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STRUCTURE OF GROUP C^*-ALGEBRAS OF LIE SEMI-DIRECT PRODUCTS $\mathbb{C}^n \rtimes \mathbb{R}$

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Abstract. In this paper we analyze the structure of group C^*-algebras of Lie semi-direct products of $\mathbb{C}^n$ by $\mathbb{R}$ to show that these C^*-algebras have finite composition series with their subquotients C^*-tensor products involving commutative C^*-algebras or the C^*-algebra of compact operators or non-commutative tori. As an application, we estimate stable rank and connected stable rank of these group C^*-algebras in terms of groups, and we deduce that group C^*-algebras of Lie semi-direct products of $\mathbb{R}^n$ by $\mathbb{R}$ have a similar structure as in the complex cases.

Keywords: *Group C^*-algebras, solvable Lie groups, stable rank.*

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1. INTRODUCTION

The problem to analyze the structure of group C^*-algebras of simply connected, solvable Lie groups has been considered by a lot of mathematicians, but it is rather mysterious although this problem is so far solved to some extent. See [6], [15], [22] and [24] for some results on the structure of group C^*-algebras of semi-direct products $\mathbb{R}^n \rtimes \mathbb{R}$ with specified actions, and [5], [7], [11], and [12] for some profound results. On the other hand, M.A. Rieffel ([14]) initiated the notion of stable rank of C^*-algebras, and raised the important question of computing the stable rank of group C^*-algebras of Lie groups. See [17], [18], [19], [20] and [21] for some partial results on this question.

In this paper we mainly consider group C^*-algebras of Lie semi-direct products of the form $\mathbb{C}^n \rtimes_\alpha \mathbb{R}$ (we often omit the symbol α). These semi-direct products contain the Mautner group which is important as an example of simply connected, non type I, solvable Lie groups. In Section 2, we first treat the case that the action α of $\mathbb{R}$ on $\mathbb{C}^n$ is diagonal, and second the case with α non-diagonal. For the analysis in some cases, we use some results by P. Green ([6], [7]) frequently, and in other cases, we use some methods of foliation C^*-algebras by A. Connes ([2]; cf. [8]). Next, the general case is examined under observation of the diagonal or non-diagonal cases. That is, we explicitly construct finite composition series of group C^*-algebras of $\mathbb{C}^n \rtimes \mathbb{R}$ as explained in the abstract. The main result (Theorem 2.3) will be the first step in analyzing the group C^*-algebras of general solvable Lie groups. As a corollary, using some formulas of stable rank and connected stable rank obtained so far, we estimate these ranks of the group C^*-algebras considered in terms of groups. This gives a partial answer to Rieffel's problem as mentioned above and a partially extended version of the main result in [21] (cf. [18], [19]). In addition, the non-splitting of some exact sequences is proved in the case of these group C^*-algebras. In Section 3, we apply the main theorem in Section 2 to the cases of Lie semi-direct products of the form $\mathbb{R}^n \rtimes \mathbb{R}$, so that we obtain the similar result with complex cases. Finally, we present an example to illustrate some results in this paper.

NOTATIONS. We now review some notations and facts used later (cf. [1], [3], [14]).

Let G be a Lie group and $C^*(G)$ its full group C^*-algebra. Denote by $\widehat{G}_1$ the space of all 1-dimensional representations of G. If G is a simply connected, solvable Lie group, then $\widehat{G}_1$ is isomorphic to $\mathbb{R}^k$ with $k = \dim \widehat{G}_1$ as a topological group (cf. [21]).

For a C^*-algebra $\mathfrak{A}$, we denote by $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ the C^*-crossed product of $\mathfrak{A}$ by the action α of $\mathbb{R}$. We often write it as $\mathfrak{A} \rtimes \mathbb{R}$ when α is specified.

Let $\mathfrak{A}$ be a C^*-algebra. Denote by $\mathrm{sr}(\mathfrak{A}), \mathrm{csr}(\mathfrak{A}) \in \{1, 2, \ldots, \infty\}$ the stable rank, connected stable rank of $\mathfrak{A}$ respectively.

