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A SPECTRAL BOUND FOR ASYMPTOTICALLY NORM-CONTINUOUS SEMIGROUPS

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Abstract. We introduce a new growth bound for C_0-semigroups giving information about the absence of norm-continuity of the semigroup and we give a corresponding spectral bound. For semigroups on general Banach spaces we prove an inequality between these bounds and we give a version of the spectral mapping theorem in terms of the new growth bound. For semigroups on Hilbert space we show that the bounds are equal and hence obtain new characterizations of asymptotically norm-continuous semigroups and semigroups norm-continuous for $t > 0$ in terms of the resolvent of the infinitesimal generator. In the last section we prove that versions of the spectral mapping theorem holds for three different definitions of the essential spectrum and give nice relationships between the new growth bound and the essential growth bound of the semigroup.

Keywords: *C_0-semigroup, asymptotically norm-continuous, norm-continuous at infinity, spectral mapping theorem, growth bound.*

MSC (2000): Primary 47D06; Secondary 47A10.

1. INTRODUCTION

Let $T(t)$ be a C_0-semigroup on a Banach space X with generator A. It is well known that if $T(t)$ is eventually norm-continuous then the spectral mapping theorem holds (cf. [14], A-III, Theorem 6.6). In [13] the wider class of semigroups that are norm-continuous at infinity is introduced and it is shown that for these semigroups a spectral mapping theorem holds for the boundary spectrum. In [18] a similar result is given for another generalization of eventually norm-continuous semigroups — semigroups that are essentially norm-continuous. Such results are of particular interest as when such a spectral mapping theorem holds the spectral bound and growth bound of the semigroup coincide and hence the asymptotic behaviour of solutions of the associated Cauchy problem is determined by the spectrum of the generator.

In this paper we study these continuity properties via the introduction of new bounds, $\delta(T)$, the growth bound of the local variation of $T(t)$ which gives information about the absence of norm-continuity, and an associated spectral bound $s_0^\infty(A)$, involving the behaviour of the resolvent away from the real axis. We relate these bounds to essentially norm-continuous and essentially norm-measurable semigroups, and show that semigroups that are norm-continuous at infinity (for which we shall use the alternative terminology asymptotically norm-continuous) have a particularly simple characterization in terms of $\delta(T)$ (Proposition 3.5). By reformulating the spectral mapping theorem in terms of $\delta(T)$, we obtain a very general version (Theorem 3.6) that has the versions given in [14], [13] and [18] as immediate corollaries.

When the underlying space is a Hilbert space we find that $\delta(T) = s_0^\infty(A)$ and that the classes of essentially norm-continuous semigroups, essentially norm-measurable semigroups and asymptotically norm-continuous semigroups coincide (Theorem 4.4). It follows that these properties are all equivalent to the purely spectral condition $s_0^\infty(A) < s(A)$. Using the same techniques we obtain sufficient conditions on the resolvent for a semigroup on a Hilbert space to be norm-continuous for $t > \alpha$ (Theorem 4.5). A new characterization of norm-continuity for $t > 0$ on Hilbert space follows.

In the final section we investigate the relationship between continuity and compactness properties of the semigroup. We show that essentially compact semigroups are asymptotically norm-continuous (Theorem 5.1), and we prove a spectral mapping theorems for the Browder essential spectrum, the upper Fredholm essential spectrum and the Fredholm essential spectrum (Theorem 5.6). These results enable us to prove that the essential growth bound and the essential spectral bound of a semigroup are nicely related to $\delta(T)$ (Corollary 5.8). Together with the results of the previous chapter these give a characterization of essentially compact semigroups on Hilbert space in terms of the resolvent and essential spectrum of the generator (Corollary 5.9).

2. LAPLACE TRANSFORM RESULTS

In this section we shall consider results that do not require T to be a semigroup. We shall assume only that the one-parameter families T that we consider should be strongly measurable, exponentially bounded functions from $\mathbb{R}_+$ into $B(X)$. We shall define the *type* of such a family, written $\omega_0(T)$ by

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} : \text{there exists } M_\omega \text{ such that } \|T(t)\| \leqslant M_\omega \mathrm{e}^{\omega t} \text{ for all } t \geqslant 0\}.$$

We shall denote the Laplace transform by $\mathcal{L}$,

$$\mathcal{L}(T)(\lambda)x := \int_0^\infty \mathrm{e}^{-\lambda t} T(t)x \,\mathrm{d}t \quad \text{for all } x \in X,$$

and $\lambda \mapsto \mathcal{L}(T)(\lambda)$ is a holomorphic function from $\mathrm{Re}(\lambda) > \omega_0(T)$ into $B(X)$. For $N \geqslant 0$ we shall denote the Nth derivative of the $\mathcal{L}(T)$ by $\mathcal{L}(T)^{(N)}$. For $1 \leqslant p < \infty$ we shall denote by $L^p(\mathbb{R}, X)$ the space of strongly measurable functions $f : \mathbb{R} \to X$

with $\|f\|_p := \Big\{ \int\limits_{-\infty}^{\infty} \|f(t)\|^p \, \mathrm{d}t \Big\}^{1/p} < \infty$. We shall identify $L^p(\mathbb{R}_+, X)$ as a subspace of $L^p(\mathbb{R}, X)$ in the obvious manner by regarding a function in $L^p(\mathbb{R}_+, X)$ as taking the value 0 on $(-\infty, 0)$.

We shall denote by $\mathcal{F}$ the Fourier transform on $L^1(\mathbb{R}, X)$ defined by

$$(\mathcal{F}f)(s) := \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}st} f(t) \, \mathrm{d}t. \tag{2.1}$$

Then by the Riemann-Lebesgue lemma, $\mathcal{F}$ maps $L^1(\mathbb{R}, X)$ into the space of continuous X-valued functions vanishing at infinity. We shall also denote by $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{R}, X)$ defined by (2.1) for $f \in L^1(\mathbb{R}, X) \cap L^2(\mathbb{R}, X)$. If X is a Hilbert space then we also have the Hilbert space valued Plancherel Theorem that $\frac{1}{\sqrt{2\pi}}\mathcal{F}$ extends to an isometry on $L^2(\mathbb{R}, X)$.

In [18], Definition 2.6, *essentially norm-continuous* and *essentially norm-measurable* C_0-semigroups are introduced, generalizing both eventually norm-continuous semigroups and essentially compact semigroups. Here we modify these definitions to make them applicable to the more general situation of exponentially bounded families $T : \mathbb{R}_+ \to B(X)$ of operators.

DEFINITION 2.1. Let $T : \mathbb{R}_+ \to B(X)$ be an exponentially bounded family of operators on a Banach space X, and let $\beta > 0$. $T(t)$ is said to be *essentially norm-continuous* (*of type* β) if for each α such that $0 < \alpha < \beta$ there exist families of operators $S_1, S_2 : \mathbb{R}_+ \to B(X)$ (depending on α) such that

$$T(t) = S_1(t) + S_2(t)$$

with S_1 right-continuous on $(0, \infty)$ in the operator norm topology and $\omega_0(S_2) \leqslant \omega_0(T) - \alpha$.

T is said to be *essentially norm-measurable (of type β)* if for each $0 < \alpha < \beta$ there exist families of operators $S_1, S_2 : \mathbb{R}_+ \to B(X)$ such that

$$T(t) = S_1(t) + S_2(t)$$

with S_1 norm-measurable on $(0, \infty)$ and $\omega_0(S_2) \leqslant \omega_0(T) - \alpha$.

DEFINITION 2.2. Let $T : \mathbb{R}_+ \to B(X)$ be an exponentially bounded family of operators on a Banach space X. Define the function $f_T : [0, \infty) \to [0, \infty)$ by

$$f_T(t) := \limsup_{h \to 0+} \|T(t+h) - T(t)\|.$$

Then we may define

$$\delta(T) := \inf\{\omega \in \mathbb{R} : \text{there exists } M_\omega \text{ such that} f_T(t) \leqslant M_\omega \mathrm{e}^{\omega t}\}$$

and we have $\delta(T) \leqslant \omega_0(T)$.

Notice that if $T : \mathbb{R}_+ \to B(X)$ is a family of operators on a Banach space X with $\omega_0(T) > -\infty$ then we can define $T'(t) := \mathrm{e}^{-\omega_0(T)t} T(t)$ and we have $\omega_0(T') = 0$, $\delta(T') = \delta(T) - \omega_0(T)$, and $T'(t)$ is essentially norm-continuous (respective essentially norm-measurable) of type β if and only if $T(t)$ also has this property. Hence by replacing $T(t)$ by $T'(t)$ if necessary we will usually be able to assume that $\omega_0(T) = 0$.

