2) *Suppose $\pi_p \in \operatorname{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$. Then:*

(i) $\|\overline{\pi_p(x)}\| = p(x)$*, $\forall x \in \mathcal{D}(p)$;*

(ii) $\pi_p(\mathcal{A})'_{\mathrm{w}} = \overline{\pi_p(\mathcal{D}(p))}'$ *and* $\pi_p(\mathcal{A})'_{\mathrm{w}}\mathcal{D}(\pi_p) \subset \mathcal{D}(\pi_p)$.

(3) π_p *satisfies the condition (2) (i) if and only if there exists an element* π_p^{WB} *of* $\operatorname{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ *which is a restriction of* π_p.

(4) *Suppose p is semifinite. Then $\pi_p \in \operatorname{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ and $\mathfrak{N}_p^2$ is total in $\mathcal{D}(p)$ with respect to p, and so p is nondegenerate.*

(5) *Suppose p is uniformly semifinite. Then:*

$$\mathcal{A}_\flat^{\pi_p} = \mathcal{A}_\flat^p := \{a \in \mathcal{A} : \exists k_a > 0 \text{ such that } p(ax) \leqslant k_a p(x),\ \forall x \in \mathfrak{N}_p\},$$
$$\|\overline{\pi_p(b)}\| = \sup\{p(bx) : x \in \mathfrak{N}_p \text{ and } p(x) \leqslant 1\}, \quad \forall b \in \mathcal{A}_\flat^p$$

for each $\pi_p \in \operatorname{Rep}(\mathcal{A}, p)$.

(6) *p is finite if and only if $\mathcal{D}(p)$ is a left ideal of $\mathcal{A}$.*

Proof. (1) Since the $\|\cdot\|_p$-closure $\overline{\mathfrak{N}_p\lceil N_p}^{\|\cdot\|_p}$ of $\{x + N_p : x \in \mathfrak{N}_p\}$ in $\mathcal{A}_p$ is a left ideal of the C^*-algebra $\mathcal{A}_p$, it follows that there exists a left approximate identity $\{E_\alpha\}$ in $\overline{\mathfrak{N}_p\lceil N_p}^{\|\cdot\|_p}$, so that $\lim_\alpha \|(x + N_p)E_\alpha - (x + N_p)\|_p = 0$ for each $x \in \mathfrak{N}_p$. For any α, it follows since $E_\alpha \in \overline{\mathfrak{N}_p\lceil N_p}^{\|\cdot\|_p}$ that there exists a sequence $\{e_\alpha^{(n)}\}$ in $\mathfrak{N}_p$ such that $\lim_{n\to\infty} \|(e_\alpha^{(n)} + N_p) - E_\alpha\|_p = 0$. Take an arbitrary $\eta \in [\Pi_p(\mathfrak{N}_p + N_p)\mathcal{H}_{\Pi_p}] \ominus [\pi_p(\mathfrak{N}_p)\Pi_p(\mathfrak{N}_p + N_p)\mathcal{H}_{\Pi_p}]$. Then we have

$$\begin{aligned}
(\Pi_p(x + N_p)\xi|\eta) &= \lim_\alpha (\Pi_p(x + N_p)\Pi_p(E_\alpha)\xi|\eta) \\
&= \lim_\alpha \lim_{n\to\infty} (\Pi_p(x + N_p)\Pi_p(e_\alpha^{(n)} + N_p)\xi|\eta) \\
&= \lim_\alpha \lim_{n\to\infty} (\pi_p(x)\Pi_p(e_\alpha^{(n)} + N_p)\xi|\eta) = 0
\end{aligned}$$

for each $x \in \mathfrak{N}_p$ and $\xi \in \mathcal{H}_{\Pi_p}$, which implies that $[\pi_p(\mathfrak{N}_p)\Pi_p(\mathfrak{N}_p + N_p)\mathcal{H}_{\Pi_p}] = [\Pi_p(\mathfrak{N}_p + N_p)\mathcal{H}_{\Pi_p}] = \mathcal{H}_{\pi_p}$. Hence π_p is strongly nondegenerate.

(2) Suppose $\pi_p \in \operatorname{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$. Since $\pi_p(b) = \Pi_p(b + N_p)\lceil\mathcal{D}(\pi_p)$, $\forall b \in \mathcal{D}(p)$ and $\mathcal{H}_{\Pi_p} = \overline{\mathcal{D}(\pi_p)}^{\|\cdot\|}$, it follows that $\overline{\pi_p(b)} = \Pi_p(b + N_p)$, $\forall b \in \mathcal{D}(p)$, which implies the statement (i). The statement (ii) follows since

$$\begin{aligned}
&C\Pi_p(x + N_p)\xi = \Pi_p(x + N_p)C\xi \in \mathcal{D}(\pi_p(a)), \\
&\pi_p(a)C\Pi_p(x + N_p)\xi = \pi_p(a)\Pi_p(x + N_p)C\xi = \Pi_p(ax + N_p)C\xi \\
&\qquad = C\pi_p(a)\Pi_p(x + N_p)\xi
\end{aligned}$$

for each $C \in \overline{\pi_p(\mathcal{D}(p))}'$, $a \in \mathcal{A}$, $x \in \mathfrak{N}_p$ and $\xi \in \mathcal{H}_{\pi_p}$.

(3) Suppose π_p satisfies condition (i) above. We put

$$\Pi_p^{\mathrm{WB}}(b+N_p)=\overline{\pi_p(b)},\quad b\in\mathcal{D}(p).$$

Since $\|\Pi_p^{\mathrm{WB}}(b+N_p)\|=\|\overline{\pi_p(b)}\|=p(b)=\|b+N_p\|_p$ for each $b\in\mathcal{D}(p)$, it follows from (1) that Π_p^{WB} can be extended to a faithful nondegenerate $*$-representation of the C^*-algebra $\mathcal{A}_p$ on the Hilbert space $\mathcal{H}_{\pi_p}$ and denote it by the same Π_p^{WB}. We also denote by π_p^{WB} the strongly nondegenerate $*$-representation of $\mathcal{A}$ induced by Π_p^{WB}. Since

$$\begin{aligned}\mathcal{D}(\pi_p^{\mathrm{WB}})&=\text{ linear span of }\{\Pi_p^{\mathrm{WB}}(x+N_p)\xi : x\in\mathfrak{N}_p,\ \xi\in\mathcal{H}_{\pi_p}\}\\&=\text{ linear span of }\{\overline{\pi_p(x)}\xi : x\in\mathfrak{N}_p,\ \xi\in\mathcal{H}_{\pi_p}\},\end{aligned}$$

it follows from (1) that $\mathcal{H}_{\pi_p^{\mathrm{WB}}}=\mathcal{H}_{\pi_p}=\mathcal{H}_{\Pi_p^{\mathrm{WB}}}$, which means that $\pi_p^{\mathrm{WB}}\in\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$. The converse follows from (2) (i).

(4) Suppose p is semifinite. Since p is semifinite, it follows that $\{\Pi_p(x+N_p) : x\in\mathfrak{N}_p\}$ is uniformly dense in the C^*-algebra $\Pi_p(\mathcal{A}_p)$, which implies by the nondegenerateness of Π_p that $\mathcal{H}_{\Pi_p}=\mathcal{H}_{\pi_p}$. Hence $\pi_p\in\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$. By (1) we have $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)=\mathrm{Rep}(\mathcal{A},p)$. Since the C^*-algebra $\mathcal{A}_p$ has a bounded approximate identity and $\mathfrak{N}_p$ is dense in $\mathcal{D}(p)$ with respect to p, it follows that $\mathfrak{N}_p^2$ is total in $\mathcal{D}(p)$ with respect to p.

(5) It is clear that $\mathcal{A}_\flat^{\pi_p}\subset\mathcal{A}_\flat^p$ without the assumption of the uniform semifiniteness of p. Suppose p is uniformly semifinite. Then we show the converse inclusion. Let $\{u_\alpha\}$ be in Definition 2.4. Take an arbitrary $a\in\mathcal{A}_\flat^p$, $\{x_k\}\subset\mathfrak{N}_p$ and $\{\xi_k\}\subset\mathcal{H}_{\Pi_p}$. Since

$$\begin{aligned}&\|\pi_p(a)\Pi_p(u_\alpha x_k+N_p)\xi_k-\pi_p(a)\Pi_p(x_k+N_p)\xi_k\|=\|\Pi_p(a(u_\alpha x_k-x_k)+N_p)\xi_k\|\\&\quad\leqslant k_a p(u_\alpha x_k-x_k)\|\xi_k\|=k_a p(x_k^*u_\alpha-x_k^*)\|\xi_k\|\underset{\alpha}{\longrightarrow}0,\end{aligned}$$

it follows that

$$\begin{aligned}&\Big\|\pi_p(a)\sum_k\Pi_p(x_k+N_p)\xi_k\Big\|=\lim_\alpha\Big\|\pi_p(a)\sum_k\Pi_p(u_\alpha x_k+N_p)\xi_k\Big\|\\&=\lim_\alpha\Big\|\pi_p(au_\alpha)\sum_k\Pi_p(x_k+N_p)\xi_k\Big\|\leqslant\overline{\lim_\alpha}\|\overline{\pi_p(au_\alpha)}\|\,\Big\|\sum_k\Pi_p(x_k+N_p)\xi_k\Big\|\\&=\overline{\lim_\alpha}p(au_\alpha)\Big\|\sum_k\Pi_p(x_k+N_p)\xi_k\Big\|\leqslant k_a\Big\|\sum_k\Pi_p(x_k+N_p)\xi_k\Big\|,\end{aligned}$$

which implies $a\in\mathcal{A}_\flat^{\pi_p}$. Hence we have $\mathcal{A}_\flat^p=\mathcal{A}_\flat^{\pi_p}$.

(6) This is trivial. This completes the proof. ∎

3. UNBOUNDED C^*-SEMINORMS DEFINED BY $*$-REPRESENTATIONS

In Section 2 we constructed a family $\mathrm{Rep}(\mathcal{A}, p)$ (respectively $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$) of strongly nondegenerate $*$-representations of $\mathcal{A}$ from an (respectively weakly semifinite) unbounded C^*-seminorm p on $\mathcal{A}$. Conversely we shall construct an unbounded C^*-seminorm r_π on $\mathcal{A}$ from a strongly nondegenerate $*$-representation π of $\mathcal{A}$ and the natural representation $\pi^N_{r_\pi}$ of $\mathcal{A}$ constructed from r_π, and investigate the relation between π and $\pi^N_{r_\pi}$. Let π be a strongly nondegenerate $*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$. We put

$$\mathcal{A}^\pi_\flat = \{x \in \mathcal{A} : \overline{\pi(x)} \in \mathcal{B}(\mathcal{H}_\pi)\} \quad \text{and} \quad \pi_\flat(x) = \overline{\pi(x)}, \quad x \in \mathcal{A}^\pi_\flat.$$

Then $\mathcal{A}^\pi_\flat$ is a $*$-subalgebra of $\mathcal{A}$ with the identity $\mathbb{1}$ and $\pi_\flat$ is a bounded $*$-representation of $\mathcal{A}^\pi_\flat$ on $\mathcal{H}_\pi$. We denote by $C^*(\pi)$ the C^*-algebra generated by $\pi_\flat(\mathcal{A}^\pi_\flat)$. We now define an unbounded C^*-seminorm r_π on $\mathcal{A}$ as follows:

$$\mathcal{D}(r_\pi) = \mathcal{A}^\pi_\flat \quad \text{and} \quad r_\pi(x) = \|\pi_\flat(x)\|, \quad x \in \mathcal{D}(r_\pi).$$

Then r_π satisfies the condition $\mathfrak{N}_{r_\pi} \not\subset N_{r_\pi}$. In fact, this follows since $\mathcal{I} \subset \mathfrak{N}_{r_\pi}$, where $\mathcal{I}$ is a left ideal of $\mathcal{A}$ contained in $\mathcal{A}^\pi_\flat$ such that $[\pi(\mathcal{I})\mathcal{D}(\pi)] = \mathcal{H}_\pi$. Here we put

$$\Pi(x + N_{r_\pi}) = \pi_\flat(x), \quad x \in \mathcal{A}^\pi_\flat.$$

Since $\|\Pi(x + N_{r_\pi})\| = r_\pi(x) = \|x + N_{r_\pi}\|_{r_\pi}$ for each $x \in \mathcal{A}^\pi_\flat$, it follows that Π can be extended to a faithful $*$-representation $\Pi^N_{r_\pi}$ of $\mathcal{A}_{r_\pi}$ on the Hilbert space $\mathcal{H}_\pi$. The $*$-representation $\pi^N_{r_\pi}$ of $\mathcal{A}$ defined by $\Pi^N_{r_\pi}$ as above is called the *natural representation* of $\mathcal{A}$ induced by π. Since $\mathcal{H}_{\Pi^N_{r_\pi}} = \mathcal{H}_\pi$, it follows that $\mathcal{H}_{\pi^N_{r_\pi}}$ is a closed subspace of $\mathcal{H}_\pi$. We simply sketch the above method of the construction of $\pi^N_{r_\pi}$:

We have the following results for the relation between π and $\pi^N_{r_\pi}$:

PROPOSITION 3.1. *Let π be a $*$-representation of $\mathcal{A}$. Suppose that π is strongly nondegenerate, that is, there exists a left ideal $\mathcal{I}$ of $\mathcal{A}$ contained in $\mathcal{A}^\pi_b$ such that $[\pi(\mathcal{I})\mathcal{D}(\pi)] = \mathcal{H}_\pi$. Then $\pi^N_{r_\pi} \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, r_\pi)$ and $\pi^N_{r_\pi} \subset \widetilde{\pi}$. Furthermore, if $\pi(\mathcal{I})\mathcal{D}(\pi)$ is total in $\mathcal{D}(\pi)$ with respect to the graph topology t_π, then $\widetilde{\pi^N_{r_\pi}} = \widetilde{\pi}$.*

Proof. Since

$$\begin{aligned}\mathcal{D}(\pi^N_{r_\pi}) &= \text{ linear span of } \left\{\Pi^N_{r_\pi}(x+N_{r_\pi})\xi : x \in \mathfrak{N}_{r_\pi},\, \xi \in \mathcal{H}_\pi\right\} \\ &= \text{ linear span of } \{\overline{\pi(x)}\xi : x \in \mathfrak{N}_{r_\pi},\, \xi \in \mathcal{H}_\pi\},\end{aligned} \tag{3.1}$$

it follows that

$$\begin{aligned}(\pi(a)^*\eta|\Pi^N_{r_\pi}(x+N_{r_\pi})\xi) &= (\pi(a)^*\eta|\overline{\pi(x)}\xi) = (\pi(x)^*\pi(a)^*\eta|\xi) \\ &= (\pi(ax)^*\eta|\xi) = (\eta|\overline{\pi(ax)}\xi) = (\eta|\pi^N_{r_\pi}(a)\Pi^N_{r_\pi}(x+N_{r_\pi})\xi)\end{aligned}$$

for each $a \in \mathcal{A}, \eta \in \mathcal{D}(\pi(a)^*), x \in \mathfrak{N}_{r_\pi}$ and $\xi \in \mathcal{H}_\pi$, which implies $\Pi^N_{r_\pi}(x+N_{r_\pi})\xi \in \mathcal{D}(\overline{\pi(a)})$ and $\overline{\pi(a)}\Pi^N_{r_\pi}(x+N_{r_\pi})\xi = \pi^N_{r_\pi}(a)\Pi^N_{r_\pi}(x+N_{r_\pi})\xi$. Hence, $\mathcal{D}(\pi^N_{r_\pi}) \subset \mathcal{D}(\widetilde{\pi})$ and $\widetilde{\pi}\lceil\mathcal{D}(\pi^N_{r_\pi}) = \pi^N_{r_\pi}$.

Since π is strongly nondegenerate and $\mathcal{A}^\pi_\flat = \mathcal{D}(r_\pi)$, it follows that $[\overline{\pi(\mathfrak{N}_{r_\pi})}\mathcal{H}_\pi] = \mathcal{H}_\pi$, which implies by (3.1) that $\mathcal{H}_{\pi^N_{r_\pi}} = \mathcal{H}_\pi = \mathcal{H}_{\Pi^N_{r_\pi}}$, so that $\pi^N_{r_\pi} \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, r_\pi)$. Suppose that $\pi(\mathcal{I})\mathcal{D}(\pi)$ is total in $\mathcal{D}(\pi)[t_\pi]$. Then it follows from (3.1) that $\widetilde{\pi^N_{r_\pi}} = \widetilde{\pi}$. This complete the proof. ∎

By Proposition 2.6 and Proposition 3.1 we have the following diagram:

And we have the following

COROLLARY 3.2. *The following statements are equivalent:*

(i) *There exists an unbounded C^*-seminorm p on $\mathcal{A}$ such that $\mathfrak{N}_p \not\subset N_p$.*
(ii) *There exists a strongly nondegenerate $*$-representation of $\mathcal{A}$.*
(iii) *There exists a well-behaved $*$-representation of $\mathcal{A}$.*

Next we investigate the relations between unbounded C^*-seminorms p and r_{π_p} and the $*$-representations π_p and $\pi^N_{r_{\pi_p}}$. We first define an order relation among unbounded seminorms as follows:

DEFINITION 3.3. Let p and q be unbounded seminorms on $\mathcal{A}$. We say that p is an *extension* of q (or q is a *restriction* of p) if $\mathcal{D}(q) \subset \mathcal{D}(p)$ and $q(x) = p(x)$ for each $x \in \mathcal{D}(q)$, and then denote by $q \subset p$.

We denote by $\mathrm{C^*N}(\mathcal{A})$ the set of all unbounded C^*-seminorms p on $\mathcal{A}$ such that $\mathfrak{N}_p \not\subset N_p$. Then $\mathrm{C^*N}(\mathcal{A})$ is a partially ordered set with the order $\subset$. For any $p \in \mathrm{C^*N}(\mathcal{A})$ we put

$$\mathrm{C^*N}(p) = \{q \in \mathrm{C^*N}(\mathcal{A}) : p \subset q\}.$$

Then it follows from Zorn's lemma that $\mathrm{C^*N}(p)$ has a maximal element. We show that if p is weakly semifinite then r_{π_p} is a maximal element of $\mathrm{C^*N}(p)$.

Proposition 3.4. *Suppose p is a weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$ and $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$. Then r_{π_p} is a maximal element of* $\mathrm{C^*N}(p)$ *and $r_{\pi_p} = r_{\pi'_p}$ for each $\pi_p, \pi'_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$.*

Proof. We show that r_{π_p} is a maximal element of $\mathrm{C^*N}(p)$. Take an arbitrary $r \in \mathrm{C^*N}(r_{\pi_p})$. By Proposition 2.6 we have $p \subset r_{\pi_p} \subset r$, and so it follows that the linear map: $x + N_p \in \mathcal{D}(p)/N_p \longmapsto x + N_r \in \mathcal{D}(p)/N_r$ is a bijection and isometry, so that $\mathcal{A}_p$ is regarded as a closed $*$-subalgebra of the C^*-algebra $\mathcal{A}_r$. By the stability of C^*-algebras ([11], Proposition 2.10.2) there exists a $*$-representation Π_r of $\mathcal{A}_r$ such that $\Pi_p \subset \Pi_r$. Then we can construct in the same way as the proof of Proposition 2.6 the $*$-representation π_r of $\mathcal{A}$ induced by Π_r which is an extention of π_p, which implies that $\pi_p(a)$ is bounded and

$$\|\overline{\pi_p(a)}\| \leqslant \|\overline{\pi_r(a)}\| \leqslant r(a), \quad \forall a \in \mathcal{D}(r). \tag{3.2}$$

Hence we have

$$\mathcal{D}(r) \subset \mathcal{D}(r_{\pi_p}). \tag{3.3}$$

On the other hand, since $r_{\pi_p} \subset r$, we have $r = r_{\pi_p}$. We next show that $r_{\pi_p} = r_{\pi'_p}$ for each $\pi_p, \pi'_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$. Since $p \subset r := r_{\pi'_p}$, it follows from (3.2) and (3.3) that $\mathcal{D}(r_{\pi'_p}) = \mathcal{D}(r) \subset \mathcal{D}(r_{\pi_p})$ and $r_{\pi_p}(x) = \|\overline{\pi_p(x)}\| \leqslant r(x) = r_{\pi'_p}(x)$ for each $x \in \mathcal{D}(r) = \mathcal{D}(r_{\pi'_p})$ Similarly we have that $\mathcal{D}(r_{\pi_p}) \subset \mathcal{D}(r_{\pi'_p})$ and $r_{\pi'_p}(x) \leqslant r_{\pi_p}(x)$ for each $x \in \mathcal{D}(r_{\pi_p})$. Hence, $r_{\pi_p} = r_{\pi'_p}$. This completes the proof. ∎

By Proposition 3.1 and Proposition 3.4 we have the following

Corollary 3.5. *Suppose π is a strongly nondegenerate $*$-representation of $\mathcal{A}$. Then r_π is maximal.*

For the relation of $*$-representations π_p and $\pi^N_{r_{\pi_p}}$ we have the following

Proposition 3.6. *Suppose p is a weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$ and $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$. Then $\pi_p \subset \pi^N_{r_{\pi_p}}$ and $\widetilde{\pi^N_{r_{\pi_p}}} = \widetilde{\pi_p}$.*

Proof. It follows from the definition of $\pi^N_{r_{\pi_p}}$ that $\mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi^N_{r_{\pi_p}}}$ and since $\mathfrak{N}_p \subset \mathfrak{N}_{r_{\pi_p}} \subset \mathcal{A}_\flat^{\pi_p}$ and

$$\Pi_p(x+N_p)\xi = \overline{\pi_p(x)}\xi = \Pi^N_{r_{\pi_p}}(x + N_{r_{\pi_p}})\xi$$

for each $x \in \mathfrak{N}_p$ and $\xi \in \mathcal{H}_{\pi_p}$, we have $\mathcal{D}(\pi_p) \subset \mathcal{D}(\pi^N_{r_{\pi_p}})$. Furthermore, since

$$\pi_p(a)\Pi_p(x+N_p)\xi = \overline{\pi_p(ax)}\xi = \pi^N_{r_{\pi_p}}(a)\Pi^N_{r_{\pi_p}}(x+N_{r_{\pi_p}})\xi = \pi^N_{r_{\pi_p}}(a)\Pi(x+N_p)\xi$$

for each $a \in \mathcal{A}, x \in \mathfrak{N}_p$ and $\xi \in \mathcal{H}_{\pi_p}$, it follows that $\pi_p = \pi^N_{r_{\pi_p}} \lceil \mathcal{D}(\pi_p)$. On the other hand, we have $\mathcal{D}(\pi^N_{r_{\pi_p}}) \subset \mathcal{D}(\widetilde{\pi_p})$ by Proposition 3.1. Therefore it follows that $\mathcal{H}_{\pi_p} = \mathcal{H}_{\pi^N_{r_{\pi_p}}}$, $\pi_p \subset \pi^N_{r_{\pi_p}}$ and $\widetilde{\pi_p} = \widetilde{\pi^N_{r_{\pi_p}}}$. This completes the proof. ∎

4. REGULAR UNBOUNDED C^*-SEMINORMS

In this section we define and characterize the notion of regular unbounded C^*-seminorms on $*$-algebras. We first prepare an unbounded C^*-seminorm $\sup_\alpha p_\alpha$ constructed by a family $\{p_\alpha\}$ of unbounded C^*-seminorms on $\mathcal{A}$ and the notion of direct sum of $*$-representations of $\mathcal{A}$. Let $\{p_\alpha\}$ be a family of unbounded C^*-seminorms on $\mathcal{A}$. We put

$$\mathcal{D}(\sup_\alpha p_\alpha) = \big\{x \in \bigcap_\alpha \mathcal{D}(p_\alpha) : \sup_\alpha p_\alpha(x) < \infty\big\},$$
$$(\sup_\alpha p_\alpha)(x) = \sup_\alpha p_\alpha(x), \quad x \in \mathcal{D}(\sup_\alpha p_\alpha).$$

Then $\sup_\alpha p_\alpha$ is an unbounded C^*-seminorm on $\mathcal{A}$, and it is an unbounded C^*-norm if and only if $p_\alpha(x) = 0$, $\forall\alpha$ implies $x = 0$.

DEFINITION 4.1. An unbounded C^*-(semi)norm p on $\mathcal{A}$ is said to be *regular* if $p \subset \sup_\alpha p_\alpha$, where $\{p_\alpha\}$ is a family of C^*-seminorms on $\mathcal{A}$.

Let $\{\pi_\alpha\}$ be a family of $*$-representations of $\mathcal{A}$. We put

$$\mathcal{D}\Big(\bigoplus_\alpha \pi_\alpha\Big) = \Big\{\xi = (\xi_\alpha) \in \bigoplus_\alpha \mathcal{H}_{\pi_\alpha} : \xi_\alpha \in \mathcal{D}(\pi_\alpha),\ \forall\alpha \text{ and } \sum_\alpha \|\pi_\alpha(a)\xi_\alpha\|^2 < \infty,\ \forall a \in \mathcal{A}\Big\},$$
$$\Big(\bigoplus_\alpha \pi_\alpha\Big)(a)(\xi_\alpha) = (\pi_\alpha(a)\xi_\alpha), \quad a \in \mathcal{A},\ (\xi_\alpha) \in \mathcal{D}\Big(\bigoplus_\alpha \pi_\alpha\Big).$$

Then $\bigoplus_\alpha \pi_\alpha$ is a $*$-representation of $\mathcal{A}$ on $\bigoplus_\alpha \mathcal{H}_{\pi_\alpha}$ such that

$$x \in \mathcal{A}_\flat^{\oplus_\alpha \pi_\alpha} \text{ iff } \pi_\alpha(x) \text{ is bounded } \forall\alpha, \quad \text{and} \quad \sup_\alpha \|\overline{\pi_\alpha(x)}\| < \infty.$$

DEFINITION 4.2. A $*$-representation π of $\mathcal{A}$ is said to be *weakly bounded* if $\pi \subset \bigoplus_\alpha \pi_\alpha$ as the same Hilbert space, where $\{\pi_\alpha\}$ is a family of bounded $*$-representations of $\mathcal{A}$.