Let X be a locally compact T_2-space and $C_0(X)$ the C^*-algebra of all complex valued, continuous functions on X vanishing at infinity. Then we have that

$$\text{(F1)}: \quad \mathrm{sr}(C_0(X)) = [\dim X/2] + 1 \equiv \dim_{\mathbb{C}} X,$$

where $\dim X$ is the covering dimension of X and $[\,\cdot\,]$ is the Gauss symbol.

For an exact sequence of C^*-algebras $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$, we have that

$$\begin{aligned} \text{(F2)}: \quad \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \leqslant \mathrm{sr}(\mathfrak{A}) &\leqslant \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}), \\ \mathrm{csr}(\mathfrak{A}) &\leqslant \mathrm{csr}(\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}) \end{aligned}$$

where $\vee$ means the maximum (see [14] and [17]).

Denote by $\mathbb{K}$ the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. For $\mathfrak{A} \otimes \mathbb{K}$ the C^*-tensor product of a C^*-algebra $\mathfrak{A}$,

$$\text{(F3)}: \quad \mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = 2 \wedge \mathrm{sr}(\mathfrak{A}), \quad \mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leqslant 2 \wedge \mathrm{csr}(\mathfrak{A})$$

where $\wedge$ means the minimum (cf. [10], [14] and [17]).

2. STRUCTURE OF GROUP C^*-ALGEBRAS OF $\mathbb{C}^n \rtimes \mathbb{R}$

We analyze the structure of group C^*-algebras of Lie semi-direct products $\mathbb{C}^n \rtimes_\alpha \mathbb{R}$ with α general. First of all, since the action α induces a Lie group homomorphism from $\mathbb{R}$ to $\mathrm{GL}_n(\mathbb{C})$, we have the following diagram:

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{\alpha} & \mathrm{GL}_n(\mathbb{C}) & & t \mapsto \alpha_t \\ \uparrow & & \uparrow \mathrm{exp} & & \\ \mathbb{R} & \xrightarrow{\mathrm{d}\alpha} & \mathrm{M}_n(\mathbb{C}) & & t \mapsto t\mathrm{d}\alpha \end{array}$$

where $\mathrm{d}\alpha \in \mathrm{M}_n(\mathbb{C})$ is the differential of α at $t = 0$. Next, we consider the Jordan canonical form of $\mathrm{d}\alpha$. By taking a suitable base of $\mathbb{C}^n$ and denoting this base as the natural base of $\mathbb{C}^n$, we may have that $\mathrm{d}\alpha$ is equal to the diagonal sum $\bigoplus_{i=1}^{s} A_i$ of Jordan blocks as follows:

$$A_i = \begin{pmatrix} \mu_i & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & \mu_i \end{pmatrix} = D_i + N_i, \quad \text{or } A_i = D_i$$

where μ_i is an eigenvalue of $\mathrm{d}\alpha$, and D_i, N_i are respectively the diagonal, off diagonal part of A_i. Then we have that

$$\alpha_t = \exp(t\mathrm{d}\alpha) = \bigoplus_{i=1}^{s} \exp(tA_i)$$

on the decomposition of $\mathbb{C}^n$ associated with the Jordan decomposition of $\mathrm{d}\alpha$. Then via Fourier transform we have $C^*(\mathbb{C}^n \rtimes_\alpha \mathbb{R}) \cong C_0(\mathbb{C}^n) \rtimes_{\widehat{\alpha}} \mathbb{R}$, where $\widehat{\alpha}_t = \bigoplus_{i=1}^{s} \exp(tA_i^*)$ with A_i^* the adjoint matrix of A_i. Then since the origin $\{0_n\}$ of $\mathbb{C}^n$ is closed in $\mathbb{C}^n$ and invariant under $\widehat{\alpha}$, we obtain the following exact sequence:

$$0 \to C_0(\mathbb{C}^n \setminus \{0_n\}) \rtimes \mathbb{R} \to C_0(\mathbb{C}^n) \rtimes_{\widehat{\alpha}} \mathbb{R} \to C_0(\mathbb{R}) \to 0.$$

We will examine the structure of $C_0(\mathbb{C}^n \setminus \{0_n\}) \rtimes \mathbb{R}$ in the diagonal case, i.e., $A_i = D_i$ first and the non-diagonal case next in the following.

