FACTORING IN NON-COMMUTATIVE
ANALYTIC TOEPLITZ ALGEBRAS

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ABSTRACT. The non-commutative analytic Toeplitz algebra is the WOT-closed algebra generated by the left regular representation of the free semigroup on n generators. The structure theory of contractions in these algebras is examined. Each is shown to have an H^∞ functional calculus. The isometries defined by words are shown to factor only as the words do over the unit ball of the algebra. This turns out to be false over the full algebra. The natural identification of WOT-closed left ideals with invariant subspaces of the algebra is shown to hold only for a proper subcollection of the subspaces.

KEYWORDS: *Analytic Toeplitz algebra, left regular representation, isometric dilation.*

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In [2] and [3], the algebraic and invariant subspace structures of the non-commutative analytic Toeplitz algebras were developed extensively. Several analogues of the analytic Toeplitz algebra were obtained. Many of these results came from a lucid characterization of the WOT-closed right ideals of these algebras. Although technical difficulties were encountered, a similar characterization of the left ideals was expected. In this paper, it is shown that, although it holds for a subcollection, the analogous characterization of the WOT-closed left ideals fails. The reason for this failure is a deep factorization problem in these algebras. Typically, when norm conditions are placed on possible factors of operators in these algebras, reasonable factorization results can be obtained. Indeed, positive results regarding isometries in the unit ball are included. However, in the general setting it turns out that even seemingly obvious unique factorizations do not hold. The examples provided go toward understanding the fabric of these algebras as well as the pathologies of factorization involved. Many of these examples rely on an understanding of the structure theory of contractions in these algebras. The minimal isometric dilation

of these contractions is determined. Further, each contraction is shown to have an H^∞ functional calculus. The author would like to thank his supervisor, Ken Davidson, for all of his assistance.

The terminology and notation used in this paper is that of [2] and [3]. The left regular representation of the unital free semigroup $\mathcal{F}_n$ on n generators $z_1, \ldots, z_n$ acts naturally on $\ell_2(\mathcal{F}_n) = \mathcal{H}_n$ by $\lambda(w)\xi_v = \xi_{wv}$, for v, w in $\mathcal{F}_n$. The non-commutative analytic Toeplitz algebra $\mathfrak{L}_n$ is the unital, WOT-closed algebra generated by the isometries $L_i = \lambda(z_i)$ for $1 \leqslant i \leqslant n$. For convenience, given w in $\mathcal{F}_n$ put $L_w = \lambda(w)$. The case $n = 1$ is exactly the analytic Toeplitz algebra. Throughout the paper, n is taken to be a finite positive integer. However, the results of ([2], [3]) on the structure of $\mathfrak{L}_n$ which are used here are also valid for $n = \infty$. So the results of this paper go through for $n = \infty$ with only minor notational changes. For ease of presentation, the paper is written as though n is finite. The algebra corresponding to the right regular representation is denoted by $\mathfrak{R}_n$. The generating isometries are defined by $\rho(w) = R_{w'}$ where $R_u \xi_v = \xi_{vu}$, and w' denotes the word w in reverse order. It is unitarily equivalent to $\mathfrak{L}_n$ and is precisely the commutant of $\mathfrak{L}_n$ (see [3] and [6]).

In the first section it is proved that the minimal isometric dilation of each non-unitary contraction in $\mathfrak{L}_n$ is a shift. This is accomplished by showing that the powers of the adjoint of such a contraction converge strongly to 0. All of these shifts have infinite multiplicity. Further, each of these contractions has an H^∞ functional calculus.

The second section contains positive factorization results for isometries over the unit ball of $\mathfrak{L}_n$. The isometries L_w factor exactly as the words w. Hence, such isometries are irreducible over the unit ball exactly when the length of the word $|w| = 1$. A characterization of a broader class of irreducible isometries over the unit ball is also included. Surprisingly, these isometries are reducible over the full algebra. Particular factors are constructed using the structure of the orthogonal complement of their ranges.

The last section contains a discussion of the WOT-closed ideals of these algebras. Unlike right ideals, the left ideal generated by the isometry L_2 is not WOT-closed. The same is true for the two-sided ideal generated by L_2. The natural identification of subspaces in $\operatorname{Lat} \mathfrak{R}_n$ with the WOT-closed right and two-sided ideals is shown to only transfer over to a proper subcollection of $\operatorname{Lat} \mathfrak{L}_n$, when considering left ideals.

1. CONTRACTIONS

The inner-outer factorization in $\mathfrak{L}_n$ (see [1], [3] and [6]) shows that much of the game in these algebras is played in the world of isometries (inner operators). Thus it is worthwhile to examine the structure theory of these operators. It turns out there is much that one can say about such isometries when they are considered in the more general setting of contractions.

For notational purposes recall from [2] that given $k \geqslant 1$, every X in $\mathfrak{L}_n$ can be uniquely written as a sum

$$(1.1) \qquad X = \sum_{|w|<k} x_w L_w + \sum_{|w|=k} L_w X_w,$$

where $x_w \in \mathbb{C}$ and $X_w \in \mathfrak{L}_n$ (for $n = \infty$, the first sum belongs to ℓ_2 and the second sum is actually a WOT-limit). The scalars $\{x_w\}_{w\in\mathcal{F}_n}$ are called the Fourier coefficients of X. This notation is justified since they determine the operator. Indeed, $X\xi_1 = \sum_w x_w \xi_w$ and hence

$$X\xi_v = X(R_v \xi_1) = R_v(X\xi_1) = \sum_w x_w \xi_{wv}.$$

One writes $X \sim \sum_w x_w L_w$. For the sake of convenience, given a word $w = z_{i_1} \cdots z_{i_k}$ also let $X_{i_1 \cdots i_k}$ (respectively $x_{i_1 \cdots i_k}$) denote the operator X_w (respectively the scalar x_w) in the expansion (1.1) of X. Lastly, the scalar x_0 is taken to be the inner product $(X\xi_1, \xi_1)$.

The starting point is a simple result which is used several times throughout the paper. In [2] it is shown that the only normal operators belonging to $\mathfrak{L}_n$ are the scalars. Thus the only unitaries in $\mathfrak{L}_n$ are scalar. The proof of the latter fact is actually quite elementary.

PROPOSITION 1.1. *The collection of unitary operators in $\mathfrak{L}_n$ is exactly the set $\{\lambda I : \lambda \in \mathbb{T}\}$.*

Proof. Let U be unitary in $\mathfrak{L}_n$. Then there is a unit vector η in $\mathcal{H}_n$ with $U\eta = \xi_1$. The scalar $\lambda = (\eta, \xi_1)$ satisfies $|\lambda| \leqslant 1$. Consider the Fourier expansion, $U \sim \sum_w a_w L_w$. Evidently then,

$$1 = \|U\xi_1\|^2 = \sum_w |a_w|^2 \geqslant |a_0|^2.$$

However, it is also true that

$$1 = |(U\eta, \xi_1)| = |a_0 \lambda|.$$

Whence $|a_0| = |\lambda| = 1$. Thus $\eta = \lambda \xi_1$, so that

$$\overline{\lambda} \xi_1 = \overline{\lambda} U\eta = U\xi_1.$$

Therefore $U = \overline{\lambda} I$, and the proof is finished. ∎

In the seminal text [9], it is shown that understanding the behaviour of the powers of the adjoint of a contraction is a key issue. In particular, strong convergence to 0 yields information on the minimal isometric dilation of the operator, as well as an H^∞ functional calculus. This condition holds for every non-unitary contraction in $\mathfrak{L}_n$.