LEMMA 4.3. *Let p be an unbounded C^*-seminorm on $\mathcal{A}$. Suppose $p \subset \sup_\alpha p_\alpha$ for a net $\{p_\alpha\}$ of weakly semifinite unbounded C^*-seminorms on $\mathcal{A}$, and further $\mathfrak{N}_p$ is dense in $\mathcal{D}(p_\alpha)$ with respect to $\{p_\alpha\}$. Then p is weakly semifinite, and for any π_{p_α} of $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p_\alpha)$ $\forall\alpha$, there exists an element π_p of $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ such that $\pi_p \subset \bigoplus_\alpha \pi_{p_\alpha}$.*

Proof. We put

$$\Pi_p(x + N_p)(\xi_\alpha) = (\Pi_{p_\alpha}(x + N_{p_\alpha})\xi_\alpha), \quad x \in \mathcal{D}(p),\ (\xi_\alpha) \in \bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}}.$$

Since

$$\|\Pi_p(x + N_p)\| = \sup_\alpha \|\Pi_{p_\alpha}(x + N_{p_\alpha})\| = \sup_\alpha p_\alpha(x) = p(x)$$

for each $x \in \mathcal{D}(p)$, it follows that Π_p can be extended to a faithful $*$-representation of $\mathcal{A}_p$ on $\bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}}$. We denote π_p the $*$-representation of $\mathcal{A}$ induced by Π_p. Then we have

$$\begin{cases} \mathcal{D}(\pi_p) = \text{ linear span of } \{\Pi_p(x+N_p)\xi : x \in \mathfrak{N}_p,\, \xi \in \mathcal{H}_{\pi_p}\} \\ \qquad\quad = \text{ linear span of } \big\{(\overline{\pi_{p_\alpha}(x)}\xi_\alpha) : x \in \mathfrak{N}_p,\, \xi = (\xi_\alpha) \in \bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}}\big\}, \\ \pi_p(a)(\overline{\pi_{p_\alpha}(x)}\xi_\alpha) = (\overline{\pi_{p_\alpha}(ax)}\xi_\alpha). \end{cases}$$

We show that p is weakly semifinite, that is, $\mathcal{D}(\pi_p)$ is dense in $\bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}}$. Take an arbitrary $\xi = (\xi_\alpha) \in \bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}} \ominus \overline{\mathcal{D}(\pi_p)}$. Take an arbitrary α. For any $\eta_\alpha \in \mathcal{H}_{\pi_{p_\alpha}}$ we have

$$(\overline{\pi_{p_\alpha}(x)}\eta_\alpha|\xi_\alpha) = (\delta_{\alpha\beta}\overline{\pi_{p_\beta}(x)}\eta_\beta|\xi) = 0 \tag{4.1}$$

for each $x \in \mathfrak{N}_p$. Since $\mathfrak{N}_p$ is dense in $\mathcal{D}(p_\alpha)$ with respect to p_α, it follows that $\overline{\pi_{p_\alpha}(\mathfrak{N}_p)}\mathcal{H}_{\pi_{p_\alpha}}$ is total in $\overline{\pi_{p_\alpha}(\mathcal{D}(p_\alpha))}\mathcal{H}_{\pi_{p_\alpha}}$, and further it follows from the weak semifiniteness of p_α that $\overline{\pi_{p_\alpha}(\mathcal{D}(p_\alpha))}\mathcal{H}_{\pi_{p_\alpha}}$ is total in $\mathcal{H}_{\pi_{p_\alpha}}$. Hence, $\overline{\pi_{p_\alpha}(\mathfrak{N}_p)}\mathcal{H}_{\pi_{p_\alpha}}$ is total in $\mathcal{H}_{\pi_{p_\alpha}}$, and so by (4.1) $\xi_\alpha = 0$. Hence, $\xi = 0$. Thus, $\mathcal{D}(\pi_p)$ is dense in $\bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}}$. By the definition of π_p we have $\pi_p \subset \bigoplus_\alpha \pi_{p_\alpha}$. This completes the proof. ∎

By Lemma 4.3 we have the following

PROPOSITION 4.4. *Let p be an unbounded C^*-seminorm on $\mathcal{A}$. Suppose p is regular, that is, $p \subset \sup_\alpha p_\alpha$ for some net $\{p_\alpha\}$ of C^*-seminorms on $\mathcal{A}$, and further $\mathfrak{N}_p$ is dense in $\mathcal{A}$ with respect to $\{p_\alpha\}$. Then there exists an element π_p of* $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$ *which is weakly bounded. Conversely suppose $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$ and it is weakly bounded. Then p is regular.*

In Section 7 we shall give several examples of regular unbounded C^*-(semi) norms.

5. UNBOUNDED GELFAND-NAIMARK C^*-SEMINORMS

In this section we construct and characterize an unbounded Gelfand-Naimark C^*-seminorm $|\cdot|_p$ from an unbounded m^*-seminorm p on a $*$-algebra $\mathcal{A}$. An unbounded m^*-seminorm p on $\mathcal{A}$ is said to be *representable* if there exists a non-zero nondegenerate bounded $*$-representation π of $\mathcal{D}(p)$ such that $\|\pi(x)\| \leqslant p(x)$ for each $x \in \mathcal{D}(p)$. Every unbounded C^*-seminorm on $\mathcal{A}$ is representable, but an unbounded m^*-seminorm is not necessarily representable (see Section 37, Example 16 in [9]). Let p be a representable unbounded m^*-seminorm on $\mathcal{A}$ and $\mathrm{Rep}(p)$ the set of all nondegenerate bounded $*$-representations π of $\mathcal{D}(p)$ on $\mathcal{H}_\pi$ such that $\|\pi(x)\| \leqslant k_\pi p(x),\ \forall x \in \mathcal{D}(p)$ for some constant k_π. Let $\pi \in \mathrm{Rep}(p)$. It is easily shown that $\|\pi(x)\| \leqslant p(x)$ for each $x \in \mathcal{D}(p)$, and so we can define an unbounded C^*-seminorm $|\cdot|_p$ on $\mathcal{A}$ by

$$\mathcal{D}(|\cdot|_p) = \mathcal{D}(p) \quad \text{and} \quad |x|_p = \sup_{\pi \in \mathrm{Rep}(p)} \|\pi(x)\|, \quad x \in \mathcal{D}(p)$$

and call it the *unbounded Gelfand-Naimark C^*-seminorm* of the unbounded m^*-seminorm p. To investigate the unbounded Gelfand-Naimark C^*-seminorm $|\cdot|_p$, we prepare another order $\leqslant$ on $\mathrm{C^*N}(p)$ as follows: $r_1 \leqslant r_2$ iff $\mathcal{D}(r_2) \subset \mathcal{D}(r_1)$ and $r_1(x) \leqslant r_2(x)$, $\forall x \in \mathcal{D}(r_2)$.

PROPOSITION 5.1. *Let p be a representable unbounded m^*-seminorm on a $*$-algebra $\mathcal{A}$. Then the following statements hold:*

(i) *$|\cdot|_p$ is the largest element of $(\mathrm{C^*N}(p), \leqslant)$.*

(ii) *If p is semifinite, then $|\cdot|_p$ is semifinite.*

(iii) *Suppose $\mathfrak{N}_p$ is dense in $\mathcal{D}(p)$ with respect to the set $\{r_\pi : \pi \in \mathrm{Rep}(p)\}$ of seminorms r_π. Then $|\cdot|_p$ is weakly semifinite and there exists a $*$-representation π_p of $\mathcal{A}$ such that $\|\pi_p(x)\| = |x|_p$ for each $x \in \mathcal{D}(p)$.*

(iv) *Suppose p is an unbounded C^*-seminorm on $\mathcal{A}$. Then $|\cdot|_p = p$.*

Proof. (i) Let r be any unbounded C^*-seminorm on $\mathcal{A}$ such that $r \leqslant p$. For any $\Pi_r \in \mathrm{Rep}(A_r)$ we define a bounded $*$-representation π_r^0 of $\mathcal{D}(r)$ by

$$\pi_r^0(x) = \Pi_r(x + N_r), \qquad x \in \mathcal{D}(r).$$

Then since $\mathcal{D}(p) \subset \mathcal{D}(r)$, it follows that $\pi_r^0\lceil\mathcal{D}(p)$ is a bounded $*$-repesentation of $\mathcal{D}(p)$ and $\|\pi_r^0(x)\| = r(x) \leqslant p(x)$ for each $x \in \mathcal{D}(p)$, which implies $\pi_r^0\lceil\mathcal{D}(p) \in \mathrm{Rep}(p)$. Hence it follows that $r(x) \leqslant |x|_p$ for each $x \in \mathcal{D}(p)$.

(ii) This follows since $\mathcal{D}(|\cdot|_p) = \mathcal{D}(p)$, $\mathfrak{N}_{|\cdot|_p} = \mathfrak{N}_p$ and $|x|_p \leqslant p(x)$, $\forall x \in \mathcal{D}(p)$.

(iii) We put

$$\Pi_p(x + N_{|\cdot|_p}) = \Big(\bigoplus_{\pi \in \mathrm{Rep}(p)} \pi\Big)(x), \quad x \in \mathcal{D}(p).$$

Then Π_p can be extended to a faithful nondegenerate $*$-representation of the C^*-algebra $\mathcal{A}_{|\cdot|_p}$ on $\bigoplus_{\pi \in \mathrm{Rep}(p)} \mathcal{H}_\pi$ and denote it by the same Π_p. Here we denote by π_p the $*$-representation of $\mathcal{A}$ defined by Π_p, that is,

$$\begin{aligned}\mathcal{D}(\pi_p) &= \text{ linear span of } \{\Pi_p(x + N_{|\cdot|_p})(\xi_\pi) : x \in \mathfrak{N}_p,\, \xi_\pi \in \mathcal{H}_\pi\}\\ &= \text{ linear span of } \{(\pi(x)\xi_\pi) : x \in \mathfrak{N}_p,\, \xi_\pi \in \mathcal{H}_\pi\},\\ \pi_p(a)(\pi(x)\xi_\pi) &= (\pi(ax)\xi_\pi), \quad a \in \mathcal{A},\, x \in \mathfrak{N}_p,\, \xi_\pi \in \mathcal{H}_\pi.\end{aligned}$$

Since $\mathfrak{N}_p$ is dense in $\mathcal{D}(p)$ with respect to r_π $(\pi \in \mathrm{Rep}(p))$ and any π is nondegenerate, it follows that $\mathcal{D}(\pi_p)$ is dense in $\bigoplus_\pi \mathcal{H}_\pi$, which implies that $|\cdot|_p$ is weakly semifinite. Hence, it follows from Proposition 2.6 that $|x|_p = \|\pi_p(x)\|$ for each $x \in \mathcal{D}(p)$.

(iv) Suppose p is an unbounded C^*-seminorm on $\mathcal{A}$. Take an arbitrary $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$. We put

$$\pi_p^0(x) = \Pi_p(x + N_p), \quad x \in \mathcal{D}(p).$$

Then it follows that $\pi_p^0 \in \mathrm{Rep}(p)$ and $\|\pi_p^0(x)\| = p(x)$ for each $x \in \mathcal{D}(p)$, which implies $|\cdot|_p = p$. This completes the proof. ∎

We next characterize the unbounded Gelfand-Naimark C^*-seminorm $|\cdot|_p$ of a representable unbounded m^*-seminorm p extending some main results in [33] about C^*-seminorms on $*$-algebras with identity to unbounded C^*-seminorms on $*$-algebras without identity. A positive linear functional f on $\mathcal{A}$ is said to be *representable* if there exists a constant $\gamma > 0$ such that $|f(x)|^2 \leqslant \gamma f(x^*x)$ for all $x \in \mathcal{A}$.