THE DIAGONAL CASE. When $n = 1$ we have the following non-trivial cases:
(1) Radial case: $\alpha_t(z) = \mathrm{e}^{\mu t + \mathrm{i}\theta t} z$ for $z \in \mathbb{C}$ and $t, \mu, \theta \in \mathbb{R}$ with $\mu \neq 0$;
(2) The rotation: $\alpha_t(z) = \mathrm{e}^{\mathrm{i}\theta t} z$ for $z \in \mathbb{C}$, $\theta, t \in \mathbb{R}$ with $\theta \neq 0$.

REMARK 1.1. We may assume that $\mu = 1 = \theta$ since every group C^*-algebra in each case has the same structure as given below.

For some uses below, we give the following definitions (cf. [6]):

DEFINITION 1.2. We say that the action α of $\mathbb{R}$ on a subset X of $\mathbb{C}^n$ is wandering if for any compact subset U of X, the set $\{g \in \mathbb{R} \mid \alpha_g(U) \cap U \neq \emptyset\}$ is relatively compact in $\mathbb{R}$.

REMARK 1.3. In the case (1) above, α is wandering and free on $\mathbb{C} \setminus \{0\}$.

DEFINITION 1.4. We say that the action α of $\mathbb{R}$ on $\mathbb{C}^n$ is diagonal if α_t is diagonal for any $t \in \mathbb{R}$, and that the diagonal action α is diagonally radial (respectively rotational) if the restriction of α to each direct factor $\mathbb{C}$ of $\mathbb{C}^n$ is radial (respectively the rotation).

Let $G = \mathbb{C}^n \rtimes_\alpha \mathbb{R}$, where α is diagonal. If α is trivial on a direct factor $\mathbb{C}$ of $\mathbb{C}^n$, then we have that $C^*(G) \cong C_0(\mathbb{C}) \otimes (C_0(\mathbb{C}^{n-1}) \rtimes_{\widehat{\alpha}} \mathbb{R})$. Thus we assume that α is non trivial on each $\mathbb{C}$ of $\mathbb{C}^n$. Since the direct product of k direct factors $\mathbb{C} \setminus \{0\}$ corresponding to each subset $\{i_j\}_{j=1}^k$ of $\{1, \ldots, n\}$, denoted by the same symbol $(\mathbb{C} \setminus \{0\})^k$, is closed in X_{k-1} defined below and invariant under $\widehat{\alpha}$, and these $(\mathbb{C} \setminus \{0\})^k$ are open and closed in their disjoint union, we have the following exact sequences $(1 \leqslant k \leqslant n)$ inductively

$$0 \to C_0(X_k) \rtimes \mathbb{R} \to C_0(X_{k-1}) \rtimes \mathbb{R} \to \bigoplus_{1 \leqslant i_1 < \cdots < i_k \leqslant n} C_0((\mathbb{C} \setminus \{0\})^k) \rtimes \mathbb{R} \to 0$$

with $X_0 = \mathbb{C}^n \setminus \{0_n\}$, $X_n = (\mathbb{C} \setminus \{0\})^n$, where $\bigoplus_{1 \leqslant i_1 < \cdots < i_k \leqslant n}$ means the direct sum of the combination $\binom{n}{k}$ direct factors. By taking refinements of the above exact sequences, $C^*(G)$ has a finite composition series $\{\mathfrak{I}_l\}_{l=1}^K$ with $\mathfrak{I}_K = C^*(G)$, $\mathfrak{I}_0 = \{0\}$ and

$$\mathfrak{I}_l / \mathfrak{I}_{l-1} \cong \begin{cases} C_0(\widehat{G}_1) = C_0(\mathbb{R}) & \text{for } l = K \equiv 1 + \sum_{k=1}^{n} \binom{n}{n-k+1}, \\ C_0((\mathbb{C} \setminus \{0\})^{n-j+1}) \rtimes \mathbb{R}, & \end{cases}$$

for $\sum_{k=1}^{j-1} \binom{n}{n-k+1} + 1 \leqslant l \leqslant \sum_{k=1}^{j} \binom{n}{n-k+1}$ with $1 \leqslant j \leqslant n$.