LEMMA 1.2. *If L is a non-unitary contraction in $\mathfrak{L}_n$, then*

$$\lim_{k \to \infty} \|(L^*)^k \xi\| = 0,$$

for all ξ in $\mathcal{H}_n$.

Proof. The key is the unique decomposition of L. Using (1.1), write

$$L = \lambda I + \sum_{i=1}^n L_i A_i,$$

with each A_i in $\mathfrak{L}_n$. Then by the previous proposition, $|\lambda| < 1$ since L is not unitary. Hence if $A = \sum_{i=1}^n L_i A_i$, then $\|A\| < 2$.

The lemma will be proved for basis vectors corresponding to words. Suppose that w is a word of length l. For $k > l$,

$$(L^*)^k \xi_w = (\overline{\lambda} I + A^*)^k \xi_w$$
$$= \left(\overline{\lambda}^k I + \binom{k}{1} \overline{\lambda}^{k-1} A^* + \cdots + \binom{k}{l} \overline{\lambda}^{k-l} (A^*)^l \right) \xi_w + 0.$$

For $0 \leqslant j \leqslant l$ define polynomials p_j by $p_j(x) = \frac{1}{j!} x(x-1) \cdots (x-j+1)$. Then one has

$$(L^*)^k \xi_w = \sum_{j=0}^l p_j(k) \overline{\lambda}^{k-j} (A^*)^j \xi_w.$$

However, the limit $\lim_{k \to \infty} \frac{k^m}{\alpha^k} = 0$ holds for any real number m and $\alpha > 1$ (see [8], p. 57). It follows that for $0 \leqslant j \leqslant l$,

$$\lim_{k \to \infty} p_j(k) |\lambda|^{k-j} = 0.$$

Now given $\varepsilon > 0$, choose $K > l$ such that $k \geqslant K$ implies that

$$p_j(k) |\lambda|^{k-j} < \varepsilon$$

for $0 \leqslant j \leqslant l$. Then for all sufficiently large k one has

$$\|(L^*)^k \xi_w\| \leqslant \sum_{j=0}^l p_j(k) |\lambda|^{k-j} \|(A^*)^j \xi_w\| < \sum_{j=0}^l \varepsilon \, 2^j = (2^{l+1} - 1)\varepsilon.$$

Hence, $\lim_{k \to \infty} \|(L^*)^k \xi_w\| = 0$ for all words w.

That it is true in full generality follows from the boundedness of the sequence $\{(L^*)^k\}$. Indeed, any uniformly bounded sequence of operators $\{A_k\}$ on $\mathcal{H}_n$ which satisfies $\lim_{k \to \infty} \|A_k \xi_w\| = 0$ for all words w, must converge in the strong operator topology to 0. This completes the proof. ∎

As a direct consequence of the lemma, one obtains the intuitive result that the powers of every non-unitary contraction in $\mathfrak{L}_n$ converge in the weak operator topology to 0. The lemma also yields deeper information on the structure theory of contractions.

THEOREM 1.3. *The minimal isometric dilation of any non-unitary contraction in $\mathfrak{L}_n$ is a shift.*

Proof. In [9] it is shown that every contraction has a minimal isometric dilation. By the Wold decomposition for isometries, every isometry is the orthogonal direct sum of a unitary and copies of the unilateral shift. It is further shown in [9] that the powers of the adjoint of the contraction converging strongly to 0 (in other words, belonging to the class $C_{\cdot 0}$) is equivalent to the unitary part of its minimal isometric dilation being vacuous. Hence the lemma yields the result. ∎

REMARK 1.4. This appears to be new for $n = 1$. The author could find no references, but it is probably known in this case.

Recall that a contraction is called *completely non-unitary* provided its restriction to any non-zero reducing subspace is never unitary.

COROLLARY 1.5. *Every non-unitary contraction L in $\mathfrak{L}_n$ is completely non-unitary.*

Proof. Any non-zero reducing subspace for which the restriction of L to it is unitary would be contained in the unitary summand of the Wold decomposition for the minimal isometric dilation of L. However, by the theorem this space is vacuous. ∎

Another important result which comes out of the Sz.-Nagy and Foiaş machinery is that every completely non-unitary contraction possesses an H^∞ functional calculus.

COROLLARY 1.6. *Every non-unitary contraction in $\mathfrak{L}_n$ has an H^∞ functional calculus.*

Given a non-unitary contraction L in $\mathfrak{L}_n$, the collection of operators defined by this H^∞ functional calculus is denoted $H^\infty(L)$.

The cardinality of the shift in the Wold decomposition of an isometry V in $\mathcal{B}(\mathcal{H})$ is given by the dimension of $\mathcal{H} \ominus V\mathcal{H}$. For isometries in the analytic Toeplitz algebra this cardinality can be both finite and infinite. In fact, if φ belongs to H^∞, the dimension of $H^2 \ominus \varphi H^2$ is finite exactly when the analytic inner symbol is continuous. That is, when φ belongs to the disk algebra (see [4]). For the non-commutative algebras there is in general more room in the orthogonal complement, and this cardinality turns out to always be infinite. To prove this, first note the following result, the proof of which is actually contained in the proof of Theorem 1.7 from [3].

Theorem 1.7. *For $n \geqslant 2$, if A in $\mathfrak{L}_n$ has proper closed range, then $\overline{\mathrm{Ran}(A)}$ has infinite codimension.*

Remark 1.8. For isometries L in $\mathfrak{L}_n$, the infinite cardinality of $\mathcal{H}_n \ominus L\mathcal{H}_n$ is actually easy to see when $(L\xi_1, \xi_1) = 0$. Indeed, let P_k be the projection of $\mathcal{H}_n$ onto $\mathrm{span}\{\xi_w : |w| = k\}$. Then,

$$P_k \mathcal{H}_n \supseteq P_k(\mathrm{Ran}(L)) = P_k\left(\sum_{i=0}^{k-1} L(P_i \mathcal{H}_n)\right).$$

The former space has dimension n^k, the latter has dimension at most $\frac{n^k - 1}{n - 1}$. Summing over $k \geqslant 1$ proves the claim.

In any event, it follows that the range of any non-outer operator has infinite codimension. Recall that A in $\mathfrak{L}_n$ is *inner* if it is an isometry and *outer* if $\mathrm{Ran}(A)$ is dense in $\mathcal{H}_n$.

Corollary 1.9. *For $n \geqslant 2$, if A in $\mathfrak{L}_n$ is not outer, then $\overline{\mathrm{Ran}(A)}$ has infinite codimension.*

Proof. Since A is not outer, by the unique inner-outer factorization it can be written as $A = LB$ for some non-scalar isometry L and outer operator B, both in $\mathfrak{L}_n$. But then,

$$\overline{A\mathcal{H}_n} = \overline{LB\mathcal{H}_n} = L(\overline{B\mathcal{H}_n}) = L\mathcal{H}_n.$$

The latter space has infinite codimension by the theorem. ∎

The general result can now be proved. The proof makes use of Fredholm theory in $\mathfrak{L}_n$.

Theorem 1.10. *Let L be a non-unitary contraction in $\mathfrak{L}_n$, for $n \geqslant 2$. If V in $\mathcal{B}(\mathcal{K})$ is its minimal isometric dilation, then*

$$\dim(\mathcal{K} \ominus V\mathcal{K}) = \infty.$$

Proof. The multiplicity of V is given by the rank of $I - VV^*$. From the construction of the minimal isometric dilation, this is at least the rank of the operator $I - LL^*$. Suppose this number is finite. Then L^* has an essential left inverse, and hence $\ker L^* = (\overline{\mathrm{Ran}\, L})^\perp$ is finite dimensional. Thus by the previous corollary, L must in fact be outer.