Let $\mathcal{F}_p$ be the set of all p-continuous representable positive linear functionals f on $\mathcal{D}(p)$ such that $|f(x)|^2 \leqslant f(x^*x)$ for each $x \in \mathcal{D}(p)$. Then we have the following

PROPOSITION 5.2. *Let p be a representable unbounded m^*-seminorms on $\mathcal{A}$. Then*

$$\mathcal{D}(p) = \{x \in \mathcal{D}(p) : \sup_{f\in\mathcal{F}_p} f(x^*x) < \infty\},$$

$$|x|_p = \sup_{f\in\mathcal{F}_p} f(x^*x)^{1/2}, \quad x \in \mathcal{D}(p).$$

Proof. Take an arbitrary $f \in \mathcal{F}_p$. Since f is p-continuous, there exists a constant $M_f > 0$ such that $|f(x)| \leqslant M_f p(x)$, $\forall x \in \mathcal{D}(p)$, which implies

$$|f(x)|^2 \leqslant f(x^*x) \leqslant M_f p(x^*x) \leqslant M_f p(x)^2$$

for each $x \in \mathcal{D}(p)$. Repeating this, we have

$$|f(x)| \leqslant M_f^{1/n} p(x), \quad \forall x \in \mathcal{D}(p),\ \forall n \in \mathbb{N}.$$

Hence we have

$$|f(x)| \leqslant p(x), \quad \forall x \in \mathcal{D}(p). \tag{5.1}$$

For any $y \in \mathcal{D}(p)$ with $f(y^*y) = 1$ we define a positive linear functional on $\mathcal{D}(p)$ by

$$f_y(x) = f(y^*xy), \quad x \in \mathcal{D}(p).$$

Then we have

$$|f_y(x)|^2 = |f(y^*xy)|^2 \leqslant f(y^*y)f(y^*x^*xy) = f_y(x^*x)$$

and by (5.1)

$$|f_y(x)| \leqslant p(y)^2 p(x)$$

for each $x \in \mathcal{D}(p)$. Hence we have

$$f_y \in \mathcal{F}_p \text{ for each } y \in \mathcal{D}(p) \text{ with } f(y^*y) = 1. \tag{5.2}$$

Here we put

$$\begin{cases} \mathcal{D}(r_{\mathcal{F}_p}) = \{x \in \mathcal{D}(p) : \sup_{f\in\mathcal{F}_p} f(x^*x) < \infty\} \\ r_{\mathcal{F}_p}(x) = \sup_{f\in\mathcal{F}_p} f(x^*x)^{1/2}, \quad x \in \mathcal{D}(r_{\mathcal{F}_p}). \end{cases}$$

By (5.1) we have

$$\mathcal{D}(r_{\mathcal{F}_p}) = \mathcal{D}(p) \quad \text{and} \quad r_{\mathcal{F}_p}(x) \leqslant p(x), \quad \forall x \in \mathcal{D}(p). \tag{5.3}$$

Let $(\pi_f, \lambda_f, \mathcal{H}_f)$ be the GNS-construction for f. We show

$$\begin{cases} \mathcal{D}(p) = \{x \in \mathcal{D}(p) : \sup_{f \in \mathcal{F}_p} \|\overline{\pi_f(x)}\| < \infty\} \\ r_{\mathcal{F}_p}(x) = \sup_{f \in \mathcal{F}_p} \|\pi_f(x)\|, \quad x \in \mathcal{D}(p). \end{cases}$$

In fact, take an arbitrary $x \in \mathcal{D}(p)$. By (5.2) we have, for any $y \in \mathcal{D}(p)$ with $f(y^*y) = 1$,

$$\|\pi_f(x)\lambda_f(y)\|^2 = f_y(x^*x) \leqslant r_{\mathcal{F}}(x)^2$$

for each $x \in \mathcal{D}(p)$, which implies that $\pi_f(x)$ is bounded and $\|\overline{\pi_f(x)}\| \leqslant r_{\mathcal{F}}(x)$ for each $x \in \mathcal{D}(p)$. Hence we have

$$\sup_{f \in \mathcal{F}_p} \|\overline{\pi_f(x)}\| \leqslant r_{\mathcal{F}_p}(x), \quad \forall x \in \mathcal{D}(p).$$

Since $|f(x)| \leqslant f(x^*x)^{1/2} = \|\lambda_f(x)\|, x \in \mathcal{D}(p)$, it follows from the Riesz theorem that there exists an element ξ_f of $\mathcal{H}_f$ such that $\|\xi_f\| \leqslant 1$ and $f(x) = (\lambda_f(x)|\xi_f)$ for all $x \in \mathcal{D}(p)$, which implies by the boundedness of $\pi_f(x)$ that $\lambda_f(x) = \overline{\pi_f(x)}\xi_f$ and

$$|f(x^*x)|^{1/2} = \|\overline{\pi_f(x)}\xi_f\| \leqslant \|\overline{\pi_f(x)}\|, \quad \forall x \in \mathcal{D}(p).$$

Hence

$$r_{\mathcal{F}_p}(x) \leqslant \sup_{f \in \mathcal{F}_p} \|\overline{\pi_f(x)}\|, \quad \forall x \in \mathcal{D}(p).$$

Thus we have

$$r_{\mathcal{F}_p}(x) = \sup_{f \in \mathcal{F}_p} \|\overline{\pi_f(x)}\|, \quad x \in \mathcal{D}(p),$$

which implies that $r_{\mathcal{F}_p}$ is an unbounded C^*-seminorm on $\mathcal{A}$ such that $\mathcal{D}(r_{\mathcal{F}_p}) = \mathcal{D}(p)$ and $r_{\mathcal{F}_p}(x) \leqslant |x|_p$ for each $x \in \mathcal{D}(p)$. On the other hands, take arbitrary $\pi \in \mathrm{Rep}(p)$ and $\xi \in \mathcal{H}_\pi$ such that $\|\xi\| = 1$. Then the positive linear functional f_ξ on $\mathcal{D}(p)$ defined by $f_\xi(x) = (\pi(x)\xi|\xi)$, $x \in \mathcal{D}(p)$ belongs to $\mathcal{F}_p$, and so

$$\|\pi(x)\| = \sup_{\|\xi\|=1} f_\xi(x^*x)^{1/2} \leqslant r_{\mathcal{F}_p}(x), \quad x \in \mathcal{D}(p).$$

Hence, we have

$$|x|_p \leqslant r_{\mathcal{F}_p}(x), \quad \forall x \in \mathcal{D}(p).$$

Thus we have $| \cdot |_p = r_{\mathcal{F}_p}$. This completes the proof. ∎

6. SPECTRAL $*$-REPRESENTATIONS AND SPECTRAL UNBOUNDED C^*-SEMINORMS

In this section we define the notion of (hereditary) spectrality of unbounded C^*-seminorms and further define the notion of stable unbounded C^*-seminorms and investigate the relation of spectrality and stability of unbounded C^*-seminorms.

Let $\mathcal{B}$ be a $*$-subalgebra of a $*$-algebra $\mathcal{A}$ with identity $\mathbb{1}$ and the $*$-algebra $\mathcal{B}_{\mathbf{1}}$ obtained by adjoining the identity $\mathbb{1}$ to $\mathcal{B}$ when $\mathcal{B}$ does not have the identity. We denote by $\mathcal{B}^{\rm qr}$ the set of all *quasi-regular* elements x of $\mathcal{B}$, that is, $\mathbb{1} - x$ is invertible in $\mathcal{B}_{\mathbf{1}}$. We have the spectrum $\mathrm{Sp}_{\mathcal{B}}(x)$ and the spectral radius $r_{\mathcal{B}}(x)$ of $x \in \mathcal{B}$ as follows:

$$\mathrm{Sp}_{\mathcal{B}}(x) = \{\lambda \in \mathbb{C} : \nexists(\lambda\mathbb{1} - x)^{-1} \text{ in } \mathcal{B}_{\mathbf{1}}\} \quad \text{and} \quad r_{\mathcal{B}}(x) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}_{\mathcal{B}}(x)\}.$$

By Theorem 3.1 of [21] we have the following

Lemma 6.1. *Let p be an unbounded m^*-seminorm on $\mathcal{A}$. Then the following statements are equivalent:*

(i) $\{x \in \mathcal{D}(p) : p(x) < 1\} \subset \mathcal{D}(p)^{\rm qr}$.

(ii) $r_{\mathcal{D}(p)}(x) \leqslant p(x)$ *for each* $x \in \mathcal{D}(p)$.

(iii) $r_{\mathcal{D}(p)}(x) = \lim_{n\to\infty} p(x^n)^{1/n}$ *for each* $x \in \mathcal{D}(p)$.

In particular, if p is an unbounded C^*-seminorm on $\mathcal{A}$, then the conditions (i) $\sim$ (iii) *are equivalent to*

(iv) $r_{\mathcal{D}(p)}(x) = p(x)$ *for each* $x \in \mathcal{D}(p)$ *with* $x^*x = xx^*$.

We remark that the equivalence of (i) and (ii) in Lemma 6.1 holds for a general unbounded seminorm p.

Definition 6.2. An unbounded m^*- (or C^*-) seminorm p on a $*$-algebra $\mathcal{A}$ is said to be *spectral* if it satisfies one of equivalent conditions (i) $\sim$ (iii) in Lemma 6.1.

Here we need a new notion of hereditary spectral unbounded m^*- (or C^*-) seminorms which plays an important rule in this section.

Definition 6.3. An unbounded m^*- (or C^*-) seminorm p on $\mathcal{A}$ is said to be *hereditary spectral* if for any $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ the restriction $p\lceil\mathcal{B}$ of p to $\mathcal{B}$ is spectral.

The hereditary spectrality of unbounded m^*- (or C^*-) seminorms implies the spectrality, but the converse does not hold in general. For example, if $\mathcal{A}$ is a C^*-algebra, there is a spectral m^*-seminorm on $\mathcal{A}$ which is not hereditary spectral ([23]). According to Palmer ([22] and [23]), a *spectral algebra* $\mathcal{A}$ is an algebra on which there is defined a spectral seminorm with $\mathcal{D}(p) = \mathcal{A}$. A spectral algebra need not be normable, however it is rich enough to admit a satisfactory spectral theory like Banach algebras. A C^*-spectral (hereditary C^*-spectral) algebra which is a $*$-algebra with a spectral (hereditary spectral) C^*-seminorm has been studied in [8]. C^*-spectral (hereditary C^*-spectral) algebras appear to be potential enough to recapture much of the algebraic theory of C^*-algebras. They also help to clarify the notion of local algebras that arises in non-commutative geometry, in particular, smooth structure in C^*-algebras ([10] and [11]). Here we define and characterize unbounded C^*-spectral algebras and unbounded hereditary C^*-spectral algebras.

DEFINITION 6.4. An unbounded C^*-spectral algebra is a $*$-algebra admitting a spectral unbounded C^*-seminorm. An unbounded hereditary C^*-spectral algebra is a $*$-algebra $\mathcal{A}$ admitting a hereditary spectral unbounded C^*-seminorm on $\mathcal{A}$.

We define the notion of (hereditary) spectral $*$-representations and characterize unbounded (hereditary) C^*-spectral algebras by the existence of (hereditary) spectral strongly nondegenerate $*$-representations.

DEFINITION 6.5. Let π be a $*$-representation of $\mathcal{A}$ and $x \in \mathcal{A}$. We define a *spectrum of the closed operator* $\overline{\pi(x)}$ in $C^*(\pi)$ as follows:

$$\mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) = \{\lambda \in \mathbb{C} : (\lambda I - \overline{\pi(x)})^{-1} \text{ does not exist in } C^*(\pi)\}.$$

If $\mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x) := \{\lambda \in \mathbb{C} : (\lambda \mathbb{1} - x)^{-1} \text{ does not exist in } \mathcal{A}_\flat^\pi\} \subset \mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \cup \{0\}$, $\forall x \in \mathcal{A}$, then π is said to be *spectral.* If for any $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ the restriction $\pi\lceil\mathcal{B}$ of π to $\mathcal{B}$ is a spectral $*$-representation of $\mathcal{B}$, then π is said to be *hereditary spectral.*

Let π be a $*$-representation of $\mathcal{A}$. It is easily shown that

$$\mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \cup \{0\} \subset \mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}(\overline{\pi(x)}) \subset \mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x), \quad \forall x \in \mathcal{A}. \tag{6.1}$$

We first characterize the spectrality of bounded $*$-repesentation $\pi_\flat$ of the $*$-algebra $\mathcal{A}_\flat^\pi$.

LEMMA 6.6. *Let π be a $*$-representation of $\mathcal{A}$. Consider the following statements:*

(i) *π is spectral;*

(ii) *$\pi_\flat$ is spectral, that is,* $\mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x) \subset \mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \cup \{0\}$, $\forall x \in \mathcal{A}_\flat^\pi$;

(iii) *r_π is spectral;*

(iv) $\mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x) = \mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}(\overline{\pi(x)})$, $\forall x \in \mathcal{A}_\flat^\pi$ *and the normed $*$-algebra $\overline{\pi(\mathcal{A}_\flat^\pi)}$ with norm r_π is a Q-algebra, that is, $\overline{\pi(\mathcal{A}_\flat^\pi)}^{\mathrm{qr}}$ is open;*

(v) $\mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x) = \mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}(\overline{\pi(x)})$, $\forall x \in \mathcal{A}$.

Then the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v) *hold.*

Proof. (i) $\Rightarrow$ (ii) This is trivial. (ii) $\Rightarrow$ (iii) Suppose $\pi_\flat$ is spectral. Take an arbitrary $x \in \mathcal{A}_\flat^\pi$ with $r_\pi(x) < 1$. Since $\|\overline{\pi(x)}\| < 1, \overline{\pi(x)}$ is quasi-regular in the C^*-algebra $C^*(\pi)$, and so $1 \notin \mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)})$. Since $\pi_\flat$ is spectral, we have $1 \notin \mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x)$, and so $x \in (\mathcal{A}_\flat^\pi)^{\mathrm{qr}}$. Therefore it follows from Lemma 6.1 that r_π is spectral.