DEFINITION 2.3. Let $T : \mathbb{R}_+ \to B(X)$ be an exponentially bounded, strongly measurable family of operators on a Banach space X. For $N \in \mathbb{N}$, $\gamma > \omega_0(T)$, define

$$\varphi_\gamma^T(N) := \Big(\limsup_{|s|\to\infty} \Big\| \frac{1}{N!} \mathcal{L}\,(T)^{(N)}\,(\gamma + \mathrm{i}s) \Big\| \Big)^{\frac{1}{N+1}}.$$

THEOREM 2.4. *Let T be a strongly measurable, exponentially bounded family of operators on a Banach space X, and let $\beta > 0$. Then we have* (i)⇒(ii)⇒(iii)⇒(iv), *and* (iii)⇒(iv′), *where*

(i) *$T(t)$ is essentially norm-continuous of type β;*

(ii) *$\delta(T) \leqslant \omega_0(T) - \beta$;*

(iii) *$T(t)$ is essentially norm-measurable of type β;*

(iv) *there is a constant $C > 0$ such that given $\gamma > \omega_0(T)$ there exists $r_\gamma > 0$ such that $\|\mathcal{L}\,(T)\,(\gamma + \mathrm{i}s)\| < C$ whenever $|s| > r_\gamma$;*

(iv′) *let $\gamma > \omega_0(T)$; for each $0 < \alpha < \beta$ there exists N_α such that $\varphi_\gamma^T(N_\alpha) \leqslant (\gamma + \alpha)^{-1}$.*

Proof. (i)⇒(ii). Suppose that $T(t)$ is essentially norm-continuous of type β, so if $0 < \alpha < \beta$ then there exist S_1, S_2 such that $T(t) = S_1(t) + S_2(t)$ where S_1 is right norm-continuous for $t > 0$ and $\omega_0(S_2) \leqslant \omega_0(T) - \alpha$. Then as S_1 is right norm-continuous,

$$\delta(T) = \delta(S_1 + S_2) = \delta(S_2) \leqslant \omega_0(S_2) \leqslant \omega_0(T) - \alpha.$$

As $0 < \alpha < \beta$ was arbitrary, we get that $\delta(T) \leqslant \omega_0(T) - \beta$.

(ii)⇒(iii). Without loss of generality we may assume that $\omega_0(T) = 0$, so, assuming (ii), we have that if $0 < \alpha < \beta$ there is a constant $C > 0$ such that for all $t > 0$ we have

$$\limsup_{h\to 0+} \|T(t+h) - T(t)\| \leqslant C\mathrm{e}^{-\alpha t}.$$

So for each $t \geqslant 0$ there exists a half open interval $U_t = [t, x_t)$ with

$$\|T(s) - T(t)\| < 2C\mathrm{e}^{-\alpha t} \quad \text{for all } s \in U_t.$$

We may also assume that the intervals have length less than d where $\mathrm{e}^{-\alpha d} > \frac{1}{2}$. As $[0, \infty)$ is Lindelöf in the Sorgenfrey topology (see [4], Example 3.8.14, p. 248) generated by the half open intervals $[a, b)$, there is a countable family $\{U_{t_i} : i \in \mathbb{N}\}$ that covers it. Let $V_{t_i} := U_{t_i} \setminus \bigcup_{j<i} U_{t_j}$ so that $V_{t_i} \subseteq U_{t_i}$, the V_{t_i} are disjoint and have union $[0, \infty)$, and define

$$S(t) = \sum_0^\infty \chi_{V_{t_i}}(t) T(t_i).$$

Then if $s \in [0, \infty)$, there is t_i such that $s \in V_{t_i}$, and

$$\|T(s) - S(s)\| = \|T(s) - T(t_i)\| \leqslant 2C\mathrm{e}^{-\alpha t_i} = 2C\mathrm{e}^{-\alpha s}\mathrm{e}^{-\alpha(t_i - s)} \leqslant 4C\mathrm{e}^{-\alpha s}.$$

$S(t)$ is a limit of step functions, so it is norm-measurable. So putting $S_1(t) = S(t)$, and $S_2(t) = T(t) - S(t)$ we have $T(t) = S_1(t) + S_2(t)$ with S_1 norm-measurable, and $\omega_0(S_2) \leqslant -\alpha$. As $0 < \alpha < \beta$ was arbitrary, it follows that $T(t)$ is essentially norm-measurable of type β.

(iii)⇒(iv). Suppose that $T(t)$ is essentially norm-measurable of type $\beta > 0$ and, without loss of generality, that $\omega_0(T) = 0$. Let $0 < \alpha < \beta$. Then we have that there exist S_1, S_2 and $K > 0$ such that $T(t) = S_1(t) + S_2(t)$ with S_1 norm-measurable and

$$\|S_2(t)\| \leqslant K\mathrm{e}^{-\alpha t}.$$

Let $\gamma > 0$. Then as S_1 is norm-measurable with $\omega_0(S_1) = 0$, we have that the function $t \mapsto \mathrm{e}^{-\gamma t}S_1(t)$ is in $L^1((0,\infty), B(X))$. Hence by the Riemann-Lebesgue lemma there exists $r_\gamma > 0$ such that $\|\mathcal{F}\left(\mathrm{e}^{-\gamma\,\cdot}S_1(\,\cdot\,)\right)(s)\| < \frac{K}{\alpha}$ whenever $|s| > r_\gamma$. Then if $x \in X$,

$$\begin{aligned}
\|\mathcal{L}\,(T)\,(\mathrm{i}s+\gamma)x\| &\leqslant \Big\|\int_0^\infty \mathrm{e}^{-(\gamma+\mathrm{i}s)t}T(t)x\,\mathrm{d}t\Big\| \\
&\leqslant \Big\|\int_0^\infty \mathrm{e}^{-(\gamma+\mathrm{i}s)t}S_1(t)x\,\mathrm{d}t\Big\| + \Big\|\int_0^\infty \mathrm{e}^{-(\gamma+\mathrm{i}s)t}S_2(t)x\,\mathrm{d}t\Big\| \\
&\leqslant \|\mathcal{F}\left(\mathrm{e}^{-\gamma\,\cdot}S_1(\,\cdot\,)\right)(-s)\|\|x\| + \int_0^\infty K\mathrm{e}^{-\alpha t}\|x\|\,\mathrm{d}t \leqslant 2\frac{K}{\alpha}\|x\|.
\end{aligned}$$

(iii)⇒(iv′). Assume (iii). Without loss of generality we may assume that $\gamma > \omega_0(T) = 0$. We may choose S_1,S_2 and $\varepsilon > 0$ such that $T(t) = S_1(t) + S_2(t)$ with S_1 norm-measurable and $\omega_0(S_2) \leqslant -(\alpha+\varepsilon)$. Then

$$\begin{aligned}
&\sum_{n=0}^\infty \int_0^\infty \Big\|(\alpha+\gamma)^{n+1}\mathrm{e}^{-(\mathrm{i}s+\gamma)t}\frac{t^n}{n!}S_2(t)\Big\|\,\mathrm{d}t \\
&= (\alpha+\gamma)\sum_{n=0}^\infty \int_0^\infty \Big\|\frac{((\alpha+\gamma)t)^n}{n!}\mathrm{e}^{-(\alpha+\gamma)t}\mathrm{e}^{\alpha t}S_2(t)\Big\|\,\mathrm{d}t \leqslant (\alpha+\gamma)\int_0^\infty \mathrm{e}^{\alpha t}\|S_2(t)\|\,\mathrm{d}t < \infty.
\end{aligned}$$

Hence it follows that there exists $N > 0$ such that for all $s \in \mathbb{R}$,

$$\Big\|\int_0^\infty \mathrm{e}^{-(\mathrm{i}s+\gamma)t}\frac{t^N}{N!}S_2(t)\,\mathrm{d}t\Big\| < \frac{1}{2}(\alpha+\gamma)^{-(N+1)}.$$

As S_1 is norm-measurable with $\omega_0(S_1) = 0$, we have that the function $t \mapsto \frac{1}{N!}\mathrm{e}^{-t}t^N S_1(t)$ is in $L^1((0,\infty), B(X))$. Therefore by the Riemann-Lebesgue lemma there exists $r > 0$ such that

$$\Big\|\int_0^\infty \mathrm{e}^{-(\mathrm{i}s+\gamma)t}\frac{t^N}{N!}S_1(t)\,\mathrm{d}t\Big\| < \frac{1}{2}(\alpha+\gamma)^{-(N+1)},$$

whenever $|s| > r$. Combining these we get, for $|s| > r$,

$$\Big\|\frac{1}{N!}\mathcal{L}\,(T)^{(N)}\,(\mathrm{i}s+\gamma)\Big\| = \Big\|\int_0^\infty \mathrm{e}^{-(\mathrm{i}s+\gamma)t}\frac{t^N}{N!}T(t)\,\mathrm{d}t\Big\| \leqslant (\alpha+\gamma)^{-(N+1)}. \quad ∎$$

3. SEMIGROUP RESULTS

We now introduce four spectral bounds, of which the first two are familiar and the last two are new.