It now follows that the operators L^*L and LL^* are unitarily equivalent. For, the partial isometry in the polar decomposition of L is really invertible and acts as the intertwining unitary. Therefore,

$$\mathrm{rank}(I - L^*L) = \mathrm{rank}(I - LL^*) < \infty,$$

so that L is an essential unitary.

As a Fredholm operator, L has closed range and is thus surjective since it is outer. From [3], every operator in $\mathfrak{L}_n$ is injective. Hence L is invertible. However, it was also shown in [3] that the essential norm of every operator in $\mathfrak{L}_n$ is the same as its original norm. Since $\mathfrak{L}_n$ is inverse closed ([3]), this implies that

$$\|L^{-1}\| = \|L^{-1}\|_{\mathrm{e}} = \|L^*\|_{\mathrm{e}} = 1.$$

As an invertible isometry in $\mathfrak{L}_n$, L must be scalar. This contradiction completes the proof. ∎

The investigation of the structure of $\mathcal{H}_n \ominus L\mathcal{H}_n$ will be revisited next section in the context of factoring.

In many ways, contractions satisfying $(L\xi_1, \xi_1) = 0$ are easier to deal with. It is thus helpful to finish off this section by observing that there is a large class of contractions in $\mathfrak{L}_n$ for which this inner product is non-zero, however these operators are unitarily equivalent to contractions in $\mathfrak{L}_n$ which have no scalar part. Observe that for any operator L in $\mathfrak{L}_n$, the space $\mathcal{H}_n \ominus \overline{L\mathcal{H}_n}$ belongs to Lat $\mathfrak{R}_n^*$.

THEOREM 1.11. *Suppose L is a contraction in $\mathfrak{L}_n$ for which $\mathcal{H}_n \ominus \overline{L\mathcal{H}_n}$ contains an eigenvector of $\mathfrak{R}_n^*$. Then L is unitarily equivalent to a contraction in $\mathfrak{L}_n$ which has no scalar part. In addition, this unitary implements an automorphism of $\mathfrak{L}_n$.*

Proof. In [3] the eigenvectors of $\mathfrak{R}_n^*$ are identified. Each scalar λ in the unit ball of n-dimensional Hilbert space defines an eigenvector v_λ. Further, for each such vector there is a unitary U_λ on $\mathcal{H}_n$ for which $\text{Ad}\, U_\lambda$ determines an automorphism of $\mathfrak{L}_n$ with

$$U_\lambda v_\lambda = \xi_1.$$

Thus, suppose some v_λ belongs to $\mathcal{H}_n \ominus \overline{L\mathcal{H}_n}$. Then the operator $U_\lambda L U_\lambda^*$ is a contraction in $\mathfrak{L}_n$ and

$$(U_\lambda L U_\lambda^* \xi_1, \xi_1) = (Lv_\lambda, v_\lambda) = 0,$$

which proves the result. ∎

2. FACTORING

In the analytic Toeplitz algebra $\mathfrak{L}_1 = \mathfrak{R}_1 = \mathcal{T}(H^\infty)$, the associated function theory yields a good factorization theory over the full algebra (see [4] and [5] for example). When moving to several non-commutative variables, the strong link to the function theory is lost and factorization becomes much more difficult to deal with. Nonetheless, positive results such as inner-outer factorization can be obtained. Other factorization results can be obtained when norm restrictions are placed on possible factors.

THEOREM 2.1. *Let $w \in \mathcal{F}_n$. Over the unit ball of $\mathfrak{L}_n$, the isometry L_w factors only in the same way as the word w, modulo scalars in $\mathbb{T}$.*

Proof. Suppose $L_w = BC$ with B and C belonging to $b_1(\mathfrak{L}_n)$. It is clear that C must in fact be an isometry. For each $k \geqslant 1$, consider the corresponding form of (1.1) for B and C. Let $\{b_v\}$ be the scalars and let $\{B_v\}$ be the operators for B in this decomposition. Use similar notation for C.

If $w = 1$, then $I = b_0 c_0 I$. Whence, $|b_0| = |c_0| = 1$ and the operators B and C are scalar unitaries. Otherwise, put $w = z_{i_1} \cdots z_{i_k}$, and note that $b_0 c_0 = 0$.

First suppose that $b_0 = 0$. Then $B = \sum_{i=1}^{n} L_i B_i$ and equating factorizations yields,

$$L_{i_1}\left(B_{i_1} C\right) = L_{i_1}\left(L_{i_2} \cdots L_{i_k}\right)$$

and

$$L_j \left(B_j C \right) = 0 \quad \text{for } j \neq i_1.$$

In particular, $B_j C = 0$ for $j \neq i_1$. But every non-zero element of $\mathfrak{L}_n$ is injective ([3]), so that $B_j = 0$ for $j \neq i_1$. Further, B_{i_1} is a contraction and $B_{i_1} C = L_{i_2} \cdots L_{i_k}$. Hence, by induction one has $B_{i_1} = \lambda L_u$ and $C = \bar{\lambda} L_v$ where $uv = z_{i_2} \cdots z_{i_k}$ and λ belongs to $\mathbb{T}$. Thus, $B = \lambda L_{i_1} L_u$ and $C = \bar{\lambda} L_v$.

Next suppose $b_0 \neq 0$, so that $c_0 = 0$. Note first that

$$(\xi_w, \xi_{z_{i_k}}) = (BC\xi_1, \xi_{z_{i_k}}) = \sum_{v,u} b_u c_v (\xi_{uv}, \xi_{z_{i_k}}) = b_0 c_{i_k} + b_{i_k} c_0 = b_0 c_{i_k}.$$

Hence if $w = z_{i_k}$, one would have $b_0 c_{i_k} = 1$. As B and C are contractions, it would follow that $|b_0| = |c_{i_k}| = 1$, and that B is a scalar unitary. Otherwise suppose $|w| > 1$. Inductively, one can show that

$$0 = c_0 = c_{i_k} = \cdots = c_{i_2 \cdots i_k}.$$

To see this, observe that $0 = (\xi_w, \xi_{z_{i_k}}) = b_0 c_{i_k}$, and hence $c_{i_k} = 0$. Then suppose $0 = c_0 = c_{i_k} = \cdots = c_{i_j \cdots i_k}$ for some j, $2 < j \leqslant k$. Equating Fourier coefficients of $L_w = BC$ shows that

$$0 = (\xi_w, \xi_{z_{j-1} \cdots z_k}) = (BC\xi_1, \xi_{z_{j-1} \cdots z_k})$$
$$= b_0 c_{i_{j-1} \cdots i_k} + b_{i_{j-1}} c_{i_j \cdots i_k} + \cdots + b_{i_{j-1} \cdots i_k} c_0 = b_0 c_{i_{j-1} \cdots i_k}.$$

Whence, $c_{i_{j-1} \cdots i_k} = 0$ as claimed. But then,

$$1 = (\xi_w, \xi_w) = (BC\xi_1, \xi_w) = b_0 c_w + b_{i_1} c_{i_2 \cdots i_k} + \cdots + b_w c_0 = b_0 c_w.$$

Thus, $|b_0| = |c_w| = 1$ and B is a scalar unitary. ∎

It is immediate that the generating isometries are irreducible over the unit ball.