(iii) $\Rightarrow$ (ii) Suppose r_π is spectral. Take arbitrary $x \in \mathcal{A}_\flat^\pi$ and $\lambda \neq 0 \in \mathbb{C}$ such that $(\lambda I - \overline{\pi(x)})^{-1} \in C^*(\pi)$. Since $C^*(\pi) = \overline{\pi(\mathcal{A}_\flat^\pi)}^{\|\cdot\|}$, there exists an element $y \in \mathcal{A}_\flat^\pi$ such that $r_\pi\big(\frac{x}{\lambda} + y - \frac{xy}{\lambda}\big) = \big\|I - \big(I - \overline{\pi\big(\frac{1}{\lambda}x\big)}\big)(I - \overline{\pi(y)})\big\| < 1$ and $r_\pi\big(\frac{x}{\lambda} + y - \frac{yx}{\lambda}\big) = \big\|I - (I - \overline{\pi(y)})\big(I - \overline{\pi\big(\frac{1}{\lambda}x\big)}\big)\big\| < 1$.

Since r_π is spectral, it follows from Lemma 6.1 that $\frac{x}{\lambda} + y - \frac{xy}{\lambda} = \mathbb{1} - \big(\mathbb{1} - \frac{1}{\lambda}x\big)(\mathbb{1} - y)$, $\frac{x}{\lambda} + y - \frac{yx}{\lambda} = \mathbb{1} - (\mathbb{1} - y)\big(\mathbb{1} - \frac{1}{\lambda}x\big)$ are contained in $(\mathcal{A}_\flat^\pi)^{\mathrm{qr}}$, and so

$\big(\mathbb{1}-\frac{1}{\lambda}x\big)(\mathbb{1}-y)$ and $(\mathbb{1}-y)\big(\mathbb{1}-\frac{1}{\lambda}x\big)$ are invertible in $\mathcal{A}_\flat^\pi$. Hence, $\mathbb{1}-\frac{1}{\lambda}x$ is invertible in $\mathcal{A}_\flat^\pi$, and so $\lambda \notin \mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x)$.

(ii) $\Rightarrow$ (iv) It follows from (6.1) and the assumption (ii) that

$$\mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \cup \{0\} = \mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}(\overline{\pi(x)}) = \mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x), \quad \forall x \in \mathcal{A}_\flat^\pi.$$

Further, it follows from Proposition 2 of [4] that

$$\begin{aligned} &\mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \cup \{0\} = \mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}(\overline{\pi(x)}), \quad \forall x \in \mathcal{A}_\flat^\pi \\ &\text{if and only if } \overline{\pi(\mathcal{A}_\flat^\pi)} \text{ is a } Q\text{-algebra.} \end{aligned} \tag{6.2}$$

Hence, the statement (iv) holds.

(iv) $\Rightarrow$ (ii) This follows from (6.2) and the assumption (iv).

(ii) $\Rightarrow$ (v) Take arbitrary $x \in \mathcal{A}$ and $\lambda \neq 0 \in \mathbb{C}$ such that $(\lambda I - \overline{\pi(x)})^{-1} \in \overline{\pi(\mathcal{A}_\flat^\pi)}$. Then there exists an element y of $\mathcal{A}_\flat^\pi$ such that $\big(I - \pi(y)\big)\big(I - \pi\big(\frac{x}{\lambda}\big)\big) = \big(I - \pi\big(\frac{x}{\lambda}\big)\big)\big(I - \pi(y)\big) = I$, and so $\pi\big(\frac{x}{\lambda} + y - \frac{yx}{\lambda}\big) = \pi\big(\frac{x}{\lambda} + y - \frac{xy}{\lambda}\big) = 0$. Hence, $1 \notin \mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}\big(\overline{\pi\big(\frac{x}{\lambda} + y - \frac{yx}{\lambda}\big)}\big) \cup \mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}\big(\overline{\pi\big(\frac{x}{\lambda} + y - \frac{xy}{\lambda}\big)}\big)$. Since $\mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}(\overline{\pi(a)}) \subset \mathrm{Sp}_{C^*(\pi)}(\overline{\pi(a)})$ for each $a \in \mathcal{A}_\flat^\pi$, it follows from (ii) that $1 \notin \mathrm{Sp}_{\mathcal{A}_\flat^\pi}\big(\frac{x}{\lambda} + y - \frac{yx}{\lambda}\big)$ and $1 \notin \mathrm{Sp}_{\mathcal{A}_\flat^\pi}\big(\frac{x}{\lambda} + y - \frac{xy}{\lambda}\big)$, and so there exist elements z_1 and z_2 of $\mathcal{A}_\flat^\pi$ such that $(\mathbb{1} - z_1)(\mathbb{1} - y)\big(\mathbb{1} - \frac{x}{\lambda}\big) = \mathbb{1}$ and $\big(\mathbb{1} - \frac{x}{\lambda}\big)(\mathbb{1} - y)(\mathbb{1} - z_2) = \mathbb{1}$. Hence we have $\frac{x}{\lambda} \in (\mathcal{A}_\flat^\pi)^{\mathrm{qr}}$ and so $\lambda \notin \mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x)$. This completes the proof. ∎

LEMMA 6.7. *Let $\mathcal{A}$ be a $*$-representation of $\mathcal{A}$. Then the following statements are equivalent:*

(i) *$\pi_\flat$ is hereditary spectral;*

(ii) *r_π is a hereditary spectral unbounded C^*-seminorm on $\mathcal{A}$.*

Proof. This is proved similarly to the proof of (ii) $\Leftrightarrow$ (iii) in Lemma 6.6. ∎

THEOREM 6.8. *The following statements are equivalent:*

(i) *There exists a strongly nondegenerate $*$-representation π of $\mathcal{A}$ such that $\pi_\flat$ is (hereditary) spectral.*

(ii) *There exists a maximal, weakly semifinite, (hereditary) spectral unbounded C^*-seminorm on $\mathcal{A}$.*

Proof. (i) $\Rightarrow$ (ii) Let π be a strongly nondegenerate $*$-representation of $\mathcal{A}$ such that $\pi_\flat$ is (hereditary) spectral. By Proposition 3.1 and Corollary 3.5, r_π is a maximal, weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$. Further, it follows from Lemmas 6.6 and 6.7 that r_π is (hereditary) spectral.

(ii) $\Rightarrow$ (i) Let p be a maximal, weakly semifinite, (hereditary) spectral unbounded C^*-seminorm on $\mathcal{A}$. Then there exists an element π of $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ such that $p = r_\pi$. By Proposition 2.6 (1), π is strongly nondegenerate. Further, it follows from Lemmas 6.6 and 6.7 that π is (hereditary) spectral. This completes the proof. ∎

We next generalize the following property (stability) of C^*-algebras ([12], Proposition 2.10.2) to general $*$-algebras, and characterize it by the hereditary spectrality of unbounded C^*-seminorms.

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ any closed $*$-subalgebra of $\mathcal{A}$. For any $*$-representation π of $\mathcal{B}$ on a Hilbert space $\mathcal{H}_\pi$ there exists a $*$-representation $\widehat{\pi}$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\widehat{\pi}}$ such that $\mathcal{H}_{\widehat{\pi}} \supset \mathcal{H}_\pi$ as a closed subspace and $\pi(x) = \widehat{\pi}(x)\lceil\mathcal{H}_\pi$ for each $x \in \mathcal{B}$.

DEFINITION 6.9. An unbounded m^*-(or C^*-)seminorms p is said to be *stable* if for any $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ and any $*$-representation π of $\mathcal{B}$ such that $\mathcal{B}\cap\mathcal{D}(p) \subset \mathcal{B}^\pi_\flat$ and $[\pi(\mathcal{B}\cap\mathcal{D}(p))\mathcal{D}(\pi)] = \mathcal{H}_\pi$ there exists a $*$-representation ϱ of $\mathcal{A}$ such that $\mathcal{D}(p) \subset \mathcal{A}^\varrho_\flat$, $[\varrho(\mathcal{D}(p))\mathcal{D}(\varrho)] = \mathcal{H}_\varrho$, $\mathcal{H}_\varrho$ contains $\mathcal{H}_\pi$ as a closed subspace and $\overline{\pi(x)} = \overline{\varrho(x)}\lceil\mathcal{H}_\pi$ for each $x \in \mathcal{B}\cap\mathcal{D}(p)$.

The following is one of main results of the paper.

THEOREM 6.10. *Let $\mathcal{A}$ be a $*$-algebra and p a semifinite unbounded C^*-seminorm on $\mathcal{A}$. Then the following statements are equivalent:*
(i) *p is hereditary spectral;*
(ii) *p is spectral and stable.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{B}$ be a $*$-subalgebra of $\mathcal{A}$ and let π be a $*$-representation of $\mathcal{B}$ such that $\mathcal{B}\cap\mathcal{D}(p) \subset \mathcal{B}^\pi_\flat$ and $[\overline{\pi(\mathcal{B}\cap\mathcal{D}(p))}\mathcal{H}_\pi] = \mathcal{H}_\pi$. Since p is hereditary spectral, it follows that

$$\overline{\lim_{n\to\infty}}\|\overline{\pi(x)}^n\|^{\frac{1}{n}} = r_{C^*(\pi)}(\overline{\pi(x)}) \leqslant r_{\overline{\pi(\mathcal{B}\cap\mathcal{D}(p))}}(\overline{\pi(x)}) = r_{\mathcal{B}\cap\mathcal{D}(p)}(x) \leqslant p(x)$$

for each $x \in \mathcal{B}\cap\mathcal{D}(p)$, which implies that $\|\overline{\pi(h)}\| \leqslant p(h)$ for each $h^* = h \in \mathcal{B}\cap\mathcal{D}(p)$. Then, for any $x \in \mathcal{B}\cap\mathcal{D}(p)$ we have

$$\|\overline{\pi(x)}\|^2 = \|\overline{\pi(x^*x)}\| \leqslant p(x^*x) = p(x)^2,$$

and so

$$\|\overline{\pi(x)}\| \leqslant p(x) \quad \text{for each } x \in \mathcal{B}\cap\mathcal{D}(p). \tag{6.3}$$

By the semifiniteness of p we have $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p) \neq \phi$. Let $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$ and put

$$\widetilde{\varrho}_0(\overline{\pi_p(x)}) = \overline{\pi(x)}, \quad x \in \mathcal{B}\cap\mathcal{D}(p).$$

It follows from Proposition 2.6 and (6.3) that

$$\|\widetilde{\varrho}_0(\overline{\pi_p(x)})\| \leqslant p(x) = \|\overline{\pi_p(x)}\| \tag{6.4}$$

for each $x \in \mathcal{B}\cap\mathcal{D}(p)$, and hence $\widetilde{\varrho}_0$ can be extended to a $*$-representation of the C^*-algebra $\overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))}^{\|\cdot\|}$ on $\mathcal{H}_\pi$ and it is denoted by the same $\widetilde{\varrho}_0$. By the stability of C^*-algebras there exists a Hilbert space $\mathcal{H}_{\widetilde{\varrho}}$ containing $\mathcal{H}_\pi$ as a closed subspace and a $*$-representation $\widetilde{\varrho}$ of the C^*-algebra $\overline{\pi_p(\mathcal{D}(p))}^{\|\cdot\|}$ on $\mathcal{H}_{\widetilde{\varrho}}$ such that $\widetilde{\varrho}(A)\lceil\mathcal{H}_\pi = \widetilde{\varrho}_0(A)$ for each $A \in \overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))}^{\|\cdot\|}$. We here put

$$\begin{cases} \mathcal{D}(\varrho) = \text{ linear span of } \{\widetilde{\varrho}(\overline{\pi_p(x)})\xi : x \in \mathfrak{N}_p,\ \xi \in \mathcal{H}_{\widetilde{\varrho}}\}, \\ \varrho(a)\widetilde{\varrho}(\overline{\pi_p(x)})\xi = \widetilde{\varrho}(\overline{\pi_p(ax)})\xi \quad \text{for } a \in \mathcal{A},\ x \in \mathfrak{N}_p,\ \xi \in \mathcal{H}_{\widetilde{\varrho}}. \end{cases}$$

Then it is easily shown that ϱ is a $*$-representation of $\mathcal{A}$ on $\mathcal{D}(\varrho)$ in $\mathcal{H}_\varrho := \overline{\mathcal{D}(\varrho)}$. Since p is semifinite, it follows that $\mathcal{H}_\varrho = [\widetilde{\varrho}(\overline{\pi_p(\mathcal{D}(p))})\mathcal{H}_{\tilde{\varrho}}]$, so that

$$\mathcal{H}_\pi = [\overline{\pi(\mathcal{B}\cap\mathcal{D}(p))}\mathcal{H}_\pi] = [\widetilde{\varrho}_0(\overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))})\mathcal{H}_\pi] = [\widetilde{\varrho}(\overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))})\mathcal{H}_\pi] \subset \mathcal{H}_\varrho.$$

By the definition of ϱ we have $\mathcal{D}(p) \subset \mathcal{A}_\flat^\varrho$ and $\overline{\varrho(x)}\lceil\mathcal{H}_\pi = \widetilde{\varrho}(\overline{\pi_p(x)})\lceil\mathcal{H}_\pi = \widetilde{\varrho}_0(\overline{\pi_p(x)}) = \overline{\pi(x)}$ for each $x \in \mathcal{B}\cap\mathcal{D}(p)$. Further, since p is semifinite, it follows from Proposition 2.6 (4) that $[\overline{\varrho(\mathcal{D}(p))}\mathcal{H}_\varrho] = \mathcal{H}_\varrho$. Thus we have that p is stable.