Definition 3.1. Let $T(t)$ be a C_0-semigroup with generator A. We define

$$\begin{aligned} s(A) &:= \inf\{\omega : a + \mathrm{i}s \in \rho(A) \text{ whenever } a > \omega \text{ and } s \in \mathbb{R}\}; \\ s_0(A) &:= \inf\{\omega : \text{there exists } C_\omega \text{ such that } a + \mathrm{i}s \in \rho(A) \text{ and} \\ &\qquad \|R(a+\mathrm{i}s, A)\| < C_\omega \text{ whenever } a > \omega \text{ and } s \in \mathbb{R}\}; \\ s^\infty(A) &:= \inf\{\omega : \text{there exists } r_\omega \text{ such that } a + \mathrm{i}s \in \rho(A) \\ &\qquad \text{whenever } a > \omega \text{ and } |s| > r_\omega\}; \\ s_0^\infty(A) &:= \inf\{\omega : \text{there exist } r_\omega, C_\omega \text{ such that } a + \mathrm{i}s \in \rho(A) \\ &\qquad \text{and } \|R(a+\mathrm{i}s, A)\| < C_\omega \text{ whenever } a > \omega \text{ and } |s| > r_\omega\}. \end{aligned}$$

For any C_0-semigroup, $T(t)$, the spectral bounds and the growth bound are related by the well-known inequality $s(A) \leqslant s_0(A) \leqslant \omega_0(T)$ (see for instance [15], Section 1.2). A corollary to the next proposition gives a similar relationship between the new spectral and growth bounds $s^\infty(A)$, $s_0^\infty(A)$ and $\delta(T)$.

Proposition 3.2. *Let $T(t)$ be a C_0-semigroup, $\gamma > \omega_0(T)$. Then for each $N \in \mathbb{N}$ we have $s_0^\infty(A) \leqslant \gamma - (\varphi_\gamma^T(N))^{-1}$.*

Proof. Without loss of generality suppose that $\omega_0(T) = 0$ and $\varphi_\gamma^T(N) < \gamma^{-1}$. Let $0 < u < 1$, and $\alpha > 0$ be such that $\varphi_\gamma^T(N) < (\gamma + \alpha)^{-1}$. We have that $\frac{1}{N!}\mathcal{L}(T)^{(N)}(\mathrm{i}s + \gamma) = (-1)^N R(\mathrm{i}s + \gamma, A)^{N+1}$, so there are constants $N, r > 0$ such that for $|s| > r$

$$\left\|R(\gamma + \mathrm{i}s, A)^{N+1}\right\| \leqslant (\gamma + \alpha)^{-(N+1)}.$$

Define

$$K := \sup_{s \in \mathbb{R}} \sum_{n=0}^{N} \|R(\gamma + \mathrm{i}s, A)^{n+1}\|(\gamma + \alpha)^n.$$

Then as $1 > s_0(A)$, $K < \infty$. For $|s| > r$ and $|a - \gamma| \leqslant (\gamma + \alpha)u$,

$$\begin{aligned} &\sum_{n=0}^{\infty} \|R(\gamma + \mathrm{i}s, A)^{n+1}(\gamma - a)^n\| \\ &\leqslant \sum_{j=0}^{\infty} \Big(\sum_{k=0}^{N} \|R(\gamma + \mathrm{i}s, A)^{k+1}\|\, |a - \gamma|^k \Big) \|R(\gamma + \mathrm{i}s, A)^{N+1}(\gamma - a)^{N+1}\|^j \leqslant \sum_{j=0}^{\infty} K u^j, \end{aligned}$$

so that the Neumann series

$$R(a + \mathrm{i}s, A) = \sum_{n=0}^{\infty} R(\gamma + \mathrm{i}s, A)^{n+1}(\gamma - a)^n$$

is uniformly convergent with respect to s and a in this region. Hence $s_0^\infty(A) \leqslant \gamma - (\gamma + \alpha)u$. As $0 < u < 1$ and $0 < \alpha < (\varphi_\gamma^T(N))^{-1} - \gamma$ were arbitrary, we obtain $s_0^\infty(A) \leqslant \gamma - (\varphi_\gamma^T(N))^{-1}$. ∎

COROLLARY 3.3. *Let $T(t)$ be a C_0-semigroup then $s^\infty(A) \leqslant s_0^\infty(A) \leqslant \delta(T)$.*

Proof. The first inequality is obvious. The second follows from the above proposition and Theorem 2.4, (ii)⇒(iv′). ∎

In [13] (Definition 1.1) the C_0-semigroups that are *norm-continuous at infinity* are introduced. We shall use the name *asymptotically norm-continuous* for these semigroups instead, and we show that this property is in fact the same as the seemingly stronger property $\delta(T) < \omega_0(T)$.

DEFINITION 3.4. A C_0-semigroup $T(t)$ with generator A is called *asymptotically norm-continuous* (*norm-continuous at infinity* in [13]) if

$$\lim_{t\to\infty} \limsup_{s\to 0} \|\mathrm{e}^{-\omega_0(T)t}T(t)(I - \mathrm{e}^{-\omega_0(T)s}T(s))\| = 0.$$

PROPOSITION 3.5. *Let $T(t)$ be a C_0-semigroup. Then the function*

$$f_T(t) := \limsup_{h\to 0+} \|T(t+h) - T(t)\|$$

is submultiplicative. Hence $\delta(T) < \omega_0(T)$ if and only if $T(t)$ is asymptotically norm-continuous.

Proof. Let $s, t, h > 0$. Then

$$\begin{aligned} &2\|T(s+t+h) - T(s+t)\| \\ &\quad = \|T(s+t+2h) - 2T(s+t+h) + T(s+t) + T(s+t) - T(s+t+2h)\| \\ &\quad \leqslant \|(T(s+h) - T(s))(T(t+h) - T(t))\| + \|T(s+t+2h) - T(s+t)\| \\ &\quad \leqslant \|T(s+h) - T(s)\|\,\|T(t+h) - T(t)\| + \|T(s+t+2h) - T(s+t)\|. \end{aligned}$$

Letting $h \to 0+$ we get

$$2f_T(s+t) \leqslant f_T(s)f_T(t) + f_T(s+t),$$

so

$$f_T(s+t) \leqslant f_T(s)f_T(t).$$

It is clear that if $\delta(T) < \omega_0(T)$ then $T(t)$ is asymptotically norm-continuous. Suppose that $T(t)$ is asymptotically norm-continuous, so that by definition $\mathrm{e}^{\omega_0(T)t}f_T(t) \to 0$ as $t \to \infty$. As f_T is submultiplicative, so is $t \mapsto \mathrm{e}^{\omega_0(T)t}f_T(t)$ so it follows that it decays exponentially fast. Hence $\delta(T) < \omega_0(T)$. ∎

One of the simplest classes of C_0-semigroups are multiplier semigroups. For these semigroups the situation is particularly simple. Let E be the Banach space $C_0(X)$ where X is a locally compact space, and let $q : X \to \mathbb{C}$ be continuous such that $\sup_{x\in X} \mathrm{Re}\,(q(x)) < \infty$. Then the multiplication operator $Af = qf$ with maximal domain generates the multiplier semigroup $T(t)$ on E where $T(t)f = \mathrm{e}^{tq}f$. Then we have that $\sigma(A) = \overline{\mathrm{Ran}\ q}$ ([14], Example 1.28), and it is easy to see that

$$s(A) = s_0(A) = \omega_0(T) = \sup_{x\in X} \mathrm{Re}\,(q(x)),$$

and

$$s^\infty(A) = s_0^\infty(A) = \delta(T) = \inf\{\omega : \mathrm{Ran}\ q \cap \{\mathrm{Re}\,(\lambda) > \omega\} \text{ is bounded}\}.$$

Hence $T(t)$ is asymptotically norm-continuous if and only if there exists $a \in \mathbb{R}$ such that Ran $q \cap \{\lambda : \mathrm{Re}\,(\lambda) > a\}$ is bounded and non-empty. [13], Example 1.11, considers the case where $q(x) = -1 + \frac{1}{x} + \mathrm{i}x$ on the space $E = C_0[1, \infty)$. This semigroup is asymptotically norm-continuous but not eventually norm-continuous, with $s(A) = s_0(A) = \omega_0(T) = 0$ and $s^\infty(A) = s_0^\infty(A) = \delta(T) = \varphi_\gamma^T(N) = -1$ for every $N \in \mathbb{N}$ and $\gamma > 0$.

It is well known that for eventually norm-continuous semigroups the *spectral mapping theorem* holds i.e. $\sigma(T(\tau)) \setminus \{0\} = \exp(\tau\sigma(A))$. Versions of the spectral mapping theorem that hold for the boundary spectrum are known for essentially norm-continuous semigroups ([18], Theorem 3.2) and more generally for asymptotically norm-continuous semigroups ([13], Theorem 1.2). We now give a modification of the proof of [13], Theorem 1.2 that generalizes these results in terms of $\delta(T)$. In [8], Herbst introduces *property P* for the generator of a C_0-semigroup which is just $s_0^\infty(A) = -\infty$. As a corollary of Gearhart's theorem, a spectral mapping theorem is given for semigroups on Hilbert spaces satisfying property P ([8], Lemma 2.1). This will follow as a corollary to the next theorem when we prove $s_0^\infty(A) = \delta(T)$ for semigroups on Hilbert space in Section 4. Recall that if we denote the spectral radius by $r(\,\cdot\,)$, then for C_0-semigroups we have the relationship $r(T(\tau)) = \mathrm{e}^{\omega_0(T)\tau}$ (see [15], Proposition 1.2.1).

