J. OPERATOR THEORY
45(2001), 377–412

THE STABLE IDEALS OF A CONTINUOUS NEST ALGEBRA

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Communicated by William B. Arveson

Abstract. This paper characterizes those closed ideals of a continuous nest algebra which are fixed by the automorphism group. This provides a framework in which to organize previously known ideals, and introduces many new examples.

Keywords: *Nest algebra, automorphism invariant, ideals.*

MSC (2000): 47L35.

We shall call a closed two-sided ideal of a Banach algebra *stable* if it is invariant under the action of the automorphism group of the algebra. In this paper we shall present a complete description of all the stable ideals of a continuous nest algebra.

Clearly a two-sided ideal is always invariant for an inner automorphism. So if all automorphisms of an algebra are inner then all (two-sided) ideals are stable. Thus, the interesting situation is when the algebra has large enough an outer automorphism group that the condition of stability is not vacuous, but which is not so free as to prevent any non-trivial examples arising (as happens, for example, in $C_0(\mathbb{R})$, in which the only stable ideals are (0) and $C_0(\mathbb{R})$ itself).

Certain ideals are stable in any Banach algebra. Trivial examples are the zero ideal and the whole algebra. Also, the Jacobson radical and the strong radical (the intersection of all maximal two-sided ideals) are stable. Of course the two radicals may be zero, or be the whole algebra.

In fact nest algebras have a very rich structure of stable ideals. The finite dimensional nest algebras and the algebra of infinite upper triangular matrices have no outer automorphisms, so all ideals in those algebras are stable. In contrast, the continuous nest algebras have many outer automorphisms, so that the stable ideals are a natural class of ideals which is large enough to furnish many examples, but which is still tractable. The compact operators in the algebra, the Jacobson radical, and ideals such as $\mathfrak{R}_0$ studied by Erdos in [11] and the strong radical $\mathfrak{J}_{\min}$

([21]) are examples of distinct, non-trivial stable ideals. Other new examples are given in Figure 1 and Section 6.

Investigation of the fairly diverse spectrum of ideals in nest algebras, and the behavior of operators in them, has played an important part in the development of the field. (See, e.g. [26], [17], [2], [16], [11], [15], [1], [12], [14], [25], [20], [4].) Some progress has been made in developing a comprehensive understanding of the ideal structure ([5], [12]), but there has been no large scale framework which allows one to tie together the ideals which have actually been used in practice. Although this work does not succeed in bringing all the ideals hitherto studied into one tent (the most notable exceptions being Larson's ideal, $\mathfrak{R}^\infty_\mathfrak{N}$, and the maximal ideals), it at least allows many ideals (including the Jacobson radical and the ideal of compact operators) to be seen as examples of the broader phenomenon of stability.

In recent years Davidson's remarkable Similarity Theorem for nests ([5]) has provided a very powerful tool for understanding nest algebras ([18], [21], [1]). This is most evident for the continuous nest algebras, where the manifestations of the similarity theory were the most surprising. The Interpolation Theorem from [23] (which rests on the Similarity Theorem) gives a computable criterion for determining the projections which lie in a given ideal of a continuous nest algebra. To this extent, it goes some of the way towards filling the gap that plagues non-selfadjoint algebra theory from the lack of spectral theory.

Classification of the stable ideals is only possible because the automorphism groups of nest algebras are so well understood. Ringrose ([27]) showed that all isomorphisms between nest algebras are spatial (that is, they are implemented by conjugation by an invertible). So every automorphism of the nest algebra $\operatorname{Alg}\mathfrak{N}$ is of the form Ad_S, where $S\operatorname{Alg}\mathfrak{N}\,S^{-1} = \operatorname{Alg}\mathfrak{N}$. The Similarity Theorem for nests ([5]) gives necessary and sufficient conditions for two nests to be similar. Together, these two results identify the outer automorphism group $\mathrm{Out}(\operatorname{Alg}\mathfrak{N})$ of a continuous nest algebra $\operatorname{Alg}\mathfrak{N}$ with $\mathrm{Hom}_0[0,1]$, the set of increasing homeomorphisms of $[0,1]$ to itself.

The classification of the stable ideals is accomplished in two parts. There is a finite family of stable ideals which are related to the compact operators, and we study these ideals first. (Note that because automorphisms of nest algebras are spatial, the set of compact operators in a nest algebra is a stable ideal.) The classification of these ideals is given in Theorem 2.16, which is the main result of Section 2.

The remaining stable ideals represent the bulk of the stable ideals. There are infinitely many, and they are all classified in terms of the asymptotic behavior of their elements near the diagonal.

To describe how this classification works, recall Ringrose's description of the Jacobson radical of a nest algebra. For each $X \in \operatorname{Alg}\mathfrak{N}$, define

$$i^+_N(X) = \inf\{\|(N'-N)X(N'-N)\| : N' > N,\, N \in \mathfrak{N}\}$$

and

$$i^-_N(X) = \inf\{\|(N-N')X(N-N')\| : N' < N,\, N \in \mathfrak{N}\}.$$

For each fixed N, the maps $X \mapsto i^\pm_N(X)$ are submultiplicative seminorms on $\operatorname{Alg}\mathfrak{N}$. For each neighborhood (N', N'') of N in $\mathfrak{N}$, the values of these seminorms depend only on $(N''-N')X(N''-N')$. This is why we say the seminorms are *local* to the

point N. Ringrose ([26]) showed that the Jacobson radical of $\mathrm{Alg}\,\mathfrak{N}$ is precisely the set of X for which
$$\max\{i_N^+(X), i_N^-(X)\} = 0$$
for all $N \in \mathfrak{N}$.

From the present point of view, the key aspect of Ringrose's result is that an ideal is described as the zero set of a family of localized seminorms parameterized by $N \in \mathfrak{N}$. In Section 3 we introduce four other possible formulas for localized seminorms; $0, e_N^+, e_N^-$, and j_N. The most general localized seminorm needed is obtained by splicing together the formulas for the six types at different points of $\mathfrak{N}$.

Proposition 3.4 shows how to use these seminorms to build general examples of stable ideals. Section 6 illustrates how old and new examples of ideals fit into this framework.

Section 4 is a collection of technical propositions. One of the key results is Proposition 4.9. As might be expected, it uses similarity theory and the results of [23]. But it also needs a combinatoric result on ordered sets. The precise theorem needed is given in [21], but the underlying ideas come from Laver's deep proof that countable scattered order types cannot be embedded one into another to form an infinite decreasing chain ([19]). (This answered an old question of Fraïssé ([13]).) This is the first example of really sophisticated order theory being used to study nest algebras. The main theorem of the paper is proved in Section 5 and is followed by a section devoted to examples.

1. PRELIMINARIES

Throughout the paper, we suppose that $\mathfrak{N}$ is a continuous nest on a separable Hilbert space $\mathfrak{H}$. We write $\mathrm{Alg}\,\mathfrak{N}$ for the nest algebra of operators in $B(\mathfrak{H})$ leaving the ranges of the projections in $\mathfrak{N}$ invariant. If $x, y \in \mathfrak{H}$ we write xy^* for the rank-1 operator:
$$\xi \mapsto \langle \xi, y\rangle x.$$
For the standard terminology and techniques of nest algebras we refer the reader to [6].

Davidson's Similarity Theorem for Nests is one of the key tools that we use to study stable ideals. We state the theorem here for reference:

Theorem 1.1. ([5]) *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be nests on separable Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively, and let $\theta : \mathfrak{L}_1 \to \mathfrak{L}_2$ be an order preserving isomorphism such that,*
$$\mathrm{rank}(\theta(L) - \theta(M)) = \mathrm{rank}(L - M)$$
for all $L > M$ in $\mathfrak{L}_1$. Then there is an invertible operator $S : \mathfrak{H}_1 \to \mathfrak{H}_2$ such that
$$SL\mathfrak{H}_1 = \theta(L)\mathfrak{H}_2$$
for all $L \in \mathfrak{L}_1$. Moreover, S can be taken to be an arbitrarily small compact perturbation of a unitary.

For the continuous nest $\mathfrak{N}$, $\mathrm{rank}(L - M) = +\infty$ for all $L > M$ in $\mathfrak{N}$. Thus, since $\mathfrak{N}$ has the same order type as $[0, 1]$, it follows that for each $\theta \in \mathrm{Hom}_0[0, 1]$ there is an invertible $S \in B(\mathfrak{H})$ such that $SL\mathfrak{H} = \theta(L)\mathfrak{H}$ for all $L \in \mathfrak{N}$. Conjugation by S gives an automorphism of $\operatorname{Alg}\mathfrak{N}$. If $f \in \operatorname{Hom}_0[0,1]$ is not the identity function then at least one of S or S^{-1} does not belong to $\operatorname{Alg}\mathfrak{N}$. It is straightforward to see that each f gives rise to a distinct class in $\operatorname{Out}(\operatorname{Alg}\mathfrak{N}) := \operatorname{Aut}(\operatorname{Alg}\mathfrak{N})/\operatorname{Inn}(\operatorname{Alg}\mathfrak{N})$ ([8]).

2. The ideals of compact character

Definition 2.1. An operator K in $\operatorname{Alg}\mathfrak{N}$ is said to be of *compact character* if, for all $0 < M < N < I$ in $\mathfrak{N}$, $(N-M)K(N-M)$ is compact. An ideal is of compact character if all of its elements are of compact character.

We define the following ideals in $\operatorname{Alg}\mathfrak{N}$:

(i) $\mathfrak{K}$ is the set of compact operators in $\operatorname{Alg}\mathfrak{N}$.

(ii) $\mathfrak{K}^+$ is the set of operators $X \in \operatorname{Alg}\mathfrak{N}$ for which $N^\perp X$ is compact for all $N > 0$ in $\mathfrak{N}$.

(iii) $\mathfrak{K}^-$ is the set of operators $X \in \operatorname{Alg}\mathfrak{N}$ for which XM is compact for all $M < I$ in $\mathfrak{N}$.

In addition we have two ideals which are obtained from $\mathfrak{K}^\pm$:

(iv) $\mathfrak{K}_0^+$ is the set of operators in $\mathfrak{K}^+$ for which $\inf_{M>0} \|XM\| = 0$.

(v) $\mathfrak{K}_0^-$ is the set of operators in $\mathfrak{K}^-$ for which $\inf_{N<I} \|N^\perp X\| = 0$.

Five more ideals can be constructed from these ideals by meets and joins. Figure 1 shows the relationships of the various ideals. The main result of this section, Theorem 2.16, is to show that these are all of the non-zero stable ideals of compact character.

If one chooses to think of the operators of $\operatorname{Alg}\mathfrak{N}$ as "upper triangular forms" with "continuous diagonals", then one would think of an operator of compact character diagrammatically as an upper triangular form which is compact "in the interior of the triangle", and with non-compactness "accumulating" at the top and side edges. Although certainly not mathematically precise, this heuristic is useful. Likewise, the diagrams associated with each ideal in Figure 1 are intended to show the places where noncompactness can "build up".

One should also note that although Figure 1 shows a lattice of ideals, it is by no means clear that the ideals on the higher branches — the ones obtained by adding elementary ideals — are norm-closed, which is part of our definition of a stable ideal. Lemma 2.13 provides the missing ingredient to deal with this, and the classification is finished in Theorem 2.15. Most of the work of classifying the stable ideals of compact character is carried out in a series of incrementally stronger propositions and their corollaries: 2.3–2.12. We begin with a couple of technical lemmas which will be immensely useful throughout the paper. Lemma 2.1 was shown to the author by Ken Davidson.

Figure 1. The stable ideals of compact character

Lemma 2.2. *Let $X \in B(\mathfrak{H})$ and let P_n, Q_n $(n \in \mathbb{N})$ be sequences of projections such that* $\operatorname{dist}(P_n X Q_n, \mathfrak{F}_{4n-4}) > 1$ *for all n, where $\mathfrak{F}_k$ is the set of operators of rank less than or equal to k. Then there are orthonormal sequences $x_i \in P_i\mathfrak{H}$ and $y_i \in Q_i\mathfrak{H}$ such that $\langle x_i, X y_j \rangle = 0$ for all $i \neq j$, and $\langle x_i, X y_i \rangle > 1$ for all $i \in \mathbb{N}$.*

Proof. Choose the vectors x_i, y_i inductively. Suppose that $x_1, \ldots, x_{n-1}$ and $y_1, \ldots, y_{n-1}$ have been chosen. Let $\mathfrak{M}$ be the span of the vectors

$$\{y_i,\ X^* P_n x_i,\ X^* P_n X y_i,\ X^* x_i : 1 \leqslant i < n\}.$$

Then $\mathfrak{M}$ has dimension no greater than $4(n-1)$, so the distance condition on $P_n X Q_n$ implies that $\|P_n X Q_n P_{\mathfrak{M}}^{\perp}\| > 1$ (where $P_{\mathfrak{M}}$ is the projection onto $\mathfrak{M}$), and this allows us to choose a unit vector y_n in the range of Q_n, which is orthogonal to $\mathfrak{M}$ and for which $\|P_n X Q_n y_n\| > 1$. We take $x_n = P_n X y_n / \|P_n X y_n\|$. We claim that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ now have the desired properties.

Certainly y_n is orthogonal to all y_i for $i < n$. Also

$$\langle x_n, x_i \rangle = \frac{\langle P_n X y_n, x_i \rangle}{\|P_n X y_n\|} = 0$$

for $i < n$. Now $\langle x_n, X y_n \rangle = \|P_n X y_n\| > 1$, while $\langle x_i, X y_n \rangle = \langle X^* x_i, y_n \rangle = 0$ for $i < n$. Finally,

$$\langle x_n, X y_i \rangle = \langle y_n, X^* P_n X y_i \rangle / \|P_n X y_n\| = 0. \quad ∎$$

Generally in order to use Lemma 2.2 we shall first use the next result, Lemma 2.3, to establish the hypotheses of Lemma 2.2.

Lemma 2.3. *Let P_n, Q_n be sequences of projections which decrease to zero, and let $X \in B(\mathfrak{H})$. If $\lim_n \|P_n X Q_n\| > 1$ then, for each k,*

$$\lim_n \operatorname{dist}(P_n^\perp X Q_n^\perp, \mathfrak{F}_k) > 1.$$

Proof. For fixed k, the sequence $\operatorname{dist}(P_n^\perp X Q_n^\perp, \mathfrak{F}_k)$ is increasing, so suppose for a contradiction that $\operatorname{dist}(P_n^\perp X Q_n^\perp, \mathfrak{F}_k) \leqslant 1$ for all $n \in \mathbb{N}$. Then we could find a sequence, F_n, of rank-k operators such that $\lim_n \|P_n^\perp X Q_n^\perp - F_n\| \leqslant 1$. The sequence F_n is bounded by $\|X\| + 1$ and so, letting F be a weak limit point of the F_n's, we would have $\|X - F\| \leqslant 1$ by weak lower semicontinuity of the norm. But the set of rank-k operators is weakly closed, so F would belong to $\mathfrak{F}_k$ and this would show $\lim_n \|P_n X Q_n\| = \lim_n \|P_n (X - F) Q_n\| \leqslant 1$, contrary to hypothesis. ∎

Proposition 2.4. *$\mathfrak{K}$ is the smallest non-zero stable ideal of* $\operatorname{Alg}\mathfrak{N}$.