COROLLARY 2.2. *For $n \geqslant 2$, each L_i is irreducible over the unit ball of $\mathfrak{L}_n$.*

In fact, many more isometries are irreducible over the unit ball of $\mathfrak{L}_n$. Indeed, one can work harder to obtain the next result which includes a large collection of isometries. For example, the isometries $L = \frac{1}{\sqrt{2}}(L_1 + L_2)$ and $L = \frac{1}{\sqrt{2}} L_1 + \frac{1}{2} L_2^2 + \frac{1}{2} L_3^3$ are irreducible.

THEOREM 2.3. *Suppose $L \sim \sum_{w \neq 1} a_w L_w$ is an isometry in $\mathfrak{L}_n$ for which there is an i with $a_{z_i} \neq 0$ and $R_i R_i^* (L\xi_1) = a_{z_i} \xi_{z_i}$. Then L is irreducible over the unit ball of $\mathfrak{L}_n$, modulo scalars in $\mathbb{T}$.*

Proof. Suppose $L = BC$ with B and C in $b_1(\mathfrak{L}_n)$. The operator C must be an isometry. As in the proof of the previous theorem, consider the expansions of B and C determined by (1.1). Recall that

$$B = b_0 I + \sum_{j=1}^{n} L_j B_j \quad \text{and} \quad C = c_0 I + \sum_{j=1}^{n} L_j C_j.$$

By equating unique factorizations of $L = BC$ one obtains, $b_0 c_0 = 0$. It is also clear that

$$L_i \left(b_0 C_i + B_i C \right) = L_i \left(a_{z_i} I \right).$$

First suppose $b_0 = 0$. Then, $B_i C = a_{z_i} I \neq 0$. Thus by the injectivity of all elements in $\mathfrak{L}_n$, the operator B_i must be invertible. Hence C is also invertible. Therefore, as an invertible isometry in $\mathfrak{L}_n$, C must be a scalar unitary.

Next suppose $b_0 \neq 0$ and so $c_0 = 0$. This corresponds to the case when B is a scalar unitary. Indeed, note first from the Fourier expansions,

$$a_{z_i} = (L\xi_1, \xi_{z_i}) = (BC\xi_1, \xi_{z_i}) = b_0 c_{z_i} + b_{z_i} c_0 = b_0 c_{z_i}.$$

As B is a contraction, $|c_{z_i}| \geqslant |a_{z_i}|$. Now by hypotheses one has

$$(L - a_{z_i} L_i)\xi_1 = \sum_{j=1}^{n} R_j R_j^* (L - a_{z_i} L_i)\xi_1 = \sum_{j \neq i} R_j R_j^* (L\xi_1).$$

Further since $L = BC$,

$$B(C - c_{z_i} L_i)\xi_1 = \sum_{j=1}^{n} R_j R_j^* (L - c_{z_i} B L_i)\xi_1 = \sum_{j \neq i} R_j R_j^* (L\xi_1) + R_i R_i^* (L - c_{z_i} B L_i)\xi_1.$$

Evidently then,

$$\|B(C - c_{z_i} L_i)\xi_1\| \geqslant \|(L - a_{z_i} L_i)\xi_1\|.$$

Thus the following is true:

$$1 = \|C\xi_1\|^2 = \|(C - c_{z_i} L_i)\xi_1\|^2 + |c_{z_i}|^2 \geqslant \|B(C - c_{z_i} L_i)\xi_1\|^2 + |c_{z_i}|^2$$
$$\geqslant \|(L - a_{z_i} L_i)\xi_1\|^2 + |a_{z_i}|^2 = \|L\xi_1\|^2 = 1.$$

Therefore, $|a_{z_i}| = |c_{z_i}|$ and $|b_0| = 1$, which shows that B is a scalar unitary. ∎

REMARK 2.4. This proof can be perturbed to include broader classes of isometries. For example, any isometry L which satisfies

$$R_i R_i^* (L\xi_1) = a_{w z_i} \xi_{w z_i},$$

for some i and word w is irreducible over the unit ball of $\mathfrak{L}_n$.

As it turns out, the unique factorizations over the unit ball of $\mathfrak{L}_n$ discussed above do not hold over the full algebra. Remarkably, even the operator L_2 has proper factorizations in $\mathfrak{L}_n$. This comes out of an interesting result from the function theory.

LEMMA 2.5. *The function* $f(z) = \sum\limits_{k \geqslant 0} \frac{z^k}{k+1}$ *belongs to* $H^2 \setminus H^\infty$. *However,* $\frac{1}{f}$ *defines a function which lies in* H^∞.

Proof. Since f is analytic on the unit disk and the Fourier coefficients of f are ℓ_2-summable, the function belongs to H^2 with

$$\|f\|_2 = \left(\sum_{k \geqslant 0} \frac{1}{(k+1)^2} \right)^{\frac{1}{2}}.$$

For $|z| < 1$, $f(z)$ is defined by the formula

$$(2.1) \qquad\qquad zf(z) = -\log(1 - z)$$

for the principal branch of the logarithm. Given r such that $0 < r < 1$, let f_r be the function on $\mathbb{T}$ defined by $f_r(e^{i\theta}) = f(r\,e^{i\theta})$. Then the identity

$$\lim_{r \to 1^-} f(r) = \lim_{r \to 1^-} \frac{-\log(1 - r)}{r} = \infty,$$

together with the continuity of f on the disk, shows that

$$\|f\|_{H^\infty} := \lim_{r \to 1} \|f_r\|_\infty = \infty.$$

Hence the function f is not in H^∞.

To prove that $\frac{1}{f}$ defines a function in H^∞, it is required to show that $\frac{1}{f}$ defines an analytic function on the unit disk with $\|\frac{1}{f}\|_\infty < \infty$. Now, $\frac{1}{f}$ is analytic on the unit disk by (2.1). To see that f is bounded below first observe the identity $1 - e^{i\theta} = (2\sin\frac{\theta}{2})e^{\frac{\theta - \pi}{2}i}$. Hence for $|\theta| \leqslant \pi$ with $\theta \neq 0$,

$$|f(e^{i\theta})|^2 = \left| \frac{-\log(1 - e^{i\theta})}{e^{i\theta}} \right|^2 = \left| \frac{\log|2\sin\frac{\theta}{2}| + i\left(\frac{\theta - \pi}{2}\right)}{e^{i\theta}} \right|^2$$

$$= \left(\log\left|2\sin\frac{\theta}{2}\right| \right)^2 + \left(\frac{\theta - \pi}{2} \right)^2.$$

But $\left(\frac{\theta - \pi}{2}\right)^2 \geqslant \frac{\pi^2}{16}$ for $-\pi \leqslant \theta \leqslant \frac{\pi}{2}$ and $|2\sin\frac{\theta}{2}| \geqslant \sqrt{2}$ for $\frac{\pi}{2} \leqslant \theta \leqslant \pi$. Thus,

$$|f(e^{i\theta})|^2 \geqslant \min\left\{ \frac{(\log 2)^2}{4}, \frac{\pi^2}{16} \right\} = \frac{(\log 2)^2}{4}$$

for $\theta \neq 0$. It now follows that $\|\frac{1}{f}\|_\infty < \infty$, and the proof is finished. ∎

This unusual function theoretic result allows one to construct explicit factorizations which are exclusive to the non-commutative setting.