(ii) $\Rightarrow$ (i) Let $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ and $\mathcal{B}$ be any $*$-subalgebra of $\mathcal{A}$. We first show that

$$\mathrm{Sp}_{\mathcal{B}\cap\mathcal{D}(p)}(b) \cap \mathbb{R} \subset \mathrm{Sp}_{\overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))}^{\|\cdot\|}}(\overline{\pi_p(b)}) \cup \{0\} \tag{6.5}$$

for each $b^* = b \in \mathcal{B}\cap\mathcal{D}(p)$. Let $b^* = b \in \mathcal{B}\cap\mathcal{D}(p)$ and $0 \neq \lambda \in \mathrm{Sp}_{\mathcal{B}\cap\mathcal{D}(p)}(b)\cap\mathbb{R}$. Let $\mathcal{C}$ be the $*$-subalgebra of $\mathcal{B}\cap\mathcal{D}(p)$ generated by b. Then $\mathcal{C}\left(\frac{1}{\lambda}b - \mathbb{1}\right)$ is a proper modular $*$-ideal of $\mathcal{C}$ with modular identity $u := \frac{1}{\lambda}b$. Hence there exists a maximal modular $*$-ideal $\mathfrak{M}$ of $\mathcal{C}$ containing $\mathcal{C}\left(\frac{1}{\lambda}b - \mathbb{1}\right)$. Then the quotient algebra $\mathcal{C}/\mathfrak{M}$ is isomorphic to $\mathbb{C}$. In fact, since $u^k - u \in \mathfrak{M}$ for all $k \in \mathbb{N}$, it follows that $x + \mathfrak{M} = \sum_k \alpha_k\lambda^k u + \mathfrak{M}$ for any $x = \sum_k \alpha_k b^k \in \mathcal{C}$. Thus $\mathcal{C}/\mathfrak{M} = \{\alpha u + \mathfrak{M} : \alpha \in \mathbb{C}\}$, and $\tau : \alpha u + \mathfrak{M} \to \alpha$ gives a $*$-isomorphism of $\mathcal{C}/\mathfrak{M}$ onto $\mathbb{C}$. Let $\imath : \mathcal{C} \to \mathcal{C}/\mathfrak{M}$, $\imath(x) = x + \mathfrak{M}$. Let $\pi = \tau\circ\imath$; thus, $\pi\Big(\sum_k \alpha_k b^k\Big) = \sum_k \alpha_k\lambda^k$. Then π is a 1-dimensional $*$-representation of $\mathcal{C}$ such that $\pi(b) = \lambda$. By the stability of p there exists a $*$-representation ϱ of $\mathcal{A}$ such that

$$\mathcal{A}_\flat^\varrho \supset \mathcal{D}(p), [\overline{\varrho(\mathcal{D}(p))}\mathcal{H}_\varrho] = \mathcal{H}_\varrho \quad \text{and} \quad \varrho(b)\lceil\mathbb{C} = \pi(b) = \lambda. \tag{6.6}$$

Since p is spectral and (6.6), we have

$$\|\overline{\varrho(h)}\| = r_{C^*(\varrho)}(\overline{\varrho(h)}) \leqslant r_{\mathcal{D}(p)}(h) \leqslant p(h)$$

for each $h^* = h \in \mathcal{D}(p)$, which implies

$$\|\overline{\varrho(x)}\|^2 = \|\overline{\varrho(x^*x)}\| \leqslant p(x^*x) \leqslant p(x)^2$$

for each $x \in \mathcal{D}(p)$. Hence it follows from Proposition 2.6 that

$$\|\overline{\varrho(x)}\| \leqslant p(x) = \|\overline{\pi_p(x)}\| \tag{6.7}$$

for each $x \in \mathcal{D}(p)$. Hence, $\overline{\pi_p(x)} \mapsto \overline{\varrho(x)}$ can be extended to a $*$-representation of the C^*-algebra $\overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))}^{\|\cdot\|}$, which implies by (6.6) that

$$\lambda = \pi(b) \in \mathrm{Sp}_{\overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))}^{\|\cdot\|}}(\overline{\pi_p(b)}).$$

We next show

$$\mathrm{Sp}_{\mathcal{B}\cap\mathcal{D}(p)}(x) \subset \{\lambda \in \mathbb{C} : |\lambda| < p(x)\}, \quad \forall x \in \mathcal{B}\cap\mathcal{D}(p). \tag{6.8}$$

Let $x \in \mathcal{B} \cap \mathcal{D}(p)$ and $|\lambda| > p(x) = \|\overline{\pi_p(x)}\|$. Then $(\lambda I - \overline{\pi_p(x)})^*(\lambda I - \overline{\pi_p(x)})$ is invertible in $\big(\overline{\pi_p(\mathcal{B} \cap \mathcal{D}(p))}^{\|\cdot\|}\big)_I$, and so

$$|\lambda|^2 \notin \mathrm{Sp}_{\overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))}^{\|\cdot\|}}(\overline{\pi_p(\lambda x^* + \overline{\lambda} x - x^* x)}).$$

Hence it follows from (6.5) that $|\lambda|^2 \notin \mathrm{Sp}_{\mathcal{B}\cap\mathcal{D}(p)}(\lambda x^* + \overline{\lambda} x - x^* x)$, which implies $(\lambda \mathbb{1} - x)^*(\lambda \mathbb{1} - x)$ is invertible in $(\mathcal{B} \cap \mathcal{D}(p))_{\mathbf{1}}$. Similarly, $(\lambda \mathbb{1} - x)(\lambda \mathbb{1} - x)^*$ is invertible in $(\mathcal{B} \cap \mathcal{D}(p))_{\mathbf{1}}$. Thus, we have $\lambda \notin \mathrm{Sp}_{\mathcal{B}\cap\mathcal{D}(p)}(x)$. It follows from (6.8) that $r_{\mathcal{B}\cap\mathcal{D}(p)}(x) \leqslant p(x)$ for each $x \in \mathcal{B} \cap \mathcal{D}(p)$, which means that p is hereditary spectral. This completes the proof. ∎

REMARK 6.11. As seen in the proof of Theorem 6.10, the implication (ii) $\Rightarrow$ (i) in Theorem 6.10 holds under the assumption of weak semifiniteness of the unbounded C^*-seminorm p instead of that of the semifiniteness.

We consider the case of unbounded m^*-seminorms.

PROPOSITION 6.12. *Let p be a semifinite representable unbounded m^*-seminorm on a $*$-algebra $\mathcal{A}$ and $|\cdot|_p$ the unbounded Gelfand-Naimark C^*-seminorm of p. Then the following statements are equivalent:*
(i) *$|\cdot|_p$ is hereditary spectral;*
(ii) *$|\cdot|_p$ is spectral and stable;*
(iii) *p is spectral and stable.*
If this is true, then p is hereditary spectral.

Proof. Since $\mathcal{D}(p) = \mathcal{D}(|\cdot|_p)$ and $|\cdot|_p \leqslant p$ on $\mathcal{D}(p)$, it follows that $|\cdot|_p$ is semifinite, and p is stable if and only if $|\cdot|_p$ is stable, which implies by Theorem 6.10 that the statements (i) and (ii) are equivalent, and the implication (ii) $\Rightarrow$ (iii) holds. We show the implication (iii) $\Rightarrow$ (ii). Since $|\cdot|_p$ is a semifinite unbounded C^*-seminorm on $\mathcal{A}$, there exists a $*$-representation π_p of $\mathcal{A}$ such that $\|\overline{\pi_p(x)}\| = |x|_p$ for each $x \in \mathcal{D}(|\cdot|_p) = \mathcal{D}(p)$. It is shown similarly to the proof of (ii) $\Rightarrow$ (i) in Theorem 6.10 that $|\cdot|_p$ is spectral. Here we note simply the proof. Take arbitrary $h^* = h \in \mathcal{D}(p)$ and $\lambda \neq 0 \in \mathrm{Sp}_{\mathcal{D}(p)}(h) \cap \mathbb{R}$. By the stablity of p there exists a $*$-representation ϱ of $\mathcal{A}$ such that $\mathcal{A}_\flat^\varrho \supset \mathcal{D}(p)$, $[\varrho(\mathcal{D}(p))\mathcal{H}_\varrho] = \mathcal{H}_\varrho$ and $\varrho(h)\lceil\mathbb{C} = \lambda$. Further, it follows from the spectrality of p that $\|\overline{\varrho(x)}\| \leqslant p(x)$ for each $x \in \mathcal{D}(p)$, which implies that $\varrho\lceil\mathcal{D}(p) \in \mathrm{Rep}(p)$. Hence we have

$$\|\overline{\varrho(x)}\| \leqslant |x|_p = \|\overline{\pi_p(x)}\|, \qquad \forall x \in \mathcal{D}(p),$$

which implies $\lambda \in \mathrm{Sp}_{\overline{\pi_p(\mathcal{D}(p))}^{\|\cdot\|}}(\overline{\pi_p(h)})$. Hence we have

$$\mathrm{Sp}_{\mathcal{D}(p)}(h) \cap \mathbb{R} \subset \mathrm{Sp}_{\overline{\pi_p(\mathcal{D}(p))}^{\|\cdot\|}}(\overline{\pi_p(h)}) \cup \{0\},$$

which implies

$$\mathrm{Sp}_{\mathcal{D}(p)}(x) \subset \{\lambda \in \mathbb{C} : |\lambda| < |x|_p\}, \qquad \forall x \in \mathcal{D}(p).$$

Hence it follows that $r_{\mathcal{D}(p)}(x) \leqslant |x|_p$ for each $x \in \mathcal{D}(p)$. Thus, $|\cdot|_p$ is spectral. This completes the proof. ∎

The implication (iii) $\Rightarrow$ (i) in Proposition 6.13 holds under a weaker assumption than that of semifiniteness of p as follows:

Corollary 6.13. *Suppose p is a spectral, stable, representable unbounded m^*-seminorm on $\mathcal{A}$ such that $\mathfrak{N}_p$ is dense in $\mathcal{D}(p)$ with respect to any r_π ($\pi \in \mathrm{Rep}(p)$). Then $|\cdot|_p$ is hereditary spectral and $\mathcal{A}$ is an unbounded hereditary C^*-spectral algebra.*

Proof. By Proposition 5.1, $|\cdot|_p$ is weakly semifinite and there exists a $*$-representation π_p of $\mathcal{A}$ such that $\|\overline{\pi_p(x)}\| = |x|_p$ for each $x \in \mathcal{D}(p)$. Hence it is shown in the same way as the proof (iii) $\Rightarrow$ (ii) in Proposition 6.12 that $|\cdot|_p$ is spectral, which implies by Proposition 6.12 that $|\cdot|_p$ is hereditary spectral. ∎

7. EXAMPLES

We give some examples of unbounded C^*-seminorms on $*$-algebras.

Example 7.1. A locally convex $*$-algebra is a $*$-algebra which is also a Hausdorff locally convex space such that the multiplication is separately continuous and the involution is continuous. Let $\mathcal{A}$ be a locally convex $*$-algebra with identity $\mathbb{1}$. We denote by $\mathfrak{B}$ the collection of closed, bounded absolutely convex subsets $\mathcal{B}$ of $\mathcal{A}$ satisfying $\mathbb{1} \in \mathcal{B}$ and $\mathcal{B}^2 \subset \mathcal{B}$. For every $\mathcal{B} \in \mathfrak{B}$, the linear span $\mathcal{A}[\mathcal{B}]$ of $\mathcal{B}$ forms a normed algebra equipped with the Minkowski functional $\|\cdot\|_{\mathcal{B}}$ of $\mathcal{B}$. If $\mathcal{A}[\mathcal{B}]$ is complete for every $\mathcal{B} \in \mathfrak{B}$, then $\mathcal{A}$ is said to be *pseudo-complete*. If $\mathcal{A}$ is sequentially complete, then it is pseudo-complete. An element x of $\mathcal{A}$ is *bounded* if $\{(\lambda x)^n : n \in \mathbb{N}\}$ is bounded for some $\lambda \in \mathbb{C}$, and denote by $\mathcal{A}_0$ the set of all bounded elements of $\mathcal{A}$. G.R. Allan ([2]) and P.G. Dixon ([13]) defined the notion of GB*-algebra which is a generalization of C^*-algebra. A pseudo-complete locally convex $*$-algebra $\mathcal{A}$ is said to be a GB**-algebra over* $\mathcal{B}_0$ if $\mathcal{B}_0$ is the greatest member in $\mathfrak{B}^* := \{\mathcal{B} \in \mathfrak{B}^* : \mathcal{B}^* = \mathcal{B}\}$ and $(\mathbb{1} + x^*x)^{-1} \in \mathcal{A}[\mathcal{B}_0]$ for every $x \in \mathcal{A}$. Then $\mathcal{A}[\mathcal{B}_0]$ is a C^*-algebra with the C^*-norm $\|\cdot\|_{\mathcal{B}_0}$. We put

$$\mathcal{D}(p_{\mathrm{GB}^*}) = \mathcal{A}[\mathcal{B}_0] \quad \text{and} \quad p_{\mathrm{GB}^*}(x) = \|x\|_{\mathcal{B}_0}, \quad x \in \mathcal{A}[\mathcal{B}_0].$$

Then p_{GB^*} is a spectral unbounded C^*-norm on $\mathcal{A}$. Hence every GB*-algebra is an unbounded C^*-spectral algebra. We consider the following questions:

(1.) When does p_{GB^*} satisfy the condition $\mathfrak{N}_{p_{\mathrm{GB}^*}} \not\subset N_{p_{\mathrm{GB}^*}}$ (equivalently $\mathfrak{N}_{p_{\mathrm{GB}^*}} \neq \{0\}$)?