Proof. The finite rank operators of $\operatorname{Alg}\mathfrak{N}$ are norm dense in $\mathfrak{K}$ and every finite rank operator in $\operatorname{Alg}\mathfrak{N}$ is a sum of rank-1's in $\operatorname{Alg}\mathfrak{N}$ ([10]), so it suffices to show that if $\mathfrak{I}$ is a non-zero stable ideal then $\mathfrak{I}$ contains every rank-1 operator in $\operatorname{Alg}\mathfrak{N}$.

Every rank-1 operator in $\operatorname{Alg}\mathfrak{N}$ is of the form $R = NRN^\perp$ for some $N \in \mathfrak{N}$. However, if $R = NRN^\perp$ is rank-1 then $MRN^\perp$ converges to R in norm as M increases to N in $\mathfrak{N}$. Thus it suffices to show that every rank-1 operator of the form $S = MSN^\perp$ with $M < N$ belongs to $\mathfrak{I}$.

Let $S := xy^*$ with $x \in M\mathfrak{H}$ and $y \in N^\perp\mathfrak{H}$ be such a rank-1 operator, and let X be a fixed non-zero operator in $\mathfrak{I}$. There are $0 < L < G < I$ in $\mathfrak{N}$ such that $(G-L)X(G-L) \neq 0$. Let T be an invertible operator which implements an order isomorphism of $\mathfrak{N}$ to itself that maps L to M and G to N. It follows that $(N-M)TXT^{-1}(N-M) \neq 0$ and so we can pick non-zero vectors $u, v \in (N-M)\mathfrak{H}$ such that $(N-M)TXT^{-1}(N-M)u = v$. Let

$$A := \frac{xv^*}{\|v\|} \quad \text{and} \quad B := \frac{uy^*}{\|v\|}.$$

Then $ATXT^{-1}B = S$, and since $A = MA(N-M)$ and $B = (N-M)BN^\perp$ belong to $\operatorname{Alg}\mathfrak{N}$, and since $TXT^{-1} \in \mathfrak{I}$ by stability, it follows that $S \in \mathfrak{I}$. ∎

The next proposition will be used in Corollary 2.7 to establish that $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$ is the next-biggest stable ideal.

PROPOSITION 2.5. *Suppose* $X \in \operatorname{Alg}\mathfrak{N}$ *and*

$$\inf_{0<M,N<I} \|MXN^{\perp}\| > a > 0.$$

Then the stable ideal generated by X *contains* $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$.

At the heart of this result is a technical lemma (Lemma 2.6) regarding the stable ideal generated by X. We shall prove this lemma first and return to Proposition 2.5.

LEMMA 2.6. *Suppose* X *is as in Proposition* 2.5 *and* $M_n < N_n$ $(n \in \mathbb{N})$ *are sequences of projections in* $\mathfrak{N}$*, with* M_n *decreasing to* 0 *and* N_n *increasing to* I*. Then the stable ideal generated by* X *contains any operator* $W := \sum_n W_n$ *where each* W_n *is a finite rank isometry of the form* $(M_n - M_{n+1})W_n(N_{n+1} - N_n)$.

Proof. Fix $0 < M < N < I$. Then

$$\inf_{\substack{0<M'<M \\ N<N'<I}} \|M'(MXN^{\perp}){N'}^{\perp}\| > a$$

and so by Lemma 2.3, for each $k \in \mathbb{N}$, there are $0 < M' < M$ and $N < N' < I$ such that

$$\operatorname{dist}((M - M')X(N' - N), \mathfrak{F}_k) = \operatorname{dist}({M'}^{\perp}(MXN^{\perp})N', \mathfrak{F}_k) > a.$$

Thus we can inductively choose sequences $M'_n < N'_n$ $(n \geqslant 0)$ in $\mathfrak{N}$, respectively decreasing to zero and increasing to I, such that

$$\operatorname{dist}((M'_{n-1} - M'_n)X(N'_n - N'_{n-1}), \mathfrak{F}_{4n-4}) > a$$

for all $n \geqslant 1$. By Lemma 2.2, there are orthonormal sequences x_n, y_n respectively in the ranges of $(M'_{n-1} - M'_n)$ and $(N'_n - N'_{n-1})$ such that $\langle x_i, Xy_j\rangle = 0$ for $i \neq j$ and $\langle x_i, Xy_i\rangle \geqslant a$ for all i. Let

$$A := \sum_{n\geqslant 1} a_n x_{n+1}x_n^* \quad \text{and} \quad B := \sum_{n\geqslant 1} y_n y_{n+1}^*$$

where $a_n = \langle Xy_n, x_n\rangle^{-1}$. The a_n's are bounded, so the series of A converges strongly.

Since $y_n \in (N'_n - N'_{n-1})\mathfrak{H}$, therefore $y_n y_{n+1}^* = N'_n y_n y_{n+1}^* {N'_n}^{\perp}$, and so $y_n y_{n+1}^* \in \operatorname{Alg}\mathfrak{N}$, hence $B \in \operatorname{Alg}\mathfrak{N}$. Likewise $A \in \operatorname{Alg}\mathfrak{N}$. Thus since the operator $U := \sum_{n\geqslant 2} x_n y_n^*$ factors as $U = AXB$, therefore U belongs to the ideal generated by X.

Now we shall show that W belongs to the stable ideal generated by U. Select a subsequence N''_n $(n \geqslant 0)$ of N'_n starting at $N''_0 = N'_0$ such that $U_n := U(N''_n - N''_{n-1})$ is a partial isometry with rank at least the rank of W_n for each $n \geqslant 1$, and let M''_n be the corresponding subsequence of M'_n's. Observe that by the construction of U, $U_n = (M''_{n-1} - M''_n)U_n(N''_n - N''_{n-1})$ and $U = \sum_{n\geqslant 1} U_n$.

By our original hypotheses, $M_1 < N_1$ and so we can pick two new points, $M_1 < M_0 < N_0 < N_1$ in $\mathfrak{N}$. For each $n \in \mathbb{N}$, the nests $(N''_n - N''_{n-1})\mathfrak{N}$ and $(N_n - N_{n-1})\mathfrak{N}$ are continuous. Thus by the Similarity Theorem, there are invertible operators T_n from $(N''_n - N''_{n-1})\mathfrak{H}$ to $(N_n - N_{n-1})\mathfrak{H}$ which map $(N''_n - N''_{n-1})\mathfrak{N}$ to $(N_n - N_{n-1})\mathfrak{N}$. Likewise there are invertible operators S_n mapping $(M''_{n-1} - M''_n)\mathfrak{N}$ onto $(M_{n-1} - M_n)\mathfrak{N}$. Since also $M''_0 < N''_0$, there is an invertible operator T_0 from $(N''_0 - M''_0)\mathfrak{H}$ to $(N_0 - M_0)\mathfrak{H}$ which maps $(N''_0 - M''_0)\mathfrak{N}$ to $(N_0 - M_0)\mathfrak{N}$.

Patching these operators together we obtain an invertible operator

$$T := \sum_{n \geqslant 0} T_n + \sum_{n \geqslant 1} S_n$$

on $\mathfrak{H}$ which maps $\mathfrak{N}$ to itself and such that $T^{-1} W_n T = S^{-1}_{n+1} W_n T_{n+1}$ for each $n \geqslant 1$. Since $\operatorname{rank}(T^{-1} W_n T) \leqslant \operatorname{rank}(U_n)$ we can factor

$$T^{-1} W_n T = S^{-1}_{n+1} W_n T_{n+1} = C_n U_n D_n$$

where $C_n = (M''_n - M''_{n+1}) C_n (M''_{n-1} - M''_n)$ and $D_n = (N''_n - N''_{n-1}) D_n (N''_{n+1} - N''_n)$.

Note that C_n and D_n belong to $\operatorname{Alg} \mathfrak{N}$. Also, because W_n and U_n are partial isometries, we can arrange that the norms of the operators C_n and D_n are uniformly bounded and so

$$W = \sum_{n \geqslant 1} W_n = \sum_{n \geqslant 1} T C_n U_n D_n T^{-1} = T(CUD)T^{-1}$$

where $C = \sum_{n \geqslant 1} C_n$ and $D = \sum_{n \geqslant 1} D_n$. Conjugation by T is an automorphism of $\operatorname{Alg} \mathfrak{N}$ and CUD belongs to the ideal generated by X since U belongs to that ideal. Thus W belongs to the stable ideal generated by X. ∎

Proof of Proposition 2.5. Let $K \in \mathfrak{K}^+_0 \cap \mathfrak{K}^-_0$ and let $\varepsilon > 0$. Recall from the definition of $\mathfrak{K}^+_0$ and $\mathfrak{K}^-_0$ that $M^\perp K$ and KN are compact for any $M > 0$ or $N < I$ in $\mathfrak{N}$. Thus we can choose sequences $M_n < N_n$ in $\mathfrak{N}$ with M_n decreasing to 0 and N_n increasing to I and such that $\|M_n K N_n\| < \varepsilon/2^n$ for all $n \geqslant 1$. (Simply choose the sequence $0 < N_n < I$ first and then, since KN_n is compact for each n, we can choose a suitable M_n.)

Also, for each n, $(M_n - M_{n+1}) K N^\perp_n$ is compact so choose a finite-rank operator $F_n = (M_n - M_{n+1}) F_n N^\perp_n$ within distance $\varepsilon/2^n$ of it. Thus

$$\begin{aligned} M_1 K = \sum_{n \geqslant 1} (M_n - M_{n+1}) K &= \sum_{n \geqslant 1} (M_n - M_{n+1}) F_n N^\perp_n \\ &+ \sum_{n \geqslant 1} (M_n - M_{n+1})(K - F_n) N^\perp_n + \sum_{n \geqslant 1} (M_n - M_{n+1}) K N_n. \end{aligned}$$

The last two sums converge in norm to values less than ε, while the first sum converges strongly, so that the second sum, which is equal to $\sum_{n \geqslant 1} F_n$, converges strongly. Write $F := \sum_{n \geqslant 1} F_n$ and observe that $\|M_1 K - F\| < 2\varepsilon$.

Let W_n be a finite-rank partial isometry with final space equal to the range of F_n and initial space contained in the range of $N_n - N_{n-1}$. Thus $W_n = (M_n - M_{n+1}) W_n (N_n - N_{n-1})$ and by Lemma 2.6, $W = \sum_{n \geqslant 1} W_n$ belongs to the stable ideal generated by X. Also $W^* F = \sum_{n \geqslant 1} (N_n - N_{n-1}) W^*_n F_n N^\perp_n$ and so belongs to

$\operatorname{Alg}\mathfrak{N}$. Thus, $F = WW^*F$ belongs to the stable ideal generated by X. Also, since $K \in \mathfrak{K}_0^+$, $M_1^\perp K$ is compact and so by Proposition 2.4, $F + M_1^\perp K$ belongs to the stable ideal generated by X. However, this last operator is within 2ε of K in norm and, since ε was arbitrary, K is a limit point of the stable ideal generated by X. Stable ideals are closed by definition, so K must also belong to this ideal. ∎

COROLLARY 2.7. *The only proper stable subideals of* $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$ *are* 0 *and* $\mathfrak{K}$.

Proof. If $K \in \mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$ and the infinum $\inf\limits_{0<M,N<I} \|MKN^\perp\|$ is equal to zero then K is the norm limit of terms $K - MKN^\perp$. Since these are of the form $KN + M^\perp KN^\perp$, they are compact and so K is compact. Thus, if an ideal of $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$ contains non-compact elements then they satisfy the hypothesis of Proposition 2.5, and the ideal must equal $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$. ∎

We shall continue in the spirit of Corollary 2.7 and find the list of all proper subideals of $\mathfrak{K}_0^+$. We shall develop this result in an analogous manner to our proof of Corollary 2.7, by first proving a proposition describing operators which generate stable ideals containing $\mathfrak{K}_0^+$.

PROPOSITION 2.8. *Suppose* $X \in \operatorname{Alg}\mathfrak{N}$ *and there is* $Q < I$ *in* $\mathfrak{N}$ *such that* $\inf\limits_{M>0} \|MXQ\| > a > 0$. *Then the stable ideal generated by* X *contains* $\mathfrak{K}_0^+$.

Just like in the proof of Proposition 2.5, we shall first prove a technical lemma:

LEMMA 2.9. *Let* X *be as in Proposition* 2.8 *and let* $M_n < N_n$ *be a sequence of projections in* $\mathfrak{N}$ *with* M_n *decreasing to* 0 *and* N_n *increasing to a projection* $N < I$. *Then the stable ideal generated by* X *contains any operator of the form* $\sum\limits_n W_n$ *where* $W_n = (M_n - M_{n+1})W_n(N_{n+1} - N_n)$ *is a finite rank partial isometry.*

Proof. The proof is quite similar to Lemma 2.6. Apply Lemma 2.3 to XQX^* to see that for each $M > 0$ and $k \in \mathbb{N}$ there is a $0 < M' < M$ such that $\operatorname{dist}((M - M')XQX^*(M - M'), \mathfrak{F}_k) > a^2$. It follows that for each $M > 0$ and $k \in \mathbb{N}$ there is a $0 < M' < M$ such that $\operatorname{dist}((M - M')XQ, \mathfrak{F}_k) > a$. Thus there is a sequence $M_n' < Q$ which decreases to zero and such that

$$\operatorname{dist}((M_{n-1}' - M_n')XQ, \mathfrak{F}_{4n-4}) > a$$

for all $n \geqslant 1$.

By Lemma 2.2 there are orthonormal sequences $x_n \in (M_{n-1}' - M_n')$ and $y_n \in Q$ such that $\langle x_i, Xy_j \rangle = 0$ when $i \neq j$ and $\langle x_i, Xy_i \rangle \geqslant a$. Pick a projection $N' \in \mathfrak{N}$ between Q and I and let $N_n' > Q$ $(n \geqslant 0)$ increase to N'. Choose unit vectors, z_n, in $N_n' - N_{n-1}'$ and let

$$A := \sum_{n\geqslant 1} a_n\, x_{n+1}x_n^* \quad \text{and} \quad B := \sum_{n\geqslant 1} y_n z_{n+1}^*$$

where $a_n := \langle Xy_n, x_n \rangle^{-1}$. Since the sequence a_n is bounded, the series for A converges strongly. Note that since

$$x_{n+1}x_n^* = M_n'\, x_{n+1}x_n^*\, {M'}_n^\perp \in \operatorname{Alg}\mathfrak{N},$$

it follows that $A \in \operatorname{Alg}\mathfrak{N}$. Since $B = QBQ^\perp$, B also belongs to $\operatorname{Alg}\mathfrak{N}$. The choice of vectors x_n and y_n ensures that we can factor: $AXB = U := \sum_i x_i z_i^*$, and so U belongs to the ideal generated by X.

As in Lemma 2.6, select subsequences M_n'' and N_n'' of M_n' and N_n', starting at M_0 and N_0 respectively, such that $U_n := U(N_n'' - N_{n-1}'') = (M_{n-1}'' - M_n'')U(N_n'' - N_{n-1}'')$ has at least the rank of W_n. Introduce two new points $M_0 < N_0$ between M_1 and N_1 and use the Similarity Theorem to get invertible maps, S_n, which map $(M_{n-1}''-M_n'')\mathfrak{N}$ to $(M_{n-1}-M_n)\mathfrak{N}$ for all $n \geqslant 1$, and T_n which map $(N_n''-N_{n-1}'')\mathfrak{N}$ to $(N_n-N_{n-1})\mathfrak{N}$ for all $n \geqslant 1$. Also find T_0 which maps $(N_0''-M_0'')\mathfrak{N}$ to $(N_0-M_0)\mathfrak{N}$ and patch these maps together to make an invertible operator

$$T := \sum_{n\geqslant 0} T_n + \sum_{n \geqslant 1} S_n$$

which maps $\mathfrak{N}$ to $\mathfrak{N}$.