In the cases with diagonally radial actions on $(\mathbb{C} \setminus \{0\})^k$ where k is as in the above setting, it is clear that α is free and wandering on it. Hence by Green's result ([6], Corollary 15), we get that

$$C_0((\mathbb{C} \setminus \{0\})^k) \rtimes_{\widehat{\alpha}} \mathbb{R} \cong C_0((\mathbb{C} \setminus \{0\})^k / \mathbb{R}) \otimes \mathbb{K}$$

where $(\mathbb{C} \setminus \{0\})^k / \mathbb{R}$ is the quotient space by $\widehat{\alpha}$, and it is homeomorphic to $\mathbb{T} \times (\mathbb{C} \setminus \{0\})^{k-1}$.

In the cases with diagonally rotational actions on $(\mathbb{C} \setminus \{0\})^k$ as above, we let

$$\alpha_t(z_{i_1}, \ldots, z_{i_k}) = (\mathrm{e}^{\mathrm{i}\theta_{i_1} t} z_{i_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_{i_k} t} z_{i_k}), \quad z_{i_j} \in \mathbb{C} \setminus \{0\},\, t \in \mathbb{R},\, \theta_{i_j} \in \mathbb{R} \setminus \{0\},$$

$(1 \leqslant j \leqslant k)$. Then $\widehat{\alpha}_t(w_{i_1}, \ldots, w_{i_k}) = (\mathrm{e}^{-\mathrm{i}\theta_{i_1} t} w_{i_1}, \ldots, \mathrm{e}^{-\mathrm{i}\theta_{i_k} t} w_{i_k})$ with $w_{i_j} \in \mathbb{C} \setminus \{0\}$. By the identification between $w \in \mathbb{C} \setminus \{0\}$ and $(|w|, w/|w|) \in \mathbb{R}_+ \times \mathbb{T}$, we obtain that

$$C_0((\mathbb{C} \setminus \{0\})^k) \rtimes_{\widehat{\alpha}} \mathbb{R} \cong C_0(\mathbb{R}_+^k) \otimes (C(\mathbb{T}^k) \rtimes_{\widehat{\alpha}} \mathbb{R}).$$

Set $\Theta(i_1, \ldots, i_k) = \{\theta_{i_j}\}_{j=1}^k$ and $\Theta = \Theta(i_1, \ldots, i_k)$.

Then we examine the structure of $C(\mathbb{T}^k) \rtimes_{\widehat{\alpha}} \mathbb{R}$ in the following. We say that the elements of Θ are rationally dependent, when there is the least positive number q such that $q\theta_{i_j} = 0 \bmod 2\pi$ for any $1 \leqslant j \leqslant k$. If elements of Θ are not rationally dependent, then $G = \mathbb{C}^n \rtimes \mathbb{R}$ for $n = 2$ is the Mautner group.

In the case $k = 1$, the action of $\mathbb{R}$ on $\mathbb{T}$ is the rotation. By Green's imprimitivity theorem ([7], Corollary 2.10), we obtain that

$$C(\mathbb{T}) \rtimes \mathbb{R} \cong C(\mathbb{R}/\mathbb{R}_1) \rtimes \mathbb{R} \cong C^*(\mathbb{R}_1) \otimes \mathbb{K}(L^2(\mathbb{T})) \cong C(\mathbb{T}) \otimes \mathbb{K}$$

where $\mathbb{R}_1 = \mathbb{Z}$ is the stabilizer of $1 \in \mathbb{T}$. Hence, we obtain that

$$C_0(\mathbb{C} \setminus \{0\}) \rtimes_{\widehat{\alpha}} \mathbb{R} \cong C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{R}) \cong C(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}.$$

For the cases of $k \geqslant 2$, we use the methods of foliation C^*-algebras (see [2]). Since the stabilizers for all elements of $\mathbb{T}^k$ are isomorphic to $\mathbb{Z}$ or zero, hence discrete, according to whether elements of Θ are rationally dependent or not. So we obtain that