THEOREM 3.6. *For $\tau > 0$ define*

$$\Gamma_\tau := \{\lambda \in \mathbb{C} : \mathrm{e}^{\delta(T)\tau} < |\lambda|\}.$$

Then the spectral mapping theorem holds for the part of the spectrum of $T(\tau)$ in Γ_τ, i.e.

$$\sigma(T(\tau)) \cap \Gamma_\tau = \exp(\tau\sigma(A)) \cap \Gamma_\tau.$$

In particular if $T(t)$ is eventually norm-continuous so that $\delta(T) = -\infty$ then

$$\sigma(T(\tau)) \setminus \{0\} = \exp(\tau\sigma(A)).$$

If $T(t)$ is asymptotically norm-continuous, i.e. if $\delta(T) < \omega_0(T)$, then the spectral mapping theorem holds for the boundary spectrum,

$$\sigma(T(\tau)) \cap \{\lambda : |\lambda| = \mathrm{e}^{\omega_0(T)\tau}\} = \exp(\tau\sigma(A)) \cap \{\lambda : |\lambda| = \mathrm{e}^{\omega_0(T)\tau}\},$$

and we have $\omega_0(T) = s(A)$.

Proof. Without loss of generality we may suppose that $\omega_0(T) = 0$. If $\delta(T) = 0$ we have nothing to prove so assume that $\delta(T) < 0$. Let $0 < \alpha < -\delta(T)$. We have to show that for any such α the spectral mapping theorem holds for the part of the spectrum of $T(\tau)$ in

$$\Gamma'_\tau := \{\lambda \in \mathbb{C} : \mathrm{e}^{-\alpha\tau} < |\lambda|\}.$$

By the spectral inclusion theorem ([14], A-III 6.2) and the spectral mapping theorem for the residual spectrum ([14], A-III 6.5) it is sufficient to prove that whenever $\tau > 0$ and $\lambda \in \sigma_{\mathrm{a}}(T(\tau)) \cap \Gamma'_\tau$ then $\lambda \in \mathrm{e}^{\tau\sigma(A)}$, where σ_{a} denotes the approximate point spectrum. Define the space

$$\ell_T^\infty(X) := \{(x_n) \in \ell^\infty(X) : \lim_{t\to 0} \|T(t)x_n - x_n\| = 0 \text{ uniformly for } n \in \mathbb{N}\}.$$

This is well-known to be a closed subspace of $\ell^\infty(X)$ containing the space $c_0(X)$. Define the space $\widehat{X}_T$ to be the quotient $\ell_T^\infty(X)/c_0(X)$. Then it is well known that $(T(t))$ induces a C_0-semigroup $(\widehat{T(t)})$ on $\widehat{X}_T$ via

$$\widehat{T(t)}((x_n)+c_0(X)) := (T(t)x_n)+c_0(X) \quad \text{for all } (x_n)\in \ell_T^\infty(X).$$

Moreover, if $\widehat{A}$ is the generator of $(\widehat{T(t)})$ then $\sigma(\widehat{A})=\sigma(A)$. Let $\tau>0$ and $\lambda\in\sigma_{\mathrm{a}}(T(\tau))\cap\Gamma'_\tau$. Then there exists a normalized sequence $(x_n)\subseteq X$ such that

$$\lim_{n\to\infty}\|T(\tau)x_n-\lambda x_n\|=0.$$

To show that $\lambda\in \mathrm{e}^{\tau\sigma(A)}$ we shall show that $(x_n)\in\ell_T^\infty(X)$ so that $(x_n)+c_0(X)$ is a λ-eigenvector of $\widehat{T(\tau)}$ as then by the spectral mapping theorem for the point spectrum ([14], A-III 6.4), $\lambda\in \mathrm{e}^{\tau\sigma(\widehat{A})}=\mathrm{e}^{\tau\sigma(A)}$. Let $\varepsilon>0$. By definition,

$$\lim_{k\to\infty}\limsup_{h\to 0+}\|\mathrm{e}^{\alpha k\tau}(T(k\tau+h)-T(k\tau))\|=0.$$

As $\lambda\in\Gamma'_\tau$, we have $|\lambda|^{-k}\leqslant \mathrm{e}^{\alpha k\tau}$, so

$$\lim_{k\to\infty}\limsup_{h\to 0+}\|\,|\lambda|^{-k}\,T(k\tau)(I-T(h))\|=0.$$

So there are $k>0$ and $\delta>0$ such that

$$|\lambda|^{-k}\,\|T(k\tau)(I-T(t))\|\leqslant\frac{\varepsilon}{2}\qquad\text{whenever } 0\leqslant t\leqslant\delta.$$

Let $M:=\sup\{\|I-T(t)\|:0\leqslant t\leqslant\delta\}$. We have

$$\lim_{n\to\infty}\|T(\tau)x_n-\lambda x_n\|=0$$

and hence

$$\lim_{n\to\infty}\|T(k\tau)x_n-\lambda^k x_n\|=0.$$

So there is an n_0 such that

$$\|T(k\tau)x_n-\lambda^k x_n\|\leqslant\frac{\varepsilon}{2M}\,|\lambda|^k\qquad\text{whenever } n\geqslant n_0.$$

Now for $0\leqslant t\leqslant\delta$ and $n\geqslant n_0$, we have

$$\begin{aligned}\|T(t)x_n-x_n\| &= |\lambda|^{-k}\,\|T(t)\lambda^k x_n-\lambda^k x_n\|\\ &\leqslant |\lambda|^{-k}\,\|(I-T(t))(\lambda^k-T(k\tau))x_n\|+|\lambda|^{-k}\,\|T(k\tau)(I-T(t))x_n\|\\ &\leqslant M\,|\lambda|^{-k}\,\|(\lambda^k-T(k\tau))x_n\|+|\lambda|^{-k}\,\|T(k\tau)(I-T(t))x_n\|\leqslant\varepsilon.\end{aligned}$$

So $(x_n)\in\ell_T^\infty(X)$ and the proof is complete. ∎

PROPOSITION 3.7. (i) *For any semigroup we have* $\omega_0(T)=\max(\delta(T),s(A))$.
(ii) *If* $s_0(A)<\omega_0(T)$, *then* $s_0^\infty(A)<\delta(T)$.

Proof. (i) Suppose for a contradiction that $\delta(T)<\omega_0(T)$ and $s(A)<\omega_0(T)$, then by the Spectral mapping Theorem 3.6 we have $s(A)=s_0(A)=\omega_0(T)$, a contradiction.

(ii) This follows immediately from (i) and the fact that $s(A)<s_0(A)$. ∎

PROPOSITION 3.8. $s_0(A) = \max(s(A), s_0^\infty(A))$.

Proof. We have that $s(A) \leqslant s_0(A)$ and $s_0^\infty(A) \leqslant s_0(A)$, so suppose for a contradiction that $s(A) < s_0(A)$ and that there exists $a \in \mathbb{R}$ such that $s_0^\infty(A) < a < s_0(A)$ and $s(A) < a$. Then there exist $N, C > 0$ such that $\lambda \in \rho(A)$ with $\|R(\lambda, A)\| \leqslant C$ whenever $\operatorname{Re}\lambda \geqslant a$ and $|\operatorname{Im}\lambda| \geqslant N$. Let

$$\begin{aligned} S_1 :=&\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant a, |\operatorname{Im}\lambda| \geqslant N\}.\\ S_2 :=&\{\lambda \in \mathbb{C} : a \leqslant \operatorname{Re}\lambda \leqslant s_0(A) + 1, |\operatorname{Im}\lambda| \leqslant N\}.\\ S_3 :=&\{\lambda \in \mathbb{C} : s_0(A) + 1 \leqslant \operatorname{Re}\lambda, |\operatorname{Im}\lambda| \leqslant N\}.\end{aligned}$$

Then $S_1 \subseteq \rho(A)$ with $\|R(\lambda, A)\|$ bounded for $\lambda \in S_1$. The same is true of S_2 as $a > s(A)$ and S_2 is compact. The same is also true of S_3 by definition of $s_0(A)$. We therefore have the same for $S_1 \cup S_2 \cup S_3 = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant a\}$. Hence $a \geqslant s_0(A)$, a contradiction. ∎

It follows from Propositions 3.7 and 3.8 that any of the standard examples of semigroups satisfying $s_0(A) < \omega_0(T)$ (respective $s(A) < s_0(A)$) also satisfy $s_0^\infty(A) < \delta(T)$ (respective $s^\infty(A) < s_0^\infty(A)$). In particular there are examples of positive C_0-semigroups on Banach lattices such that $s_0(A) = s(A) < \omega_0(T)$ ([16], A-IV, Ex 1.2 or [15], Example 1.4.4), and it is easy to check that for these example we actually have $s(A) = s_0(A) = s_0^\infty(A) < \delta(T) = \omega_0(T)$. Moreover, there exists a C_0-semigroup on a Hilbert space where $s^\infty(A) = s(A) < s_0(A) = s_0^\infty(A) = \delta(T) = \omega_0(T)$ ([15], Example 1.2.4).

4. HILBERT SPACE RESULTS

The main results of this section all follow almost immediately from one, namely Lemma 4.3. The proof of this is an adaptation of [13], Lemma 2.3 and requires two preliminary results.

