J. OPERATOR THEORY
45(2001), 335–355

COMPACT COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES OF THE UNIT BALL

DANA D. CLAHANE

Communicated by Norberto Salinas

ABSTRACT. For $p > 0$ and $\alpha \geqslant 0$, let $A^p_\alpha(B_n)$ be the weighted Bergman space of the unit ball B_n in $\mathbb{C}^n$, and denote the Hardy space by $H^p(B_n)$. Suppose that $\varphi : B_n \to B_n$ is holomorphic. We show that if the composition operator C_φ defined by $C_\varphi(f) = f \circ \varphi$ is bounded on $A^p_\alpha(B_n)$ and satisfies

$$\lim_{|z|\to 1^-} \Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^{\alpha+2} \|\varphi'(z)\|^2 = 0,$$

then C_φ is compact on $A^p_\beta(B_n)$ for all $\beta \geqslant \alpha$. Along the way we prove some comparison results on boundedness and compactness of composition operators on $H^p(B_n)$ and $A^p_\alpha(B_n)$, as well as a Carleson measure-type theorem involving these spaces and more general weighted holomorphic Sobolev spaces.

KEYWORDS: *Composition operators, compact operators, Bergman spaces, Hardy spaces, several complex variables, Carleson measures, unit ball.*

MSC (2000): Primary 47B33; Secondary 30H05, 32A10, 32A35.

1. INTRODUCTION

Let $n \in \mathbb{N}$. Given a set Ω, a linear space of functions X defined on Ω, and a map $\varphi : \Omega \to \Omega$, we define the linear operator C_φ on X by $C_\varphi(f) = f \circ \varphi$ for all $f \in X$. C_φ is called the *composition operator* induced by the *symbol* φ.

The purpose of this paper is to give sufficient conditions for compactness of composition operators on weighted Bergman spaces of the unit ball in several complex variables. Much effort in the study of composition operators on analytic function spaces such as these has been devoted to relating boundedness, compactness, and other properties of C_φ to function-theoretic properties of φ. For example, it has been known for some time now [11] that if $\alpha > -1$, $p > 0$, and

$\varphi : D \to D$ is holomorphic on the unit disk $D \subset \mathbb{C}$, then C_φ is compact on the weighted Bergman space $A^p_\alpha(D)$ iff

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0. \tag{1.1}$$

It follows from the Julia-Caratheodory Theorem ([16], p. 57) that the above equation can be restated as non-existence of a finite angular derivative for φ on the boundary of D ([20], Chapter 10). The proof of the necessity of equation (1.1) for compactness is essentially due to J. Shapiro and P. Taylor [17], and analogous necessary conditions hold on a large class of function spaces and domains in $\mathbb{C}^n$ (cf. [7], p. 171–172) and [6]). The sufficiency of equation (1.1) for compactness in the one-variable case was originally proven by B. MacCluer and J. Shapiro in [11]. In the same paper, MacCluer and Shapiro constructed counterexamples essentially showing that equation (1.1) is not sufficient for compactness of C_φ on the Hardy space $H^p(D)$. These authors also gave explicit holomorphic maps $\varphi : B_n \to B_n$ that induce bounded but non-compact operators on $A^p_\alpha(B_n)$ for each $\alpha > -1$ and $n > 1$, with no finite angular derivative at any point of the boundary of B_n. However, as is the case for D, the Julia-Caratheodory Theorem for B_n ([7], p. 105) can be used to show that non-existence of a finite angular derivative for φ at all points on the surface of B_n is equivalent to equation (1.1) above.

If the image of φ has compact closure in B_n, it is not difficult to show that C_φ is compact on a wide variety of spaces, including $H^p(B_n)$ and $A^p_\alpha(B_n)$. We are therefore primarily concerned with self-maps φ that have unit supremum norm. In the case of the unit disk D in $\mathbb{C}$, Shapiro and Taylor in [17] proved that if C_φ is compact on the Hardy space $H^p(D)$, then φ has no finite angular derivative at any point of the boundary of the unit disk; that is, equation (1.1) above holds. MacCluer and Shapiro extended this result to the unit ball in [11]. They also showed that equation (1.1) does not imply compactness of C_φ on $H^p(D)$, unlike the situation for $A^p_\alpha(D)$.

It turns out that equation (1.1) does imply compactness of C_φ on $H^p(B_n)$ for $n \geqslant 1$ if one places additional hypotheses on φ, such as univalence of φ along with boundedness of the so-called dilation ratio $\|\varphi'(z)\|^2/|J_\varphi(z)|^2$, where $\|\varphi'(z)\|$ is the operator norm of the Frechét derivative $\varphi'(z)$ and J_φ is the Jacobian determinant of φ' ([7], p. 171).

MacCluer in [10] gave measure-theoretic characterizations of the holomorphic maps φ that induce compact (and bounded) composition operators on $H^p(B_n)$, and analogous results involving Carleson-measure conditions also hold on the spaces $A^p_\alpha(B_n)$, $\alpha > -1$ (cf. [7], p. 161–164). In this paper, however, we give a purely function-theoretic condition on φ (Theorem 1.1) so that C_φ will be compact on $A^p_\alpha(B_n)$ for $\alpha \geqslant 0$. The condition that we give is a generalization of one given previously by K. Madigan and A. Matheson in [12], wherein it is shown that such a condition is equivalent to compactness of C_φ on $\mathcal{B}_0(D)$, the little Bloch space of D.

Our proof of Theorem 1.1 uses a Carleson measure-type result, Theorem 2.11, and the comparison result of Theorem 3.3. These results are perhaps of independent interest. Theorem 2.11 states that if μ is an α-Carleson measure ([7], Chapter 2) then functions in given weighted holomorphic Sobolev spaces satisfy a certain integral inequality with respect to μ. This result is a variation of the multivariable Carleson measure theorem of J. Cima and W. Wogen, [4]. Theorem 3.3 is a comparison result for boundedness and compactness of composition operators. The theorem states that the bounded composition operators on $H^p(B_n)$, the bounded composition operators on $A^p_\alpha(B_n)$, and the compact composition operators on $A^p_\alpha(B_n)$ are increasing sets in the parameter $\alpha \in [-1,\infty)$, where we associate $H^p(B_n)$ with $\alpha = -1$.

Fixing a positive integer n, we denote the *unit ball* of $\mathbb{C}^n$ by B_n, which inherits the norm $|\cdot|$ induced by the standard inner product on $\mathbb{C}^n$ ([14], Chapter 1). Let $\mathcal{O}(B_n)$ denote the space of complex-valued, holomorphic functions on B_n. In this paper φ will always be a holomorphic map ([14], Chapter 1) from B_n to itself. Let $\mathrm{d}v(z)$ denote Lebesgue volume measure on B_n. Let $p > 0$ and $\alpha > -1$. $f \in \mathcal{O}(B_n)$ is said to be in the *weighted Bergman space* $A^p_\alpha(B_n)$ iff

$$\|f\|_{A^p_\alpha(B_n)} := \left\{ \int\limits_{B^n} |f(z)|^p (1-|z|^2)^\alpha \,\mathrm{d}v(z) \right\}^{1/p} < \infty.$$

Let $\mathrm{d}\sigma$ represent normalized surface area measure on B_n ([14], Chapter 1). $f \in \mathcal{O}(B_n)$ is said to be a member of the *Hardy space* $H^p(B_n)$ iff

$$\|f\|_{H^p(B_n)} := \left(\sup_{0<r<1} \int\limits_{\partial B_n} |f(r\xi)|^p \,\mathrm{d}\sigma(\xi) \right)^{1/p} < \infty.$$

It is a well-known fact that $\|\cdot\|_{H^p(B_n)}$ and $\|\cdot\|_{A^p_\alpha(B_n)}$ are norms only for $p \geqslant 1$ (cf. [15], Chapter 7). It is also known that the spaces $A^p(B_n)$ and $H^p(B_n)$ are Banach spaces with the above norms, and that these spaces are complete, non-locally convex topological vector spaces for $0 < p < 1$ ([7], Chapter 2). $A^2_\alpha(B_n)$ and $H^2(B_n)$ are Hilbert spaces (see [7], Chapter 2) with inner products respectively given by

$$\langle f, g\rangle_{A^2_\alpha(B_n)} = \int\limits_{B_n} f(z)\overline{g(z)}(1-|z|^2)^\alpha \,\mathrm{d}v(z),$$

and

$$\langle f, g\rangle_{H^2(B_n)} = \lim_{r\to 1^-} \int\limits_{B_n} f(r\xi)\overline{g(r\xi)} \,\mathrm{d}\sigma(\xi).$$

For $\gamma \in (0,1]$ we define the *analytic Lipschitz space* ([7], Chapter 4) $\mathrm{Lip}_\gamma(D)$ to be the Banach space of functions $f \in \mathcal{O}(D)$ satisfying

$$\sup_{z,w\in\partial D} \frac{|f(z)-f(w)|}{|z-w|^\gamma} < \infty.$$

C will be used to represent positive constants whose values may change from line to line.

The goal of this paper is to prove the following theorem:

THEOREM 1.1. (MAIN RESULT) *Let $p > 0$ and $\alpha \geqslant 0$. Suppose that $\varphi : B_n \to B_n$ is a holomorphic map such that C_φ is bounded on $A^p_\alpha(B_n)$ and*

$$\lim_{|z|\to 1^-} \Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^{\alpha+2} \|\varphi'(z)\|^2 = 0. \tag{1.2}$$

Then C_φ is compact on $A^p_\beta(B_n)$ for all $\beta \geqslant \alpha$.