(2.) When is p_{GB^*} semifinite or weakly semifinite?

(3.) When does there exist a family $\{p_\lambda\}_{\lambda \in \Lambda}$ of seminorms determining the topology such that $p_{\mathrm{GB}^*} = \sup_{\lambda \in \Lambda} p_\lambda$?

Let $\mathfrak{M}$ be a left ideal of a GB*-algebra $\mathcal{A}$ contained in $\mathcal{A}[\mathcal{B}_0]$. Suppose $\mathfrak{M}$ is dense in the C^*-algebra $\mathcal{A}[\mathcal{B}_0]$. By standard C^*-algebra theory, $\mathfrak{M}$ contains a bounded approximate identity $\{u_\alpha\}$ for the C^*-algebra $\mathcal{A}[\mathcal{B}_0], u_\alpha^* = u_\alpha, \|u_\alpha\|_{\mathcal{B}_0} \leqslant 1$ for all α. By the proof of Theorem 3.6 in [5] (see also [24], Proposition 3.11 for a particular case), $\{u_\alpha\}$ is a bounded approximate identity for $\mathcal{A}$. Since $\mathfrak{M} \subset \mathfrak{N}_{p_{\mathrm{GB}^*}}$, it follows that p_{GB^*} is uniformly semifinite. Let π be any $*$-representation of $\mathcal{A}$ having $\mathcal{A}_\flat^\pi = \mathcal{A}[\mathcal{B}_0]$. Let $r_\pi(x) = \|\pi(x)\|$ for $x \in \mathcal{D}(r_\pi) = \mathcal{A}_\flat^\pi$. Since $\mathfrak{M} \subset \mathfrak{N}_{r_\pi}$, it follows from Proposition 3.1 that $\widetilde{\pi_{r_\pi}^N} = \widetilde{\pi}$. Here we consider the cases of pro-C^*-algebras and C^*-like locally convex $*$-algebras which are important in GB*-algebras.

(1) A complete locally convex $*$-algebra $\mathcal{A}[\tau]$ is said to be a pro-C^*-algebra ([24]) if the topology τ is determined by a direct family $\{p_\lambda\}_{\lambda\in\Lambda}$ of C^*-seminorms. Then $\mathcal{A}$ is a GB*-algebra over $\mathcal{B}_0 = \mathcal{U}(\sup_{\lambda\in\Lambda} p_\lambda) := \{x \in \mathcal{A} : \sup_{\lambda\in\Lambda} p_\lambda(x) \leqslant 1\}$ with $p_{\mathrm{GB}^*} = \sup_{\lambda\in\Lambda} p_\lambda$

(a) Let X be a locally compact non-compact Hausdorff-space and $\mathcal{A} = C(X)$ is a locally convex $*$-algebra of all complex-valued continuous functions on X with the compact open topology. The compact open topology is defined by a family $\{p_M : M \text{ is a compact subset of } X\} : p_M(f) = \sup_{x\in M} |f(x)|$, $f \in C(X)$. Then $\mathcal{A}$ is a pro-C^*-algebra and $\mathcal{A}[\mathcal{B}_0]$ equals the C^*-algebra $(C_\flat(X), \|\cdot\|_\infty)$ of all bounded continuous functions on X. Since $C_c(X) := \{f \in C_\flat(X) : \mathrm{supp} f \text{ is compact}\} \subset \mathfrak{N}_{p_{\mathrm{GB}^*}}$, it follows that $\mathfrak{N}_{p_{\mathrm{GB}^*}}$ is dense in $\mathcal{D}(p_{\mathrm{GB}^*})$ with respect to the compact open topology, but p_{GB^*} is not semifinite in general. For example, when $X = \mathbb{R}$, p_{GB^*} is maximal and weakly semifinite, but not semifinite.

(b) Let X be a σ-finite measure space and $\mathcal{A} = L^\infty_{\mathrm{loc}}(X)$ is a locally convex $*$-algebra of all measurable functions which are essentially bounded on every set of finite measure equipped with the topology defined by the family of C^*-seminorms $\{\|\cdot\|_A : \|f\|_A = \operatorname{ess\,sup}_{x\in A} |f(x)|$, where $A \subset X$ is any set of finite measure$\}$. Then $\mathcal{A}$ is a pro-C^*-algebra and a GB*-algebra having $\mathcal{A}[\mathcal{B}_0] = L^\infty(X)$ and $p_{\mathrm{GB}^*}(f) = \|f\|_\infty := \sup_A \|f\|_A$, $f \in L^\infty(X)$. Since

$$L^\infty_{\mathrm{c}}(X) := \{f \in L^\infty_{\mathrm{loc}}(X) : \mathrm{supp}\, f \text{ is contained in some set of finite measure}\} \subset \mathfrak{N}_{p_{\mathrm{GB}^*}},$$

it follows that $\mathfrak{N}_{p_{\mathrm{GB}^*}}$ is dense in $\mathcal{D}(p_{\mathrm{GB}^*})$ with respect to the locally convex topology and p_{GB^*} is maximal and weakly semifinite.

(c) Let $\mathcal{B}$ be a C^*-algebra without identity. Let $K_{\mathcal{B}}$ be the Pedersen ideal of $\mathcal{B}$, $M(\mathcal{B})$ be the C^*-algebra of all multipliers of $\mathcal{B}$, and $\mathcal{A} = \Gamma(K_{\mathcal{B}})$ be the $*$-algebra of all multipliers of $K_{\mathcal{B}}$ ([15] and [25]). Let p be any C^*-seminorm on $\mathcal{B}$. Then p can be regarded as an unbounded C^*-seminorm on $\mathcal{A}$ with $\mathcal{D}(p) = \mathcal{B}$. Since $K_{\mathcal{B}}$ is a $*$-ideal of $\mathcal{A}$ and it is dense in $\mathcal{B}$, it follows that $K_{\mathcal{B}} \subset \mathfrak{N}_p$ and p is uniformly semifinite. In fact, $\mathcal{A}$ is a pro-C^*-algebra with appropriate topology.

(2) A complete locally convex $*$-algebra $\mathcal{A}[\tau]$ is said to be *C^*-like* if there exists a C^*-like family $\{p_\lambda\}_{\lambda\in\Lambda}$ of seminorms determining the topology τ such that $\mathcal{D}\big(\sup_{\lambda\in\Lambda} p_\lambda\big) := \big\{x \in \mathcal{A} : \sup_{\lambda\in\Lambda} p_\lambda(x) < \infty\big\}$ is τ-dense in $\mathcal{A}$. Here we say that $\{p_\lambda\}_{\lambda\in\Lambda}$ is *C^*-like* if for any $\lambda \in \Lambda$ there exists $\lambda' \in \Lambda$ such that $p_\lambda(xy) \leqslant p_{\lambda'}(x)p_{\lambda'}(y)$, $p_\lambda(x^*) = p_\lambda(x)$ and $p_\lambda(x)^2 \leqslant p_{\lambda'}(x^*x)$ for each $x, y \in \mathcal{A}$. It follows from ([18], Theorem 2.1) that $\mathcal{A}$ is a GB*-algebra over $\mathcal{B}_0 = \mathcal{U}(\sup_{\lambda\in\Lambda} p_\lambda)$ with $p_{\mathrm{GB}^*} = \sup_{\lambda\in\Lambda} p_\lambda$. Let $\mathcal{A} = L^\omega[0,1] := \bigcap_{1\leqslant p<\infty} L^p[0,1]$ be the Arens GB*-algebra equipped with the topology defined by the family of L^p-norms ([3]). Then $\mathcal{A}$ is a C^*-like locally convex $*$-algebra with the C^*-like family $\{\|\cdot\|_p : 1 \leqslant p < \infty\}$ of seminorms, and

$\mathcal{A}[\mathcal{B}_0] = L^\infty[0,1]$ and $p_{\mathrm{GB}^*} = \sup_{1 \leqslant p < \infty} \| \cdot \|_p$. But, $L^\omega[0,1]$ is not a pro-C^*-algebra and $\mathfrak{N}_{p_{\mathrm{GB}^*}} = \{0\}$. Here is a non-commutative analogue of this ([17]). Let $\mathcal{M}_0$ be a von Neumann algebra with a faithful normal tracial state φ. Let $L^p(\mathcal{M}_0, \varphi)(1 \leqslant p \leqslant \infty)$ be the Segal L^p-space ([30]). Then $L^p(\mathcal{M}_0, \varphi)$ is a Banach space of closed operators in $\mathcal{H}$ affiliated with $\mathcal{M}_0$ with L^p-norm $\|X\|_p := \varphi(|X|^p)^{1/p}$. For $1 \leqslant r \leqslant p$, $L^\infty(\mathcal{M}_0, \varphi) = \mathcal{M}_0 \subset L^p(\mathcal{M}_0, \varphi) \subset L^r(\mathcal{M}_0, \varphi) \subset L^1(\mathcal{M}_0, \varphi)$. By using non-commutative Hölder's inequality it follows that $L^\omega(\mathcal{M}_0, \varphi) := \bigcap_{1 \leqslant p < \infty} L^p(\mathcal{M}_0, \varphi)$ is a $*$-algebra with identity and with strong operators : $\overline{X+Y}, \overline{\lambda X}, \overline{XY}$ and operator adjoint as the involution. Let τ_ω be the topology on $L^\omega(\mathcal{M}_0, \varphi)$ defined by the C^*-like family $\Gamma = \{\| \cdot \|_p : 1 \leqslant p < \infty\}$. Then $L^\omega(\mathcal{M}_0, \varphi)$ is a C^*-like locally convex $*$-algebra with $p_{\mathrm{GB}^*}(X) = \sup_{n \in \mathbb{N}} \|X\|_n = \|X\|_\infty$ (operator-norm).

Example 7.2. We consider Köthe sequence spaces and convolution algebras.

(1) Let ω denote the set of all sequences of complex numbers. Let $\mathcal{P}$ be a set of positive sequences $a = \{a_n\}$ in ω satisfying

(i) $\forall \{a_n\}, \{b_n\} \in \mathcal{P}$, $\exists \{c_n\} \in \mathcal{P}$; $a_n \leqslant c_n$, $b_n \leqslant c_n$, $n \in \mathbb{N}$;
(ii) $a_n > 0$, $\forall n \in \mathbb{N}$ for $\forall \{a_n\} \in \mathcal{P}$;
(iii) $a_{n+1} \leqslant a_n$, $\forall n \in \mathbb{N}$ for $\forall \{a_n\} \in \mathcal{P}$;
(iv) ${}^\forall\{a_n\} \in \mathcal{P}$, $\exists \{d_n\} \in \mathcal{P}$; $a_n \leqslant d_n^2$, $\forall n \in \mathbb{N}$.

Let $1 \leqslant q < \infty$. The Köthe sequence space $\ell^q(\mathcal{P})$ is defined as

$$\ell^q(\mathcal{P}) = \Big\{ x = \{x_n\} \in \omega : p_a^q(x) := \Big(\sum_n |x_n|^q a_n^q \Big)^{1/q} = \|xa\|_q < \infty, \forall a \in \mathcal{P} \Big\}.$$

$\ell^q(\mathcal{P})$ is a complete locally convex $*$-algebra (pointwise operations, complex conjugation) with respect to the topology $\tau_{\mathcal{P}}^q$ defined by seminorms $\{p_a^q : a \in \mathcal{P}\}$ ([6]). It is clear that $\mathcal{P} \subset \ell^\infty$ and $\ell^q(\mathcal{P})$ contains ℓ^q as a dense $*$-subalgebra. Further, it follows from (iv) that for any $a \in \mathcal{P}$, $p_a^q(xy) \leqslant p_d^q(x) p_d^q(y)$ and $p_a^q(x^*) = p_a^q(x)$ for each $x, y \in \ell^q(\mathcal{P})$, which implies that $\sup_{a \in \mathcal{P}} p_a^q$ is a spectral unbounded m^*-norm on $\ell^q(\mathcal{P})$. Let $q = \infty$. Then

$$\ell^\infty(\mathcal{P}) := \big\{ x = \{x_n\} \in \omega : p_a^\infty(x) = \|xa\|_\infty < \infty, \forall a \in \mathcal{P} \big\}$$

is a C^*-like locally convex $*$-algebra with the C^*-like direct family $\{p_a^\infty : a \in \mathcal{P}\}$ of seminorms. Hence $\sup_{a \in \mathcal{P}} p_a^\infty$ is a spectral unbounded C^*-norm on $\ell^\infty(\mathcal{P})$.