$$C(\mathbb{T}^k) \rtimes \mathbb{R} \cong C^*(\mathbb{T}^k, \mathfrak{F}_k)$$

which is the foliation C^*-algebra with the foliation $\mathfrak{F}_k$ consisting of all orbits in $\mathbb{T}^k$ by $\widehat{\alpha}$ (cf. [8], Proposition 6.5). Then the graph $\mathcal{G}$ of $\mathfrak{F}_k$ is the groupoid $\mathbb{T}^k \times \mathbb{R}$ with the range, source maps $\mathrm{r}, \mathrm{s} : \mathcal{G} \to \mathbb{T}^k$ respectively defined by $\mathrm{r}(w,t) = \widehat{\alpha}_t(w)$, $\mathrm{s}(w,t) = w$ for $w \in \mathbb{T}^k$, $t \in \mathbb{R}$. Moreover, we see that $(\mathbb{T}^k, \mathfrak{F}_k)$ is a foliated $\mathbb{T}^{k-1}$-bundle over $\mathbb{T}$ with $\mathfrak{F}_k$ transversal to each fibers $\mathbb{T}^{k-1}$. Therefore, we have that

$$C^*(\mathbb{T}^k, \mathfrak{F}_k) \cong C^*(\mathcal{G}_N) \otimes \mathbb{K} \cong (C(\mathbb{T}^{k-1}) \rtimes \mathbb{Z}) \otimes \mathbb{K}$$

where $C^*(\mathcal{G}_N)$ is the reduced groupoid C^*-algebra of the reduced graph $\mathcal{G}_N$ which is the groupoid $N \times \mathbb{Z} = (\{1\} \times \mathbb{T}^{k-1}) \times \mathbb{Z}$ with the range, source maps $\mathrm{r}_N, \mathrm{s}_N : \mathcal{G}_N \to N$ respectively defined by $\mathrm{r}_N(w,n) = \widehat{\alpha}_{2\pi\theta_{i_1}^{-1} n}(w)$, $\mathrm{s_N}(w,n) = w$ for $w \in N$, $n \in \mathbb{Z}$. The case of $k = 1$ is recovered since $C(\{1\}) \rtimes \mathbb{Z} \cong C(\mathbb{T})$ with $\{1\} = N \subset \mathbb{T}$. Moreover, we see that $C(\mathbb{T}^{k-1}) \rtimes \mathbb{Z}$ is a special case of noncommutative tori, say $\mathfrak{A}_{\Theta(i_2,\ldots,i_k)}$ (cf. [1], [5]). Note that elements of $\Theta(i_1,\ldots,i_k)$ are not rationally dependent if and only if $\widehat{\alpha}$ is minimal on $\mathbb{T}^{k-1}$, i.e. any orbit by $\widehat{\alpha}$ is dense in $\mathbb{T}^{k-1}$. In this case, we see that $\mathfrak{A}_{\Theta(i_2,\ldots,i_k)}$ is simple (cf. [13]). As a remarkable fact, it is known that if $\mathfrak{A}_{\Theta(i_2,\ldots,i_k)}$ is simple, then it is an inductive limit of direct sums of matrix algebras over $C(\mathbb{T})$ (cf. [5]).

We next treat the general cases with diagonal actions. By taking a suitable base of $\mathbb{C}^n$, we have a decomposition $\mathbb{C}^n = \prod_{i=0}^{2} \mathbb{C}^{n_i}$ for which α is trivial on $\mathbb{C}^{n_0}$, diagonally radial on $\mathbb{C}^{n_1}$, and diagonally rotational on $\mathbb{C}^{n_2}$. Then $C^*(G)$ is obtained by iterating finitely many extensions by the C^*-algebras of the following form:

$$C_0(\mathbb{C}^{n_0} \times (\mathbb{C} \setminus \{0\})^{k_1} \times (\mathbb{C} \setminus \{0\})^{k_2}) \rtimes \mathbb{R}$$

where $0 \leqslant k_i \leqslant n_i$ $(i = 1, 2)$. Denote by $C_0(X) \rtimes \mathbb{R}$ the C^*-algebra of the above form. If $k_1 \neq 0$, that is, an eigenvalue of $\mathrm{d}\alpha$ on X is nonzero and not purely imaginary, then α is free and wandering on X. By [6], Corollary 15, $C_0(X) \rtimes \mathbb{R} \cong C_0(X/\mathbb{R}) \otimes \mathbb{K}$, where $X/\mathbb{R}$ is the quotient space of X by $\mathbb{R}$, and homeomorphic to the following product space:

$$\mathbb{C}^{n_0} \times \mathbb{T} \times (\mathbb{C} \setminus \{0\})^{k_1 - 1} \times (\mathbb{C} \setminus \{0\})^{k_2}$$

since any orbit under $\widehat{\alpha}$ on X has the same type. If $k_1 = 0$, then $C_0(X) \rtimes \mathbb{R}$ is isomorphic to $C_0(\mathbb{C}^{n_0}) \otimes C_0((\mathbb{C} \setminus \{0\})^{k_3}) \rtimes \mathbb{R}$. If $k_i = 0$ for $i = 1, 2$, then we obtain that $C_0(X) \rtimes \mathbb{R} \cong C_0(\mathbb{C}^{n_0} \times \mathbb{R})$.