LEMMA 4.1. *Let $T(t)$ be a C_0-semigroup on a Hilbert space H such that $\omega_0(T) = 0$. Then there exists a constant $C > 0$ such that, for all $x \in H$ and $\varphi \in H^*$,*

$$\Big(\int_{-\infty}^{\infty} \|R(1 + \mathrm{i}s, A)x\|^2 \,\mathrm{d}s\Big)^{1/2} \leqslant C\|x\|$$

and

$$\Big(\int_{-\infty}^{\infty} \|R(1 + \mathrm{i}s, A)^*\varphi\|^2 \,\mathrm{d}s\Big)^{1/2} \leqslant C\|\varphi\|.$$

Proof. As $\omega_0(T) = 0$ we have that for $x \in H$

$$\int_0^\infty \|\mathrm{e}^{-t}T(t)x\|^2 \,\mathrm{d}t \leqslant \Big(\int_0^\infty \|\mathrm{e}^{-t}T(t)\|^2 \,\mathrm{d}t\Big)\|x\|^2 < \infty.$$

So by the Hilbert space valued Plancherel theorem,

$$\begin{aligned}\Big(\int_{-\infty}^{\infty}\|R(1+\mathrm{i}s,A)x\|^2\,\mathrm{d}s\Big)^{1/2} &= \sqrt{2\pi}\Big(\int_0^{\infty}\|\mathrm{e}^{-t}T(t)x\|^2\,\mathrm{d}t\Big)^{1/2}\\ &\leqslant \sqrt{2\pi}\Big(\int_0^{\infty}\|\mathrm{e}^{-t}T(t)\|^2\,\mathrm{d}t\Big)^{1/2}\|x\|.\end{aligned}$$

As H is a Hilbert space, $T(t)^*$ is a C_0-semigroup on $H^*=H$ and $\|T(t)\|=\|T(t)^*\|$, so we also have

$$\Big(\int_{-\infty}^{\infty}\|R(1+\mathrm{i}s,A)^*\varphi\|^2\,\mathrm{d}s\Big)^{1/2}\leqslant\sqrt{2\pi}\Big(\int_0^{\infty}\|\mathrm{e}^{-t}T(t)\|^2\,\mathrm{d}t\Big)^{1/2}\|\varphi\|. \quad ∎$$

LEMMA 4.2. *Let $a,b,N,K\in\mathbb{R}$ be such that $\lambda\in\rho(A)$ with $\|R(\lambda,A)\|<K$ whenever $a\leqslant\operatorname{Re}\lambda\leqslant b$ and $|\operatorname{Im}\lambda|\geqslant N$. Let $n\geqslant 1$, $x\in X$. Then*

$$\lim_{|s|\to\infty}\|R(\mu+\mathrm{i}s)^n x\|\to 0$$

uniformly with respect to $\mu\in[a,b]$, and

$$\lim_{|s|\to\infty}\int_{a+\mathrm{i}s}^{b+\mathrm{i}s}\mathrm{e}^{\lambda t}R(\lambda,A)^n x\,\mathrm{d}\lambda=0.$$

Proof. Let $\varepsilon>0$. Then as $D(A)$ is dense in X, there exists $y\in D(A)$ such that $\|x-y\|<\varepsilon$. We have that if $|s|>N$ and $a\leqslant\mu\leqslant b$,

$$(\mu+\mathrm{i}s)R(\mu+\mathrm{i}s,A)y=y+R(\mu+\mathrm{i}s,A)Ay$$

so

$$\|R(\mu+\mathrm{i}s,A)y\|\leqslant\frac{1}{|\mu+\mathrm{i}s|}(\|y\|+K\|Ay\|).$$

Hence there exists $t>N$ such that $\|R(\mu+\mathrm{i}s,A)y\|<\varepsilon$ whenever $|s|\geqslant t$ and $a\leqslant\mu\leqslant b$. Then for these s and μ,

$$\|R(\mu+\mathrm{i}s,A)x\|\leqslant\|R(\mu+\mathrm{i}s,A)(x-y)\|+\|R(\mu+\mathrm{i}s,A)y\|\leqslant(K+1)\varepsilon,$$

and

$$\|R(\mu+\mathrm{i}s,A)^n x\|\leqslant\|R(\mu+\mathrm{i}s,A)\|^{n-1}\|R(\mu+\mathrm{i}s,A)x\|\leqslant K^{n-1}(K+1)\varepsilon.$$

Hence

$$\Big\|\int_{a+\mathrm{i}s}^{b+\mathrm{i}s}\mathrm{e}^{\lambda t}R(\lambda,A)^n x\,\mathrm{d}\lambda\Big\|\leqslant\mathrm{e}^{bt}(b-a)K^{n-1}(K+1)\varepsilon. \quad ∎$$

LEMMA 4.3. *Let $T(t)$ be a C_0-semigroup on a Hilbert space H. Suppose that $\omega_0(T)=0$ and that there exist constants $a\in\mathbb{R}$ and $N,K>0$ such that whenever*

$\operatorname{Re}\lambda \geqslant a$ and $|\operatorname{Im}\lambda| \geqslant N$ we have $\lambda \in \rho(A)$ with $\|R(\lambda, A)\| \leqslant K$. Let $b > 0$ and define the operator $S_{a,b,N}(t)$ by

$$S_{a,b,N}(t)x := \frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma_{a,b,N}} \mathrm{e}^{\lambda t} R(\lambda, A)x \, \mathrm{d}\lambda$$

where $\Gamma_{a,b,N}$ is the contour

$$[a - \mathrm{i}N, b - \mathrm{i}N] \cup [b - \mathrm{i}N, b + \mathrm{i}N] \cup [b + \mathrm{i}N, a + \mathrm{i}N].$$

Then $\{S_{a,b,N}(t)\}_{t\geqslant 0}$ is a norm-continuous family of operators such that there exists a constant $C > 0$, independent of a, b, and N, with

$$\|T(t) - S_{a,b,N}(t)\| \leqslant \frac{\mathrm{e}^{at}}{\pi t}(K + C(1 + |1 - a|\, K)^2,$$

for all t.

Proof. Let $x \in H$, $\varphi \in H^*$, then we have by [19], Theorem 1.1,

$$\varphi(T(t)x) = \frac{1}{2\pi \mathrm{i}} \lim_{M\to\infty} \int\limits_{b-\mathrm{i}M}^{b+\mathrm{i}M} \mathrm{e}^{\lambda t} \varphi(R(\lambda, A)x) \, \mathrm{d}\lambda.$$

By Lemma 4.2

$$\lim_{|M|\to\infty} \int\limits_{a+\mathrm{i}M}^{b+\mathrm{i}M} \mathrm{e}^{\lambda t} \varphi(R(\lambda, A)x) \, \mathrm{d}\lambda = 0,$$

so by using Cauchy's theorem we have that

$$\begin{aligned} &\varphi((T(t) - S_{a,b,N}(t))x) \\ &\quad = \lim_{M\to\infty} \frac{1}{2\pi \mathrm{i}} \Big(\int\limits_{a+\mathrm{i}N}^{a+\mathrm{i}M} \mathrm{e}^{\lambda t} \varphi(R(\lambda, A)x) \, \mathrm{d}\lambda + \int\limits_{a-\mathrm{i}M}^{a-\mathrm{i}N} \mathrm{e}^{\lambda t} \varphi(R(\lambda, A)x) \, \mathrm{d}\lambda \Big). \end{aligned}$$

Integrating by parts gives

$$\begin{aligned} \int\limits_{a+\mathrm{i}N}^{a+\mathrm{i}M} \mathrm{e}^{\lambda t} \varphi(R(\lambda, A)x) \, \mathrm{d}\lambda &= \frac{\mathrm{e}^{(a+\mathrm{i}M)t}}{t} \varphi(R(a + \mathrm{i}M, A)x) - \frac{\mathrm{e}^{(a+\mathrm{i}N)t}}{t} \varphi(R(a + \mathrm{i}N, A)x) \\ &\quad + \int\limits_{a+\mathrm{i}N}^{a+\mathrm{i}M} \frac{\mathrm{e}^{\lambda t}}{t} \varphi(R(\lambda, A)^2 x) \, \mathrm{d}\lambda. \end{aligned}$$

But by Lemma 4.2,

$$\lim_{M\to\infty} |\varphi(R(a + \mathrm{i}M, A)x)| = 0$$

and

$$\Big|\int_{a+\mathrm{i}N}^{a+\mathrm{i}M}\frac{\mathrm{e}^{\lambda t}}{t}\varphi(R(\lambda,A)^2x)\,\mathrm{d}\lambda\Big| \leqslant \frac{\mathrm{e}^{at}}{t}\int_N^M \|R(a+\mathrm{i}s,A)^*\varphi\|\,\|R(a+\mathrm{i}s,A)x\|\,\mathrm{d}s$$

$$\leqslant (1+|1-a|\,K)^2\frac{\mathrm{e}^{at}}{t}\int_N^M \|R(1+\mathrm{i}s,A)^*\varphi\|\,\|R(1+\mathrm{i}s,A)x\|\,\mathrm{d}s$$