The organization of this paper is as follows: In Section 2, we list some notation and preliminary facts, culminating in the statement and proof of our Carleson measure-type theorem relating α-Carleson measures and weighted holomorphic Sobolev spaces. In Section 3 we prove the comparison theorems on boundedness and compactness of composition operators. Our main result for compactness of composition operators, Theorem 1.1, is proved in Section 4. We conclude with a discussion of examples and open problems in Section 5.

2. NOTATION AND PRELIMINARY FACTS

We will call two positive variable quantities x and y *comparable* (and write $x \sim y$) iff their ratio is bounded above and below by positive constants. We define a finite, positive, Borel regular measure v_φ on B_n by

$$v_\varphi(E) = v[\varphi^{-1}(E)],$$

where dv is Lebesgue volume measure on B_n ([7], p. 164). We use the notation φ^* for the radial limit of the mapping φ ([7], p. 161). We define another finite, positive, and Borel regular measure μ_φ on $\overline{B}_n$ by $\mu_\varphi(E) = \sigma[\{\varphi^*\}^{-1}(E) \cap \partial B_n]$. We denote the Lebesgue measure of a set E by $|E|$, and for $\alpha > -1$ we define the weighted measure $\mathrm{d}\nu_\alpha(z) = (1-|z|^2)^\alpha \,\mathrm{d}v(z)$ on B_n, as in [4]. $\mathrm{d}\nu_\alpha$ induces on B_n another finite, positive, Borel regular measure, $\mathrm{d}\mu^\alpha_\varphi$, defined by $\mu^\alpha_\varphi(E) = \nu_\alpha[\varphi^{-1}(E)]$ ([7], p. 164). Now suppose that $0 < p < \infty$ and that $\alpha > -1$. A finite, positive, Borel regular measure μ on B_n is called an *α-Carleson measure* if and only if there exists a constant $K \in (0,\infty)$ such that for all $\xi \in \partial B_n$, $h \in (0,1)$,

$$\mu[S(\xi,h)] \leqslant K h^{n+\alpha+1}.$$

For $s \in \mathbb{Z}_+$, $q > 0$, and $0 < p \leqslant \infty$, we define the weighted spaces $A^p_{q,s}$ and the *weighted holomorphic Sobolev spaces* $\mathcal{W}^p_{q,s}$ as in [2]. We define the partial differential operators $D_j = \partial/\partial z_j$, $j = 1, 2, \ldots, n$, as in Chapter 1 of [14].

Let $\xi \in \mathbb{C}^n$, $|\xi| = 1$, and $h > 0$. We define the *Carleson sets* $S(\xi,h)$ and $\mathcal{S}(\xi,h)$ ([7], p. 42), by

$$S(\xi,h) = \{z \in B_n : |1 - \langle z, \xi\rangle| < h\} \quad \text{and} \quad \mathcal{S}(\xi,h) = \{z \in \overline{B}_n : |1 - \langle z, \xi\rangle| < h\}.$$

Note that for $h \geqslant 2$ and $\xi \in \partial B_n$, $S(\xi,h) = B_n$. Combining this statement with the fact that if $\alpha > -1$ and $h \in (0,2]$, then $\nu_\alpha[S(\xi,h)] \sim h^{n+\alpha+1}$ as h and ξ vary [11], it is obvious that for any fixed $M \geqslant 2$,

$$\nu_\alpha[S(\xi,h)] \sim h^{n+\alpha+1} \tag{2.1}$$

as $h \in [0,M]$ and $\xi \in \partial B_n$ vary. Define $d : \overline{B}_n \times \overline{B}_n :\to \mathbb{R}$ by $d(z,w) = |1 - \langle z,w\rangle|^{1/2}$.

Following [14], pages 11, 23, 25ff, for each $a \in B_n$, $a \neq 0$, define φ_a to be the involutive automorphism (called a *Moebius transformation*) of B_n that takes 0 to a, with explicit form given in page 2 of [14]. Let ρ denote the *pseudohyperbolic metric* on B_n, and for $0 < r < 1$ and $a \in B_n$, define the *pseudohyperbolic ball* $E(a, r)$ and the set $S(a)$ as in [9]. We define the *Carleson maximal operator* M by

$$(Mf)(z) = \sup_{h>1-|z|} \frac{1}{\nu_\alpha[S(\pi(z), h)]} \int\limits_{S(\pi(z),h)} |f(w)| \, \mathrm{d}\nu_\alpha(w),$$

for all locally ν_α-integrable functions $f : B_n \to \mathbb{C}$, where $\pi(z) = z/|z|$ $\forall z \neq 0$ in B_n. For any $f : B_n \to \mathbb{C}$, Mf is called the *Carleson maximal function of* f. We then define an *uncentered Carleson maximal operator* $\widetilde{M}$ by

$$\begin{aligned} \widetilde{M}f(z) &= \sup_{\{(\xi,h)\in\partial B_n\times(0,\infty): z\in S(\xi,h)\}} \frac{1}{\nu_\alpha[S(\xi, h)]} \int\limits_{S(\xi,h)} |f(w)| \, \mathrm{d}\nu_\alpha(w) \\ &= \sup_{\{(\xi,h)\in\partial B_n\times(0,2): z\in S(\xi,h)\}} \frac{1}{\nu_\alpha[S(\xi, h)]} \int\limits_{S(\xi,h)} |f(w)| \, \mathrm{d}\nu_\alpha(w), \end{aligned}$$

where $f : B_n \to \mathbb{C}$ is locally ν_α-integrable. In this case we call $\widetilde{M}f$ the *uncentered Carleson maximal function of* f.

The following proposition will be referred to frequently:

Proposition 2.1. *If $\alpha > -1$, then the norms on $A^2_\alpha(B_n)$ and $\mathcal{W}^2_{\alpha+3,1}$ are equivalent, and the spaces are the same. That is,*

$$A^2_\alpha(B_n) = \mathcal{W}^2_{\alpha+3,1}. \tag{2.2}$$

Proof. This result can be proven by combining the relations $A^2_{\alpha+1,0} = A^2_{\alpha+3,1}$ ([2], p. 35) and $A^2_{\alpha+3,1} = \mathcal{W}^2_{\alpha+3,1}$ ([2], p. 44). ∎

The results that we have used here from [2] also hold more generally on smoothly bounded, strictly pseudoconvex domains in Stein manifolds [1], thus giving substantial support for our belief that analogues of the results in the present paper can be proven for those domains. The question of whether or not these results hold for bounded symmetric domains is also natural and interesting.

The following well-known result, which characterizes the compact composition operators on $A^p_\alpha(B_n)$, is well-known. The interested reader is referred to [11], where it is stated that this result holds on the so-called Dirichlet spaces, and therefore for the weighted Bergman spaces as well (by Example 3.5.9 of [7]). For many function spaces other than $A^p_\alpha(B_n)$, in one and several variables, the result can be proven by modifying the proof given for D in page 128 of [7] (also, see [6]).

Theorem 2.2. *Let $\varphi : B_n \to B_n$ be holomorphic, and suppose that $0 < p < \infty$ and $\alpha > -1$. Then C_φ is compact on $A^p_\alpha(B_n)$ iff for each bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $A^p_\alpha(B_n)$ converging to 0 uniformly on compacta in B_n, it follows that $C_\varphi(f_k) \to 0$ in $A^p_\alpha(B_n)$-metric.*

An application of the triangle inequality and the reverse triangle inequality shows that if $h > 0$, $z \in \mathbb{C}$, $|1 - z| < h$, and $1 \leqslant r \leqslant 1/|z|$, then $|1 - rz| < 2h$. Using this fact, we now prove the following lemma:

LEMMA 2.3. *If $\xi \in \partial B_n$, $z \in B_n$, $0 < h < 1$, and $|1 - \langle z, \xi\rangle| < h$, then*

$$\Big|1 - \Big\langle \frac{z}{|z|}, \xi \Big\rangle\Big| < 2h.$$

Proof. If $\xi = e_1 := (1, 0, 0, \dots, 0)$ then the remark preceding the statement of the lemma with $r = 1/|z|$ clearly implies that $|1-\langle z/|z|, e_1\rangle| = \|1-z_1/|z|\| < 2h$. To extend this result to arbitrary $\xi \in \partial B_n$, one can clearly choose a unitary map $U : \mathbb{C}^n \to \mathbb{C}^n$ that sends ξ to e_1. It follows that

$$|1 - \langle z, \xi\rangle| = |1 - \langle Uz, U\xi\rangle| = |1 - \langle Uz, e_1\rangle| < h,$$

since U preserves inner products. Hence,

$$\Big|1 - \Big\langle \frac{z}{|z|}, \xi \Big\rangle\Big| = \Big|1 - \Big\langle \frac{Uz}{|Uz|}, e_1 \Big\rangle\Big|.$$

Since $|1 - \langle Uz, e_1\rangle| < h$ we must have that the quantity above is less than $2h$, by the preliminary result that we proved for $\xi = e_1$. ∎

We will now show that the sets $S(\xi, h)$ are "homogeneous" by modifying arguments in pages 8 and 37 of [18].

LEMMA 2.4. *Let $\alpha > -1$. Then there exist constants c_1 and c_2 such that for all $h > 0$ and ξ, $\eta \in \partial B_n$ the following properties:*
(i) *if $S(\xi, h) \cap S(\eta, h) \neq \emptyset$, then $S(\xi, h) \subset S(\eta, c_1h)$;*
(ii) $\nu_\alpha[S(\xi, c_1h)] \leqslant c_2\nu_\alpha[S(\xi, h)]$.