Observe that $T^{-1}W_nT = S_{n+1}^{-1}W_nT_{n+1}$, which factors as $S_{n+1}^{-1}W_nT_{n+1} = C_nU_nD_n$, because $\operatorname{rank}(W_n) \leqslant \operatorname{rank}(U_n)$. Note that we can arrange that $C_n = (M_n'' - M_{n+1}'')C_n(M_{n-1}'' - M_n'') \in \operatorname{Alg}\mathfrak{N}$ and that $D_n = (N_n'' - N_{n-1}'')D_n(N_{n+1}'' - N_n'') \in \operatorname{Alg}\mathfrak{N}$, and that C_n and D_n have uniformly bounded norms. Thus $W = T(CUD)T^{-1}$, where $C := \sum_{n\geqslant 1} C_n$ and $D := \sum_{n\geqslant 1} D_n$ belong to $\operatorname{Alg}\mathfrak{N}$. Thus CUD belongs to the ideal generated by X, and W belongs to the stable ideal generated by X, since conjugation by T is an automorphism. ∎

Proof of Proposition 2.8. This proof is very similar to the proof of Proposition 2.5. Let $K \in \mathfrak{K}_0^+$ and let $\varepsilon > 0$. Choose $N > 0$ in $\mathfrak{N}$ such that $\|KN\| < \varepsilon$. Now let $M_n < N_n$ ($n \geqslant 0$) be two sequences in $\mathfrak{N}$, M_n decreasing to zero and N_n increasing to N. For each n, $(M_{n-1} - M_n)KN^\perp$ is compact, so find finite rank operators $F_n = (M_{n-1} - M_n)F_nN^\perp$ such that $\|(M_{n-1} - M_n)KN^\perp - F_n\| < \varepsilon/2^n$ for all $n \geqslant 1$. Now

$$\begin{aligned} M_1K = M_1KN + M_1KN^\perp &= M_1KN + \sum_{n\geqslant 1}(M_{n-1} - M_n)KN^\perp \\ &= M_1KN + \sum_{n\geqslant 1}(M_{n-1} - M_n)(K - F_n)N^\perp + \sum_{n\geqslant 1} F_n. \end{aligned}$$

Since all the sums other than $\sum_n F_n$ are known to converge at least in the strong operator topology, $F := \sum_{n\geqslant 1} F_n$ converges strongly, and differs from M_1K by less than 2ε.

For each n, let W_n be a finite-rank partial isometry with range space equal to the range projection of F_n, and initial space contained in the range of $N_{n+1} - N_n$. Thus $W_n = (M_{n-1} - M_n)W_n(N_{n+1} - N_n)$ and by Lemma 2.9, $W = \sum W_n$ belongs to the stable ideal generated by X. Also $W^*F = \sum_{n\geqslant 1}(N_{n+1} - N_n)W_n^*F_nN^\perp$ and so since $N_n \leqslant N$, $W^*F = NW^*FN^\perp \in \operatorname{Alg}\mathfrak{N}$. Thus $F = W(W^*F)$ belongs to the stable ideal generated by X. Also, since $K \in \mathfrak{K}_0^+$, $M_1^\perp K$ is compact and so by Proposition 2.4, $F + M_1^\perp K$ belongs to the stable ideal generated by X. However this last operator is within 2ε of K in norm and, since ε was arbitrary, K is a limit point of the stable ideal generated by X. Stable ideals are closed by definition, so K must also belong to this ideal. ∎

COROLLARY 2.10. *The only proper stable subideals of $\mathfrak{K}_0^+$ are $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$, $\mathfrak{K}$ and* 0.

Proof. Let $K \in \mathfrak{K}_0^+$ and consider the stable ideal it generates. If

$$\inf_{N>0} \|NKQ\| = 0 \quad \text{for all } Q < I \text{ in } \mathfrak{N},$$

then KQ is compact for all $Q < I$ and so $K \in \mathfrak{K}_0^+ \cap \mathfrak{K}^- = \mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$. Thus, if a stable subideal of $\mathfrak{K}_0^+$ is not contained in $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$ then it must contain some element which satisfies the hypotheses of Proposition 2.8, and hence be equal to $\mathfrak{K}_0^+$. On the other hand, the case of ideals which are contained in $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$ has already been dealt with in Corollary 2.7. ∎

The dual results hold for $\mathfrak{K}_0^-$, in which the ordering is reversed.

COROLLARY 2.11. *The only proper stable subideals of $\mathfrak{K}_0^-$ are $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$, $\mathfrak{K}$ and* 0.

The proof can be constructed by applying Corollary 2.10 to the nest algebra $\operatorname{Alg} \mathfrak{N}^\perp = (\operatorname{Alg} \mathfrak{N})^*$. The "plus ideals" in $\operatorname{Alg} \mathfrak{N}^\perp$ are in one-to-one correspondance with the corresponding "minus ideals" in $\operatorname{Alg} \mathfrak{N}$ by means of adjoints.

The proof of the next proposition is very similar to its precursors, and so we only sketch out its main points.

PROPOSITION 2.12. *Suppose $X \in \operatorname{Alg} \mathfrak{N}$ and $\inf\limits_{M>0} \|MXM\| > a > 0$. Then the stable ideal generated by X contains $\mathfrak{K}^+$.*

Proof. We can use Lemma 2.3 exactly as in Lemma 2.6 to show that, for any choice of a sequence (M_n) in $\mathfrak{N}$ that decreases to zero, the stable ideal generated by X contains all partial isometries of the form $W = \sum\limits_n W_n$ where W_n is a finite rank partial isometry mapping from $M_n - M_{n+1}$ into $M_{n+2} - M_{n+3}$.

Let $K \in \mathfrak{K}^+$ and $\varepsilon > 0$ be given. We can use a compactness argument to construct a sequence M_n that decreases to zero and such that $\|(M_n - M_{n+3})K(M_n - M_{n+3})\| < \varepsilon$ for each n. To see this, start by finding an open cover of $(0, I]$ by intervals $(G - L)$ with $L > 0$ and $\|(G - L)K(G - L)\| < \varepsilon$. From this, select a sequence of overlapping intervals $G_n - L_n$ with $L_n < G_{n+1} < L_{n-1} < G_n$ and such that G_n and L_n decrease to zero. Then pick the terms of (M_n) in threes, by choosing $M_{3n} > M_{3n+1} > M_{3n+2}$ to lie in the interval between L_{n-1} and G_n for each n. From this construction it is clear that each interval $(M_n - M_{n+3})$ is subordinate to one of the intervals $(G_i - L_i)$ and so has the desired properties.

The strips $(M_{n+2} - M_{n+3})KM_n^\perp$ are compact so build $F := \sum\limits_n F_n$ from finite rank operators $F_n = (M_{n+2} - M_{n+3})F_n M_n^\perp$, where each F_n has been chosen to be sufficiently close to the nth strip that $\|K - F\| < 3\varepsilon$. Take W_n to map from $M_n - M_{n+1}$ onto the range of F_n. Then $W := \sum\limits_n W_n$ is in $\operatorname{Alg} \mathfrak{N}$, as is W^*F, and so $F = W(W^*F)$ is in the stable ideal generated by X. Since ε was arbitrary, K belongs to this ideal. ∎

We use Proposition 2.12 to classify all the stable subideals of $\mathfrak{K}^+$ and $\mathfrak{K}^-$. The proof is directly analogous to the proofs of Corollaries 2.10, 2.11, and 2.7, and is left to the reader.

Corollary 2.13. *The only proper stable subideals of* $\mathfrak{K}^+$ *(respectively* $\mathfrak{K}^-$*) are* $\mathfrak{K}_0^+$, $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$, $\mathfrak{K}$ *and* 0 *(respectively* $\mathfrak{K}_0^-$, $\mathfrak{K}_0^+ \cap \mathfrak{K}_0^-$, $\mathfrak{K}$ *and* 0*).*

In order to complete the classification of the stable ideals of compact character, we need to understand the ideals obtained as sums of $\mathfrak{K}_0^+$, $\mathfrak{K}_0^-$, $\mathfrak{K}^+$, and $\mathfrak{K}^-$. It turns out that such ideals account for the remainder of the stable ideals of compact character. The next proposition will be used to prove this assertion by decomposing the remaining ideals as sums.

Lemma 2.14. *If* $K \in \operatorname{Alg}\mathfrak{N}$ *is of compact character, then* $K = K^+ + K^-$ *where* $K^+ \in \mathfrak{K}^+$ *and* $K^- \in \mathfrak{K}^-$.

For the proof of this lemma we need the following result from [3]:

Theorem 2.15. ([3]) *Let* A *be an operator and let* B_n *be a sequence of operators which converge strong-$*$ to zero. Then, given* $\varepsilon > 0$*, there is an* n_0 *such that, for all* $n \geqslant n_0$,

$$\|A + B_n\| \leqslant \max\{\|A\|, \|A\|_{\text{ess}} + \|B_n\|\} + \varepsilon.$$

Proof of Lemma 2.14. Suppose K is of compact character. Pick sequences M_n and N_n ($n \in \mathbb{N}$) in $\mathfrak{N}$, respectively decreasing to zero and increasing to I. We shall construct a subsequence N_{n_k} of N_n such that if

$$K_r := \sum_{k=1}^{r} (M_{k-1} - M_k) K N_{n_k}^{\perp}$$

then $\|K_r\| \leqslant 2\|K\| + 1 - 1/2^{r-1}$. (In the last sum, set $M_0 = I$.)

We shall construct the subsequence, n_k, inductively starting with $n_1 = 1$. Supposing $n_1, \ldots, n_{r-1}$ have been chosen, observe that $(M_{r-1} - M_r) K N_n^{\perp}$ converges strong-$*$ to zero in n and thus, by Theorem 2.15, we can choose $n_r > n_{r-1}$ such that

$$\|K_r\| = \|K_{r-1} + (M_{r-1} - M_r) K N_{n_r}^{\perp}\| \leqslant \max\{\|K_{r-1}\|, \|K_{r-1}\|_{\text{ess}} + \|K\|\} + \frac{1}{2^{r-1}}.$$

Now observe that $\|K_{r-1}\|_{\text{ess}} \leqslant \|K\|$, since

$$M_{r-1}^{\perp} K = K_{r-1} + \sum_{k=1}^{r-1} (M_{k-1} - M_k) K N_{n_k}$$

and, since K is of compact character, the last term is compact, so that $\|K\| \geqslant \|M_{r-1}^{\perp} K\|_{\text{ess}} = \|K_{r-1}\|_{\text{ess}}$. Thus, by the induction hypothesis,

$$\|K_r\| \leqslant 2\|K\| + 1 - \frac{1}{2^{r-2}} + \frac{1}{2^{r-1}}$$

as required.

We have shown that the sequence K_r is bounded, so let K^- be a weak limit point. We claim that $K^- \in \mathfrak{K}^-$. For, if $N < I$, there is an r_0 such that $N_{n_r} > N$ for all $r \geqslant r_0$ and so $K_r N$ is constant and compact for all $r \geqslant r_0$. Thus $K^- N$ is compact. Since N was arbitrary, K^- belongs to $\mathfrak{K}^-$. A similar argument shows $K^+ := K - K^-$ belongs to $\mathfrak{K}^+$ and we are finished. ∎

THEOREM 2.16. *The stable ideals of compact character are precisely the ideals listed in Figure* 1.

Proof. It is not evident *a priori* that those ideals listed which are the sum of two stable ideals are themselves stable ideals. They are certainly automorphism invariant ideals but we need to show that they are norm closed. However, by Lemma 2.14, $\mathfrak{K}^+ + \mathfrak{K}^-$ is the set all operators of compact character, which is a closed set. The other ideals in question can be expressed as the intersection of $\mathfrak{K}^+ + \mathfrak{K}^-$ with one or both of the closed sets:

$$\{X \in \operatorname{Alg}\mathfrak{N} : \inf_{N>0} \|XN\| = 0\} \quad \text{and} \quad \{X \in \operatorname{Alg}\mathfrak{N} : \inf_{M<I} \|M^\perp X\| = 0\}.$$

Since the list of ideals in Figure 1 is a complete lattice of ideals, it will suffice to show that each singly generated stable ideal belongs to this list (because each stable ideal of compact character is a join of singly generated stable ideals of compact character). Let K be of compact character and let $\mathfrak{I}$ be the stable ideal it generates.

Suppose that for each $Q < I$ in $\mathfrak{N}$, $\inf_{N>0} \|NKQ\| = 0$. Then for each $Q < I$, KQ is a norm limit of compact operators, $N^\perp KQ$, and so is compact. Thus K belongs to $\mathfrak{K}^-$ and $\mathfrak{I}$ is one of the ideals listed in Corollary 2.13.

So we next suppose that the infimum above is non-zero for some $Q < I$. By Lemma 2.14 we can write $K = K^+ + K^-$ with $K^+ \in \mathfrak{K}^+$ and $K^- \in \mathfrak{K}^-$. We claim that K^+ and K^- belong to $\mathfrak{I}$. To see this, fix on a projection $P \in \mathfrak{N}$ strictly between 0 and I. Since $K^+ \in \mathfrak{K}^+$, therefore $K^+P^\perp \in \mathfrak{K}_0^+$ and since, by Lemma 2.8, $\mathfrak{I}$ contains $\mathfrak{K}_0^+$, we see that $K^+P^\perp \in \mathfrak{I}$. On the other hand, $K^+P = KP - K^-P$, and K^-P is compact, and so belongs to $\mathfrak{I}$. Thus both $K^+P^\perp$ and K^+P belong to $\mathfrak{I}$, so $\mathfrak{I}$ contains K^+ and hence also contains $K^- = K - K^+$.

It folows that $\mathfrak{I}$ is the join of the stable ideals generated by K^+ and K^-. But then $\mathfrak{I}$ is the join of two of the ideals listed in Corollary 2.13, and we saw in the first paragraph that this means $\mathfrak{I}$ is in fact the sum of two such ideals. The result follows. ∎

3. DIAGONAL SEMINORM FUNCTIONS

In this section we introduce the technical machinery which is used to construct and classify the remaining stable ideals. In contrast with the ideals of compact character, we will now study ideals which are described by local conditions (applied at each point of $\mathfrak{N}$). Still, at some level, the asymptotic conditions which we now start to study reflect the same spectrum of behavior as the ideals of compact character exhibited in the large scale at the endpoints 0 and I.

DEFINITION 3.1. For $X \in \operatorname{Alg}\mathfrak{N}$ and $N \in \mathfrak{N}$ define

$$i_N^+(X) = \begin{cases} \lim\limits_{M\downarrow N} \|(M-N)X(M-N)\|, & \text{if } N < I, \\ 0, & \text{if } N = I, \end{cases}$$

and

$$i_N^-(X) = \begin{cases} \lim\limits_{M\uparrow N} \|(N-M)X(N-M)\|, & \text{if } N > 0, \\ 0, & \text{if } N = 0, \end{cases}$$

where limits are taken as $M \to N$ in $\mathfrak{N}$ with the order topology.