(using the resolvent identity)

$$\leqslant (1+|1-a|\,K)^2\frac{\mathrm{e}^{at}}{t}\Big(\int_{-\infty}^{\infty}\|R(1+\mathrm{i}s,A)^*\varphi\|^2\,\mathrm{d}s\Big)^{1/2}\Big(\int_{-\infty}^{\infty}\|R(1+\mathrm{i}s,A)x\|^2\,\mathrm{d}s\Big)^{1/2}$$

$$\leqslant C(1+|1-a|\,K)^2\frac{\mathrm{e}^{at}}{t}\|x\|\,\|\varphi\|,$$

where we have used Lemma 4.1 and C is a constant. Similarly we have

$$\Big|\int_{a-\mathrm{i}M}^{a-\mathrm{i}N}\frac{\mathrm{e}^{\lambda t}}{t}\varphi(R(\lambda,A)^2x)\,\mathrm{d}\lambda\Big| \leqslant C(1+|1-a|\,K)^2\frac{\mathrm{e}^{at}}{t}\|x\|\,\|\varphi\|.$$

So combining these gives

$$\begin{aligned}&|\varphi((T(t)-S_{a,b,N}(t))x)|\\ &\quad\leqslant \frac{\mathrm{e}^{at}}{2\pi t}(|\varphi(R(a+\mathrm{i}N)x)|+|\varphi(R(a-\mathrm{i}N)x)|+2C(1+|1-a|\,K)^2\|x\|\,\|\varphi\|)\\ &\quad\leqslant \frac{\mathrm{e}^{at}}{\pi t}(K+C(1+|1-a|\,K)^2)\|x\|\,\|\varphi\|.\end{aligned}$$

So we have

$$\|T(t)-S_{a,b,N}(t)\| \leqslant \frac{\mathrm{e}^{at}}{\pi t}(K+C(1+|1-a|\,K)^2).$$

It is clear that the family $\{S_{a,b,N}(t)\}_{t\geqslant 0}$ is norm-continuous for $t \geqslant 0$ by the dominated convergence theorem. ∎

We now come to the main results of this section.

THEOREM 4.4. *Let $T(t)$ be a C_0-semigroup on a Hilbert space, and let $\gamma > \omega_0(T)$, $\beta > 0$. The following properties are equivalent:*
(i) *$T(t)$ is essentially norm-continuous of type β;*
(ii) *$\delta(T) \leqslant \omega_0(T) - \beta$;*
(iii) *$T(t)$ is essentially norm-measurable of type β;*
(iv) *given $0 < \alpha < \beta$, there exists N_α such that $\varphi^T_\gamma(N_\alpha) \leqslant (\gamma+\alpha)^{-1}$;*
(v) *$s_0^\infty(A) \leqslant \omega_0(T) - \beta$;*
(vi) *$s_0^\infty(A) \leqslant s(A) - \beta$;*
It follows that $s_0^\infty(A) = \delta(T)$.

Proof. We have (i)⇒(ii)⇒(iii)⇒(iv) by Theorem 2.4, and (iv)⇒(v) by Proposition 3.2, so it suffices to show that (v)⇒(i), (vi)⇒(v) and (ii)⇒(vi). (v)⇒(i).

Assume (v), and without loss of generality we may suppose that $\omega_0(T) = 0$, so that $s_0^\infty(A) \leqslant -\beta$. Let $0 < \alpha < \beta$. Then $-\alpha > s_0^\infty(A)$ so by Lemma 4.3 (taking $a = -\alpha$ and any $b > 0$) there exist a norm-continuous family of operators $\{S(t)\}_{t\geqslant 0}$ and a constant C (depending on α) such that

$$\|T(t) - S(t)\| \leqslant C\frac{\mathrm{e}^{-\alpha t}}{t}.$$

Hence $T(t)$ is essentially norm-continuous of type β.

(vi)⇒(v). This follows immediately from the fact that $s(A) \leqslant \omega_0(T)$.

(ii)⇒(vi). Assume (ii). Then $T(t)$ is asymptotically norm-continuous, hence $s(A) = \omega_0(T)$ (by Proposition 3.6, or [13], Corollary 1.4), and (vi) holds. To see that $s_0^\infty(A) = \delta(T)$ we only need to show that $\delta(T) \leqslant s_0^\infty(A)$ (by Corollary 3.3), and this follows from (v)⇒(ii) by taking $\beta = \omega_0(T) - s_0^\infty(A)$. ∎

So we have that on a Hilbert space the classes of essentially norm-continuous semigroups, essentially norm-measurable semigroups and of asymptotically norm-continuous semigroups coincide, and by (iv)⇔(ii), they have a characterization in terms of the behaviour of the resolvent along vertical lines: $T(t)$ is asymptotically norm-continuous if and only if for some (every) $\gamma > \omega_0(T)$ there exists N such that $\varphi_\gamma^T(N) < (\gamma + \omega_0(T))^{-1}$. Similar results for eventually norm-continuous semigroups and semigroups norm-continuous for $t > 0$ are also known. In our notation, the result of Blasco and Martinez in [1] becomes that a semigroup on a Hilbert space is eventually norm-continuous if and only if $(N\varphi_\gamma^T(N))_{n=0}^\infty$ is bounded for some (all) $\gamma > \omega_0(T)$, and the result of You Puhong ([17], with a more elementary proof given in [3]) that $T(t)$ is norm-continuous for $t > 0$ if and only if $\varphi_\gamma^T(0) = 0$ for some (all) $\gamma > \omega_0(T)$. By (vi)⇔(ii) above we have a second characterization of asymptotically norm-continuous semigroups on Hilbert space in terms of the resolvent being bounded in a region: $T(t)$ is asymptotically norm-continuous if and only if $s_0^\infty(A) < s(A)$. We now use Lemma 4.3 to obtain a similar result for semigroups norm-continuous for $t > 0$.

Theorem 4.5. *Let $T(t)$ be a C_0-semigroup with generator A on a Hilbert space H.*

(i) *Suppose that there exist $K > 0$, $\alpha \geqslant 0$, $c > \omega_0(T)$ and a decreasing function $\psi : (0, \infty) \to \mathbb{R}$ such that $\psi(M) \to -\infty$ as $M \to \infty$, and*

$$S := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \psi(|\operatorname{Im}\lambda)|\} \subseteq \rho(A)$$

with $\|R(\lambda, A)\| \leqslant K\mathrm{e}^{-\alpha \operatorname{Re}\lambda}$ whenever $\lambda \in S$ and $\operatorname{Re}\lambda \leqslant c$. Then $T(t)$ is norm-continuous for $t > 2\alpha$.

(ii) *$T(t)$ is norm-continuous for $t > 0$ if and only if there exist $K > 0$ and a decreasing function $\psi : (0, \infty) \to \mathbb{R}$ such that $\psi(M) \to -\infty$ as $M \to \infty$, and*

$$S := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \psi(\operatorname{Im}|\lambda|)\} \subseteq \rho(A)$$

with $\|R(\lambda, A)\| \leqslant K$ for $\lambda \in S$.

Proof. (i) Assume, without loss of generality, that $\omega_0(T) = 0$, and suppose that such a function ψ exists. Suppose R is sufficiently large that $\psi(R) < 0$. By Lemma 4.3, with $a = \psi(R)$, $b = 1$, $N = R$, there exist exponentially bounded norm-continuous families $S_R : \mathbb{R}_+ \to B(X)$, and $C > 0$ (independent of R) such that

$$\|T(t) - S_R(t)\| \leqslant \frac{\mathrm{e}^{\psi(R)t}}{\pi t}(K\mathrm{e}^{-\alpha\psi(R)} + C(1 + |1 - \psi(R)|\, K\mathrm{e}^{-\alpha\psi(R)})^2)$$

and hence

$$\limsup_{h\to 0} \|T(t+h) - T(t)\| \leqslant 2\frac{\mathrm{e}^{\psi(R)(t-2\alpha)}}{\pi t}(K\mathrm{e}^{\alpha\psi(R)} + C(\mathrm{e}^{\alpha\psi(R)} + |1 - \psi(R)|\, K)^2).$$

Fixing $t > 2\alpha$, and letting $R \to \infty$, we get

$$\limsup_{h\to 0} \|T(t+h) - T(t)\| = 0.$$

Hence $T(t)$ is norm-continuous for $t > 2\alpha$.

(ii) If $T(t)$ is norm-continuous for $t > 0$, then the existence of a function ψ with the required properties follows from the Riemann-Lebesgue lemma and a simple expansion of the resolvent argument (see [17], Lemma 2). The converse follows from (i) with $\alpha = 0$. ∎

5. ESSENTIALLY COMPACT SEMIGROUPS AND ESSENTIAL SPECTRA

Let $\mathcal{K}(X)$ be the two-sided ideal of compact operators in $B(X)$. Then we define the *essential norm* of $T \in B(X)$ by

$$\|T\|_{\mathrm{ess}} := \inf\{\|T - K\| : K \in \mathcal{K}(X)\}.$$

We define the *essential growth bound* of a semigroup $T(t)$ by

$$\omega_0^{\mathrm{ess}}(T) := \inf\{\omega \in \mathbb{R} : \text{there exists } M_\omega \text{ such that} \|T(t)\|_{\mathrm{ess}} \leqslant M_\omega \mathrm{e}^{\omega t} \text{ for all } t \geqslant 0\}$$

(see [15], Section 3.6). It is clear that $\omega_0^{\mathrm{ess}}(T) \leqslant \omega_0(T)$. A semigroup is said to be *essentially compact* if $\omega_0^{\mathrm{ess}}(T) < \omega_0(T)$.