Further, suppose
(v) $\|a\|_\infty \leqslant 1$ for $\forall a \in \mathcal{P}$.
Then since $\mathcal{D}(\sup_{a \in \mathcal{P}} p_a^q) \supset \ell^q$ and

$$\mathfrak{N}_{\sup_{a \in \mathcal{P}} p_a^q} \supset \mathcal{F} := \{x = \{x_n\} \in \omega : x_n = 0 \text{ except for finite many } n\},$$

it follows that $\sup_{a \in \mathcal{P}} p_a^q$ is semifinite. Similarly, $\sup_{a \in \mathcal{P}} p_a^\infty$ is semifinite. Here is an important special case. Let

$$s = \{x = \big\{x_n\big\} \in \omega : \{n^k x_n\} \in \ell^\infty, \forall k \in \mathbb{N}\}$$

be the $*$-algebra consisting of all rapidly decreasing sequences. Then

$$\mathcal{P} := \Big\{\{|x_n|\} : \{x_n\} \in s, \sup_n |x_n| \leqslant 1 \text{ and } |x_{n+1}| \leqslant |x_n|, \forall n \in \mathbb{N}\Big\}$$

satisfies the condition (i)-(v). Then we have

$$\begin{aligned}\ell^1(\mathcal{P}) &= \{x = \{x_n\} \in \omega : \{x_n y_n\} \in \ell^1, \forall y = \{y_n\} \in \mathcal{P}\} \\ &= s' \text{ (the set of all tempered sequences)} \\ &= \{x \in \omega : \sup_n |x_n| n^{-m} < \infty \text{ for some } m \in \mathbb{N}\}, \\ \mathcal{D}(\sup_{y\in\mathcal{P}} p_y^1) &= \{x \in s' : \sup_{y\in\mathcal{P}} \|xy\|_1 < \infty\}, \\ (\sup_{y\in\mathcal{P}} p_y^1)(x) &= \sup_{y\in\mathcal{P}} \|xy\|_1, \quad x \in \mathcal{D}(\sup_{y\in\mathcal{P}} p_y^1)\end{aligned}$$

and $\sup_{y\in\mathcal{P}} p_y^1$ is a semifinite spectral unbounded m^*-norm on s'.

We can define the following unbounded m^*-norms p^q and p^q_∞ on $\ell^q(\mathcal{P})$ by

$$\begin{array}{lll}\mathcal{D}(p^q) = \ell^q(\mathcal{P}) \cap \ell^q = \ell^q & \text{and} & p^q(x) = \|x\|_q, \quad x \in \mathcal{D}(p^q); \\ \mathcal{D}(p^q_\infty) = \ell^q(\mathcal{P}) \cap \ell^\infty & \text{and} & p^q_\infty(x) = \|x\|_\infty, \quad x \in \mathcal{D}(p^q_\infty).\end{array}$$

Since $(\ell^q, \|\cdot\|_q)$ is a Banach $*$-algebra and $\mathfrak{N}_{p^q}$ contains a dense subspace $\mathcal{F}$ in ℓ^q, it follows that p^q is a semifinite spectral unbounded m^*-norm on $\ell^q(\mathcal{P})$, and p^q_∞ is the unbounded Gelfand-Naimark C^*-norm defined by the unbounded m^*-norm p^q, and it is semifinite.

(2) The above (1) can be used to model certain convolution algebra as illustrated below. Let $\triangle = \{z \in \mathbb{C} : |z| < 1\}, H(\triangle)$ be the nuclear Fréchet space of all functions holomorphic on $\triangle$. $H(\triangle)$ is a $*$-algebra with involution $f^*(z) = \overline{f(\overline{z})}$ and Hadamard product $(f * g)(x) = \frac{1}{2\pi \mathrm{i}} \int f(z) g(xz^{-1}) z^{-1}\, \mathrm{d}z$, $|x| < r < 1$. The function $e(z) = (1-z)^{-1}$ is the identity of $H(\triangle)$. The algebra $H(\triangle)$ is $*$-isomorphic to $\ell^1(\mathcal{P})$ with $\mathcal{P} = \{\{r^n\}_{n=0}^\infty : 0 < r < 1\}$ via the isomorphism $\psi : H(\triangle) \to \ell^1(\mathcal{P})$, $\psi(f) = \left\{\frac{f^{(n)}(0)}{n!}\right\}_{n=0}^\infty$. It follows that $a^q(f) = \sup_{0<r<1} \left[\sum_n \left|\frac{f^{(n)}(0)}{n!} r^n\right|^q\right]^{\frac{1}{q}}$ $(1 \leqslant q \leqslant \infty)$ defines a semifinite unbounded norm on $H(\triangle)$. Let $T = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. The Fréchet space $C^\infty(T)$ of C^∞-functions on T with the topology τ defined by the seminorms $p_n(f) = \sum_{k=0}^n \frac{1}{k!} \sup_{t\in T} |f^{(k)}(t)|$ is a convolution $*$-algebra with involuiton $f^*(z) = \overline{f(\overline{z})}$. $C^\infty(T)$ is isomorphic to the sequence algebra $s(\mathbb{Z}) := \big\{x = \{x_n\}_{-\infty}^\infty : \{|n|^k x_n\}_{-\infty}^\infty \in \ell^\infty, \forall k \in \mathbb{N}\big\}$. The dual of $C^\infty(T)$ is the commutative convolution algebra $\mathcal{D}(T)$ of all distributions on T, the identity being the Dirac delta δ and the involution being $u \to u^*$, $< u^*, f >= \overline{\langle u, f^* \rangle}$ $(f \in C^\infty(T))$. Let $u \to \widehat{u}$, $\widehat{u}(n) = \langle u, \exp(-\mathrm{i}nt)\rangle$ $(n \in \mathbb{Z})$ be the Fourier-Schwarz transform that map $\mathcal{D}(T)$-$*$-isomorphically onto the $*$-algebra $s'(\mathbb{Z}) = \big\{a = \{a_n\}_{-\infty}^\infty : a_n = \mathrm{O}(|n|^m)$ for some m depending on $a\big\}$ having pointwise operations and complex conjugation as the involution. Under this map, the $*$-subalgebra $\mathrm{PM}(T)$ (*pseudo measures on* T) of $\mathcal{D}(T)$ is mapped onto $\ell^\infty(\mathbb{Z})$. By (1) we can define a semifinite spectral unbounded m^*-norm on $\mathcal{D}(T)$ and a semifinite spectral unbounded C^*-norm on $\mathrm{PM}(T)$. In fact, $\mathcal{D}(T)$ is a sequentially

complete GB*-algebra with sequentially jointly continuous multiplication and having bounded part $\mathcal{A}[\mathcal{B}_0] = \mathrm{PM}(T)$. For the unbounded C^*-norm p_{GB^*}, we have $\mathcal{D}(p_{\mathrm{GB}^*}) = \mathrm{PM}(T)$ and $p_{\mathrm{GB}^*}(x) = \sup_{n\in\mathbb{Z}} |\widehat{x}(n)| = \|\widehat{x}\|_\infty$. Further, by (12.6.2, p.74) in [15] $C^\infty(T)$ is an ideal of $\mathcal{D}(T)$ and so $C^\infty(T) \subset \mathfrak{N}_{p_{\mathrm{GB}^*}}$.

Example 7.3. We consider unbounded C^*-norms on O*-algebras. We put

$$\mathcal{D}(p_b) = \mathcal{M}_b := \{X \in \mathcal{M} : \overline{X} \text{ is bounded }\}$$

and $p_b(X) = \|\overline{X}\|$, $X \in \mathcal{D}(p_b)$. Then p_b is an unbounded C^*-norm on $\mathcal{M}$.

(1) Let $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ be a family of bounded $*$-algebras $\mathcal{M}_\lambda$ on Hilbert spaces $\mathcal{H}_\lambda$ with identity operator and $\prod_{\lambda\in\Lambda} \mathcal{M}_\lambda$ be the product of $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$. We put

$$\mathcal{D}\Big(\prod_{\lambda\in\Lambda} \mathcal{M}_\lambda\Big) = \Big\{(\xi_\lambda) \in \bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda : \sum_{\lambda\in\Lambda} \|X_\lambda\xi_\lambda\|^2 < \infty, \forall (X_\lambda) \in \prod_{\lambda\in\Lambda} \mathcal{M}_\lambda\Big\},$$

$$(X_\lambda)(\xi_\lambda) = (X_\lambda\xi_\lambda), \quad (X_\lambda) \in \prod_{\lambda\in\Lambda} \mathcal{M}_\lambda, \quad (\xi_\lambda) \in \mathcal{D}\Big(\prod_{\lambda\in\Lambda} \mathcal{M}_\lambda\Big).$$

Then $\prod_{\lambda\in\Lambda} \mathcal{M}_\lambda$ is an O*-algebra on $\mathcal{D}(\prod_{\lambda\in\Lambda} \mathcal{M}_\lambda)$ in $\bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda$. A $*$-subalgebra of such an O*-algebra is said to be *weakly bounded*. Let $\mathcal{M}$ be a weakly bounded O*-algebra, that is, a $*$-subalgebra of the O*-algebra $\prod_{\lambda\in\Lambda} \mathcal{M}_\lambda$. Then

$$\mathcal{D}(p_\flat) = \big\{(X_\lambda) \in \mathcal{M} : \sup_\lambda \|X_\lambda\| < \infty\big\},$$

$$p_\flat((X_\lambda)) = \sup_\lambda \|X_\lambda\|, \quad (X_\lambda) \in \mathcal{D}(p_\flat).$$

Suppose that $\mathcal{M}$ contains the family $\{E_\lambda\}_{\lambda\in\Lambda}$ of the projection E_λ of $\bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda$ onto $\mathcal{H}_\lambda$, in particular, $\mathcal{M} = \prod_{\lambda\in\Lambda} \mathcal{M}_\lambda$. Then $p_\flat$ is a maximal, regular and semifinite unbounded C^*-norm on $\mathcal{M}$. Schmüdgen ([28]) has given necessary and sufficient conditions under which a closed O*-algebra is weakly bounded.

(2) Let $\mathcal{M}$ be an O*-algebra on $\mathcal{D}$ in $\mathcal{H}$. Suppose $\mathcal{M} \supset \{\xi_n \otimes \overline{\xi_n} : n \in \mathbb{N}\}$, where $\{\xi_n\}$ is an orthonormal basis in $\mathcal{H}$ contained in $\mathcal{D}$. Then $p_\flat$ is a maximal and weakly semifinite unbounded C^*-norm on $\mathcal{M}$.

(3) Let $\mathcal{M}_0$ be the O*-algebra on the Schwartz space $\mathcal{S}(\mathbb{R})$ generated by the momentum operator P and the position operator Q. Then $\mathcal{D}(p_\flat) = \mathbb{C}I$ and $\mathfrak{N}_{p_\flat} = \{0\}$. Let $\mathcal{M}$ be an O*-algebra on $\mathcal{S}(\mathbb{R})$ generated by $\mathcal{M}_0$ and $\{f_n \otimes \overline{f_n} : n = 0, 1, \ldots\}$, where $\{f_n\}$ is an orthonormal basis in $L^2(\mathbb{R})$ consisting of the normalized Hermite functions. Then it follows that $\mathfrak{N}_{p_\flat}$ equals the $*$-algebra generated by $\{A(f_n \otimes \overline{f_n}) : A \in \mathcal{M}_0,\ n = 0, 1, \ldots\}$, so that $p_\flat$ is a maximal and weakly semifinite unbounded C^*-norm on $\mathcal{M}$.

We intend to study unbounded m^*-(or C^*-)seminorms on *locally convex $*$-algebras*. In particular, it seems important to define and study the notions of topologically (hereditary) C^*-spectral algebras, topologically (hereditary) spectral $*$-representations and topological stability in case of locally convex $*$-algebras.

Acknowledgements. Much of this work has been done since S.J. Bhatt visited Department of Applied Mathematics, Fukuoka University, Fukuoka, during September-October 1995. He is thankful to Fukuoka University for supporting this visit and stay at Fukuoka, as well as to the Sardar Patel University for granting leave. He also expresses his gratitute to Professor A. Inoue, Dr. H. Ogi and Professor Kurose, as well as other colleagues at Fukuoka for warm hospitality. A. Inoue talked a part of this work at the International Workshop on Operator Theory in Cefalù, and at the Department Wiskunde, Katholieke Universiteit Leuven in 1997, and at the Institute di Fisica, Università Palermo in 1998 and at the Mathematisches Institut, Universität Leipzig in 1999. He would like to thank Professor C. Trapani, F. Bagarello and G. Epifanio in Palermo and Professor Van Daele in Leuven, and Professor K. D. Kürsten and K. Schmüdgen in Leipzig for their warm hospitality.

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SUBHASH J. BHATT
Department of Mathematics
Sardar Patel University
Vallabh Vidyanagar 388120
INDIA
E-mail: sjb@spu.ernet.in

ATSUSHI INOUE
Department of Applied Mathematics
Fukuoka University
Nanakuma, Jonan-ku
Fukuoka 814-0180
JAPAN
E-mail: a-inoue@fukuoka-u.ac.jp

HIDEKAZU OGI
Dept. Functional Materials Engineering
Fukuoka Institute of Technology
Wazirohigashi, Higashi-ku
Fukuoka 811-0295
JAPAN
E-mail: ogi@fit.ac.jp

Received July 6, 1998; revised January 10, 1999.