DEFINITION 3.2. For $X \in \operatorname{Alg}\mathfrak{N}$ and $N \in \mathfrak{N}$ define

$$e_N^+(X) = \begin{cases} \lim\limits_{M\downarrow N} \sup\limits_{N<L<M} \|(M-L)X(M-L)\|_{\text{ess}}, & \text{if } N < I, \\ 0, & \text{if } N = I, \end{cases}$$

and

$$e_N^-(X) = \begin{cases} \lim\limits_{M\uparrow N} \sup\limits_{M<L<N} \|(L-M)X(L-M)\|_{\text{ess}}, & \text{if } N > 0, \\ 0, & \text{if } N = 0. \end{cases}$$

DEFINITION 3.3. For $X \in \operatorname{Alg}\mathfrak{N}$ and $N \in \mathfrak{N}$ define

$$j_N(X) = \begin{cases} \lim\limits_{M\downarrow N, L\uparrow N} \|(M-L)X(M-L)\|, & \text{if } 0 < N < I, \\ i_N^+(X), & \text{if } N = 0, \\ i_N^-(X), & \text{if } N = I. \end{cases}$$

It is straightforward to check that for each fixed $N \in \mathfrak{N}$, each of the formulas we have just defined give *submultiplicative seminorms* on $\operatorname{Alg}\mathfrak{N}$. (See, e.g., [26].)

REMARK 3.4. The values of $i_N^\pm(X)$ and $j_N(X)$ are not altered if the operator norm is replaced by the essential norm in their respective definitions.

For each $N \in \mathfrak{N}$ the seminorms:

$$0,\ e_N^+,\ i_N^+,\ j_N$$

are called the *positive elementary seminorms at* N and

$$0,\ e_N^-,\ i_N^-,\ j_N$$

are called the *negative elementary seminorms* at N. Although we have been talking about a seminorm at a point, what we often really consider is a seminorm-valued function, such as $N \mapsto e_N^-$. In this spirit we make the following definition.

DEFINITION 3.5. Let S_N^+ (respectively S_N^-) denote the set of the four positive (respectively negative) elementary seminorms at N. Suppose that $N \mapsto a_N$ is a map $\mathfrak{N} \to \bigcup\limits_{N\in\mathfrak{N}} S_N^+$ such that $a_N \in S_N^+$ for all $N \in \mathfrak{N}$. Likewise suppose that $b_n \in S_N^-$ for all N. Then the formula

$$\|X\|_N := \max\{a_N(X), b_N(X)\}$$

defines a seminorm-valued function $N \mapsto \|\cdot\|_N$ on $\mathfrak{N}$. Such a function is called a *diagonal seminorm function* (dsf).

It is very important to note that the elementary seminorms a_N and b_N will typically vary with N. Nevertheless, by a slight abuse of notation we shall write $\|\cdot\|_N$ both for the diagonal seminorm function $N \mapsto \|\cdot\|_N$ as well as for its particular values. We shall also write $a_N b_N$ for the seminorm

$$(a_N b_N)(X) = \max\{a_N(X), b_N(X)\}$$

used in the definition above.

EXAMPLE 3.6. The diagonal seminorm function $\|\cdot\|_N := i_N^+ i_N^-$ is the seminorm used by Ringrose ([26]) to describe the Jacobson radical of a nest algebra. We call this diagonal seminorm function the *Ringrose seminorm.* Here the positive and negative parts of the dsf($a_N = i_N^+$ and $b_N = i_N^-$) are all of the same type as N varies. Ringrose proved that the Jacobson radical of $\operatorname{Alg}\mathfrak{N}$ is the *kernel* of $\|\cdot\|_N$; i.e. the set of $X \in \operatorname{Alg}\mathfrak{N}$ for which $\|X\|_N = 0$ for all $N \in \mathfrak{N}$.

EXAMPLE 3.7. Let F be a finite subset of $\mathfrak{N} \setminus \{0\}$. For each $N \in \mathfrak{N}$, let

$$\|\cdot\|_N^F := \begin{cases} i_N^+ i_N^-, & \text{if } N \in F, \\ j_N, & \text{if } N \notin F. \end{cases}$$

This describes a family of diagonal seminorm functions indexed by subsets of $\mathfrak{N}$. The kernel of $\|\cdot\|_N^F$ is strictly smaller than the Jacobson radical of $\operatorname{Alg}\mathfrak{N}$. A nilpotent ideal of the form $MB(\mathfrak{H})M^\perp$ ($M \in \mathfrak{N}$) only belongs to the kernel of this diagonal seminorm function if $M \in F$.

The next lemma will be very useful since it describes how dsf's transform under the action of an automorphism. However, its proof is quite elementary, and is left to the reader. (See, e.g., [17].)

LEMMA 3.8. *Let $X \in \operatorname{Alg}\mathfrak{N}$ and let T be an invertible operator inducing an order isomorphism θ of $\mathfrak{N}$ onto itself. Then if $G > L$ in $\mathfrak{N}$, $E = G - L$ and $\widehat{E} = \theta(G) - \theta(L)$ then $\widehat{E}\,TXT^{-1}\,\widehat{E}$ is equal to*

$$\widehat{E}\,T\,EXE\,T^{-1}\,\widehat{E}.$$

Hence, if a_N is an elementary seminorm function then

$$k^{-1}a_N(X) \leqslant a_{\theta(N)}(TXT^{-1}) \leqslant k a_N(X)$$

where k is the condition number $\|T\|\,\|T^{-1}\|$ of T. Also, if T is a compact perturbation of a unitary then

$$a_N(X) = a_{\theta(N)}(TXT^{-1}).$$

The diagonal seminorm functions are ordered by pointwise comparison (i.e. $\|\cdot\|_N \leqslant |\cdot|_N$ if $\|X\|_N \leqslant |X|_N$ for all $X \in \operatorname{Alg}\mathfrak{N}$ and $N \in \mathfrak{N}$). Figure 2 shows the orderings at a fixed N. The lower nine elements of the lattice are just the product order on the product of two linearly ordered three-element sets. The lattice as a whole is a complete, distributive lattice and so the family of all diagonal seminorm functions with the pointwise ordering is also such a lattice.

If N is fixed and a_N is a positive elementary seminorm at N and b_N is a negative elementary seminorm at N then, provided neither a_N nor b_N is equal to j_N, one can recover a_N and b_N from the values of $a_N b_N$. (To see this, note that if X is an arbitrary test operator of the form $X = N^\perp X N^\perp$, then $(a_N b_N)(X) =$

$a_N(X)$. One then substitutes in such an X for which, in addition, one arranges that $e_N^+(X) = 0$ and $i_N^+(X) = 1$, to detect whether $a_N = i_N^+$. Likewise, substituting an X for which $e_N^+(X) = 1$ will detect whether $a_N = 0$.)

Thus we can define

$$(a_N b_N)^+ = \begin{cases} j_N & \text{if } a_N b_N = j_N, \\ a_N & \text{otherwise,} \end{cases}$$

and

$$(a_N b_N)^- = \begin{cases} j_N & \text{if } a_N b_N = j_N, \\ b_N & \text{otherwise.} \end{cases}$$

If $\|\cdot\|_N$ is a diagonal seminorm function, define $\|\cdot\|_N^+ = (\|\cdot\|_N)^+$ for each N, and likewise for $\|\cdot\|_N^-$. Of course $\|\cdot\|_N^+$ and $\|\cdot\|_N^-$ are themselves diagonal seminorm functions.

Figure 2. The diagonal seminorm functions

Definition 3.9. Let $\mathfrak{F}$ be a family of diagonal seminorm functions. We shall call $\mathfrak{F}$ a *stable family of seminorms* if, whenever $\|\cdot\|_N$ and $|\cdot|_N$ belong to $\mathfrak{F}$ and θ is an order isomorphism of $\mathfrak{N}$ onto itself, then the diagonal seminorm functions defined by

$$\|\cdot\|_N \wedge |\cdot|_N \quad \text{and} \quad \|\cdot\|_{\theta(N)}$$

also belong to $\mathfrak{F}$.

The next proposition shows how stable families of seminorms induce stable ideals in $\operatorname{Alg}\mathfrak{N}$. The main result of this paper is that the converse to Proposition 3.10 holds — that all stable ideals (except the ideals of compact character) arise from some stable family of seminorms in the way shown. In Lemma 3.15 we shall show how to obtain the appropriate stable family of seminorms from a given stable ideal.

Proposition 3.10. *Let $\mathfrak{F}$ be a stable family of seminorms and let $\mathfrak{I}$ be the set of operators Y in $\operatorname{Alg}\mathfrak{N}$ such that, for each $\varepsilon > 0$ there is a diagonal seminorm function, $\|\cdot\|_N$, in $\mathfrak{F}$ with $\|Y\|_N < \varepsilon$ for all $N \in \mathfrak{N}$. Then $\mathfrak{I}$ is a stable ideal of $\operatorname{Alg}\mathfrak{N}$.*

Proof. Suppose Y_1, Y_2 are in $\mathfrak{I}$, let $\varepsilon > 0$ and let $\|\cdot\|_N$, $|\cdot|_N$ be diagonal seminorms in $\mathfrak{F}$ making respectively Y_1 and Y_2 less than $\varepsilon/2$. Then letting

$$|||\cdot|||_N = \|\cdot\|_N \wedge |\cdot|_N$$

it follows that for any $N \in \mathfrak{N}$

$$|||Y_1 + Y_2|||_N \leqslant |||Y_1|||_N + |||Y_2|||_N \leqslant \|Y_1\|_N + |Y_2|_N < \varepsilon.$$

Thus $Y_1 + Y_2 \in \mathfrak{I}$. If $Y \in \mathfrak{I}$ and $X \in \operatorname{Alg}\mathfrak{N}$, to see that XY (or YX) is in $\mathfrak{I}$, simply take a diagonal seminorm in $\mathfrak{F}$ such that $\|Y\|_N < \varepsilon/\|X\|$ for all N and note that $\|XY\|_N \leqslant \|X\|\,\|Y\|_N$. So $\mathfrak{I}$ is an ideal.

If S is an invertible operator implementing an automorphism of $\operatorname{Alg}\mathfrak{N}$, let $\theta = \theta_S$. For any diagonal seminorm

$$\|S^{-1}XS\|_{\theta(N)} \leqslant k\|X\|_N$$

for all N, where k is the condition number of S. To see that $S^{-1}XS \in \mathfrak{I}$, simply find a diagonal seminorm function, $\|\cdot\|_N$, in $\mathfrak{F}$ such that $\|X\|_N < \varepsilon/k$ for all $N \in \mathfrak{N}$. Finally, to see that $\mathfrak{I}$ is closed, let X be a limit point of $\mathfrak{I}$. Given $\varepsilon > 0$, find a $Y \in \mathfrak{I}$ such that $\|X - Y\| < \varepsilon/2$. Find a diagonal seminorm function, $\|\cdot\|_N \in \mathfrak{F}$, such that $\|Y\|_N < \varepsilon/2$ for all N. Then $\|X\|_N \leqslant \|Y\|_N + \|X - Y\| < \varepsilon$. ∎

Example 3.11. If the family $\mathfrak{F}$ contains only the Ringrose seminorm, then it is trivially a stable family. The stable ideal arising from this stable family by the construction of Proposition 3.10 coincides with the kernel of the Ringrose seminorm, which, as we have observed, is the Jacobson radical of $\operatorname{Alg}\mathfrak{N}$.

Remark 3.12. If $\mathfrak{F}$ is a stable family then the set of $X \in \operatorname{Alg}\mathfrak{N}$ for which there is $\|\cdot\|_N \in \mathfrak{F}$ such that $\|X\|_N = 0$ for all N is an automorphism invariant ideal. However, it need not be closed, which is the reason we are required to use the slightly more technically complex construction of Proposition 3.10. The next example illustrates this phenomenon.

Example 3.13. Let $\mathfrak{F}$ be the set of diagonal seminorms $\|\cdot\|_N^F$ from Example 3.7, as F ranges over all finite subsets of $\mathfrak{N}$. Let $\mathfrak{I}$ be the automorphism invariant ideal this gives rise to by the construction from the last remark. We claim that $\mathfrak{I}$ is not norm closed.

For if N_n is an increasing sequence in $\mathfrak{N}$ and

$$X_n = (N_n - N_{n-1})X_n(N_{n+1} - N_n)$$

is chosen so that $j_{N_n}(X_n)$ is always greater than 0 while $\|X_n\| \to 0$, then $X = \sum_{n\geqslant 1} X_n$ is in the norm closure of $\mathfrak{I}$ (because each X_n belongs to $\mathfrak{I}$) but clearly does not itself belong to $\mathfrak{I}$ (because $j_N(X) > 0$ for infinitely many values of N, and $\|X\|_N^F = j_N(x)$ for all but finitely many values of N). Of course X, and all the X_n, belong to the ideal given by Proposition 3.10, which is, in fact, the closure of $\mathfrak{I}$.

Definition 3.14. If $X \in \operatorname{Alg}\mathfrak{N}$ and $a > 0$, we say that the dsf, $\|\cdot\|_N$, is the *greatest seminorm for X and a*, if it is the largest diagonal seminorm function for which

$$\|X\|_N < a \quad \text{for all } N \in \mathfrak{N}.$$

We shall say $\|\cdot\|_N$ is *a greatest seminorm* if there are $X \in \operatorname{Alg}\mathfrak{N}$ and $a \geqslant 0$ for which it is the greatest seminorm.

Our ultimate goal is to show that all stable ideals (except the ones of compact character) arise from stable families of seminorms by means of the construction of Proposition 3.10. In the next lemma we show how a stable ideal gives rise to a stable family of seminorms in a natural way. In Theorem 5.7 we shall show that every stable ideal (not of compact character) is recovered from its associated stable family.

LEMMA 3.15. *Let $\mathfrak{I}$ be a stable ideal and let $\mathfrak{F}$ be the set of the greatest seminorms arising from $X \in \mathfrak{I}$ and $a > 0$. Then $\mathfrak{F}$ is a stable family of seminorms.*

Proof. Let $\|\cdot\|_N$ be a diagonal seminorm in $\mathfrak{F}$ and let θ be an order isomorphism of $\mathfrak{N}$ onto itself. We must show that $\|\cdot\|_{\theta(N)}$ is also in $\mathfrak{F}$. By the Similarity Theorem we can find an invertible operator, T, implementing θ which is the sum of a unitary and a compact operator. Then

$$\|TXT^{-1}\|_{\theta(N)} = \|X\|_N$$

for all $X \in \operatorname{Alg}\mathfrak{N}$ and $N \in \mathfrak{N}$ since, by Lemma 3.8, the same is true for all of the elementary seminorms composing it. Now since $\|\cdot\|_N \in \mathfrak{F}$, then it is the greatest seminorm for some $a > 0$ and $X \in \mathfrak{I}$. Since $\mathfrak{I}$ is stable, TXT^{-1} belongs to $\mathfrak{I}$ and clearly $\|\cdot\|_{\theta(N)}$ is the greatest seminorm for a and TXT^{-1}. Thus $\|\cdot\|_{\theta(N)}$ is also in $\mathfrak{F}$.

Now suppose that $\|\cdot\|_N$ and $|\cdot|_N$ are the greatest seminorms respectively for X and a, and for Y and b. We must show that $\|\cdot\|_N \wedge |\cdot|_N$ is a greatest diagonal seminorm. It is a fact that has been thoroughly exploited in [18] and [23] — and which is described in both these papers — that there are core projections F and $F^\perp$ such that

$$(G-L)F \quad \text{and} \quad (G-L)F^\perp$$

are both non-zero for all $G > L$ in $\mathfrak{N}$. Furthermore, there are operators A, B, C and D in $\operatorname{Alg}\mathfrak{N}$ such that $AB = F$, $CD = F^\perp$ and $BA = DC = I$. In fact, these operators can be taken to be compact perturbations of partial isometries whose initial and final projections are each one of $F, F^\perp$ or I. Thus, as in the last paragraph,

$$|||AXB|||_N = |||X|||_N \quad \text{and} \quad |||CYD|||_N = |||Y|||_N$$

for any diagonal seminorm $|||\cdot|||_N$.