The conditions (i) and (ii) above are called the *engulfing* and the *doubling properties*, respectively. Families of subsets $B(x, r)$ of Euclidean space satisfying properties such as these are called *spaces of homogeneous type* ([18], Chapter 1).

Proof. (ii) is an immediate consequence of relation (2.1). To prove (i), we will show that $c_1 = 9$. Suppose that $z \in S(\xi, h)$. By hypothesis, $\exists w \in S(\xi, h) \cap S(\eta, h)$. We claim that $|1 - \langle z, \eta\rangle| < 9h$. It is known that d satisfies the triangle inequality in $\overline{B}_n$ ([14], p. 66). We have that $d(z, \xi) < h^{1/2}$, so that $|1 - \langle z, \eta\rangle|^{1/2} = d(z, \eta) \leqslant d(z, \xi) + d(\xi, \eta) \leqslant d(z, \xi) + d(\xi, w) + d(w, \eta) < h^{1/2} + h^{1/2} + h^{1/2} = 3h^{1/2}$. Our claim and the desired set inclusion follow. ∎

We will also need the following consequence of Lemmas 2.4 and 2.3:

LEMMA 2.5. *If $\xi \in \partial B_n$, $h > 0$, and $z \in S(\xi, h)$, then*

$$S(\xi, h) \subset S(\pi(z), 18h).$$

Proof. First, if $h \geqslant 1/9$, then the proof is trivial. Therefore, assume that $0 < h < 1/9$. We first show that under the above hypotheses,

$$S(\pi(z), 2h) \cap S(\xi, 2h) \neq \emptyset.$$

By assumption, $d(z, \xi) < h^{1/2}$, and it follows from Lemma 2.3 that $d(\pi(z), \xi) < (2h)^{1/2}$. Therefore, there exists an $\varepsilon \in (0, [2h]^{1/2})$ such that

$$d\Big(\frac{z}{|z|}, \xi\Big) = (2h)^{1/2} - \varepsilon. \tag{2.3}$$

It follows that ε and its square must be less than 1, so that $\frac{1-\varepsilon^2}{|z|} < \frac{1}{|z|}$. We now choose $r \in ([1-\varepsilon^2]/|z|, 1/|z|)$. It is easy to see that $0 < 1 - r|z| < \varepsilon^2$. Putting $w = rz$, we obtain that

$$d(w, \pi(z)) = |1 - \langle w, \pi(z)\rangle|^{1/2} = \left|1 - \left\langle rz, \frac{z}{|z|}\right\rangle\right|^{1/2} = |1 - r|z||^{1/2} < \varepsilon < (2h)^{1/2}.$$

Squaring, one obtains $w \in S(\pi(z), 2h)$. In particular from the line above,

$$d(w, \pi(z)) < \varepsilon. \tag{2.4}$$

We will now show that $w \in S(\xi, 2h)$. By the triangle inequality for d ([14], p. 66), we have that $d(w, \xi) \leqslant d(w, \pi(z)) + d(\pi(z), \xi)$, which by relations (2.4) and (2.3) is less than $\varepsilon + (2h)^{1/2} - \varepsilon = (2h)^{1/2}$. Therefore, $w \in S(\xi, 2h)$, as we claimed. Since $w \in S(\pi(z), 2h)$ we have that $S(\xi, 2h) \cap S(\pi(z), 2h) \neq \emptyset$. By the engulfing property,

$$S(\xi, 2h) \subset S(\pi(z), 18h),$$

and the statement of the lemma immediately follows. ∎

The proof of the following lemma involves modifications of the argument given for an analogous result in page 13 of [18]. This lemma justifies the final step in the proof of Theorem 2.11.

LEMMA 2.6. *We have that*

$$\widetilde{M}f(z) \sim Mf(z)$$

as $z \in B_n$ and the locally integrable functions $f : B_n \to \mathbb{C}$ vary.

Proof. It is clear from the definitions that $Mf(z) \leqslant \widetilde{M}f(z)$ for all locally integrable functions f on B_n and $z \in B_n$. Therefore, it suffices to find a $C > 0$ such that

$$\widetilde{M}f(z) \leqslant CMf(z)$$

for all such z and f. From relation (2.1), it is clear that

$$\begin{aligned}
\widetilde{M}f(z) &\leqslant C \sup_{\{(\xi,h)\in\partial B_n\times(0,2):z\in S(\xi,h)\}} \frac{1}{\nu_\alpha[S(\pi(z), 18h)]} \int_{S(\xi,h)} |f(w)|\,\mathrm{d}\nu_\alpha(w) \\
&\leqslant C \sup_{\{(\xi,h)\in\partial B_n\times(0,\infty):z\in S(\xi,h)\}} \frac{1}{\nu_\alpha[S(\pi(z), 18h)]} \int_{S(\xi,h)} |f(w)|\,\mathrm{d}\nu_\alpha(w) \\
&\leqslant C \sup_{\{(\xi,h)\in\partial B_n\times(0,\infty):z\in S(\xi,h)\}} \frac{1}{\nu_\alpha[S(\pi(z), 18h)]} \int_{S(\pi(z),18h)} |f(w)|\,\mathrm{d}\nu_\alpha(w),
\end{aligned}$$

by Lemma 2.5. Now $z \in S(\xi, h)$ implies that $1 - |z| < h$. By reducing restrictions on the supremum we obtain that the above quantity is surely

$$\begin{aligned}
&\leqslant C \sup_{h>1-|z|} \frac{1}{\nu_\alpha[S(\pi(z), 18h)]} \int_{S(\pi(z),18h)} |f(w)|\,\mathrm{d}\nu_\alpha(w) \\
&\leqslant C \sup_{18h>1-|z|} \frac{1}{\nu_\alpha[S(\pi(z), 18h)]} \int_{S(\pi(z),18h)} |f(w)|\,\mathrm{d}\nu_\alpha(w) = CMf(z). \quad ∎
\end{aligned}$$

Using the engulfing property and duplicating the proof of an analogous Vitali-type covering lemma proven in Chapter 1 of [18] (with the sets $S(\xi, h)$ playing the role of the balls $B(x, r)$ there), it is not difficult to verify the following Vitali-type covering lemma, which will be used to show in Lemma 2.8 that the Carleson maximal operator M is type $(2,2)$ from $L^2(B_n,\, \mathrm{d}\nu_\alpha)$ to $L^2(B_n,\, \mathrm{d}\mu)$ when μ is an α-Carleson measure. See page 179 of [19] for the definition of weak-type and type (i.e., strong type).

LEMMA 2.7. *There is a positive constant c_1 such that if $E \subset B_n$ is a measurable set that is the union of a finite collection of Carleson sets $\{S(\xi_j, h_j)\}_{j=1}^k$, then one can choose a disjoint subcollection $\{S(\xi_j', h_j')\}_{j=1}^m$ such that*

$$E \subset \bigcup_{j=1}^{m} S(\xi_j', c_1 h_j').$$

We are now ready to prove that when μ is an α-Carleson measure, the Carleson maximal operator M behaves nicely as an operator from $L^2(dv)$ to $L^2(\,\mathrm{d}\mu_\alpha)$. We have modified similar ideas used in pages 13–14 of [18] and [4].

LEMMA 2.8. *Suppose that μ is an α-Carleson measure, where $\alpha > -1$. Then the Carleson maximal operator M is "type $(2,2)$"; that is, M is a bounded sublinear operator from $L^2(B_n,\, \mathrm{d}\nu_\alpha)$ to $L^2(B_n,\, \mathrm{d}\mu)$.*

Proof. Lemma 2.7 shows that the subadditive map M is a weak type $(1,1)$ operator from $L^1(B_n,\, \mathrm{d}\nu_\alpha)$ to $L^1(B_n,\, \mathrm{d}\mu)$. It is obvious that M is type (∞,∞) from $L^\infty(B_n,\, \mathrm{d}\nu_\alpha)$ to $L^\infty(B_n,\, \mathrm{d}\mu)$, and that M is subadditive. We then appeal to the Marcinkiewicz interpolation theorem ([19], p. 184) which shows that M is bounded from $L^2(B_n,\, \mathrm{d}\nu_\alpha)$ to $L^2(B_n,\, \mathrm{d}\mu)$. Therefore, if we show that the operator M is weak type $(1,1)$ from $L^2(B_n,\, \mathrm{d}\nu_\alpha)$ to $L^2(B_n,\, \mathrm{d}\mu)$, then the proof of the lemma will be complete.