It is well-known that eventually compact semigroups are eventually norm-continuous. In [18], Proposition 3.1, Thieme shows that essentially compact semigroups are essentially norm-continuous. It then follows by Theorem 2.4 that essentially compact semigroups are asymptotically norm continuous. We now show how Thieme's proof in fact gives the quantified version of this result, $\delta(T) \leqslant \omega_0^{\mathrm{ess}}(T)$. It is well-known that $\omega_0(T) = \max(\omega_0^{\mathrm{ess}}(T), s(A))$ (see for instance [15], Theorem 3.6.1) — in the light of the relationship between $\delta(T)$ and $\omega_0^{\mathrm{ess}}(T)$ this can be seen as a corollary of $\omega_0(T) = \max(\delta(T), s(A))$ (Proposition 3.7).

PROPOSITION 5.1. *$\delta(T) \leqslant \omega_0^{\mathrm{ess}}(T)$.*

Proof. Without loss of generality we may suppose that $\omega_0(T) = 0$. If $\omega_0^{\mathrm{ess}}(T) = \omega_0(T)$, then the result follows from the fact that $\delta(T) \leqslant \omega_0(T)$, so suppose $\omega_0^{\mathrm{ess}}(T) < \omega < \omega_0(T)$. Then there exist compact linear operators K_n and $M > 0$ such that

$$\|T(n) - K_n\| \leqslant M\mathrm{e}^{\omega n} \quad \text{for all } n \in \mathbb{N}.$$

Define

$$S(t) = \begin{cases} I & 0 \leqslant t < 1; \\ T(t-n)K_n & n \leqslant t < n+1. \end{cases}$$

Then $S(t)$ is right norm-continuous, and if $n \leqslant t < n+1$,

$$\begin{aligned}\|T(t) - S(t)\| = \|T(t-n)(T(n) - K_n)\| &\leqslant CM\mathrm{e}^{\omega n} \\ &\leqslant CM\mathrm{e}^{\omega t}\mathrm{e}^{\omega(n-t)} \leqslant CM\mathrm{e}^{-\omega}\mathrm{e}^{\omega t},\end{aligned}$$

where $C := \sup_{0 \leqslant r \leqslant 1} \|T(r)\|$. As $\omega_0^{\mathrm{ess}}(T) < \omega < \omega_0(T)$ was arbitrary, it follows that $\delta(T) \leqslant \omega_0^{\mathrm{ess}}(T)$. ∎

DEFINITION 5.2. Let S be a closed linear operator. We define four different notions of the essential spectrum of S: the *Browder essential spectrum*, the *upper-Fredholm essential spectrum*, the *lower-Fredholm essential spectrum*, and the *Fredholm essential spectrum* respectively by

$$\begin{aligned}
\sigma_{\mathrm{b}}^{\mathrm{ess}}(S) :=& \{\lambda \in \mathbb{C} : \lambda \text{ is a limit point of } \sigma(S) \text{ or } \lambda \text{ is an eigenvalue of} \\
& \text{infinite algebraic multiplicity or } \mathrm{Ran}\,(\lambda - S) \text{ is not closed}\}; \\
\sigma_{\mathrm{f+}}^{\mathrm{ess}}(S) :=& \{\lambda \in \mathbb{C} : \mathrm{Ker}\,(\lambda - S) \text{ has infinite dimension or } \mathrm{Ran}\,(\lambda - S) \\
& \text{is not closed}\}; \\
\sigma_{\mathrm{f-}}^{\mathrm{ess}}(S) :=& \{\lambda \in \mathbb{C} : \mathrm{Ran}\,(\lambda - S) \text{ has infinite codimension}\}; \\
\sigma_{\mathrm{f}}^{\mathrm{ess}}(S) :=& \sigma_{\mathrm{f+}}^{\mathrm{ess}} \cup \sigma_{\mathrm{f-}}^{\mathrm{ess}}.
\end{aligned}$$

See [7], Section 3, for more on the relationships between these spectra, and the other definitions of "essential spectrum" available.

REMARK 5.3. (i) Some authors explicitly include that points λ, such that $\mathrm{Ran}\,(\lambda - S)$ is not closed, are in $\sigma_{\mathrm{f-}}^{\mathrm{ess}}(S)$ as part of the definition. But if a closed operator has finite codimension then it automatically follows that it has closed range, so the additional clause is not required.

(ii) The definition of the given Browder essential spectrum is equivalent to the following (See [2], Theorem A.3.3.)

$$\begin{aligned}
\sigma_{\mathrm{b}}^{\mathrm{ess}}(S) :=& \sigma(A) \setminus \{\lambda \in \mathbb{C} : \lambda \text{ is an isolated point of } \sigma(A) \text{ such that the} \\
& \text{spectral projection of } \{\lambda\} \text{ has finite rank}\}.
\end{aligned}$$

DEFINITION 5.4. Let $L \in B(X)$. We define the *essential spectral radius* of L by

$$r^{\mathrm{ess}}(L) := \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{b}}^{\mathrm{ess}}(L)\}.$$

Let A be a closed linear operator. We define the *essential spectral bound* of A by

$$s^{\text{ess}}(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma_{\text{b}}^{\text{ess}}(A)\}.$$

Note that we could have defined the quantities above in term of any of the four essential spectra given in Definition 5.2, potentially giving four different essential spectral radii $r_b^{\text{ess}}(L)$, $r_{f+}^{\text{ess}}(L)$, $r_{f-}^{\text{ess}}(L)$ and $r_f^{\text{ess}}(L)$, and similarly four different essential spectral bounds. The next result shows that we do not need to worry about this.

Proposition 5.5. (i) *For any bounded operator L, the four quantities $r_b^{\text{ess}}(L)$, $r_{f+}^{\text{ess}}(L)$, $r_{f-}^{\text{ess}}(L)$, and $r_f^{\text{ess}}(L)$ are equal.*

(ii) *If A is the generator of a C_0-semigroup we have that $s_b^{\text{ess}}(A)$, $s_{f+}^{\text{ess}}(A)$, $s_{f-}^{\text{ess}}(A)$, and $s_f^{\text{ess}}(A)$ are equal.*

(iii) *We shall denote the common value in* (i) *by $r^{\text{ess}}(L)$, and in* (ii) *by $s^{\text{ess}}(A)$. If A generates a C_0-semigroup $T(t)$, then $r^{\text{ess}}(T(\tau)) = \mathrm{e}^{\tau\omega_0^{\text{ess}}(T)}$ for every $\tau \geqslant 0$.*

Proof. (i) This follows from [12], Theorem 6.5.

(ii) From [7], Section 6, it follows that

$$\sigma^{\text{ess}}((\beta I - A)^{-1}) \setminus \{0\} = \{(\beta - \lambda)^{-1} : \lambda \in \sigma^{\text{ess}}(A)\}$$

where σ^{ess} is any of the four essential spectra. Hence we have that if $s^{\text{ess}}(\,\cdot\,)$ and $r^{\text{ess}}(\,\cdot\,)$ are the corresponding spectral bounds and radii,

$$s^{\text{ess}}(A) = \lim_{\beta\to\infty}\Big\{\beta - \frac{1}{r^{\text{ess}}((\beta I - A)^{-1})}\Big\},$$

and the result follows from (i).

(iii) This is given on [15], page 106. ∎

In [13], Martinez and Mazon give a sketch proof of how to modify their spectral mapping theorem to obtain a spectral mapping theorem for the boundary spectrum of $\sigma_{f+}^{\text{ess}}(A)$ for asymptotically norm-continuous semigroups. We now observe that the improvements made to the spectral mapping theorem of Martinez and Mazon in Theorem 3.6 carry through to give a corresponding spectral mapping theorem for the part of $\sigma_{f+}^{\text{ess}}(A)$ in $\{\operatorname{Re}\lambda > \delta(T)\}$. Moreover, Theorem 3.6 allows us to give easy proofs that the same result holds for $\sigma_f^{\text{ess}}(A)$ and $\sigma_b^{\text{ess}}(A)$.

Theorem 5.6. *For $\tau > 0$ define $\Gamma_\tau := \{\lambda \in \mathbb{C} : \mathrm{e}^{\delta(T)\tau} < |\lambda|\}$. Then the spectral mapping theorem holds for the part of each of the Browder essential spectrum, the upper-Fredholm essential spectrum, and the Fredholm essential spectrum of $T(\tau)$ in Γ_τ, i.e.*

$$\sigma^{\text{ess}}(T(\tau)) \cap \Gamma_\tau = \exp(\tau\sigma^{\text{ess}}(A)) \cap \Gamma_\tau,$$

where

(i) $\sigma^{\text{ess}} = \sigma_{\text{b}}^{\text{ess}}$,

(ii) $\sigma^{\text{ess}} = \sigma_{\text{f+}}^{\text{ess}}$; *or*

(iii) $\sigma^{\text{ess}} = \sigma_{\text{f}}^{\text{ess}}$.