Let $|||\cdot|||_N$ be the greatest seminorm for $\frac{1}{a}AXB + \frac{1}{b}CYD$ and 1. We shall show that $|||\cdot|||_N = \|\cdot\|_N \wedge |\cdot|_N$ and so, since $|||\cdot|||_N$ belongs to $\mathfrak{F}$, the desired result will follow.

Note that $AXB = F(AXB)F$ and $CYD = F^\perp(CYD)F^\perp$ so that for each $N \in \mathfrak{N}$:

$$\begin{aligned}\max\{\tfrac{1}{a}|||X|||_N, \tfrac{1}{b}|||Y|||_N\} &= \max\{\tfrac{1}{a}|||AXB|||_N, \tfrac{1}{b}|||AYB|||_N\} \\ &= |||\tfrac{1}{a}AXB + \tfrac{1}{b}CYD|||_N < 1.\end{aligned}$$

Thus, $|||X|||_N < a$ and $|||Y|||_N < b$ for all N and so $|||\cdot|||_N \leqslant \|\cdot\|_N \wedge |\cdot|_N$. The reverse inequality follows on observing that $\|\cdot\|_N \wedge |\cdot|_N$, evaluated at $\frac{1}{a}AXB + \frac{1}{b}CYD$, is less than 1 for all N. Thus, $\|\cdot\|_N \wedge |\cdot|_N$ belongs to $\mathfrak{F}$. ∎

We have seen how to obtain a stable family of seminorms from a stable ideal $\mathfrak{I}$. It is clear that the stable ideal obtained from this family by Proposition 3.10 contains $\mathfrak{I}$. The next two sections will be devoted to proving the reverse inclusion.

We conclude this present section with a technical lemma, which characterizes the greatest seminorm functions among all diagonal seminorm functions.

LEMMA 3.16. *The* dsf, $\|\cdot\|_N$*, is a greatest diagonal seminorm if and only if it enjoys the following lower semicontinuity property: that* $\|\cdot\|_N^- = 0$ *whenever there is a sequence* $N_n \uparrow N$ *with* $\|\cdot\|_{N_n} \neq j_{N_n}$ *for all* n*, and* $\|\cdot\|_N^+ = 0$ *whenever there is a sequence* $N_n \downarrow N$ *with* $\|\cdot\|_{N_n} \neq j_{N_n}$ *for all* n*.*

Proof. Suppose first that $\|\cdot\|_N$ is the greatest seminorm for $X \in \operatorname{Alg}\mathfrak{N}$ and $a > 0$. Suppose that N_n increases to N (the case of decreasing N_n is directly analogous). Now if $\|\cdot\|_{N_n} \neq j_{N_n}$ for all n then by maximality, $j_{N_n}(X) \geqslant a$ for all n. Thus, choosing M_n in $\mathfrak{N}$ with $N_n < M_n < N_{n+1}$ for each n it follows by Remark 3.4 that

$$\|(M_{n+1} - M_n)X(M_{n+1} - M_n)\|_{\text{ess}} \geqslant j_{N_{n+1}}(X) \geqslant a$$

for all n. Thus $e_N^-(X) \geqslant a$ and so, since e_N^- is the smallest non-zero negative elementary seminorm at N, $\|\cdot\|_N^-$ must be equal to zero.

On the other hand, suppose that $\|\cdot\|_N$ has the limiting behavior described. We shall construct an operator, X, so that $\|\cdot\|_N$ is the greatest seminorm for X and 1. The assumptions on $\|\cdot\|_N$ imply that the set $S = \{N : \|\cdot\|_N \leqslant i_N^+ i_N^-\}$ is closed, so we may decompose $\mathfrak{N} \setminus S$ as the union of disjoint open order intervals (L_n, G_n).

It is routine to construct contractions $X_n = (G_n - L_n)X_n(G_n - L_n) \in \operatorname{Alg}\mathfrak{N}$ that satisfy the following criteria:

(i) $\|X_n\| = \|X_n\|_{\text{ess}} = 1$;

(ii) $j_N(X_n) = 0$ for all $G_n > N > L_n$;

(iii) In the case that $\|\cdot\|_{L_n}^+ = 0$ then $e_{L_n}^+(X_n) = 1$, in the case that $\|\cdot\|_{L_n}^+ = e_{L_n}^+$ then $e_{L_n}^+(X_n) = 0$ and $i_{L_n}^+(X_n) = 1$, and in the case that $\|\cdot\|_{L_n}^+ = i_{L_n}^+$ then $i_{L_n}^+(X_n) = 0$;

(iv) In the case that $\|\cdot\|_{G_n}^- = 0$ then $e_{G_n}^-(X_n) = 1$, in the case that $\|\cdot\|_{G_n}^- = e_{G_n}^-$ then $e_{G_n}^-(X_n) = 0$ and $i_{G_n}^-(X_n) = 1$, and in the case that $\|\cdot\|_{G_n}^- = i_{G_n}^-$ then $i_{G_n}^-(X_n) = 0$.

To see this, let M_i ($i \in \mathbb{Z}$) be a sequence which increases from L_n to G_n. We shall build X_n as the sum of partial isometries T_i of the form $T_i = (M_{i+1} - M_i)T_i(M_{i+3} - M_{i+2})$, which will ensure that $j_N(X_n) = 0$ for all $G_n > N > L_n$. Choose the T_i as follows: Let T_0 be an infinite rank partial isometry, so that X_n will satisfy the first condition. Take the T_i ($i < 0$) to be infinite rank if $\|\cdot\|_{L_n}^+ = 0$, take them to be finite rank if $\|\cdot\|_{L_n}^+ = e_{L_n}^+$ and take them to be zero if $\|\cdot\|_{L_n}^+ = i_{L_n}^+$. Choose T_i ($i > 0$) to be infinite rank, finite rank, or zero according to the corresponding conditions on $\|\cdot\|_{G_n}^-$.

We shall show that $\|\cdot\|_N$ is the greatest diagonal seminorm function for

$$X := \sum_n X_n + \Big(\sum_n G_n - L_n\Big)^{\perp} \quad \text{and } 1.$$

First, consider a value of N that is not in S. Then N belongs to one of the open order intervals (L_n, G_n), and so $j_N(X) = 0$. Thus, j_N is the largest diagonal seminorm less than 1 at that value of N.

Next consider an N in S. We shall check that $\|X\|_N^+$ is the largest positive elementary seminorm which measures X as less than 1 at N. By symmetry, the same will hold for $\|\cdot\|_N^-$ and the result will follow.

There are three cases to consider. First, suppose that S contains the interval $[N, M)$ for some $M > N$. Then $(M-N)X(M-N) = (M-N)$ and so $e_N^+(X) = 1$, so at N the only positive elementary seminorm, a_N, for which $a_N(X) < 1$ is $a_N = 0$. Also, since N is a limit from above of elements of S, it follows from the hypotheses on $\|\cdot\|_N$ that $\|\cdot\|_N^+$ is also zero at this value of N. Next, if (N, M) lies in $\mathfrak{N} \setminus S$ for some $M > N$ then $N = L_n$ for some n and by the construction of X, $\|\cdot\|_N^+$ is the largest positive elementary seminorm making X less than 1 at N. Finally, if neither of these happens then there must be a sequence L_{n_k} decreasing to N. The condition on the essential norms of X_n ensures that $e_N^+(X) = 1$ in this case and so the only positive elementary seminorm, a_N, for which $a_N(X) < 1$ is $a_N = 0$. The hypotheses on $\|\cdot\|_N$ also require that $\|X\|_N^+ = 0$ since N is the limit of a decreasing sequence in S, and so we are done. ∎

REMARK 3.17. We have shown that if $\|\cdot\|_N$ is a greatest seminorm function then

$$S := \{N : \|\cdot\|_N = j_N\}$$

is open. We can list its components as (L_n, G_n). Then $\|\cdot\|_{L_n}^+$ can be any positive elementary seminorm except j_{L_n} and $\|\cdot\|_{G_n}^-$ can be any negative elementary seminorm except j_{G_n}. All other values of $\|\cdot\|_N^+$ and $\|\cdot\|_N^-$ in $\mathfrak{N} \setminus S$ are zero. So in fact the greatest diagonal seminorm functions, which are the dsf 's which we will be most interested in, have quite a constrained structure.

4. SOME SUB-IDEALS OF STABLE IDEALS

The aim in Section 5 is to show that any given stable ideal coincides with the ideal obtained by Proposition 3.10 when using the stable family constructed in Lemma 3.15. As has been observed, the original ideal is necessarily contained in this derived ideal. The crux of the problem is to show the reverse inclusion.

The way this is done is to suppose we have a dsf, $\|\cdot\|_N$, and an operator $Y \in \operatorname{Alg}\mathfrak{N}$ such that $\|Y\|_N$ is small for all $N \in \mathfrak{N}$. The dsf is supposed to be the greatest seminorm for some $X \in \mathfrak{I}$ (the original, given ideal), and $a > 0$. Then one shows that Y belongs to, or at least, is close to, the stable ideal generated by X. The information available to us about X is that at each $N \in \mathfrak{N}$, we know the largest positive and negative elementary seminorms a_n and b_n satisfying $a_N(X), b_N(X) < a$.

In other words (looking at the immediate successors of a_N and b_N) for each positive or negative elementary seminorm s_N we know the set

$$S = \{N : s_N(X) \geqslant a\}.$$

The purpose of this section is to show that useful standard ideals which can be described in terms of S are contained in the stable ideal generated by X.

We start by studying an ideals, $\mathfrak{R}_0$ which will play an analogous role among the ideals of non-compact character as the ideal of compact operators did in the study of the ideals of compact character. In particular, $\mathfrak{R}_0$ is the smallest stable ideal not of compact character (Corollary 4.3). This ideal was first studied by Erdos in [11].

DEFINITION 4.1. Let $\mathfrak{R}_0$ be the set of operators $X \in \operatorname{Alg}\mathfrak{N}$ for which $j_N(X) = 0$ for all N.

PROPOSITION 4.2. *Suppose $X \in \operatorname{Alg}\mathfrak{N}$ is not of compact character. Then the stable ideal generated by X contains $\mathfrak{R}_0$.*

Proof. Let $\varepsilon > 0$ and let $Y \in \mathfrak{R}_0$. Since $\mathfrak{N}$ is compact, there is an increasing sequence of projections $0 = N_0 < N_1 < \cdots < N_n = I$ in $\mathfrak{N}$ such that

$$\|(N_i - N_{i-2})Y(N_i - N_{i-2})\| < \varepsilon$$

for each i. Let $Y' := \sum_{i=1}^{n-1} (N_i - N_{i-1})YN_{i+1}^{\perp}$ and note that

$$\begin{aligned}\|Y - Y'\| &\leqslant \Big\| \sum_i (N_{2i+2} - N_{2i})Y(N_{2i+2} - N_{2i}) \Big\| \\ &\qquad + \Big\| \sum_i (N_{2i+2} - N_{2i+1})Y(N_{2i+3} - N_{2i+2}) \Big\| \\ &\leqslant \Big\| \sum_i (N_{2i+2} - N_{2i})Y(N_{2i+2} - N_{2i}) \Big\| \\ &\qquad + \Big\| \sum_i (N_{2i+3} - N_{2i+1})Y(N_{2i+3} - N_{2i+1}) \Big\| < 2\varepsilon.\end{aligned}$$

Thus it suffices to show that every operator Z of the form

$$Z = MZN^{\perp},$$

where $M < N$ in $\mathfrak{N}$, belongs to the stable ideal generated by X. This is because Y' is a sum of such operators and so this will show that Y is a limit of operators in the stable ideal generated by X.

Since X is not of compact character, there are $0 < L < G < I$ in $\mathfrak{N}$ such that $(G - L)X(G - L)$ is not compact. Thus we can find operators $A = LA(G - L)$ and $B = (G - L)BG^{\perp}$ such that $W := AXB$ is a partial isometry mapping $G^{\perp}$ onto L. It follows that an arbitrary operator of the form $Z = LZG^{\perp}$ factors as $Z = W(W^*Z) = AXB(W^*Z)$. Since $W^*Z = G(W^*Z)G^{\perp}$, the terms A, B, and W^*Z of this factorization all belong to $\operatorname{Alg}\mathfrak{N}$, and so $Z \in \mathfrak{R}_0$.

Finally, given an arbitary operator of the form $Z = MZN^{\perp}$, choose an order isomorphism of $\mathfrak{N}$ into itself which maps L to M and G to N. By the Similarity Theorem this is implemented by an automorphism Ad_T, and $Z' := \operatorname{Ad}_{T^{-1}}(Z)$ satisfies $Z' = LZ'G^{\perp}$. By the last paragraph, Z' belongs to $\mathfrak{R}_0$ and so, since $\mathfrak{R}_0$ is automorphism invariant, $Z \in \mathfrak{R}_0$. ∎

COROLLARY 4.3. $\mathfrak{R}_0$ *is the smallest stable ideal which is not of compact character.*

DEFINITION 4.4. Let S be a subset of $\mathfrak{N}$. We define

$$L^+ = \bigwedge\{N \in S : N > L\} \quad \text{and} \quad L^- = \bigvee\{N \in S : N < L\}$$

for L in S (where the empty join is 0 and the empty meet is I).

We shall always make explicit mention of the set S with respect to which these operations are taken. Note that if S is closed then L^+ and L^- belong to S, but otherwise need not.

DEFINITION 4.5. Let S be a subset of $\mathfrak{N}$. Taking operations L^+ and L^- with respect to S, we let $Z(S)$ be the set of $Y \in \operatorname{Alg}\mathfrak{N}$ such that

$$(L - L^-)Y(L - L^-) = (L^+ - L)Y(L^+ - L) = 0$$

for all $L \in S$.

PROPOSITION 4.6. *Let X belong to* $\operatorname{Alg}\mathfrak{N}$ *and let $S = \{N \in \mathfrak{N} : j_N(X) \geqslant a > 0\}$. Then the closed ideal generated by X contains $Z(S)$.*

The proof of Proposition 4.6 makes use of the following theorem, which is Theorem 2.1 of [23]:

THEOREM 4.7. (Interpolation Theorem) *Let $X \in$* $\operatorname{Alg}\mathfrak{N}$ *and let $E_{\mathfrak{N}}$ be the spectral measure for $\mathfrak{N}$. If $j_N(X) \geqslant a > 0$ for all N in the Borel set S, then there are $A, B \in$* $\operatorname{Alg}\mathfrak{N}$ *such that $AXB = E_{\mathfrak{N}}(S)$.*

REMARK 4.8. For technical reasons the original theorem in [23] uses the dsf i_N^- instead of j_N. The distinction is not important for the theorem, because [21], Lemma 1.1 shows that the two seminorms agree almost everywhere.

Proof of Proposition 4.6. Let $E_{\mathfrak{N}}$ be the spectral measure for $\mathfrak{N}$ and let $Q = E_{\mathfrak{N}}(S)$. Theorem 4.7 tells us that Q is in the ideal generated by X. Thus it only remains to show that every $Y \in Q^\perp Z(S)$ factors through X.