For each $\gamma \geqslant 0$, let

$$E_\gamma = \{z \in B_n : \widetilde{M} f(z) > \gamma\},$$

and let E be any compact subset of E_γ. By definition of E_γ, for each $z \in E$, there are $\xi \in \partial B_n$ and $h \geqslant 1 - |z|$ such that $z \in S(\xi, h)$ and

$$\nu_\alpha[S(\xi, h)] < \frac{1}{\gamma} \int\limits_{S(\xi,h)} |f(w)|\, \mathrm{d}\nu_\alpha(w). \tag{2.5}$$

Now the sets $S(\xi, h)$ are obviously open by continuity of the function $|1 - \langle \xi, \cdot\rangle|$ for each $\xi \in \partial B_n$. Since each z is in some $S(\xi, h)$, then by compactness of E we can select a finite collection of sets $S(\xi, h)$ covering E. By Lemma 2.7, there is a disjoint subcollection $S(\xi_1, h_1), \ldots, S(\xi_m, h_m)$ of this cover such that

$$E \subset \bigcup_{j=1}^{m} S(\xi_j, c_1 h_j).$$

We then obtain by properties of the measure μ that

$$\begin{aligned}\mu E &\leqslant \sum_{j=1}^{m}\mu[S(\xi_j,c_1h_j)] \leqslant C\sum_{j=1}^{m}(c_1h_j)^{n+1+\alpha}\\ &\leqslant C\sum_{j=1}^{m}(h_j)^{n+1+\alpha} \leqslant C\sum_{j=1}^{m}\nu_\alpha[S(\xi_j,h_j)].\end{aligned}$$

The second inequality above is due to the assumption that μ is an α-Carleson measure, and the final inequality follows from relation (2.1). Since the $S(\xi_j,h_j)$'s are disjoint and satisfy inequality (2.5), we have that

$$\begin{aligned}\mu E &\leqslant C\sum_{j=1}^{m}\frac{1}{\gamma}\int\limits_{S(\xi_j,h_j)}|f(w)|\,\mathrm{d}\nu_\alpha(w)\\ &\leqslant \frac{C}{\gamma}\int\limits_{\bigcup_{j=1}^{m}S(\xi_j,h_j)}|f(w)|\,\mathrm{d}\nu_\alpha(w) \leqslant \frac{C}{\gamma}\int\limits_{B_n}|f(w)|\,\mathrm{d}\nu_\alpha(w).\end{aligned} \tag{2.6}$$

We can write $E_\gamma = \bigcup\limits_{k=1}^{\infty} E_k$, where $(E_k)_{k=1}^{\infty}$ is an increasing sequence of compact sets. Using this decomposition and inequality (2.6), along with the fact that μ is a Borel regular measure, we obtain that

$$\mu E_\gamma \leqslant \frac{C}{\gamma}\int\limits_{B_n}|f(w)|\,\mathrm{d}\nu_\alpha(w).$$

It follows that

$$\mu\{z\in B_n : \widetilde{M}f(z)>\gamma\} \leqslant \frac{C}{\gamma}\int\limits_{B_n}|f(w)|\,\mathrm{d}\nu_\alpha(w).$$

By Lemma 2.6, one obtains

$$\mu\{z\in B_n : Mf(z)>\gamma\} \leqslant \frac{C}{\gamma}\int\limits_{B_n}|f(w)|\,\mathrm{d}\nu_\alpha(w). \quad \blacksquare$$

Next, we outline a proof of the fact that the sets $S(\xi,h)$ can be used to approximate the sets $E(a,r)$, so that one obtains a submean value property over Carleson sets in the proof of Theorem 2.11 from a submean value property over pseudohyperbolic balls (Proposition 2.10 below).

Our outline is as follows: following pages 321–322 of [9], let $r\in(0,1)$ and $\alpha>-1$ be fixed. Then $\nu_\alpha[S(b)]\sim(1-|b|)^{n+1+\alpha}$ as b varies through B_n, and $\nu_\alpha[E(z,r)]\sim\nu_\alpha[S(z)]$ as z varies in B_n. Combining these two facts, it is not difficult to see that $\nu_\alpha[E(a,r)]\sim(1-|a|^2)^{n+1+\alpha}$ as $a\in B_n$ varies. For each $z\in B_n$ there is a $b(z)=:b\in B_n$ such that $1-|b|\sim1-|z|$ and $E(z,r)\subset S(b)$, and if $\xi\in\partial B_n$, $h\in(0,1)$, and $b=(1-h)\xi$, then $S(\xi,h)\subset S(b)\subset S(\xi,2h)$. All of these facts, together with relation (2.1) can be combined to show that the following proposition holds:

PROPOSITION 2.9. *Let $\alpha > -1$ and $r \in (0,1)$ be fixed. For each $z \in B_n$, there is a Carleson set $S(\xi, 2h)$ with*

$$E(z,r) \subset S(\xi, 2h).$$

Furthermore,

$$\nu_\alpha[E(z,r)] \sim \nu_\alpha[S(\xi, 2h)]$$

as z varies in B_n.

For $r > 0$ fixed, it is known that $1 - |w|^2 \sim 1 - |z|^2$ as $w \in E(z,r)$ and $z \in B_n$ vary ([9], p. 232). Using this fact, the submean value property for squares of moduli of holomorphic functions, the fact that $\varphi_a[E(a,r)] = B(0,r)$ ([14], p. 26 and 29), a change of variables via the map φ_a, the relation $|1-\langle w,z\rangle| \sim 1-|z|^2$ for $z \in B_n$ and $w \in E(z,r)$ ([9], p. 324), and a well-known formula for the Jacobian determinant of $\varphi_a'(z)$ in B_n ([14], p. 28), one can obtain the following result, which appears in a more general weighted form in [9] (however, the statement $r > 1$ there should be changed to $r \in (0,1)$):

LEMMA 2.10. *Let $\alpha > -1$. For each fixed $0 < r < 1$ there is a constant $C_r > 0$ such that for all $f \in \mathcal{O}(B_n)$ and $a \in B_n$,*

$$|\nabla f(a)|^2 \leqslant \frac{C_r}{\nu_\alpha[E(a,r)]} \int\limits_{E(a,r)} |\nabla f(z)|^2 \, \mathrm{d}\nu_\alpha(z).$$

The fact that we are not assuming univalence of φ in Theorem 1.1 makes it necessary for us to use Carleson measure conditions to estimate integrals with respect to v_φ. Therefore, the following Carleson-measure type theorem is crucial in the proof of Theorem 1.1. This theorem is the main result of this section.

THEOREM 2.11. *For $\alpha > -1$, suppose that μ is an α-Carleson measure. Then given $\beta > 0$ there is a $C > 0$ such that for all $f \in \mathcal{W}^2_{\alpha+\beta+1,1}$, we have*

$$\int\limits_{B_n} |\nabla f(z)|^2 (1-|z|^2)^\beta \, \mathrm{d}\mu(z) \leqslant C \int\limits_{B_n} |\nabla f(z)|^2 (1-|z|^2)^\beta \, \mathrm{d}\nu_\alpha(z). \tag{2.7}$$

Proof. We first show that there is a constant $C > 0$ such that for all $f \in W^2_{\alpha+\beta+1,1}$,

$$|\nabla f(z)|(1-|z|^2)^{\beta/2} \leqslant CM[|\nabla f|(\cdot)(1-|\cdot|^2)^{\beta/2}](z). \tag{2.8}$$

We proceed as follows. By Lemma 2.10 with r any fixed positive number smaller than say, $1/2$, there is a positive constant C such that for all $f \in \mathcal{O}(B_n)$ and $z \in B_n$,

$$|\nabla f(z)| \leqslant \frac{C}{\nu_\alpha[E(z,r)]} \int\limits_{E(z,r)} |\nabla f(w)| \, \mathrm{d}\nu_\alpha(w).$$

Therefore, there is a positive constant C such that for all $f \in \mathcal{O}(B_n)$ and $z \in B_n$,

$$|\nabla f(z)|(1-|z|^2)^{\beta/2} \leqslant \frac{C}{\nu_\alpha[E(z,r)]} \int\limits_{E(z,r)} |\nabla f(w)|(1-|z|^2)^{\beta/2} \, \mathrm{d}\nu_\alpha(w).$$

Since $1-|z|^2 \sim 1-|w|^2$ as $z \in B_n$ and $w \in E(z,r)$ vary ([9], p. 322) it follows that there is a positive constant C such that for all $f \in \mathcal{O}(B_n)$, $z \in B_n$,

$$|\nabla f(z)|(1-|z|^2)^{\beta/2} \leqslant \frac{C}{\nu_\alpha[E(z,r)]} \int\limits_{E(z,r)} |\nabla f(w)|(1-|w|^2)^{\beta/2}\,\mathrm{d}\nu_\alpha(w). \tag{2.9}$$

It follows from Proposition 2.9 that the right-hand side of inequality (2.9) is

$$\begin{aligned}
&\leqslant \frac{C'}{\nu_\alpha[S(\xi,2h)]} \int\limits_{E(z,r)} |\nabla f(w)|(1-|w|^2)^{\beta/2}\,\mathrm{d}\nu_\alpha(w)\\
&\leqslant \frac{C}{\nu_\alpha[S(\xi,2h)]} \int\limits_{S(\xi,2h)} |\nabla f(w)|(1-|w|^2)^{\beta/2}\,\mathrm{d}\nu_\alpha(w)\\
&\leqslant C \sup_{\{(\xi,2h)\in\partial B_n\times(0,\infty):z\in S(\xi,2h)\}} \frac{1}{\nu_\alpha[S(\xi,2h)]} \int\limits_{S(\xi,2h)} |\nabla f(w)|(1-|w|^2)^{\beta/2}\,\mathrm{d}\nu_\alpha(w)\\
&= C\widetilde{M}[|\nabla f|(\cdot)(1-|\cdot|^2)^{\beta/2}](z) \leqslant C' M[|\nabla f|(\cdot)(1-|\cdot|^2)^{\beta/2}](z).
\end{aligned}$$

The last inequality above follows from Lemma 2.6. Therefore, the pointwise estimate (2.8) holds. It follows from inequality (2.8) that

$$\begin{aligned}
\int\limits_{B_n} |\nabla f(z)|^2(1-|z|^2)^\beta\,\mathrm{d}\mu(z) &\leqslant C' \int\limits_{B_n} (M[|\nabla f|(\cdot)(1-|\cdot|^2)^{\beta/2}](z))^2\,\mathrm{d}\mu(z)\\
&\leqslant C \int\limits_{B_n} |\nabla f(z)|^2(1-|z|^2)^\beta\,\mathrm{d}\nu_\alpha(z).
\end{aligned}$$

The final inequality above follows from Lemma 2.8. ∎

3. A comparison theorem for $H^p(B_n)$ and $A^p_\alpha(B_n)$

In this section we show that for holomorphic maps $\varphi : B_n \to B_n$, boundedness of C_φ on $H^p(B_n)$ implies boundedness of C_φ on $A^p_\alpha(B_n)$ for all $\alpha > -1$ and that boundedness of C_φ on $A^p_\alpha(B_n)$ for some $\alpha > -1$ implies boundedness of C_φ on $A^p_\beta(B_n)$ for all $\beta \geqslant \alpha$. Both of these facts hold with "boundedness" replaced by "compactness". We begin with a few preliminary facts for the Hardy spaces.