THE NON-DIAGONAL CASE. We next consider the case $s = 1$ in the above setting. We suppose that $A_1 = D_1 + N_1$. Let λ be the complex conjugate of μ_1. By direct computation, we have that

$$\exp(tA_1^*) = \exp(tD_1^* + tN^*) = \exp(tD_1^*)\exp(tN^*)$$
$$= \begin{pmatrix} e^{t\lambda} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda} \end{pmatrix} \begin{pmatrix} 1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & t^2/2! \\ & & & \ddots & t \\ 0 & & & & 1 \end{pmatrix}.$$

If λ is not purely imaginary and $\lambda \neq 0$, $\widehat{\alpha}$ is free and wandering on $\mathbb{C}^n \setminus \{0_n\}$. In fact, for any $z = (z_i) \in \mathbb{C}^n \setminus \{0_n\}$, since $z_k \neq 0$ for some $k \geqslant 1$ and $z_j = 0$ for all $j > k$, we have $w_k = e^{t\lambda} z_k$ where $\widehat{\alpha}_t(z) = w = (w_i)$. Now, we use the following decomposition:

$$\mathbb{C}^n \setminus \{0_n\} = \bigcup_{k=1}^{n} \mathbb{C}^{k-1} \times (\mathbb{C} \setminus \{0\}) \times \{0_{n-k}\},$$

and $\mathbb{C}^{k-1} \times (\mathbb{C} \setminus \{0\}) \times \{0_{n-k}\}$ is closed in $(\mathbb{C}^n \setminus \{0_n\}) \setminus \bigcup_{l=1}^{k-1} \mathbb{C}^{l-1} \times (\mathbb{C} \setminus \{0\}) \times \{0_{n-l}\}$ and invariant under $\widehat{\alpha}$. Then $C_0(\mathbb{C}^n \setminus \{0_n\}) \rtimes \mathbb{R}$ has a finite composition series $\{\mathfrak{L}_k\}_{k=1}^n$ with $\mathfrak{L}_0 = \{0\}$ such that $\mathfrak{L}_k/\mathfrak{L}_{k-1} = C_0(\mathbb{C}^{n-k} \times (\mathbb{C} \setminus \{0\})) \rtimes \mathbb{R}$. Then the quotient space of $\mathbb{C}^{n-k} \times (\mathbb{C} \setminus \{0\})$ by $\mathbb{R}$ is homeomorphic to $\mathbb{C}^{n-k} \times \mathbb{T}$. Hence, by [7], Corollary 15, we have $C_0(\mathbb{C}^{n-k} \times (\mathbb{C} \setminus \{0\})) \rtimes \mathbb{R} \cong C_0(\mathbb{C}^{n-k} \times \mathbb{T}) \otimes \mathbb{K}$.

Next suppose λ is nonzero and purely imaginary, or $\lambda = 0$. Since the set $\{(z, 0_{n-1}) \in \mathbb{C}^n \mid z \in \mathbb{C} \setminus \{0\}\}$ is closed in $\mathbb{C}^n \setminus \{0_n\}$ and invariant under $\widehat{\alpha}$, we have that

$$0 \to C_0(\mathbb{C} \times (\mathbb{C}^{n-1} \setminus \{0_{n-1}\})) \rtimes \mathbb{R} \to C_0(\mathbb{C}^n \setminus \{0_n\}) \rtimes \mathbb{R} \to C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R} \to 0.$$