Let $G_n - L_n$ be an enumeration of the non-zero intervals of the form $L^+ - L$ or $L - L^-$ (with respect to S). For each n let $P_{n,i}$ be the projections onto an orthonormal basis of $G_n - L_n$ chosen so that, for each n and i there is an $N_{n,i} \in \mathfrak{N}$ such that $L_n + P_{n,i} < N_{n,i} < G_n$. Fix $Y \in Q^\perp Z(S)$ and $\varepsilon > 0$. Since $Y \in Z(S)$, $P_{n,i}YM \to 0$ as $M \downarrow G_n$ in $\mathfrak{N}$. For each fixed n and i pick a decreasing sequence $M_{n,i} > G_n$ such that

$$\|P_{n,i}YM_{n,i}\| < \frac{\varepsilon}{2^{n+i}},$$

whence $\|Y - Y'\| < \varepsilon$ if $Y' = \sum_{n,i} P_{n,i}YM_{n,i}^\perp$.

It is straightforward to show that j_N is upper semicontinuous, and so S is closed. Thus $G_n \in S$ and $j_{G_n}(X) \geqslant a$. Thus $\|(M_{n,i} - N_{n,i})X(M_{n,i} - N_{n,i})\|_{\text{ess}} \geqslant a$ for all n, i. By Lemma 2.2 there are orthonormal sequences $x_{n,i}$, $y_{n,i}$, where each $x_{n,i}$, $y_{n,i}$ belongs to the range of $(M_{n,i} - N_{n,i})$, such that $\langle x_{n,i}, Xy_{n',i'}\rangle > a/2$ if $(n,i) = (n',i')$ and is zero otherwise. Let

$$A = \sum_{n,i} \frac{a_{n,i}x_{n,i}^*}{\langle Xy_{n,i}, x_{n,i}\rangle} \quad \text{and} \quad B = \sum_{n,i} y_{n,i}a_{n,i}^*$$

where $a_{n,i}$ is a unit vector in the range of $P_{n,i}$. It is straightforward to check that BY' and A are in the nest algebra and that $AXBY' = Y'$. Since ε was arbitrary, the result follows. ∎

DEFINITION 4.9. Let S be a subset of $\mathfrak{N}$. Taking operations L^+ and L^- with respect to S, we define:

(i) $K^+(S)$ is the set of all Y in $\operatorname{Alg}\mathfrak{N}$ for which

$$(L^+ - M)Y(L^+ - M) \text{ is compact}$$

for all $L \in S$ and $M \in \mathfrak{N}$ with $L < M < L^+$, and

$$(L - L^-)Y(L - L^-) = 0$$

for all $L \in S$ with $L^- \notin S$;

(ii) $K^-(S)$ is the set of all Y in $\operatorname{Alg}\mathfrak{N}$ for which

$$(M - L^-)Y(M - L^-) \text{ is compact}$$

for all $L \in S$ and $M \in \mathfrak{N}$ with $L^- < M < L$, and

$$(L^+ - L)Y(L^+ - L) = 0$$

for all $L \in S$ with $L^+ \notin S$.

Note the connection with the ideals of compact character. For if $L \in S$ then $(L^+ - L)K^+(S)(L^+ - L)$ coincides with the ideal $\mathfrak{K}^+$ in the nest algebra $(L^+ - L)(\operatorname{Alg}\mathfrak{N})(L^+ - L)$, while the compression of $K^-(S)$ coincides with the corresponding $\mathfrak{K}^-$.

PROPOSITION 4.10. *Let $X \in \operatorname{Alg}\mathfrak{N}$ and let $S = \{N \in \mathfrak{N} : i_N^+(X) \geqslant a > 0\}$. Then the stable ideal generated by X contains $K^+(S)$.*

Proof. Let Y be an arbitrary operator in $\operatorname{Alg}\mathfrak{N}$ which satisfies

$$(L^+ - L)Y(L^+ - L) = 0 \quad \text{or} \quad (L - L^-)Y(L - L^-) = 0 \quad \text{for } L \in S.$$

We claim that $Y \in Z(S_0)$, where $S_0 := \{N : j_N(X) \geqslant a\}$, and hence by Proposition 4.6 that Y is in the closed ideal generated by X. For, given $G > L$ in S_0 and assuming G is the immediate successor of L, let M be the meet of all elements in S that are greater than or equal to G. Clearly $M \in S$ and, since $S \subseteq S_0$, $M^- \leqslant L$. Thus $M - M^- \geqslant G - L$ and so $(G - L)Y(G - L) = 0$.

It remains to show that if we take L_n to be an enumeration of those $L \in S$ with $L^+ > L$, and consider an operator $Y := \sum_n Y_n$ where $Y_n = (L_n^+ - L_n)Y_n(L_n^+ - L_n)$ and $(L_n^+ - M)Y_n(L_n^+ - M) \in \mathfrak{K}$ for all $M < L_n^+$, then Y is in the stable ideal generated by X.

For each $n \in \mathbb{N}$, by Proposition 2.12 there are operators A_n and B_n in $\operatorname{Alg}(L_n^+ - L_n)\mathfrak{N}$, and an invertible operator T_n on $(L_n^+ - L_n)\mathfrak{H}$ which implements an automorphism of $\operatorname{Alg}(L_n^+ - L_n)\mathfrak{N}$ such that

$$\|Y_n - \operatorname{Ad}_{T_n}(A_n X_n B_n)\| < \varepsilon$$

where $\varepsilon > 0$ is an arbitrary fixed positive quantity independent of n and $X_n = (L_n^+ - L_n)X(L_n^+ - L_n)$. Moreover, on closer examination of the proof of Proposition 2.12, the reader can observe that the operators A_n and B_n are uniformly bounded independent of n. Likewise, since T_n can be a small perturbation of a unitary, the condition number of T_n can be uniformly bounded. Thus, taking $P := \sum(L_n^+ - L_n)$, let

$$T := P^\perp + \sum_n T_n, \quad A := \sum_n A_n, \quad B := \sum_n B_n,$$

so that $\|Y - \mathrm{Ad}_T(AX'B)\| < \varepsilon$, where $X' := \sum_n X_n$. However, $X - X'$ belongs to the stable ideal generated by X, by Proposition 4.6, and so X' also belongs to this stable ideal. We have showed that Y is within ε of the stable ideal generated by X. Since ε was arbitrary, Y must belong to this ideal. ∎

The proof of the next proposition is directly analogous to the one we have just completed, and is left to the reader.

PROPOSITION 4.11. *Let $X \in \mathrm{Alg}\,\mathfrak{N}$ and let $S = \{N \in \mathfrak{N} : i_N^-(X) \geqslant a > 0\}$. Then the stable ideal generated by X contains $K^-(S)$.*

DEFINITION 4.12. Let S be a subset of $\mathfrak{N}$. Taking operations L^+ and L^- with respect to S, we define:

(i) $R^+(S)$ is the set of all Y in $\mathrm{Alg}\,\mathfrak{N}$ for which

$$(L^+ - M)Y(L^+ - M) \text{ belongs to } \mathfrak{R}_0$$

for all $L \in S$ and $M \in \mathfrak{N}$ with $L < M < L^+$ and,

$$(L - L^-)Y(L - L^-) = 0$$

for all $L \in S$ with $L^- \notin S$.

(ii) $R^-(S)$ is the set of all Y in $\mathrm{Alg}\,\mathfrak{N}$ for which

$$(M - L^-)Y(M - L^-) \text{ belongs to } \mathfrak{R}_0$$

for all $L \in S$ and $M \in \mathfrak{N}$ with $L^- < M < L$ and,

$$(L^+ - L)Y(L^+ - L) = 0$$

for all $L \in S$ with $L^+ \notin S$.

PROPOSITION 4.13. *Let $X \in \mathrm{Alg}\,\mathfrak{N}$ and let $S = \{N \in \mathfrak{N} : e_N^+(X) \geqslant a > 0\}$. Then the stable ideal generated by X contains $R^+(S)$.*

Proof. This proof is very similar to the proof of Proposition 4.10. However, we do not have a previous result analogous to Proposition 2.12 which we can quote. So we will need to incorporate a factorization argument into this proof. Let L_n be an enumeration of those $L \in S$ with $L^+ > L$ and write $\mathfrak{J}$ for the stable ideal generated by X.

Since, again, $S \subseteq \{N : j_N(X) \geqslant a\}$, it follows by Proposition 4.6 and a similar argument to the first paragraph of the proof of Proposition 4.10 that every $Y \in \mathrm{Alg}\,\mathfrak{N}$ which satisfies

$$(L^+ - L)Y(L^+ - L) = 0 \quad \text{or} \quad (L - L^-)Y(L - L^-) = 0 \quad \text{for all } L \in S$$

belongs to $\mathfrak{I}$. Thus we need only to show that an arbitrary operator of the form $Y := \sum_n Y_n$ satisfying

$$Y_n = (L_n^+ - L_n)Y_n(L_n^+ - L_n) \quad \text{and} \quad (L^+ - M)Y_n(L^+ - M) \in \mathfrak{R}_0$$

for all n, always belongs to $\mathfrak{I}$.

Set $X_n := (L_n^+ - L_n)X(L_n^+ - L_n)$, and let $X' := \sum_n X_n$. Since Proposition 4.6 shows that $X - X'$ belongs to $\mathfrak{I}$, it follows that $X' \in \mathfrak{I}$. We shall show that Y belongs to the stable ideal generated by X', and so belongs to $\mathfrak{I}$.

Because $e^+_{L_n}(X_n) = e^+_{L_n}(X) \geqslant a$, for each n we can choose a sequence $M_{n,i} < L_n^+$ decreasing to L_n such that

$$\|(M_{n,i} - M_{n,i+1})X_n(M_{n,i} - M_{n,i+1})\|_{\text{ess}} > \frac{a}{2}.$$

By a standard application of Lemma 2.2 we can construct operators A_n, B_n of norm no more than $2/a^{\frac{1}{2}}$ in $\operatorname{Alg}\mathfrak{N}$ so that

$$A_n X_n B_n = \sum_i W_{n,i}$$

where $W_{n,i}$ is a partial isometry mapping $M_{n,i-1} - M_{n,i}$ onto $M_{n,i+1} - M_{n,i+2}$. Thus

$$W := \sum_n W_n = \Big(\sum_n A_n\Big)X'\Big(\sum_n B_n\Big)$$

belongs to the ideal generated by X', and hence to $\mathfrak{I}$.

Since $j_N(Y_n) = 0$ for all $L_n < N \leqslant L_n^+$, it follows by a compactness argument similar to the one used in Proposition 4.2 that there are projections $M'_{n,i}$ strictly between L_n and L_n^+ such that, for each n, $M'_{n,i}$ decreases to L_n and

$$\|(M'_{n,i} - M'_{n,i+2})Y_n(M'_{n,i} - M'_{n,i+2})\| < \varepsilon$$

for all n, i (where ε is a fixed arbitrary positive quantity independent of n and i). Moreover, by an estimate similar to the one used in the proof of Proposition 4.2, if we take

$$Y' := \sum_{n,i}(M'_{n,i+1} - M'_{n,i+2})Y_n {M'_{n,i}}^{\perp},$$

then $\|Y - Y'\| < 2\varepsilon$.

By now it should be a routine application of the Similarity Theorem to piece together an invertible operator S which implements an automorphism of $\operatorname{Alg}\mathfrak{N}$ and such that for each n and i, $\theta_S(M'_{n,i}) = M_{n,2i-1}$. But then, writing $Y'' := \operatorname{Ad}_S(Y')$,

$$(M_{n,2i+4} - M_{n,2i})Y''(M_{n,2i+4} - M_{n,2i}) = 0$$

for all n and i, and so one can check that the shift implemented by W still allows that $W^*Y'' \in \operatorname{Alg}\mathfrak{N}$. Thus,

$$Y' = \operatorname{Ad}_{S^{-1}}(W(W^*Y'')) = \operatorname{Ad}_{S^{-1}}(W)(\operatorname{Ad}_{S^{-1}}(W^*Y''))$$

belongs to the stable ideal generated by W. Thus $Y' \in \mathfrak{I}$. Since $\|Y - Y'\| < \varepsilon$ and ε was arbitrary, it follows that $Y \in \mathfrak{I}$, and we are done. ∎

PROPOSITION 4.14. *Let $X \in \operatorname{Alg}\mathfrak{N}$ and let $S = \{N \in \mathfrak{N} : e_N^-(X) \geqslant a > 0\}$. Then the stable ideal generated by X contains $R^-(S)$.*

As the final results of this section, we can considerably strengthen Proposition 4.6.

DEFINITION 4.15. Let S be a subset of $\mathfrak{N}$. Taking operations L^+ and L^- with respect to S, we let $R(S)$ be the set of $Y \in \operatorname{Alg}\mathfrak{N}$ such that $(L-L^-)Y(L-L^-)$ and $(L^+ - L)Y(L^+ - L)$ belongs to $\mathfrak{R}_0$ for all $L \in S$.

PROPOSITION 4.16. *Let $X \in \operatorname{Alg}\mathfrak{N}$ and let $S = \{N \in \mathfrak{N} : j_N(X) \geqslant a > 0\}$. If X is not of compact character then the stable ideal generated by X contains $R(S)$.*

REMARK 4.17. The only case in which it is a possibility X might be of compact character is when $S = \{0, I\}$. In that case, Proposition 4.6 offered no information at all about the stable ideal generated by X. But of course we already know that if X is of compact character then it generates one of the ideals characterized in Section 2. And if X is not of compact character then, by Proposition 4.2, the stable ideal generated by X contains $\mathfrak{R}_0$. Proposition 4.16 extends this observation in the greatest possible generality.

The proof of this final proposition needs the following non-trivial order-theoretic fact, which is Theorem 1 of [22]:

THEOREM 4.18. *Let S be a closed subset of $[0,1]$ and suppose the set $M \subseteq [0,1]$ meets each component of S^{c} in at most finitely many points. Then there is an increasing bijection f of the unit interval to itself which maps S onto all but finitely many points of M.*

Proof of Proposition 4.16. Let $G_n - L_n$ be an enumeration of the non-zero intervals of the form $L^+ - L$ and $L - L^-$ for $L \in S$. Write $\mathfrak{I}$ for the stable ideal generated by X. By Proposition 4.6 we need only to show that $Y := \sum Y_n$ belongs to $\mathfrak{I}$, where $Y_n = (G_n - L_n)Y_n(G_n - L_n) \in \mathfrak{R}_0$.

Let $\varepsilon > 0$. Since $\mathfrak{N}$ is compact we can find finite sequences

$$L_n \leqslant M_{n,i} \leqslant G_n, \quad i = 0, \dots, k_n$$

increasing from L_n to G_n such that

$$\|(M_{n,i+1} - M_{n,i})Y_n(M_{n,i+1} - M_{n,i})\| < \varepsilon$$

and so, taking

$$Y' := \sum_{n,i}(M_{n,i+1} - M_{n,i})Y_n(I - M_{n,i+1})$$

we have $\|Y - Y'\| < \varepsilon$.

By Theorem 4.18 there is an order isomorphism θ of $\mathfrak{N}$ to itself such that $\theta(S)$ contains all but perhaps finitely many points of $M := \{M_{n,i} : n \in \mathbb{N},\, 0 \leqslant i \leqslant k_n\}$. Using the Similarity Theorem, we can find an automorphism which transforms X to X' where $j_N(X') \geqslant a' > 0$ for all $N \in \theta(S)$ (by Lemma 3.8). By another application of Proposition 4.6, this time using X' in place of X,

$$\sum_{(n,i):M_{n,i+1}\in\theta(S)} (M_{n,i+1} - M_{n,i})Y_n(I - M_{n,i+1})$$

belongs to $\mathfrak{I}$. The sum of the finitely many remaining terms of Y' belongs to $\mathfrak{R}_0$, and since X is not of compact character, the sum of these remaining terms also belongs to $\mathfrak{I}$ by Proposition 4.2. Since $\|Y - Y'\| < \varepsilon$ and $\mathfrak{I}$ is closed, we are done. ∎

5. CHARACTERIZATION OF THE STABLE IDEALS

In this section we prove the Stable Ideal Theorem (Theorem 5.7) which is the main result of this paper. This theorem builds on the propositions proved in the last section, and uses Proposition 5.6, which is a synthesis of these results. The other ingredient we need in the proof of Theorem 5.7 is an approximation lemma (Lemma 5.5), which we prove in the first part of the section.