Lemma 3.1. *For each $a \in B_n$, C_{φ_a} is bounded on $H^p(B_n)$ for $p \geqslant 1$.*

Proof. Recall that $H^p(B_n)$ is a Moebius-invariant function space ([14], p. 84–85). It is easy to show (in fact, a more general result holds — see [7], Chapter 1, Example 1.1.1) that C_{φ_a} defines a closed linear operator on the set of all $f \in H^2(B_n)$ such that $f \circ \varphi \in H^2(B_n)$. An application of the Closed Graph theorem completes the proof. ∎

We also provide the details of the proof of the following lemma, whose proof closely follows the line of reasoning used in pages 161ff of [7] to prove the same result for a single composition operator.

LEMMA 3.2. *Let $p > 0$. Suppose $\{\varphi_\beta : \beta \in I\}$ is an indexed family of holomorphic self-maps of B_n such that:*
(i) *C_{φ_β} is bounded on $H^p(B_n)$ for each $\beta \in I$; and*
(ii) *the operator norms $\|C_{\varphi_\beta}\|$ are bounded uniformly in β.*
Then there exists a constant $C > 0$ such that for all $\beta \in I$, $\xi \in \partial B_n$, $h \in (0,1)$,

$$\mu_{\varphi_\beta}[\mathcal{S}(\xi,h)] \leqslant Ch^n.$$

Proof. Boundedness of C_{φ_β} on $H^p(B_n)$ for $p > 0$ is equivalent to boundedness on $H^2(B_n)$ ([7], p. 161–162). Fix $\xi \in \partial B_n$ and $h \in (0,1)$. Consider the family of test functions f_w defined by

$$f_w(z) = (1 - \langle z, w\rangle)^{-2n},$$

where $w = (1-h)\xi$ ([7], p. 162). It can be shown ([7], p. 162) that

$$\|f_w \circ \varphi_\beta\|^2 = \int_{\partial B_n} |(f_w \circ \varphi_\beta^*)(\eta)|^2 \,\mathrm{d}\sigma(\eta) = \int_{\overline{B}_n} |f_w(z)|^2 \,\mathrm{d}\mu_{\varphi_\beta}(z), \tag{3.1}$$

by the fact that $(f_w \circ \varphi_\beta)^*(\xi) = (f_w^* \circ \varphi_\beta^*)(\xi)$ for almost every $\xi \in \partial B_n$ (since f_w clearly extends continuously to $\overline{B}_n$ and the C_{φ_β}'s are bounded (see [7], Chapter 3). By assumption, there exists a constant $C > 0$ such that for all $\beta \in I$, $h \in (0,1)$, and $\xi \in \partial B_n$,

$$\int_{\partial B_n} |(f_w^* \circ \varphi_\beta^*)(\eta)|^2 \,\mathrm{d}\sigma(\eta) \leqslant C \int_{\partial B_n} |f_w^*(\eta)|^2 \,\mathrm{d}\sigma(\eta). \tag{3.2}$$

From relation (3.1) and inequality (3.2), we have that

$$\begin{aligned} \int_{\mathcal{S}(\xi,h)} |f_w(z)|^2 \,\mathrm{d}\mu_{\varphi_\beta}(z) &\leqslant \int_{\overline{B}_n} |f_w(z)|^2 \,\mathrm{d}\mu_{\varphi_\beta}(z) \qquad (3.3)\\ &= \int_{\partial B_n} |(f_w \circ \varphi_\beta^*)(\eta)|^2 \,\mathrm{d}\sigma(\eta) \\ &\leqslant C \int_{\partial B_n} |f_w^*(\eta)|^2 \,\mathrm{d}\sigma(\eta). \qquad (3.4) \end{aligned}$$

It can be shown ([7], Example 3.5.2, p. 172) that for all $z \in \mathcal{S}(\xi,h)$, $\xi \in \partial B_n$, and $h \in (0,1)$ satisfying $w = (1-h)\xi$,

$$|f_w(z)|^2 \geqslant (2h)^{-4n}. \tag{3.5}$$

It is also not difficult to show ([14], p. 18) that

$$\|f_w\|^2_{H^2(B_n)} \sim h^{-3n}. \tag{3.6}$$

Collapsing inequalities (3.3) and (3.4) gives rise to the inequality

$$\int_{\mathcal{S}(\xi,h)} |f_w(z)|^2 \,\mathrm{d}\mu_{\varphi_\beta}(z) \leqslant C\|f_w\|^2_{H^2(B_n)}.$$

Applying inequality (3.5) to the left side of this inequality and equation (3.6) to the right side, one obtains

$$\int\limits_{\mathcal{S}(\xi,h)} (2h)^{-4n}\,\mathrm{d}\mu_{\varphi_\beta}(z) \leqslant C' h^{-3n}.$$

The above inequality can be rewritten as

$$\mu_{\varphi_\beta}[\mathcal{S}(\xi,h)] \leqslant C 2^n h^n. \quad \blacksquare \tag{3.7}$$

The author is grateful to B. MacCluer for pointing out that the proof of the boundedness portion of part (i) of the following theorem can be obtained by an identical argument that she and Carl Cowen previously used in [8] to prove that for the special case of linear fractional maps φ of B_n, boundedness of C_φ on $H^p(B_n)$ implies boundedness on $A^p_\alpha(B_n)$ for each $\alpha > -1$. Part (ii) of Theorem 3.3 will be needed in the proof of Theorem 1.1.

Theorem 3.3. *Suppose that $p > 0$, and let $\varphi : B_n \to B_n$ be holomorphic.*

(i) *If C_φ is bounded (respectively, compact) on $H^p(B_n)$, then it is also bounded (respectively, compact) on $A^p_\alpha(B_n)$ for all $\alpha > -1$.*

(ii) *If C_φ is bounded (compact) on $A^p_\alpha(B_n)$ for some $\alpha > -1$, then it is bounded (compact) on $A^p_\beta(B_n)$ for all $\beta \geqslant \alpha$.*

Proof. Part (i): For this part, we prove the boundedness portion only. Since C_φ is compact on $H^p(B_n)$ for some $p \in (0,\infty)$ if and only if C_φ is compact on $H^p(B_n)$ for all $p \in (0,\infty)$ ([7], p. 162), we can let $p = 2$. Secondly, suppose for the moment that we have proved the result for maps φ that fix the origin. If $\varphi(0) = a \in B_n$, then $\psi := \varphi_a \in \mathrm{Aut}(B_n)$ maps a to 0. Clearly, $C_\varphi = C_\psi C_{\varphi\psi^{-1}}$. The first factor is bounded on $H^2(B_n)$ by Lemma 3.1. Since $\varphi\psi^{-1}(0) = \varphi(a) = 0$, the second factor $C_{\varphi\psi^{-1}} = C_{\psi^{-1}} C_\varphi$ is also bounded by Lemma 3.1 and the preliminary result for maps fixing the origin. The statement of the theorem follows. Therefore, it remains only to consider maps φ sending 0 to 0.

Next, we claim that for $r \in (0,1)$, the operators C_{φ_r} are bounded on $H^2(B_n)$ and satisfy the uniform bound $\|C_{\varphi_r}\| \leqslant \|C_\varphi\|$. First, note that $C_{\varphi_r} = C_{\psi_r} C_\varphi$, where ψ_r is the map sending $z \in B_n$ to rz. Therefore, the claimed statement will follow if we can prove that the composition operators C_{ψ_r} are bounded with operator norm less than or equal to 1. (Actually, these operators will then have norm *equal* to 1 since the constant functions are in the Hardy space.) Letting $r \in (0,1)$ and $f \in H^2(B_n)$, we have

$$\begin{aligned}\|C_{\psi_r} f\|^2_{H^2(B_n)} &= \sup_{s\in(0,1)} \int\limits_{\partial B_n} |f(rs\xi)|^2\,\mathrm{d}\sigma(\xi) = \sup_{t\in(0,r)} \int\limits_{\partial B_n} |f(t\xi)|^2\,\mathrm{d}\sigma(\xi)\\ &\leqslant \sup_{t\in(0,1)} \int\limits_{\partial B_n} |f(t\xi)|^2\,\mathrm{d}\sigma(\xi) = \|f\|^2_{H^2(B_n)}.\end{aligned}$$

Therefore, C_{ψ_r} is bounded and $\|\psi_r\| \leqslant 1$ (hence, $= 1$), as claimed.