THEOREM 2.6. *Suppose L is an isometry in $\mathfrak{L}_n$ for which $\mathcal{H}_n \ominus L\mathcal{H}_n$ contains the range of an isometry X in $\mathfrak{L}_n$. Then L has proper factorizations in $\mathfrak{L}_n$.*

Proof. As the ranges of the isometries $X^k L$ are pairwise orthogonal for $k \geqslant 0$, the operator $A = \sum_{k \geqslant 0} \frac{1}{k+1} X^k L$ belongs to $\mathfrak{L}_n$ with

$$\|A\| = \left(\sum_{k \geqslant 0} \frac{1}{(k+1)^2} \right)^{\frac{1}{2}}.$$

Let $g = \frac{1}{f}$ be the H^∞ function obtained in the previous lemma. By the H^∞ functional calculus for X (Corollary 1.6), $g(X)$ defines an operator in $\mathfrak{L}_n$. The claim is that $g(X)A = L$. This comes as a result of a more general fact.

Note that given h in H^2, an operator $h(X)L$ can be naturally defined in $\mathfrak{L}_n$. Indeed, one can set

$$h(X)L = \sum_{k \geqslant 0} \widehat{h}(k) X^k L$$

where the $\widehat{h}(k)$ are the Fourier coefficients for h. Clearly, the map from H^2 to $\mathcal{B}(\mathcal{H}_n)$ which sends h to $h(X)L$ is isometric. The key is that this map is also continuous from the topology of weak vector convergence in H^2 to the WOT in $\mathfrak{L}_n$. To see this, suppose h_m converges weakly to h in H^2. Without loss of generality assume $h = 0$. Then $\widehat{h}_m(k)$ converges to 0 for each k and

$$\sup_m \|h_m\|_2 = c < \infty,$$

for some constant c. Let x and y be unit vectors in $\mathcal{H}_n$ and let S_k be the orthogonal projection onto $\mathrm{Ran}(X^k L)$. Then

$$\sum_{k \geqslant 0} |(X^k Lx, y)|^2 = \sum_{k \geqslant 0} |(X^k Lx, S_k y)|^2 \leqslant \sum_{k \geqslant 0} \|S_k y\|^2 \leqslant \|y\|^2 = 1.$$

Thus, given $\varepsilon > 0$ one can choose $N = N(\varepsilon)$ for which the Nth ℓ_2 tail of the above series is smaller than ε. Then, for each m, the Cauchy-Schwarz inequality shows that

$$|(h_m(X)Lx, y)| = \left| \sum_{k \geqslant 0} \widehat{h}_m(k)(X^k Lx, y) \right|$$

$$\leqslant \sum_{0 \leqslant k \leqslant N} |\widehat{h}_m(k)| + \left(\sum_{k > N} |\widehat{h}_m(k)|^2 \right)^{\frac{1}{2}} \varepsilon$$

$$\leqslant \sum_{0 \leqslant k \leqslant N} |\widehat{h}_m(k)| + c\varepsilon.$$

As $\widehat{h}_m(k)$ converges to 0 for each k, it follows that $h_m(X)L$ converges WOT to 0.

Recall that the analytic trigonometric polynomials are weak* dense in H^∞ ([4]). Let g_m be such a sequence converging weak* to g. From the definition of this weak* topology, the sequence g_m converges weakly to g in H^∞. Thus, the sequence $g_m f$ converges weakly to $gf = 1$ in H^2. Hence, since each g_m is a polynomial,

$$g_m(X)A = g_m(X)\,(f(X)L) = (g_m f)(X)L \xrightarrow{\mathrm{WOT}} (gf)(X)L = L.$$

Further, by the H^∞ functional calculus for X, $g_m(X)$ converges WOT to $g(X)$ ([9]). Therefore,

$$g_m(X)A \xrightarrow{\mathrm{WOT}} g(X)A.$$

Whence $g(X)A = L$.

It remains to observe that $g(X)$ and A are both not invertible. The invertibility of $g(X)$ in $\mathfrak{L}_n$ would imply the invertibility of g in H^∞, contradicting the previous lemma. If A was invertible, it would be the scalar multiple of an invertible isometry in $\mathfrak{L}_n$, hence scalar itself by Proposition 1.1. The proof is now complete. ∎

REMARK 2.7. The theorem really is exclusive to the non-commutative setting. The hypothesis of the theorem cannot be satisfied when $n = 1$. For if φ and ψ are inner functions in H^∞, then the function $\varphi\psi = \psi\varphi$ belongs to φH^2 and ψH^2.

As a surprising consequence of the theorem, the reducibility of the generating isometries is revealed in the non-commutative setting.

COROLLARY 2.8. *For $n \geqslant 2$, each L_i has proper factorizations over $\mathfrak{L}_n$.*

Proof. The isometry X can be taken to be L_j for $j \neq i$. ∎

In fact, there is a large collection of isometries which can be seen to be reducible in this manner. Note that by unique factorization, every operator L in $\mathfrak{L}_n$ with $(L\xi_1, \xi_1) = 0$ can be written as

$$L = \sum_{|w|=k} L_w A_w,$$

for some $k \geqslant 1$ and A_w in $\mathfrak{L}_n$.

COROLLARY 2.9. *Let L be an isometry in $\mathfrak{L}_n$ with $L = \sum\limits_{|w|=k} L_w A_w$, for some $k \geqslant 1$ and A_w in $\mathfrak{L}_n$. Suppose either one of the following conditions holds:*

(i) there is an A_w for which there exists a $B \neq 0$ in $\mathfrak{L}_n$ with the range of B orthogonal to that of A_w, or

(ii) some A_w is a scalar multiple of an isometry.

Then L properly factors in $\mathfrak{L}_n$.

Proof. Let ξ and η be vectors belonging to $\mathcal{H}_n$ throughout the proof. To prove (i), set $A = L_w B$. Write the inner-outer factorization for A as $A = L_A C$. Since C is outer, there are vectors ζ_m such that $\eta = \lim\limits_{m \to \infty} C\zeta_m$. It follows that

$$(L\xi, L_A\eta) = \lim_{m \to \infty} (L\xi, L_A C\zeta_m) = \lim_{m \to \infty} (L\xi, L_w B\zeta_m)$$
$$= \lim_{m \to \infty} (L_w A_w\xi, L_w B\zeta_m) = \lim_{m \to \infty} (A_w\xi, B\zeta_m) = 0.$$

Thus, L_A is an isometry with range orthogonal to the range of L.

Lastly, suppose that A_w is a scalar multiple of an isometry. Write L as $L = L_w A_w + A$. Let $A = L_A B$ be the inner-outer factorization of A. Since

$$\overline{A\mathcal{H}_n} = \overline{L_A B\mathcal{H}_n} = L_A\mathcal{H}_n,$$

the ranges of L_w and L_A are orthogonal. Further, one has

$$A_w^* A_w + B^* B = I.$$

So B is also a scalar multiple of an isometry, and it is therefore scalar since it is outer. Suppose that $A_w^* A_w = a^2 I$ and $B = \beta I$. Let $c = a^2 |\beta|^{-2}$, and let X be the operator

$$X = L_w A_w - c\beta L_A.$$

Then X is an operator with range orthogonal to the range of L. Indeed,

$$(L\xi, X\eta) = (L_w A_w\xi, L_w A_w\eta) - (A\xi, c\beta L_A\eta) = a^2(\xi, \eta) - c|\beta|^2(\xi, \eta) = 0.$$

As in the proof of (i), using the inner part of X yields a desired isometry. ∎

It is worthwhile to point out a striking special case of the first condition in the corollary.