Let $\|\cdot\|_N$ be the Ringrose seminorm. If $\|X\|_N < \varepsilon$ for all N then X is approximately equal to an operator X' for which $\|X'\|_N$ is identically zero. (This was observed in [26].) In Lemma 5.5 we shall obtain the analogous result for a general dsf.

In large part the proof follows the same lines as for the Ringrose seminorm. However, there is an important exception. The seminorms $e_N^{\pm}$ present problems which call for careful constructions with compact operators. To this end we quote the well-known matrix completion theorem of Parrot, ([24], which was independently discovered by Davis, Kahane and Weinberger, [9]). We will use this theorem to prove Lemma 5.2.

Theorem 5.1. ([24], [9]) *Let P and Q be projections on $\mathfrak{H}$ and suppose that X is an operator on $\mathfrak{H}$. Then there is an operator $X_0 = P^{\perp}X_0Q^{\perp}$ such that*

$$\|X + X_0\| \leqslant \max\{\|PX\|, \|XQ\|\}.$$

Lemma 5.2. *Let P_n be an increasing sequence of projections with $P_0 = 0$ and let $X \in B(\mathfrak{H})$ satisfy $\|P_nX\|_{\text{ess}} < 1$ and $P_nX = P_nXP_{n-1}$ for all n. Then there is a sequence K_n of compact operators such that for all n:*

$$K_n = (P_n - P_{n-1})K_nP_{n-1} \quad \textit{and} \quad \Big\|P_nX - \sum_{i=1}^{n} K_i\Big\| < 1.$$

Proof. Construct the operators K_i inductively by taking $K_1 := 0$ and supposing that $K_1, \ldots, K_{n-1}$ have already been chosen. Since $\|P_nX\|_{\text{ess}} < 1$, we can find a compact operator C such that $\|P_nX - C\| < 1$. Write $C_{n-1} := \sum_{i=1}^{n-1} K_i$ and choose a finite rank projection Q such that $\|(C - C_{n-1})Q^{\perp}\| < 1 - \|P_nX - C\|$. Then

$$\|P_n(X - C_{n-1})Q^{\perp}\| \leqslant \|P_nX - C\| + \|(C - C_{n-1})Q^{\perp}\| < 1$$

and, of course,

$$\|P_{n-1}(X - C_{n-1})\| = \|P_{n-1}X - C_{n-1}\| < 1.$$

Thus, by Theorem 5.1 applied to the operator $P_n(X - C_{n-1})$ and the projections P_{n-1} and $Q^{\perp}$, we can find an operator $K = P_{n-1}^{\perp}KQ$ such that $\|P_n(X - C_{n-1}) - K\| < 1$. Take $K_n := P_nKP_{n-1}$. Since Q is finite rank, K_n is finite rank and satisfies $K_n = (P_n - P_{n-1})K_nP_{n-1}$, and

$$\Big\|P_nX - \sum_{i=1}^{n} K_i\Big\| = \|P_n(X - C_{n-1} - K)P_{n-1}\| < 1. \quad ∎$$

LEMMA 5.3. *Suppose $X \in \operatorname{Alg}\mathfrak{N}$ and $e_N^+(X) < \varepsilon$. Then there is a $G > N$ in $\mathfrak{N}$ and an $A \in \operatorname{Alg}\mathfrak{N}$ such that*

$$\|(G-N)(X-A)(G-N)\| < 3\varepsilon$$

and $L^\perp A$ is compact for all $L > N$.

Proof. Since $e_N^+(X) < \varepsilon$, we can choose a projection G in $\mathfrak{N}$ such that $\sup\limits_{N<L<G} \|(G-L)X(G-L)\|_{\mathrm{ess}} < \varepsilon$. Then pick a sequence N_n in $\mathfrak{N}$, decreasing to N, with $N_0 = G$ and, for each n, use the essential norm condition to pick a compact operator $C_n = (N_{n-1}-N_n)C_n(N_{n-1}-N_n)$ such that

$$\|(N_{n-1}-N_n)X(N_{n-1}-N_n) - C_n\| < \varepsilon.$$

By [7], Theorem 5.1, the operators C_n can be taken to be in $\operatorname{Alg}\mathfrak{N}$. Let $A_0 := \sum\limits_{n=1}^{\infty} C_n$, let

$$X' := \sum_{n=1}^{\infty}(N_{n-1}-N_n)X(N_{n-1}-N_n), \quad \text{and} \quad X'' := \sum_{n=1}^{\infty} N_n X(N_{n-1}-N_n),$$

and let $P_n := G - N_n$. Then $P_n X'' = P_n X'' P_{n-1}$ and

$$\begin{aligned}\|P_n X''\|_{\mathrm{ess}} &= \|(G-N_n)X''\|_{\mathrm{ess}} = \|(G-N_n)X''(G-N_n)\|_{\mathrm{ess}} \\ &= \|(G-N_n)(X-(X'-A_0))(G-N_n)\|_{\mathrm{ess}} < 2\varepsilon.\end{aligned}$$

Thus, by Lemma 5.2, we can find compact operators $K_n = (P_n - P_{n-1})K_n P_{n-1}$ such that $\left\|P_n X'' - \sum\limits_{i=1}^{n} K_i\right\| < 2\varepsilon$. Since, in particular, $\left\|\sum\limits_{i=1}^{n} K_i\right\| < \|X''\| + 2\varepsilon$, the partial sums, $\sum\limits_{i=1}^{n} K_i$, are uniformly bounded, so we can find a weak-$*$ limit point, A_1, of the partial sums. Moreover, for each fixed n, $P_n \sum\limits_{i=1}^{k} K_i = \sum\limits_{i=1}^{n} K_i$ for all $k \geqslant n$, and so $P_n A_1 = \sum\limits_{i=1}^{n} K_i$. It follows that

$$\|P_n(X - A_0 - A_1)P_n\| \leqslant \|P_n(X' - A_0 + X'' - A_1)P_n\| < 3\varepsilon$$

for all n. By lower semicontinuity of the norm, this inequality still holds if we take the weak-$*$ limit as $n \to +\infty$. Thus

$$\|(G-N)(X - A_0 - A_1)(G-N)\| < 3\varepsilon$$

and the result follows by taking $A := A_0 + A_1$. ∎

LEMMA 5.4. *Let $\|\cdot\|_N$ be a diagonal seminorm function and suppose that for a fixed $X \in \operatorname{Alg}\mathfrak{N}$ and $N_0 \in \mathfrak{N}$, $\|X\|_{N_0} < \varepsilon$. Then provided that both $\|\cdot\|_{N_0}^+$ and $\|\cdot\|_{N_0}^-$ are non-zero, there are projections $G > N_0 > L$ in $\mathfrak{N}$ and $A \in \operatorname{Alg}\mathfrak{N}$ such that*

$$\|(G-L)(X-A)(G-L)\| < 3\varepsilon$$

and $\|A\|_N = 0$ for all $N \in \mathfrak{N}$.

Proof. If $\|\cdot\|_{N_0} = j_{N_0}$ then there are $G > N_0 > L$ in $\mathfrak{N}$ such that

$$\|(G-L)X(G-L)\| < \varepsilon$$

and we are done. Otherwise, $\|\cdot\|^+_{N_0}$ must be $e^+_{N_0}$ or $i^+_{N_0}$. If it is $e^+_{N_0}$, then by Lemma 5.3, there is a $G > N_0$ and an $A \in \operatorname{Alg}\mathfrak{N}$ such that

$$\|(G-N_0)(X-A)(G-N_0)\| < 3\varepsilon$$

and $\|A\|_{N_0} = 0$. In fact, we can arrange that $\|A\|_N = 0$ for all $N \in \mathfrak{N}$, since we can stipulate that $A = (G-N_0)A(G-N_0)$, so that $\|A\|_N = 0$ for $N < N_0$, and we know that $L^\perp A$ is compact for all $L > N$, and that the diagonal seminorms all map compact operators to zero, so that $\|A\|_N = 0$ for $N > N_0$. On the other hand, if $\|\cdot\|^+_{N_0}$ is $i^+_{N_0}$ then, by definition, there is a $G > N_0$ such that

$$\|(G-N_0)X(G-N_0)\| < \varepsilon.$$

Thus, in each case we have found $G > N_0$ and an A^+ such that

$$\|(G-N_0)(X-A^+)(G-N_0)\| < 3\varepsilon$$

and $\|A^+\|_N = 0$ for all N.

Likewise, $\|\cdot\|^-_{N_0}$ must be $e^-_{N_0}$ or $i^-_{N_0}$ and by the same token we can find $L < N_0$ and an A^- such that

$$\|(N_0-L)(X-A^-)(N_0-L)\| < 3\varepsilon$$

and $\|A^-\|_N = 0$ for all N. Thus, if we set

$$A := A^+ + N_0XN_0^\perp + A^-$$

then we obtain the desired result. ∎

Lemma 5.5. *Let $\|\cdot\|_N$ be a greatest diagonal seminorm function and let $X \in \operatorname{Alg}\mathfrak{N}$. Suppose that $\|X\|_N < \varepsilon$ for all N in $\mathfrak{N}$. Then there is an $X' \in \operatorname{Alg}\mathfrak{N}$ such that $\|X'\|_N = 0$ for all N and $\|X - X'\| < 6\varepsilon$.*

Proof. Suppose first that neither $\|\cdot\|^+_N$ nor $\|\cdot\|^-_N$ is zero except possibly for $N = 0$ or I. Fix an arbitrary $N \in \mathfrak{N}$ at which neither $\|\cdot\|^+_N$ nor $\|\cdot\|^-_N$ is zero. By Lemma 5.4, there are $G > N > L$ in $\mathfrak{N}$ and A in $\operatorname{Alg}\mathfrak{N}$ such that

$$\|(G-L)(X-A)(G-L)\| < 3\varepsilon$$

and $\|A\|_M = 0$ for all $M \in \mathfrak{N}$. (This interval is one-sided if $N = 0$ or I.) In this way we obtain an open cover of either $\mathfrak{N}$, $\mathfrak{N}\setminus\{0\}$, $\mathfrak{N}\setminus\{I\}$, or $\mathfrak{N}\setminus\{0, I\}$, depending on whether, possibly, $\|\cdot\|^+_0 = 0$ or $\|\cdot\|^-_i = 0$. Thus, by the compactness of $\mathfrak{N}$, we can find an increasing sequence N_i in $\mathfrak{N}$ satisfying $\sum_i (N_i - N_{i-1}) = I$, and such that each interval $N_{i+1} - N_{i-1}$ is dominated by an interval $G - L$ from the open cover. Thus we can find operators A_i in $\operatorname{Alg}\mathfrak{N}$ such that, for all i,

$$\|(N_{i+1}-N_{i-1})(X-A_i)(N_{i+1}-N_{i-1})\| < 3\varepsilon$$

and $\|A_i\|_N = 0$ for all N. We patch these overlapping blocks together in strips as follows:

$$X' := \sum_i (N_i - N_{i-1})(A_iN_{i+1} + XN_{i+1}^\perp).$$

Note that since the terms of this sum that involve X are entirely off the diagonal, $\|X'\|_N$ is equal to the value of $\|\cdot\|_N$ at one or possibly two A_i's, and so is zero. Also,

$$\begin{aligned}\|X - X'\| &= \Big\|\sum_i (N_i - N_{i-1})(X - A_i)N_{i+1}\Big\| \\ &\leqslant \Big\|\sum_i (N_i - N_{i-1})(X - A_i)(N_i - N_{i-1}))\Big\| \\ &\quad + \Big\|\sum_i (N_i - N_{i-1})(X - A_i)(N_{i+1} - N_i))\Big\| \\ &< 3\varepsilon + 3\varepsilon\end{aligned}$$

and we are done.

In the general case, let $S := \{N : \|\cdot\|_N^+ = 0 \text{ or } \|\cdot\|_N^- = 0\}$. By Lemma 3.16, S is a closed subset of $\mathfrak{N}$. Write the components of $\mathfrak{N} \setminus S$ as the open order intervals (L_n, G_n). Now, for fixed n, the function $\|Y\|_N' = \|Y\|_N$ for Y in $(G_n - L_n)\operatorname{Alg}\mathfrak{N}(G_n - L_n)$ and $L_n \leqslant N \leqslant G_n$ defines a diagonal seminorm on $\operatorname{Alg}(G_n - L_n)\mathfrak{N}$ which satisfies the conditions of the first paragraph. Thus there are operators X_n in $\operatorname{Alg}\mathfrak{N}$ with $\|(G_n - L_n)(X - X_n)(G_n - L_n)\| < 6\varepsilon$ and $\|X_n\|_N = 0$ for all N. Let

$$X' := X - \sum_n (G_n - L_n)(X - X_n)(G_n - L_n)$$

and from this it is routine to check that $\|X'\|_N = 0$ for all N. ∎

PROPOSITION 5.6. *Let $\|\cdot\|_N$ be the greatest diagonal seminorm for some $X \in \operatorname{Alg}\mathfrak{N}$ and $a > 0$ and assume that X is not of compact character. Then the stable ideal generated by X contains all $Y \in \operatorname{Alg}\mathfrak{N}$ for which $\|Y\|_N = 0$ for all $N \in \mathfrak{N}$.*

Proof. Suppose $\|Y\|_N = 0$ for all N. Write $\mathfrak{I}$ for the stable ideal generated by X. Let $S := \{N : \|\cdot\|_N \leqslant i_N^+ i_N^-\}$. This set is equal to

$$\{N : j_N(X) \geqslant a\} \cup \{0, I\}.$$

If $G_n - L_n$ is an enumeration of all the non-zero intervals of the form $L - L^-$ or $L^+ - L$ for $L \in S$ (where $\pm$-operations are taken with respect to S), then, by Proposition 4.6, $Y - Y' \in \mathfrak{I}$ when $Y' := \sum_n (G_n - L_n)Y(G_n - L_n)$. We shall show that Y' also belongs to $\mathfrak{I}$.

Note that, by the maximality of $\|\cdot\|_N$, for each $N < I$ in $\mathfrak{N}$,

(1) if $\|\cdot\|_N^+ = 0$ then $e_N^+(X) \geqslant a$,

(2) if $\|\cdot\|_N^+ = e_N^+$ then $i_N^+(X) \geqslant a$,

and similar remarks hold for $\|\cdot\|_N^-$ when $N > 0$. Let

$$A_1 := \{n : \|\cdot\|_{L_n}^+ = 0\}, \quad A_2 := \{n : \|\cdot\|_{L_n}^+ = e_{L_n}^+\}, \quad A_3 := \{n : \|\cdot\|_{L_n}^+ = i_{L_n}^+\}$$

and let B_1, B_2 and B_3 be the corresponding sets for $\|\cdot\|_N^-$. Since each of the projections L_n, G_n belong to S (because S is closed, since $j_N(X)$ is upper semi-continuous in N) it is clear that the A_i and the B_i both partition $\mathbb{Z}^+$. For each n choose a projection N_n in $\mathfrak{N}$ lying strictly between G_n and L_n and let

$$Y_i = \sum_{n\in A_i} (N_n - L_n)Y(G_n - L_n), \quad Y_i' = \sum_{n\in B_i} (G_n - N_n)Y(G_n - L_n).$$

Note that

$$Y' = \sum_{i=1}^{3} Y_i + \sum_{i=1}^{3} Y_i'.$$

Thus the result will follow on showing that Y_1, Y_2, Y_3 and Y_1', Y_2', Y_3' all belong to $\mathfrak{I}$.