Letting $I = (0,1)$ and writing $\beta = r$ in Lemma 3.2, we obtain that there is a constant $C > 0$ such that for all $\xi \in \partial B_n$, $h \in (0,1)$, and $r \in (0,1)$,

$$\mu_{\varphi_r}[\mathcal{S}(\xi,h)] \leqslant C h^n. \tag{3.8}$$

Since $\varphi(0)=0$, it follows from Schwarz' Lemma in B_n ([7], p. 96) that

$$1-|z| \leqslant 1-|\varphi(z)|. \tag{3.9}$$

If we let $z \in \varphi^{-1}[S(\xi,h)]$, then it is easy to see that $1-|\varphi(z)|<h$. Therefore, if $|z|=r$, inequality (3.9) implies that $r>1-h$. We use this inequality to calculate $\mu_\varphi^\alpha[S(\xi,h)]$ by transformation to polar coordinates ([14], p. 13). We write

$$\mu_\varphi^\alpha[S(\xi,h)] \leqslant \mu_\varphi^\alpha[\mathcal{S}(\xi,h)] = \int\limits_{\varphi^{-1}[\mathcal{S}(\xi,h)]} (1-|z|^2)^\alpha \,\mathrm{d}v(z) \tag{3.10}$$

$$= 2n \int\limits_{1-h}^{1} \int\limits_{\partial B_n \cap \varphi^{-1}[\mathcal{S}(\xi,h)]} (1-r^2)^\alpha r^{2n-1} \,\mathrm{d}\sigma(\eta)\,\mathrm{d}r$$

$$= 2n \int\limits_{1-h}^{1} (1-r^2)^\alpha r^{2n-1} \int\limits_{\partial B_n} \chi_{\varphi^{-1}[\mathcal{S}(\xi,h)]}(r\eta)\,\mathrm{d}\sigma(\eta)\,\mathrm{d}r. \tag{3.11}$$

φ_r^*, the radial limit function associated to φ, satisfies $\varphi_r^*(\eta)=\varphi_r(\eta)$ for $\eta \in \partial B_n$. Combining this fact with the fact that $\chi_{\varphi^{-1}[\mathcal{S}(\xi,h)]}(r\eta)=1$ iff $\varphi(r\eta)=\varphi_r(\eta)=\varphi_r^*(\eta) \in \mathcal{S}(\xi,h)$, it is not difficult to see that the inner integral in quantity (3.11) can be written as

$$\int\limits_{(\varphi_r^*)^{-1}[\mathcal{S}(\xi,h)]} \mathrm{d}\sigma(\eta) \leqslant \sigma\{(\varphi_r^*)^{-1}[\mathcal{S}(\xi,h)]\} = \mu_{\varphi_r}[\mathcal{S}(\xi,h)] \leqslant Ch^n$$

by Lemma 3.2. Therefore, quantity (3.11) is

$$= 2nCh^n \int\limits_{1-h}^{1} (1-r^2)^\alpha r^{2n-1}\,\mathrm{d}r \leqslant Ch^n \int\limits_{1-h}^{1} (1-r)^\alpha\,\mathrm{d}r = Ch^{\alpha+n+1}.$$

Therefore, $\mu_\varphi^\alpha[S(\xi,h)] \leqslant Ch^{\alpha+n+1}$, and the constant C is independent of ξ and h. Since μ_φ^α is an α-Carleson measure, we conclude that C_φ is bounded on all of the spaces $A_\alpha^p(B_n)$ ([7], p. 164).

Part (ii): By Lemma 3.1, the automorphisms φ_a induce bounded composition operators on any $A_\alpha^p(B_n)$. Therefore, as in part (i), we can assume that $\varphi(0)=0$. It is well-known ([7], p. 164) that boundedness of C_φ on $A_\alpha^p(B_n)$ is equivalent to the condition that there exists a positive constant C such that for all $\xi \in \partial B_n$ and $h>0$,

$$\mu_\varphi^\alpha[S(\xi,h)] \leqslant Ch^{\alpha+N+1}.$$

Therefore, by the converse direction of this result, it suffices to show that the condition above implies that there is a positive constant C' such that for all $\xi \in \partial B_n$ and $h>0$,

$$\mu_\varphi^\beta[S(\xi,h)] \leqslant C'h^{\alpha+N+1}.$$

The simple argument that we will use appears in [11], where an analogous result for the Dirichlet space of D is proven. By definition, we have that

$$\mu_\varphi^\beta[S(\xi,h)] = \int\limits_{\varphi^{-1}[S(\xi,h)]} (1-|z|^2)^\beta \,\mathrm{d}v(z). \tag{3.12}$$

Now if $z \in \varphi^{-1}[S(\xi,h)]$, then $|1-\langle\varphi(z),\xi\rangle| < h$. It follows that $1-|\varphi(z)| < h$, and, since $\varphi(0)=0$, Schwarz' Lemma in B_n ([7], p. 96) implies that $1-|z| < h$. We then rewrite the right side of equation (3.12) as

$$\begin{aligned}\int\limits_{\varphi^{-1}[S(\xi,h)]} (1-|z|^2)^\alpha(1-|z|^2)^{\beta-\alpha}\,\mathrm{d}v(z) &\leqslant 2^{\beta-\alpha}h^{\beta-\alpha}\int\limits_{\varphi^{-1}[S(\xi,h)]}(1-|z|^2)^\alpha\,\mathrm{d}v(z)\\ &\leqslant 2^{\beta-\alpha}\int\limits_{\varphi^{-1}[S(\xi,h)]}(1-|z|^2)^\alpha\,\mathrm{d}v(z) = 2^{\beta-\alpha}\mu_\varphi^\alpha[S(\xi,h)] \leqslant 2^{\beta-\alpha}Ch^{\alpha+n+1}.\end{aligned}$$

Note in particular that

$$\mu_\varphi^\beta[S(\xi,h)] \leqslant 2^{\beta-\alpha}\mu_\varphi^\alpha[S(\xi,h)]. \tag{3.13}$$

Letting $C' = 2^{\beta-\alpha}C$ completes the proof of the boundedness portion of part (ii).

We now prove the compactness portion. Since C_φ is compact on $A_\alpha^p(B_n)$, it follows that C_φ is bounded on $A_\alpha^p(B_n)$. Since $\beta \geqslant \alpha$, we have that inequality (3.13) holds. It follows from compactness of C_φ that the right side of inequality (3.13) tends to 0 uniformly in ξ as h tends to 0 ([7], p. 164), so that the left side must tend to 0 similarly. ∎

We are now prepared to prove our main result.

4. PROOF OF THEOREM 1.1

Proof. Since compactness of C_φ on $A_\alpha^p(B_n)$ is independent of $p>0$ ([7], p. 164), it suffices to prove the result for $p=2$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a norm-bounded sequence in $A_\alpha^2(B_n)$ such that $f_k \to 0$ uniformly on compacta in B_n. By Theorem 2.2, it suffices to show that $\|C_\varphi f_k\|_{A_\alpha^2(B_n)} \to 0$ as $k\to\infty$. By equation (2.2), there is a constant $C>0$ such that for all $k \geqslant 1$,

$$\|C_\varphi f_k\|^2_{A_\alpha^2(B_n)} \leqslant C\|C_\varphi f_k\|^2_{\mathcal{W}^2_{\alpha+3,1}} = C\sum_{i=1}^n \int\limits_{B_n}\Big|\frac{\partial}{\partial z_i}(f_k\circ\varphi)(z)\Big|^2(1-|z|^2)^{\alpha+2}\,\mathrm{d}v(z).$$

By hypothesis, we can choose $\delta\in(0,1)$ such that $\forall z$ satisfying $\delta<|z|<1$,

$$\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\|\varphi'(z)\|^2 < \varepsilon. \tag{4.1}$$

If $B_\delta := \{z \in \mathbb{C}^n : |z| < \delta\}$, then

$$\begin{aligned}\sum_{i=1}^{n} \int_{B_n} \Big|\frac{\partial}{\partial z_i}(f_k \circ \varphi)(z)\Big|^2 (1-|z|^2)^{\alpha+2}\,\mathrm{d}v(z) &= \sum_{i=1}^{n} \int_{B_\delta} \Big|\frac{\partial}{\partial z_i}(f_k \circ \varphi)(z)\Big|^2 (1-|z|^2)^{\alpha+2}\,\mathrm{d}v(z) \\ &\quad + \sum_{i=1}^{n} \int_{\delta<|z|<1} \Big|\frac{\partial}{\partial z_i}(f_k \circ \varphi)(z)\Big|^2 (1-|z|^2)^{2}\,\mathrm{d}\nu_\alpha(z).\end{aligned}$$

The first sum in the right side of the above equation can be made arbitrarily small by noting that

$$\sum_{i=1}^{n} \int_{B_\delta} \Big|\frac{\partial}{\partial z_i}(f_k \circ \varphi)(z)\Big|^2 (1-|z|^2)^{\alpha+2}\,\mathrm{d}v(z) \leqslant \sum_{i=1}^{n} \int_{B_\delta} \Big|\frac{\partial}{\partial z_i}(f_k \circ \varphi)(z)\Big|^2 \mathrm{d}v(z).$$

The right side above tends to zero as $k \to \infty$ because $f_k \to 0$ uniformly on compact subsets of B_n, implying in turn that $f_k \circ \varphi$, and the partial derivatives $\frac{\partial}{\partial z_i}(f_k \circ \varphi)$ converge to 0 uniformly on compact subsets ([14], p. 5). Therefore, it remains to show that the quantity

$$\sum_{i=1}^{n} \int_{\delta<|z|<1} \Big|\frac{\partial}{\partial z_i}(f_k \circ \varphi)(z)\Big|^2 (1-|z|^2)^{2}\,\mathrm{d}\nu_\alpha(z) \tag{4.1}$$

can be made arbitrarily small.