COROLLARY 2.10. *Let L be an isometry in $\mathfrak{L}_n$ with $L = \sum\limits_{|w|=k} L_w A_w$, for some $k \geqslant 1$ and A_w in $\mathfrak{L}_n$. If any $A_w = 0$, then L properly factors in $\mathfrak{L}_n$.*

REMARK 2.11. Obviously there are many isometries which satisfy the first condition. Further, all of the isometries shown to be irreducible over the unit ball of the algebra in Theorem 2.3 and Remark 2.4 are reducible over the full algebra since they satisfy the second condition. Other isometries which satisfy the second condition include the collection of all operators which are the sum of pairwise orthogonal words. For, in this case every non-zero A_w would necessarily be a scalar multiple of an isometry.

There are also other more specialized classes of isometries which can be factored using this method. As an example, let f and g belong to H^∞ with

$$|f|^2 + |g|^2 = 1$$

on $\mathbb{T}$. Such functions can be found by using the logmodularity of H^∞ ([5]). Then

$$L = L_1 f(L_1) + L_2 g(L_1)$$

is an isometry in $\mathfrak{L}_2$ which apparently does not satisfy the conditions in the corollary. Let $\alpha = f(0)$ and $\beta = g(0)$ and choose λ in $\mathbb{T}$ such that

$$\lambda(\alpha \overline{\beta}) = \overline{\alpha} \beta.$$

Then L and the isometry

$$X = \frac{1}{|\alpha|^2 + |\beta|^2} \left(\beta L_1 L_2 - \lambda \alpha L_2^2 \right)$$

have orthogonal ranges. Indeed, for ξ and η in $\mathcal{H}_n$ one has

$$\left(|\alpha|^2 + |\beta|^2 \right) (L\xi, X\eta) = (f(L_1)\xi, \beta L_2 \eta) - (g(L_1)\xi, \lambda \alpha L_2 \eta)$$
$$= \alpha \overline{\beta} \, (\xi, L_2 \eta) - \beta \overline{\alpha} \overline{\lambda} \, (\xi, L_2 \eta) = 0.$$

The reducibility of this large collection of isometries, together with the fact that the orthogonal complement of the range is always infinite dimensional (Theorem 1.7), leads one to believe that perhaps the theorem can be applied to every isometry L with $(L\xi_1, \xi_1) = 0$. However, this is not the case. The trouble is that difficulties arise when the space $\mathcal{H}_n \ominus L\mathcal{H}_n$ is too "thin" at each level of the $\mathcal{H}_n$ tree. That is, the dimension of $P_k(\mathcal{H}_n \ominus L\mathcal{H}_n)$ remains small as k increases (recall Remark 1.8).

THEOREM 2.12. *There are isometries L in $\mathfrak{L}_n$ with $(L\xi_1, \xi_1) = 0$ for which $\mathcal{H}_n \ominus L\mathcal{H}_n$ does not contain the range of an isometry in $\mathfrak{L}_n$.*

Proof. For $k \geqslant 0$, put $x_k = R_1^k R_2 \sum\limits_{|w|=k} R_w \xi_w$, and let x be the unit vector

$$x = \sum_{k \geqslant 0} 2^{\frac{-k-1}{2}} \|x_k\|^{-1} x_k.$$

Suppose y is in $\mathcal{H}_n$ with $(R_u^* x, y) = 0$ for all words u in $\mathcal{F}_n$. Now given $k \geqslant 0$, choose a word u with $|u| = k$. Then

$$0 = \left(2^{\frac{k+1}{2}} \|x_k\|^1 R_u^* R_2^* (R_1^k)^* x, y\right) = \left(R_u^* \left(\sum_{|w|=k} R_w \xi_w\right), y\right) = (\xi_u, y).$$

Therefore, $y = 0$.

Next, write x as $x = L\eta$, where L is an isometry in $\mathfrak{L}_n$ and η is an $\mathfrak{R}_n$ cyclic vector (every vector in $\mathcal{H}_n$ can be written in this form ([2])). The claim is that L is the desired isometry. As $(\eta, \xi_1) \neq 0$ and $(x, \xi_1) = 0$, one has $(L\xi_1, \xi_1) = 0$. Suppose X is an isometry in $\mathfrak{L}_n$ with range contained in $\mathcal{H}_n \ominus L\mathcal{H}_n$. Then the vectors $X\xi_u = X(R_u \xi_1) = R_u(X\xi_1)$ are orthogonal to $L\eta = x$ for every u in $\mathcal{F}_n$.

In other words, $(X\xi_1, R_u^* x) = 0$ for every word u. Thus, by the above argument one would have $X\xi_1 = 0$, whence $X = 0$. This contradiction completes the proof. ∎

So, this method cannot be applied to all isometries in $\mathfrak{L}_n$. Nevertheless, with the large body of examples, it is still reasonable to make the guess that every isometry L in $\mathfrak{L}_n$ with $(L\xi_1, \xi_1) = 0$ properly factors over the full algebra.

3. IDEALS AND INVARIANT SUBSPACES

The characterization of the WOT-closed right and two sided ideals $(\mathrm{Id}_r(\mathfrak{L}_n)$ and $\mathrm{Id}(\mathfrak{L}_n))$ in [2] and [3] is complete. The main theorem from [2] is stated as follows.

THEOREM 3.1. *Let* $\mu : \mathrm{Id}_r(\mathfrak{L}_n) \to \mathrm{Lat}(\mathfrak{R}_n)$ *be given by* $\mu(\mathcal{I}) = \overline{\mathcal{I}\xi_1}$. *Then* μ *is a complete lattice isomorphism. The restriction of* μ *to the set* $\mathrm{Id}(\mathfrak{L}_n)$ *is a complete lattice isomorphism onto* $\mathrm{Lat}(\mathfrak{L}_n) \cap \mathrm{Lat}(\mathfrak{R}_n)$. *The inverse map* i *sends a subspace* $\mathcal{M}$ *to*

$$i(\mathcal{M}) = \{J \in \mathfrak{L}_n : J\xi_1 \in \mathcal{M}\}.$$

The maps μ and i are still well defined when considering $\mathrm{Id}_l(\mathfrak{L}_n)$ and $\mathrm{Lat}(\mathfrak{L}_n)$. Although technical difficulties were encountered by the authors, but a similar characterization was expected for left ideals. The key observation for right and two-sided ideals is that the subspace $\mu(\mathcal{I})$ is the full range of the ideal $\mathcal{I}$. Indeed, $\overline{\mathcal{I}\xi_1} = \overline{\mathcal{I}\mathfrak{L}_n \xi_1} = \overline{\mathcal{I}\mathcal{H}_n}$. This is not true for left ideals, and is why the methods of the authors cannot be applied in this setting.

Towards the identification of right ideals it is first proved that $\mu i = \mathrm{id}$. It is then shown that this leads to the conclusion that $\overline{\mathcal{I}\xi} = \overline{i\mu(\mathcal{I})\xi}$ for every vector ξ in $\mathcal{H}_n$. The proof that $i\mu = \mathrm{id}$ exploits this fact together with the following more general result about ideals in $\mathfrak{L}_n$.