Proof. Without loss of generality we may assume that $\delta(T) < \omega_0(T) = 0$. Let $\tau > 0$.

(i) By the spectral inclusion theorem for the Browder essential spectrum ([2], Proposition 8.4) it is sufficient to prove that whenever $\lambda \in \sigma_{\mathrm{b}}^{\mathrm{ess}}(T(\tau))$ with $|\lambda| > \mathrm{e}^{\delta(T)\tau}$, then $\lambda = \mathrm{e}^{\mu\tau}$ for some $\mu \in \sigma_{\mathrm{b}}^{\mathrm{ess}}(A)$.

Let $\lambda \in \sigma_{\mathrm{b}}^{\mathrm{ess}}(T(\tau))$ with $|\lambda| > \mathrm{e}^{\delta(T)\tau}$. Choose $\mu_0 \in \mathbb{C}$ such that $\mathrm{e}^{\mu_0\tau} = \lambda$ and define $S := \{\mu \in \sigma(A) : \mathrm{e}^{\mu\tau} = \lambda\} \subseteq \{\mu_0 + 2\pi\mathrm{i}n/\tau : n \in \mathbb{N}\}$. As $\mathrm{Re}\,(\mu_0) > s^{\infty}(A)$ (Corollary 3.3), S must be bounded and hence finite, and as $\mathrm{Re}\,(\mu_0) > \delta(T)$, the spectral mapping theorem (Theorem 3.6) gives that S is non-empty.

Suppose first that λ is a limit point of $\sigma(T(\tau))$. Then by the spectral mapping theorem we may take a sequence $(\nu_n)_{n=0}^{\infty}$ such that $\mathrm{e}^{\nu_n\tau} \to \lambda$. Then $\mathrm{Re}\,(\nu_i) \longrightarrow \mathrm{Re}\,(\mu_0) > s^{\infty}(A)$, so we have that $\{|\mathrm{Im}\,(\nu_i)|\}$ is bounded, and hence by taking a subsequence if necessary, we may assume that (ν_n) converges to some μ. Then $\mu \in S \cap \sigma_{\mathrm{b}}^{\mathrm{ess}}(A)$ and $\lambda = \mathrm{e}^{\mu\tau}$.

Now suppose that λ is an isolated point of $\sigma(T(\tau))$. Suppose for contradiction that no $\mu \in S$ is in $\sigma_{\mathrm{b}}^{\mathrm{ess}}(A)$. Then each $\mu \in S$ is isolated in $\sigma(A)$, and the spectral projection of $\{\mu\}$ has finite rank. Thus the spectral projection P corresponding to S has finite rank. $\{\mu_0 + \frac{2\pi\mathrm{i}n}{\tau} : n \in \mathbb{N}\} \cap \sigma(A|\ker P) = \emptyset$, so by applying the spectral mapping theorem (Theorem 3.6) to the semigroup $T(t)|\ker P$, it follows that λ is not in $\sigma(T(\tau)|\ker P)$. Therefore P dominates the spectral projection corresponding to λ in $\sigma(T(\tau))$, so this spectral projection must have finite rank, and λ is not in $\sigma_{\mathrm{b}}^{\mathrm{ess}}(T(\tau))$, a contradiction.

(ii) A sketch proof is given in [13] for a spectral mapping theorem for the boundary of $\sigma_{\mathrm{f}+}^{\mathrm{ess}}(T(\tau))$ that holds for asymptotically norm-continuous semigroups. We show that with minor modifications this proof also gives our stronger result. Let $m(X) := \{(x_n) \in \ell^{\infty}(X) : (x_n) \text{ is relatively compact}\}$. We use the following characterization of $\sigma_{\mathrm{f}+}^{\mathrm{ess}}$, from [5], Theorem 3.3. For a closed operator S,

$$\begin{aligned}\sigma_{\mathrm{f}+}^{\mathrm{ess}}(S) = \{\lambda \in \mathbb{C} : \text{there exists } (x_n) \in \ell^{\infty}(X) \setminus m(X) \text{ such} \\ \text{that } x_n \in D(S) \text{ and } ((\lambda - S)x_n) \in m(X)\}.\end{aligned}$$

From the relationship

$$(\mathrm{e}^{\lambda\tau} - T(\tau))x = (\lambda - A)\int_0^{\tau} \mathrm{e}^{\lambda(\tau-s)}T(s)x\,\mathrm{d}s \tag{5.1}$$

(see [14], A-I, (3.1)), we have that $((\mathrm{e}^{\lambda\tau} - T(\tau))x_n) \in m(X)$ whenever $((\lambda - A)x_n) \in m(X)$ so that we always have $\exp(\tau\sigma_{\mathrm{f}+}^{\mathrm{ess}}(A)) \subseteq \sigma_{\mathrm{f}+}^{\mathrm{ess}}(T(\tau))$. For the other inclusion, we can use the same proof as in Theorem 3.6, using the space $\ell_T^{\infty}(X)/m(X)$ instead of $\ell_T^{\infty}(X)/c_0(X)$.

(iii) From (5.1) we have that $\mathrm{Ran}\,(\mathrm{e}^{\lambda\tau} - T(\tau))$ has infinite codimension whenever $\mathrm{Ran}\,(\lambda - A)$ does. Hence the "easy" inclusion holds for the whole of the lower-Fredholm essential spectrum and hence by (ii) for the whole of the Fredholm essential spectrum. Let $|\lambda| > \mathrm{e}^{\delta(T)\tau}$. By the inclusion we already have, and (ii), it is sufficient to prove that whenever $\mathrm{Ran}\,(T(\tau) - \lambda)$ is closed with infinite codimension then there exists $\mu \in S := \{\mu \in \sigma(A) : \mathrm{e}^{\mu\tau} = \lambda\}$ such that $\overline{\mathrm{Ran}\,(A - \mu)}$ has infinite codimension. Suppose this is false for a contradiction. Let $\mu \in S$. Then we have that

$$\big(X/\overline{\mathrm{Ran}\,(A - \mu)}\big)^* \cong \big(\overline{\mathrm{Ran}\,(A - \mu)}\big)^{\perp} = \mathrm{Ker}\,(A^* - \mu) \text{ has finite dimension.}$$

As in the proof of (i), S is finite, so that $K := \operatorname{span} \bigcup_{\mu\in S} \operatorname{Ker}(A^* - \mu)$ is finite dimensional. By [14], A-III, Corollary 6.4, $\operatorname{Ker}(T^*(\tau) - \lambda)$ is the closure of K in the weak* topology, so also has finite dimension. As

$$\operatorname{Ker}(T(\tau)^* - \lambda) = \left(\overline{\operatorname{Ran}(T(\tau) - \lambda)}\right)^{\perp} \cong \left(X/\overline{\operatorname{Ran}(T(\tau) - \lambda)}\right)^{*}$$

we have that $\overline{\operatorname{Ran}(T(\tau) - \lambda)} = \operatorname{Ran}(T(\tau) - \lambda)$ has finite codimension, a contradiction. ∎

REMARK 5.7. We have that for a densely defined closed operator S, $\lambda \in \sigma^{\mathrm{ess}}_{\mathrm{f}-}(S)$ if and only if $\lambda \in \sigma^{\mathrm{ess}}_{\mathrm{f}+}(S^*)$ (see [5], Corollary 3.5). It therefore follows, by duality, that above spectral mapping theorem holds for the lower-Fredholm essential spectrum for C_0-semigroups on reflexive spaces.

COROLLARY 5.8. $\omega_0^{\mathrm{ess}}(T) = \max(\delta(T), s^{\mathrm{ess}}(A))$.

Proof. By Proposition 5.1, either $\omega_0^{\mathrm{ess}}(T) = \delta(T)$ in which case we are done, or we have $\delta(T) < \omega_0^{\mathrm{ess}}(T)$. In the second case, we have $\mathrm{e}^{\omega_0^{\mathrm{ess}}(T)} = r^{\mathrm{ess}}(T(1)) = \mathrm{e}^{s^{\mathrm{ess}}(A)}$ by the spectral mapping theorem proved above. ∎

As a corollary of these results we obtain the following, which characterizes essentially compact semigroups on Hilbert spaces in terms of the resolvent and essential spectrum of the generator.

COROLLARY 5.9. *A C_0-semigroup on a Hilbert space is essentially compact if and only if $s_0^{\infty}(A) < s(A)$ and $s^{\mathrm{ess}}(A) < s(A)$.*

Proof. As $s_0^{\infty}(A) = \delta(T)$ holds on a Hilbert space (Theorem 4.4), we have $\omega_0^{\mathrm{ess}}(T) = \max(s_0^{\infty}(A), s^{\mathrm{ess}}(A))$. The result now follows from the fact that $s(A) \leqslant \omega_0(T)$, with equality when the semigroup is essentially compact ([15], Theorem 3.6.1). ∎

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