Let $\|\varphi'(z)\|_2$ denote the Hilbert-Schmidt norm of the linear transformation $\varphi'(z)$. Using the chain rule, the triangle inequality, the fact that $|D_i\varphi_j|^2 \leqslant \|\varphi'(z)\|_2^2$ for all i, j, and the fact that the Hilbert-Schmidt norm of $\varphi'(z)$ is equivalent to its operator norm, we obtain that quantity (4.2)

$$\begin{aligned}&= \int_{\delta<|z|<1} \sum_{i=1}^{n} |D_i(f_k \circ \varphi)(z)|^2 (1-|z|^2)^2\,\mathrm{d}\nu_\alpha(z) \\ &\leqslant \int_{\delta<|z|<1} \sum_{i=1}^{n} \Big|\sum_{j=1}^{n} [D_j f_k][\varphi(z)](D_i\varphi_j)(z)\Big|^2 (1-|z|^2)^2\,\mathrm{d}\nu_\alpha(z) \\ &\leqslant n \int_{\delta<|z|<1} \Big|\sum_{j=1}^{n} [D_j f_k][\varphi(z)]\Big|^2 \|\varphi'(z)\|_2^2 (1-|z|^2)^2\,\mathrm{d}\nu_\alpha(z) \\ &\leqslant C \int_{\delta<|z|<1} \Big|\sum_{j=1}^{n} [D_j f_k][\varphi(z)]\Big|^2 \|\varphi'(z)\|^2 (1-|z|^2)^2\,\mathrm{d}\nu_\alpha(z).\end{aligned}$$

From (4.1) and a measure-theoretic change of variables (proven by considering simple functions and applying an appropriate convergence theorem), it follows that the above quantity is

$$\begin{aligned} &\leqslant n\varepsilon \int\limits_{\delta<|z|<1} \Big|\sum_{j=1}^{n}[D_j f_k][\varphi(z)]\Big|^2 (1-|\varphi(z)|^2)^2 \,\mathrm{d}\nu_\alpha(z) \\ &= n\varepsilon \int\limits_{\varphi(\delta<|z|<1)} \Big|\sum_{j=1}^{n}[D_j f_k][w]\Big|^2 (1-|w|^2)^2 \,\mathrm{d}\mu_\varphi^\alpha(w) \\ &\leqslant n\varepsilon \int\limits_{B_n} \Big|\sum_{j=1}^{n}[D_j f_k][w]\Big|^2 (1-|w|^2)^2 \,\mathrm{d}\mu_\varphi^\alpha(w). \end{aligned}$$

Since C_φ is bounded on $A^2_\alpha(B_n)$, μ^α_φ is an α-Carleson measure ([7], p. 164). Therefore, Theorem 2.11 with $\mu = \mu^\alpha_\varphi$ and $\beta = 2$ implies that there is a constant $C' > 0$ (which we relabel C as usual) such that the above quantity is

$$\begin{aligned} &\leqslant C\varepsilon \int\limits_{B_n} \Big|\sum_{j=1}^{n}[D_j f_k][w]\Big|^2 (1-|w|^2)^2 \,\mathrm{d}\nu_\alpha(w) \\ &= C\varepsilon \|f_k\|_{\mathcal{W}^2_{\alpha+3,1}} \leqslant C\varepsilon \|f_k\|^2_{A^2_\alpha(B_n)}. \end{aligned}$$

The final inequality above follows from Proposition 2.1. We note once again that $\{f_k : k \in \mathbb{N}\}$ is norm bounded, independent of k. Therefore, C_φ is compact on $A^p_\alpha(B_n)$. Next, let $\beta \geqslant \alpha$. By Theorem 3.3, C_φ is also bounded and compact on $A^p_\beta(B_n)$ for all $\beta \geqslant \alpha$, so that Theorem 1.1 is completely proven. ∎

5. EXAMPLES AND QUESTIONS FOR FURTHER INVESTIGATION

It is natural to consider whether or not the following extension of Theorem 1.1 holds: if $\alpha = -1$ (respectively, $-1 < \alpha < 0$), (1.2) holds, and C_φ is bounded on $H^p(B_n)$ (respectively, bounded on A^p_α), then C_φ is compact on $H^p(B_n)$ (respectively, compact on $A^p_\alpha(B_n)$) and also on $A^p_\beta(B_n)$ for all $\beta > -1$ (respectively, $\beta > \alpha$). Indeed it can be shown with results and techniques identical to those in this paper that this result holds; however, we now show for $n = 1$ (Proposition 5.1) that any map φ under these conditions must have image with sup norm less than one, so that in this case φ automatically induces a compact composition operator. The proof of the proposition uses ideas from Chapter 4 of [7]:

Proposition 5.1. *Let $\alpha \in [-1, 0)$, and suppose that $\varphi : D \to D$ is holomorphic and satisfies*

$$\lim_{|z|\to 1^-} \Big\{\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^{\alpha+2} |\varphi'(z)|^2\Big\} = 0. \tag{5.1}$$

Then $\|\varphi\|_\infty < 1$.

Proof. Letting $\gamma = (\alpha+2)/2$, we obtain

$$\lim_{|z|\to 1^-} \Big\{\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^{\gamma} |\varphi'(z)|\Big\} = 0. \tag{5.2}$$

We first claim that C_φ is compact on $\mathrm{Liph}_{1-\gamma}(D)$. Suppose that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathrm{Liph}_{1-\gamma}(D)$ such that $f_k \to 0$ uniformly on compacta. Since $\mathrm{Liph}_{1-\gamma}(D)$ is a functional Banach space ([7], Chapter 4), it suffices to show that $\|f_k \circ \varphi\|_{\mathrm{Liph}_{1-\gamma}(D)} \to 0$. Let $\varepsilon > 0$. By equation (5.2), there exists an $r \in (0,1)$ such that for $z \in D$ such that $|z| > r$,

$$\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^\gamma |\varphi'(z)| < \varepsilon. \tag{5.3}$$

Since $\mathrm{Liph}_{1-\gamma}(D)$ is a functional Banach space, its point evaluation functionals are continuous. Combining this fact with ([7], Theorem 4.1, p. 176) (Note: there is a typographical error here — ∂B_n should be replaced by ∂D, although an analogue does hold for B_n, [5]), we have that there is a $C > 0$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned}\|f_k \circ \varphi\|_{1-\gamma} &\leqslant C \sup_{z\in D}\{|(f_k\circ\varphi)'(z)|(1-|z|^2)^\gamma\} \\ &\leqslant C\Big(\sup_{|z|\leqslant r}\{|f_k'[\varphi(z)]|\,|\varphi'(z)|(1-|z|^2)^\gamma\} + \sup_{|z|>r}\{|f_k'[\varphi(z)]|\,|\varphi'(z)|(1-|z|^2)^\gamma\}\Big).\end{aligned}$$

Clearly, the above quantity is

$$\leqslant C\Big(M\sup_{|z|\leqslant r}\{|f_k'[\varphi(z)]|\} + \sup_{|z|>r}\Big\{|f_k'[\varphi(z)]|\,|\varphi'(z)|\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^\gamma (1-|\varphi(z)|^\gamma\Big\}\Big).$$

The quantity above is

$$\begin{aligned}&\leqslant C\Big(M\sup_{|z|\leqslant r}\{|f_k'[\varphi(z)]|\} + \varepsilon\sup_{|z|>r}\{|f_k'[\varphi(z)]|(1-|\varphi(z)|^\gamma\}\Big) \\ &\leqslant C\Big(M\sup_{|z|\leqslant r}\{|f_k'[\varphi(z)]|\} + C'\varepsilon\|f_k\|_{1-\gamma}\Big) \leqslant C\Big(M\sup_{|z|\leqslant r}\{|f_k'[\varphi(z)]|\} + C''\varepsilon\Big),\end{aligned}$$

since (f_k) is bounded by hypothesis. The left summand above can be made arbitrarily small by uniform convergence of (f_k) on compacta. Since ε was chosen arbitrarily, our claim of compactness of C_φ on $\mathrm{Liph}_{1-\gamma}(D)$ holds. Since $\mathrm{Liph}_{1-\gamma}(D)$ is an automorphism-invariant, boundary regular, small space, it follows that $\|\varphi\|_\infty < 1$ ([7], p. 177). ∎

Although the above proposition shows that allowing α to be in $[-1,0)$ even for $n = 1$ in Theorem 1.1 yields $\|\varphi\|_\infty < 1$, Proposition 5.3 below shows that $\alpha = 0$ is a critical value in the sense that for $\alpha \geqslant 0$, *there are* self-maps φ of B_2 with unit modulus that satisfy the hypotheses of Theorem 1.1. Our example for B_2 will be constructed from the following single-variable existence result:

PROPOSITION 5.2. *For $\alpha \geqslant 0$, there exist holomorphic maps $\psi : D \to D$ with $\|\psi\|_\infty = 1$ satisfying*

$$\lim_{|z|\to 1^-}\left(\frac{1-|z|^2}{1-|\psi(z)|^2}\right)^{\alpha+2}|\psi'(z)|^2 = 0. \tag{5.4}$$

Proof. A region $G \subset D$ is said to have a *generalized cusp* ([13], p. 256) at $\xi \in \partial D$ iff

$$d(w,\partial G) = \mathrm{o}(|\xi - w|)$$

as $w \to \xi$ in G. This cusp is called *nontangential* iff it is contained in a non-tangential approach region at ξ defined by

$$\Gamma(\xi, M) = \{z \in D : |z - \xi| < M(1 - |z|)\},$$

where $M > 1$. These regions are shaped like petals, with a cusp at ξ and boundary curves making an angle (at ξ) whose measure is less than π and is related to the value of M ([7], p. 50–51, 60). For example, $\Gamma(1, 2)$ is a petal-shaped region in D whose cusp at 1 makes an angle of $\pi/2$, symmetric about the real axis.