PROPOSITION 3.2. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ both be* WOT-*closed right, left or two-sided ideals in $\mathfrak{L}_n$. If $\overline{\mathcal{I}_1\xi} = \overline{\mathcal{I}_2\xi}$ for all ξ in $\mathcal{H}_n$, then $\mathcal{I}_1 = \mathcal{I}_2$.*

Proof. In [3] it was shown that the weak* and weak operator topologies on $\mathfrak{L}_n$ coincide. Suppose that φ is a WOT-continuous functional on $\mathfrak{L}_n$ which annihilates $\mathcal{I}_1$. Then again from [3], there are vectors ξ and η in $\mathcal{H}_n$ such that

$$\varphi(J) = (J\xi, \eta)$$

for J in $\mathfrak{L}_n$. But then η is orthogonal to $\overline{\mathcal{I}_1\xi} = \overline{\mathcal{I}_2\xi}$, and hence φ annihilates $\mathcal{I}_2$ as well. Repeating the argument by exchanging the rôles of $\mathcal{I}_1$ and $\mathcal{I}_2$ shows that the two ideals are identical. ∎

REMARK 3.3. Now, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be WOT-closed left ideals of $\mathfrak{L}_n$. Notice that, $\overline{\mathcal{I}_1\xi_1} = \overline{\mathcal{I}_2\xi_1}$ implies $\overline{\mathcal{I}_1\xi} = \overline{\mathcal{I}_2\xi}$ when $\xi = R\xi_1$ for some isometry R in $\mathfrak{R}_n$. Indeed, one would have

$$\overline{\mathcal{I}_1\xi} = \overline{\mathcal{I}_1 R\xi_1} = R\overline{\mathcal{I}_1\xi_1} = R\overline{\mathcal{I}_2\xi_1} = \overline{\mathcal{I}_2 R\xi_1} = \overline{\mathcal{I}_2\xi}.$$

These vectors form a dense collection of vectors in $\mathcal{H}_n$. Whether this implies the same is true for all vectors in $\mathcal{H}_n$ is unclear. As Remark 3.3 points out, this requires an understanding of unbounded WOT-convergence. Thus, it becomes apparent that there are difficulties encountered when considering left ideals.

Upon further investigation, concrete differences become evident. In particular, the WOT-closed right ideal generated by a finite collection of isometries with pairwise orthogonal ranges is exactly the algebraic right ideal they generate. For left and two-sided ideals the corresponding result turns out to be false even for one isometry with norm closure.

THEOREM 3.4. *The algebraic two-sided ideal in $\mathfrak{L}_n$ generated by L_2 is not norm closed.*

Proof. The operator

$$A = \sum_{k \geqslant 0} \frac{1}{k+1} L_1^k L_2$$

clearly belongs to the norm closure of the algebraic two-sided (in fact left) ideal generated by L_2. Suppose that A could be written as

$$A = \sum_{i=1}^{p} B_i L_2 C_i$$

with each B_i and C_i in $\mathfrak{L}_n$. Put $B_i \sim \sum_w b_w^i L_w$ and $C_i \sim \sum_w c_w^i L_w$. Then the unique factorization in $\mathfrak{L}_n$ shows for each $k \geqslant 0$,

$$\frac{1}{k+1} = (A\xi_1, \xi_{z_1^k z_2}) = \sum_{i=1}^{p} (B_i L_2 C_i \xi_1, \xi_{z_1^k z_2}) = \sum_{i=1}^{p} b_{z_1^k}^i c_0^i.$$

By compressing the operators B_i to the subspace $\text{span}\{\xi_{z_1^k} : k \geqslant 0\}$, one sees that each of the operators

$$h_i(L_1) = \sum_{k \geqslant 0} b_{z_1^k}^i L_1^k \quad \text{for } 1 \leqslant i \leqslant p$$

must be in $H^\infty(L_1) \cong H^\infty$. Hence the function

$$\sum_{k \geqslant 0} \frac{z^k}{k+1} = \sum_{i=1}^{p} c_0^i h_i$$

would belong to H^∞, a contradiction (see Lemma 2.5). Therefore A does not belong to the algebraic two-sided ideal generated by L_2. ∎

As an immediate corollary of the proof, the corresponding fact about left ideals is proved.

COROLLARY 3.5. *The algebraic left ideal $\mathfrak{L}_n L_2$ is not norm closed.*

Proof. Consider the same operator A. Simply use the proof of the theorem with $p = 1$ and $C_1 = I$. ∎

REMARK 3.6. It seems reasonable to expect that the norm and WOT closures of the preceding ideals are distinct. It also becomes apparent that proving this would be quite subtle. Indeed, it is difficult to construct a bounded operator belonging to the weak closure of $\mathfrak{L}_n L_2$ without being in the norm closure of $\mathfrak{L}_n L_2$.

Even with these differences, it is still surprising that the analogous identification of left ideals does not hold. It turns out that the subspaces $\mathcal{M}$ in $\text{Lat}(\mathfrak{L}_n)$ for which $\mu i(\mathcal{M}) = \mathcal{M}$ do not fill out the entire subspace lattice.

THEOREM 3.7. *There exists $\mathcal{M} \neq \{0\}$ in $\text{Lat}(\mathfrak{L}_n)$ for which the associated left ideal $i(\mathcal{M})$ is trivial.*

Proof. Define an isometry R in $\mathfrak{R}_n$ by

$$R = \sum_{k \geqslant 0} \lambda_k R_1^k R_2,$$

where the scalars λ_k satisfy $\sum_{k \geqslant 0} |\lambda_k|^2 = 1$ but $\sum_{k \geqslant 0} \lambda_k z^k$ is not in H^∞. For example,

$$\lambda_k = \frac{c}{k+1} \quad \text{where } c = \left(\sum_{k \geqslant 1} \frac{1}{k^2} \right)^{-\frac{1}{2}}.$$

Let $\mathcal{M}$ be the subspace in $\text{Lat}(\mathfrak{L}_n)$ given by $\mathcal{M} = R\mathcal{H}_n$. Actually, every cyclic $\mathfrak{L}_n$-invariant subspace is of this form for some isometry in $\mathfrak{R}_n$ (see [1], [2], [6] and [7]). Now,

$$i(\mathcal{M}) = \{J \in \mathfrak{L}_n : J\xi_1 \in R\mathcal{H}_n\}.$$

Suppose there is a non-zero J in $\mathfrak{L}_n$ and ξ in $\mathcal{H}_n$ for which $J\xi_1 = R\xi$. Put $\xi = \sum_w a_w \xi_w$ and let v be a word of minimal length such that $a_v \neq 0$. Since $R\xi = \sum_w a_w L_w(R\xi_1)$ one has

$$J \sim \sum_{|w| \geqslant |v|,\, k \geqslant 0} a_w \lambda_k L_w L_2 L_1^k.$$

Let Q be the projection onto the subspace $\text{span}\{\xi_{z_1^k} : k \geqslant 0\}$. Evidently then,

$$Q L_2^* L_v^* J Q = a_v \sum_{k \geqslant 0} \lambda_k L_1^k Q,$$

incorrectly implying that J would be unbounded. Therefore, it follows that $i(\mathcal{M}) = 0$. ∎

There is still a strong relation between $\mathrm{Id}_l(\mathfrak{L}_n)$ and $\mathrm{Lat}(\mathfrak{L}_n)$. Essentially, it is determined by those isometries in $\mathfrak{R}_n$ which do not have the qualities of those used in the proof of the theorem.

DEFINITION 3.8. An isometry R in $\mathfrak{R}_n$ is called a *flip* if there is a non-zero vector ξ in $\mathcal{H}_n$ and an operator J in $\mathfrak{L}_n$ with

$$R\xi = J\xi_1.$$

Call R a *cyclic flip* if there are operators J_α in $\mathfrak{L}_n$ such that $J_\alpha\xi_1 \in R\mathcal{H}_n$ and

$$R\xi_1 = \lim_\alpha J_\alpha\xi_1.$$

The motivation for these definitions is when $R\xi_1 = J\xi_1$, for some J in $\mathfrak{L}_n$. This means that the Fourier coefficients of R can be *flipped* into an element of $\mathfrak{L}_n$.