As motivation for the argument, we note that the reason we split the blocks $(G_n - L_n)Y(G_n - L_n)$ into two parts (i.e. $(N_n - L_n)Y(G_n - L_n)$ and $(G_n - N_n)Y(G_n - L_n)$) is so as to separate the parts of the operator with “accumulation from above” (i.e. the Y_i) from the parts with “accumulation from below” (i.e. the Y_i'). We shall show that the Y_i belong to $\mathfrak{I}$ by demonstrating they belong to “plus” ideals such as R^+ and K^+ which are in turn contained in $\mathfrak{I}$. By the same token we will show that Y_i' are in $\mathfrak{I}$, by showing they belong to “minus” ideals.

First we will show that Y_1 belongs to $\mathfrak{I}$. If n is in A_1 then $\|\cdot\|_{L_n}^+ = 0$ and so $e_{L_n}^+(X) \geqslant a$. Thus L_n belongs to $S_1 = \{N : e_N^+(X) \geqslant a\}$. By Proposition 4.13, $\mathfrak{I}$ contains $R^+(S_1)$. We shall show that Y_1 belongs to this ideal.

To do this, we need to show that for $n \in A_1$

$$(L_n^+ - M)Y_1(L_n^+ - M) \in \mathfrak{R}_0$$

for all $M \in \mathfrak{N}$ with $L_n < M < L_n^+$ (where L_n^+ is the successor of L_n with respect to S_1). But note that $j_N(X) = \|X\|_N < a$ for all N strictly between L_n and G_n, and so L_n^+ must be at least G_n. On the other hand L_n^+ is not greater than any of the L_m's ($m \in A_1$) that dominate L_n (because each such L_m belongs to S_1). Thus

$$(L_n^+ - L_n)Y_1(L_n^+ - L_n) = (N_n - L_n)Y(G_n - L_n).$$

But when $L_n < N < G_n$, we know that $\|\cdot\| = j_N$ and so $j_N(Y) = \|Y\|_N = 0$. Thus for any fixed $M \in \mathfrak{N}$ with $L_n < M < L_n^+$, since

$$j_N((L_n^+ - M)Y_1(L_n^+ - M)) = j_N((N_n - M)Y(G_n - M)) \leqslant j_N(N_n - M)j_N(Y),$$

it follows that $j_N((L_n^+ - M)Y_1(L_n^+ - M)) = 0$ for all N and that $(L_n^+ - M)Y_1(L_n^+ - M) \in \mathfrak{R}_0$. Thus $Y_1 \in \mathfrak{R}^+(\mathfrak{N}, S_1)$.

Next we must show that Y_2 belongs to $\mathfrak{I}$. If n is in A_2 then $\|\cdot\|_{L_n}^+ = e_{L_n}^+$ and so $i_{L_n}^+(X) \geqslant a$. Thus L_n belongs to $S_2 = \{N : i_N^+(X) \geqslant a\}$ and one shows, as for Y_1, that $(L_n^+ - L_n)Y_2(L_n^+ - L_n) = (N_n - L_n)Y(G_n - L_n)$ (where L_n^+ is now taken with respect to S_2).

Also, for $n \in A_2$, we know that $e_{L_n}^+(Y) = 0$ and so, by Lemma 5.3, for any $\eta > 0$ there is an M_n in $\mathfrak{N}$ between N_n and L_n and $A_n = (M_n - L_n)A_n(M_n - L_n)$ in $\operatorname{Alg}\mathfrak{N}$ such that

$$\|(M_n - L_n)(Y - A_n)(M_n - L_n)\| < \eta$$

and $N^\perp A_n$ is compact for all $N > L_n$. Then $A = \sum A_n$ belongs to $K^+(S_2)$ which, by Proposition 4.10, is a subset of $\mathfrak{I}$ and

$$B := Y_2 - \sum (M_n - L_n)Y(M_n - L_n)$$

belongs to $R(S)$ which is a subset of $\mathfrak{I}$ by Proposition 4.16. Thus $\mathfrak{I}$ contains $A+B$ where $\|Y_2 - (A+B)\| < \eta$ and, since η was arbitrary, Y_2 belongs to $\mathfrak{I}$.

Finally, we must show that Y_3 belongs to $\mathfrak{I}$. Since $i^+_{L_n}(Y_3) = 0$ for all n in A_3, it follows that Y_3 is in $R(S)$ and so belongs to $\mathfrak{I}$, by Proposition 4.16.

The corresponding results for Y_i' follow, *mutatis mutandis*, using Propositions 4.11, 4.14 and 4.16. ∎

Theorem 5.7. *Let $\mathfrak{I}$ be a stable ideal of* $\operatorname{Alg}\mathfrak{N}$. *Then either $\mathfrak{I}$ is one of the stable ideals of compact character or else there is a stable family $\mathfrak{F}$ of seminorms and $\mathfrak{I}$ is the set of all $Y \in$* $\operatorname{Alg}\mathfrak{N}$ *such that, for each $\varepsilon > 0$, there is a diagonal seminorm $\|\cdot\|_N$ in $\mathfrak{F}$ with $\|Y\|_N < \varepsilon$ for all $N \in \mathfrak{N}$.*

Proof. If $\mathfrak{I}$ is of compact character then it is one of the ideals listed in Figure 1, so assume that $\mathfrak{I}$ is not of compact character. Let $\mathfrak{F}$ be the set of those diagonal seminorms which are greatest diagonal seminorm functions for some $X \in \mathfrak{I}$ and $\varepsilon > 0$. By Lemma 3.15, $\mathfrak{F}$ is a stable family of diagonal seminorm functions. Write $\mathfrak{I}_0$ for the stable ideal specified by $\mathfrak{F}$ as in Proposition 3.10. Clearly $\mathfrak{I}_0 \supseteq \mathfrak{I}$ so we need only prove the reverse inclusion. Let $Y \in \mathfrak{I}_0$ and let $\varepsilon > 0$. Then there is a diagonal seminorm $\|\cdot\|_N$ in $\mathfrak{F}$ such that $\|Y\|_N < \varepsilon$ for all N. By Lemma 5.5, there is a Y' in $\operatorname{Alg}\mathfrak{N}$ with $\|Y - Y'\| < 6\varepsilon$ and $\|Y'\|_N = 0$ for all N. Also, $\|\cdot\|_N$ is the greatest diagonal seminorm for some $X \in \mathfrak{I}$ and $a > 0$. If X is not of compact character we can apply Proposition 5.6 at once to conclude that Y' is in $\mathfrak{I}$.

It appears that there is an obstacle to using Proposition 5.6 if the operator X that we found in the last paragraph, is of compact character. But because the ideal $\mathfrak{I}$ is definitely not of compact character, we can always find another operator X' which is not of compact character, such that $\|\cdot\|_N$ is still the greatest diagonal seminorm function for X' and a. Thus the apparent obstacle is easily finessed.

To do this, observe that $\mathfrak{I}$ must contain some operator Z which is not of compact character. Take $0 < L < G < I$ in $\mathfrak{N}$ so that $(G-L)Z(G-L)$ is non-compact and choose projections $0 < M < N < L$ in $\mathfrak{N}$. Multiplying by a partial isometry from the range of $G-L$ to the range of $N-M$, we obtain a non-compact operator $Z' = (N-M)Z'(G-L)$ in $\mathfrak{I}$. Since $\|X+Z'\|_N = \|X\|_N$ for all N, $\|\cdot\|_N$ is the greatest seminorm for $X' := X + Z'$ and a, and we are done. ∎

6. EXAMPLES

EXAMPLE 6.1. Let $\|\cdot\|_N = i_N^+ i_N^-$ for all N. This is the Ringrose seminorm. The set $\mathfrak{F}$ consisting of $\|\cdot\|_N$ alone is a stable family of seminorms, and the stable ideal it gives rise to is the Jacobson radical of $\operatorname{Alg}\mathfrak{N}$ ([26]).

EXAMPLE 6.2. Likewise, if $\|\cdot\|_N = j_N$ for all N then the set consisting of $\|\cdot\|_N$ alone is a stable family. The ideal this induces is $\mathfrak{R}_0$, which has been studied in [11].

DEFINITION 6.3. For each $S \subseteq \mathfrak{N}$ write

$$\|\cdot\|_N^S = \begin{cases} 0, & N \in S, \\ j_N, & N \notin S. \end{cases}$$

EXAMPLE 6.4. Let $\mathfrak{F}$ be the set of all $\|\cdot\|_N^S$ as S runs through the closed null sets of $\mathfrak{N}$ (with respect to the spectral measure). Using the formula from Proposition 3.10, $\mathfrak{F}$ induces the ideal $\mathfrak{R}_{\mathfrak{N}}^\infty$, which was introduced by Larson ([17]). However, $\mathfrak{F}$ is not a *stable* family of seminorms, because it is not closed under composition with a non-absolutely continuous isomorphism. This, of course, goes hand-in-hand with the fact that $\mathfrak{R}_{\mathfrak{N}}^\infty$ is not invariant under conjugating by similarities, as was discovered by Larson ([17]).

EXAMPLE 6.5. Let $\mathfrak{F}$ be the set of all $\|\cdot\|_N^S$ as S runs through the closed nowhere dense subsets of $\mathfrak{N}$. The ideal induced by $\mathfrak{F}$ is the ideal $\mathfrak{J}_{\min}$ of [21]. In that paper we showed that $\mathfrak{J}_{\min}$ is the intersection of the maximal ideals of $\operatorname{Alg}\mathfrak{N}$ (i.e. the *strong radical* of $\operatorname{Alg}\mathfrak{N}$) and classified the ideals containing it.

Not surprisingly, none of the maximal ideals themselves are stable. For in [21] we showed that every maximal ideal arises by the construction of Proposition 3.10 from a family of dsf's

$$\mathfrak{F} = \{\|\cdot\|_N^S : S \in \mathfrak{A}\}$$

where $\mathfrak{A}$ is a maximal filter of open sets which contains all the open dense subsets of $\mathfrak{N}$. But if N_i $(i \in \mathbb{Z})$ is a sequence strictly increasing from 0 to I then exactly one of

$$\bigcup_{i\,\text{even}} (N_i, N_{i+1}) \quad \text{or} \quad \bigcup_{i\,\text{odd}} (N_i, N_{i+1})$$

must belong to $\mathfrak{A}$. These two sets are conjugate by an order isomorphism of $\mathfrak{N}$, which shows that $\mathfrak{F}$, and hence $\mathfrak{I}$, is not stable.

The next two examples show something of the diversity of new stable ideals which we can construct.

Example 6.6. Let $S = \{N_n\}$ where N_n is a sequence in $\mathfrak{N}$ increasing to I. Define

$$\|\cdot\|_N^{S,k} = \begin{cases} e_N^+, & \text{whenever } N = N_n,\ n \leqslant k, \\ i_N^+ i_N^-, & \text{if } N = N_n \text{ for } n > k, \\ 0, & N = I, \\ j_N, & N \notin S \cup \{I\}. \end{cases}$$

If we let $\mathfrak{F}$ be the set of $\|\cdot\|_N^{S,k}$ for all S and $k \geqslant 0$, then $\mathfrak{F}$ is a stable family.

The way to visualize the ideal, $\mathfrak{I}$, induced by $\mathfrak{F}$ in this example is to imagine a representative element. Let N_n be a fixed sequence increasing to I and let $k \in \mathbb{N}$. Let X_0 be an arbitrary block strictly upper triangular operator of the form:

$$X_0 = \sum_{n \geqslant 1} (N_n - N_{n-1}) X_0 N_n^\perp.$$

Because only "corners" of X_0 touch the diagonal, and only at the points, N_n, at which $\|\cdot\|_N^{S,k}$ is two-sided (and so ignores corners), $\|X_0\|_N^{S,k} = 0$ for all N, and X_0 belongs to $\mathfrak{I}$. Visually, this says that we can include any operator in $\mathfrak{I}$ that has empty diagonal blocks at each interval $N_i - N_{i-1}$.

Let

$$X_1 := \sum_{n \geqslant k} (N_n - N_{n-1}) R_n (N_n - N_{n-1})$$

where R_n belongs to $\mathfrak{R}_0$. Then each R_n satisfies $\|R_n\|_N^{S,k}$, and so $X_0 + X_1$, is in $\mathfrak{I}$. Visually, this says that we can fill in the empty diagonal blocks at each interval $N_i - N_{i-1}$ for $i \geqslant k$ with arbitrary elements of $\mathfrak{R}_0$.

Finally, the k blocks $N_i - N_{i-1}$ $(i = 1, \dots, k)$ can be filled in with arbitrary blocks $S_i + A_i$ such that

$$S_i \in \mathfrak{R}_0 \text{ and } L^\perp A_i \text{ is compact for all } L > N_{i-1}.$$

Example 6.7. Again, let $S = \{N_n\}$ where N_n is a sequence in $\mathfrak{N}$ increasing to I. This time, define

$$\|\cdot\|_N^{S,k} = \begin{cases} i_N^+, & \text{whenever } N = N_n,\ n \leqslant k, \\ 0, & N = I, \\ j_N, & N \notin S \cup \{I\}. \end{cases}$$

If we let $\mathfrak{F}$ be the set of $\|\cdot\|_N^{S,k}$ for all S and $k \geqslant 0$, then $\mathfrak{F}$ is a stable family.

As in the last example, one can visualize a representative element of this ideal as an arbitrary block strictly upper triangular operator of the form:

$$X_0 = \sum_{n \geqslant 1} (N_n - N_{n-1}) X_0 N_n^\perp$$

in which the empty diagonal blocks have been filled in with operators $X_n = (N_n - N_{n-1}) X_n (N_n - N_{n-1})$. For the first finitely many n, $(G - N_{n-1}) X_n (G - N_{n-1})$ must be in $\mathfrak{R}_0$ for all $G < N_n$. Thereafter we can fill the empty blocks with any operators for which $(G - L) X_n (G - L) \in \mathfrak{R}_0$ for all $N_{n-1} < L < G < N_n$. (Of course, the sequence of X_n's must be uniformly bounded.)

Acknowledgements. This work was started while I was a guest of David Larson at Texas A&M University in the Spring of 1990, and also while visiting the University of

Waterloo, in the same year. I am very grateful to Dave Larson, Ken Davidson and the faculty at both of these institutions for their hospitality.

The paper grew out of ideas which David Larson, David Pitts, Xingde Dai and Elias Katsoulis kindly shared with me. Foremost I must thank David Larson and David Pitts. The concept of automorphism invariant ideal was suggested to me by Larson, and both Larson and Pitts encouraged me in this work with many stimulating discussions. Their ideas run throughout this paper, and are too integral with the work to be acknowledged piecemeal. I am also indebted to Dai and Katsoulis, who showed me their examples of new stable ideals, which I have termed "compact character" in this paper.

Finally, I would like to thank the referee, for a truly meticulous reading of the paper with many valuable suggestions.

This work was supported by NSF grants DMS-9500839 and DMS-9204811.

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Received November 4, 1998; revised June 28, 2000.