Let G be the region bounded by the graphs of the equations $x = 0.75$, $y = 0.5(x-1)^2$, and the real axis. It is easy to see that this region lies completely in $\Gamma(1, 2)$. As w tends to 1 in G, it is also easy to see that $d(w, G) \leqslant 0.5(\operatorname{Re} w-1)^2 \leqslant C|1 - w|^2$. Therefore, $d(w, G)/|1 - w| \to 0$ as w tends to 1, and G has a non-tangential, generalized cusp at 1. Obviously, $\overline{G}$ touches the unit circle only at 1.

By the Riemann Mapping Theorem, there is a univalent ([13], p. 4) map ψ of D onto G. By results of K. Madigan and A. Matheson ([12]), any univalent map ψ of D onto a region $G \subset D$ that has a non-tangential, generalized cusp at 1 and touches the unit circle at no other point must induce a compact composition operator on $\mathcal{B}_0$, and, for *any* analytic $\psi : \Delta \to$, C_ψ is compact on $\mathcal{B}_0$ iff

$$\lim_{|z|\to 1^-} \frac{1 - |z|^2}{1 - |\psi(z)|^2}\psi'(z) = 0. \tag{5.5}$$

It follows that the Riemann map ψ constructed above has sup norm 1 and satisfies equation (5.4) for $\alpha = 0$. However, as we will now show, ψ actually satisfies equation (1.2) for all $\alpha \geqslant 0$. By Theorem 1.1, C_ψ must be compact on $A^2(D)$. Therefore, equation (1.1) holds (see [20], p. 218)). Squaring both sides of equation (5.5) and multiplying the fractional quantity in the resulting equation by the αth power of the fractional quantity in equation (1.1), we obtain equation (5.4). ∎

We can now prove the following existence result for $n = 2$:

PROPOSITION 5.3. *For all $p > 0$ and $\alpha \geqslant 0$, there exist holomorphic maps $\varphi : B_2 \to B_2$ such that $\|\varphi\|_\infty < 1$, C_φ is bounded on $A^p_\alpha(B_2)$, and*

$$\lim_{|z|\to 1^-} \Big(\frac{1 - |z|^2}{1 - |\varphi(z)|^2}\Big)^{\alpha+2} |\nabla\varphi_i(z)|^2 = 0, \quad i = 1, 2. \tag{5.6}$$

That is, there are maps φ with unit modulus in more than one variable that satisfy the hypotheses of Theorem 1.1.

Proof. Let ψ be any map as guaranteed by Proposition 5.2, and define the clearly holomorphic map $\varphi : B_2 \to B_2$ by

$$\varphi(z) = (\psi(z_1), 0).$$

Clearly, $\|\varphi\|_\infty = 1$, since $\|\psi\|_\infty = 1$. Since the ith coordinate of φ depends only on z_i for $i = 1, 2$, C_φ is bounded on $H^2(B_2)$ ([3], Proposition 1). Theorem 3.3 (i) then shows that C_φ is bounded on $A^p_\alpha(B_2)$ for all $\alpha \in (-1, \infty)$. Equation (5.6) trivially holds for $i = 2$.

To complete the proof of the proposition, it remains to show that

$$\lim_{|z|\to 1^-} \Big(\frac{1 - |z|^2}{1 - |\varphi(z)|^2}\Big)^{\alpha+2} |\nabla\varphi_1(z)|^2 = 0.$$

The above equation holds iff for each point $(\xi_1, \xi_2) \in \partial B_2$ and all sequences $(z^{(j)}, w^{(j)}) \in B_2$ that converge to (ξ_1, ξ_2),

$$\lim_{j\to\infty} \Big(\frac{1-|(z^{(j)}, w^{(j)})|^2}{1-|\varphi(z^{(j)}, w^{(j)})|^2}\Big)^{\alpha+2} |\nabla\varphi_1(z^{(j)}, w^{(j)})|^2 = 0. \tag{5.7}$$

Let $(z^{(j)}, w^{(j)})_{j\in\mathbb{N}}$ in B_2 and $(\xi_1, \xi_2) \in \partial B_2$ be as described above. The left side of equation (5.7) is

$$\leqslant \lim_{j\to\infty} \Big(\frac{1-|z^{(j)}|^2}{1-|\psi(z^{(j)})|^2}\Big)^{\alpha+2} |\psi'(z^{(j)})|^2. \tag{5.8}$$

First, suppose that $|\xi_1| = 1$. We then have that $|z^{(j)}| \to 1^-$ as $j \to \infty$ and equation (5.4) therefore applies. It follows that quantity (5.8) is zero. If $|\xi_1| < 1$, then $1 - |\psi(z^{(j)})|$ converges to a positive quantity. In addition, $|\psi'(z^{(j)})| \to M$ for some non-negative real number M as $j \to \infty$, since ψ' is continuous in D. Therefore, quantity (5.8) is 0. ∎

There are compact composition operators on $A^2_\alpha(B_n)$ that do not satisfy the hypotheses given in Theorem 1.1. Consider the map $\varphi(z) = 1 - ([1-z]/2)^{1/2}$, on D. The image of φ is contained in a suitable non-tangential approach region (a *lens*) at 1 and touching the boundary at no other point but 1, so it has no finite angular derivative at any point of ∂D. Therefore, C_φ is compact on $A^2_\alpha(D)$ for $\alpha > -1$ (see [16], p. 27). However, using the sequence $\{k/(k+1)\}_{k\in\mathbb{N}}$, it is easy to show that the limit in Theorem 1.1 is non-zero.

Can the results of this paper be extended to other domains? Are there relationships between the hypotheses of Theorem 1.1 and the condition that $\varphi(B_n)$ is contained in a Koranyi approach region ([10])? Is it possible to remove the hypothesis that C_φ is bounded from Theorem 1.1?

Acknowledgements. The author would like to thank his thesis supervisor, B. Russo, for his invaluable advice and encouragement, and S.-Y. Li, for several helpful conversations and inspiration. The author is also indebted to W. Wogen, who pointed out errors and oversights in earlier versions of this project. Thanks are also extended to J. Cima, A. Matheson, B. MacCluer, P. Mercer and W. Rudin for their friendly and generous correspondences.

This work was included in the author's Ph.D. Thesis, at the University of California, Irvine, under the direction of Bernard Russo.

REFERENCES

1. F. Beatrous, Estimates for derivatives of holomorphic functions in pseudoconvex domains, *Math. Z.* **191**(1986), 91–116.
2. F. Beatrous, J. Burbea, Holomorphic Sobolev spaces of the unit ball, *Dissertationes Math. (Rozprawy Mat.)* **276**(1989), 1–57.
3. J. Cima, C. Stanton, W. Wogen, On boundedness of composition operators on $H^2(B_2)$, *Proc. Amer. Math. Soc.* **91**(1984), 217–222.

4. J. CIMA, W. WOGEN, A Carleson measure theorem for the Bergman space of the unit ball, *J. Operator Theory* **7**(1982), 157–165.
5. D. CLAHANE, Bounded composition operators on holomorphic Lipschitz spaces of the unit ball, preprint.
6. D. CLAHANE, Spectra of power-compact composition operators on function spaces of bounded symmetric domains, preprint.
7. C. COWEN, B. MACCLUER, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Roton, 1995.
8. C. COWEN, B. MACCLUER, Linear fractional maps of the unit ball and their composition operators, *Acta Sci. Math. (Szeged)* **66**(2000), 351–376.
9. D. LUECKING, Representation and duality in weighted spaces of analytic functions, *Indiana Univ. Math. J.* **34**(1985), 319–336.
10. B. MACCLUER, Compact composition operators on $H^p(B_n)$, *Michigan Math. J.* **32**(1985), 237–248.
11. B. MACCLUER, J. SHAPIRO, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.* **38**(1986), 878–906.
12. K. MADIGAN, A. MATHESON, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.* **347**(1995), 2679–2687.
13. C. POMMERENKE, *Boundary Behavior of Conformal Maps*, Springer-Verlag, New York 1992.
14. W. RUDIN, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York 1980.
15. W. RUDIN, *Real and Complex Analysis*, McGraw-Hill, New York 1987.
16. J. SHAPIRO, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York 1993.
17. J. SHAPIRO, P. TAYLOR, Compact, nuclear, and Hilbert-Schmidt composition operators on H^2, *Indiana Univ. Math. J.* **125**(1973), 471–496.
18. E. STEIN, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton 1993.
19. E. STEIN, G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton 1972.
20. K. ZHU, *Operator Theory in Function Spaces*, Marcel Dekker, New York 1990.

DANA D. CLAHANE
Department of Mathematics
University of California
Irvine, California 92717
USA

E-mail: dclahane@math.uci.edu

Received October 19, 1998; revised February 1, 1999 and April 5, 2000.