PROPOSITION 3.9. *Let R be an isometry in $\mathfrak{R}_n$ with $\mathcal{M} = R\mathcal{H}_n$. The following are equivalent:*
  (i) *R is a flip;*
  (ii) *$i(\mathcal{M}) \neq 0$.*

Proof. This is straight from the definitions of $i(\mathcal{M})$ and flip isometries. ∎

REMARK 3.10. It should be noted that when R is a flip, the subspace $\mu i(\mathcal{M})$ is *large*. Indeed, suppose $J \neq 0$ belongs to $i(\mathcal{M})$. Then there is a vector ξ in $\mathcal{H}_n$ with $R\xi = J\xi_1$. Write ξ as $\xi = S\eta$, where S is an isometry in $\mathfrak{R}_n$ and η is an $\mathfrak{L}_n$-cyclic vector (this can be done for any vector in $\mathcal{H}_n$ ([3])). Note that the set $\mathfrak{L}_n J$ is contained in $i(\mathcal{M})$. Thus,

$$RS\mathcal{H}_n = \overline{\mathfrak{L}_n RS\eta} = \overline{\mathfrak{L}_n R\xi} = \overline{\mathfrak{L}_n J\xi_1} \subseteq \mu i(\mathcal{M}),$$

which shows that $\mu i(\mathcal{M})$ contains the range of the isometry RS.

PROPOSITION 3.11. *Let R be an isometry in $\mathfrak{R}_n$ with $\mathcal{M} = R\mathcal{H}_n$. The following are equivalent:*
  (i) *R is a cyclic flip;*
  (ii) *$\mu i(\mathcal{M}) = \mathcal{M}$.*

Proof. It is always true that $\mu i(\mathcal{M}) \subseteq \mathcal{M}$. Suppose that R is a cyclic flip, so that $R\xi_1 = \lim_\alpha J_\alpha\xi_1$ with each $J_\alpha\xi_1 \in \mathcal{M} \cap \mathfrak{L}_n\xi_1$. Then $R\xi_1$ lies in $\mu i(\mathcal{M})$, and hence

$$\mathcal{M} = R\mathcal{H}_n = \overline{\mathfrak{L}_n R\xi_1} \subseteq \mu i(\mathcal{M}).$$

If this latter inclusion holds, then $R\xi_1$ is such a limit since

$$R\xi_1 \in \mathcal{M} = \mu i(\mathcal{M}) = \overline{\{J \in \mathfrak{L}_n : J\xi_1 \in R\mathcal{H}_n\}\xi_1}. \quad ∎$$

This gives a good one vector characterization of cyclic subspaces $\mathcal{M}$ in $\mathrm{Lat}(\mathfrak{L}_n)$ for which $\mu i(\mathcal{M}) = \mathcal{M}$. The following corollary shows that the above condition is satisfied for a wealth of examples. For instance, consider the situation below even for $\xi = \xi_1$.

COROLLARY 3.12. *Let* $\mathcal{M} = R\mathcal{H}_n$, *where* R *is an isometry in* $\mathfrak{R}_n$. *If there is an* $\mathfrak{L}_n$-*cyclic vector* ξ *such that* $R\xi = J\xi_1$ *for some* J *in* $\mathfrak{L}_n$, *then* $\mu i(\mathcal{M}) = \mathcal{M}$. *Further, if* $J = L_w$ *for some word* w, *then* $i(\mathcal{M})$ *is exactly the* WOT-*closed left ideal generated by* L_w.

Proof. The condition in the previous proposition is satisfied since,

$$R\xi_1 \in R\mathcal{H}_n = \overline{\mathfrak{L}_n R\xi} = \overline{\mathfrak{L}_n J\xi_1},$$

which shows that R is a cyclic flip.

In general, given A in $\mathfrak{L}_n$,

$$AJ\xi_1 = AR\xi = RA\xi \in R\mathcal{H}_n = \mathcal{M}.$$

Thus, the WOT-closed left ideal generated by J is contained in $i(\mathcal{M})$. The other inclusion is true for the words $J = L_w$. Indeed, in this case it is easy to see that every A in $i(\mathcal{M})$ has a Fourier expansion of the form

$$A \sim \sum_{v \in \mathcal{F}_n} a_{vw} L_{vw}.$$

Actually, an analogous claim can be made for the right and two-sided WOT-closed ideals generated by L_w. However, in [3] it was shown that the Cesàro sums for A,

$$\Sigma_k(A) = \sum_{|v| < k} \left(1 - \frac{|v|}{k}\right) a_v L_v \in \mathfrak{L}_n L_w$$

converge in the strong* topology to A. Hence $i(\mathcal{M})$ is contained in the WOT-closed left ideal generated by L_w. ∎

It has been mentioned that every vector $J\xi_1$ with J in $\mathfrak{L}_n$ factors as $J\xi_1 = R\xi$ for some isometry R in $\mathfrak{R}_n$ and $\mathfrak{L}_n$-cyclic vector ξ. As the corollary observes, one has that $i(R\mathcal{H}_n)$ always contains the WOT-closed left ideal generated by J. The other inclusion holds for words $J = L_w$. Proving the other inclusion holds in full generality would require an understanding of unbounded WOT-convergence in these algebras. This is discussed further below.

REMARK 3.13. The corollary shows that the image in $\mathrm{Id}_l(\mathfrak{L}_n)$ contains the left ideals generated by the words L_w. It is not clear whether the image in $\mathrm{Id}_l(\mathfrak{L}_n)$ is surjective. Given a WOT-closed left ideal $\mathcal{I}$, it is always true that $\mathcal{I} \subseteq i\mu(\mathcal{I})$. In general, the other inclusion requires an understanding of unbounded WOT-convergence. For example, suppose $\mathcal{I}$ belongs to $\mathrm{Id}_l(\mathfrak{L}_n)$ with $\mu(\mathcal{I}) = \mathcal{H}_n$. It is not even known if one must have $\mathcal{I} = \mathfrak{L}_n$. For, one would like to say that the identity I belongs to $\mathcal{I}$, but all that can be said is $\xi_1 = \lim_\alpha J_\alpha \xi_1$ for some J_α in $\mathcal{I}$. For bounded nets, WOT-convergence amounts to strong convergence on the vector ξ_1 ([2]). However, this is not true for unbounded nets. Indeed, as an example consider the sequence J_m of operators in $\mathfrak{L}_n$ given by

$$J_m = \sum_{k=0}^{m} \frac{1}{k+1} L_1^m.$$

It is clear that

$$\lim_{m \to \infty} J_m \xi_1 = \sum_{k \geqslant 0} \frac{1}{k+1} \xi_{z_1^k},$$

but the latter vector does not represent the Fourier coefficients of any operator in $\mathfrak{L}_n$.

Nonetheless, it has been shown that the maps μ and i define a bijective correspondence between the cyclic subspaces of $\mathrm{Lat}(\mathfrak{L}_n)$ determined by cyclic flips on the one hand, and the image under i in $\mathrm{Id}_l(\mathfrak{L}_n)$ of these subspaces on the other. Concerning the lattice properties of these maps, it is not hard to show that μ sends closed spans to WOT-closed sums and i sends intersections to intersections. However, the behaviour of μ on intersections and i on sums again comes back to requiring an understanding of unbounded WOT-convergence.

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