Note that the action of $\mathbb{R}$ on $\mathbb{C} \setminus \{0\}$ is trivial or the rotation. We show that $\widehat{\alpha}$ is free and wandering on $\mathbb{C} \times (\mathbb{C}^{n-1} \setminus \{0_{n-1}\})$. Take an element $z = (z_i) \in \mathbb{C} \times (\mathbb{C}^{n-1} \setminus \{0_{n-1}\})$ and set $w = (w_i) = \widehat{\alpha}_t(z)$. Then there is $k \geqslant 2$ such that $z_k \neq 0$ and $z_j = 0$ for any $j > k$. If $z_{k-1} = 0$, then $w_{k-1} = te^{t\lambda} z_k$. Hence, the claim holds. If $z_{k-1} \neq 0$, then $w_{k-1} = e^{t\lambda} z_{k-1} + te^{t\lambda} z_k$. The claim follows in the case $\lambda = 0$. In the cases $\lambda \neq 0$, if $\widehat{\alpha}_t(z) = z$ for some $t \neq 0$, then we have $te^{t\lambda} z_k = (1 - e^{t\lambda}) z_{k-1}$. This equation holds for integral multiples of t, which is impossible. Indeed, for any $l \in \mathbb{Z}$, we have $t(e^{-lt\lambda} - 1) = lt(e^{-t\lambda} - 1)$, which implies $2 \geqslant |e^{-lt\lambda} - 1| = |l(e^{-t\lambda} - 1)|$.

Now, we apply the same argument above to view $\mathbb{C} \times (\mathbb{C}^{n-1} \setminus \{0_{n-1}\})$ as the disjoint union of Y_k, where $Y_k = \mathbb{C} \times (\mathbb{C}^{k-2} \times (\mathbb{C} \setminus \{0\}) \times \{0_{n-k}\})$ $(2 \leqslant k \leqslant n)$. Then we see that the orbit space of Y_k is homeomorphic to $\mathbb{C}^{k-2} \times (\mathbb{C} \setminus \{0\}) \times \mathbb{R}$.

Moreover, the quotient space $(\mathbb{C} \times (\mathbb{C} \setminus \{0\}))/\mathbb{R}$ by $\widehat{\alpha}$ is obtained by identifying points in the subset

$$\{(\mathrm{e}^{t\lambda} z_{k-1} + t\mathrm{e}^{t\lambda} z_k, \mathrm{e}^{t\lambda} z_k) \in \mathbb{C} \times (\mathbb{C} \setminus \{0\}) \mid t \in \mathbb{R}\}.$$

If $\lambda = 0$, we have the following fiber structure of $(\mathbb{C}\times(\mathbb{C}\setminus\{0\}))/\mathbb{R}$ with $\mathbb{R}$ the base space and $(\mathbb{C}\setminus\{0\})$ fibers since the orbit of (z_{k-1}, z_k) by $\widehat{\alpha}$ corresponds to the point $(t_0, z_k) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\})$ such that $(z_{k-1} + t_0 z_k, z_k)$ is in the line orthogonal to the line $\{(tz_k, z_k) \mid t \in \mathbb{R}\}$ and containing $\{(0, z_k)\}$. Since any orbit under $\widehat{\alpha}$ in $\mathbb{C} \times (\mathbb{C} \setminus \{0\})$ has the same type, we obtain $(\mathbb{C} \times (\mathbb{C} \setminus \{0\}))/\mathbb{R} \approx \mathbb{R} \times (\mathbb{C} \setminus \{0\})$.

If λ is nonzero and purely imaginary, we see that the orbit of (z_{k-1}, z_k) by $\widehat{\alpha}$ corresponds to the point $(t_0, \mathrm{e}^{t_0\lambda} z_k) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\})$ such that $(\mathrm{e}^{t_0\lambda}(z_{k-1} + t_0 z_k), \mathrm{e}^{t_0\lambda} z_k)$ is in the product space $L \times (\mathbb{C} \setminus \{0\})$, where L is the line orthogonal to the line $\{tz_k \in \mathbb{C} \mid t \in \mathbb{R}\}$ and containing $\{0\}$. Then we have that $(\mathbb{C} \times (\mathbb{C} \setminus \{0\}))/\mathbb{R} \approx \mathbb{R} \times (\mathbb{C} \setminus \{0\})$.

Therefore, when λ is nonzero and purely imaginary, or $\lambda = 0$, we get that

$$Y_k/\mathbb{R} \approx \mathbb{C}^{k-2} \times ((\mathbb{C} \times (\mathbb{C} \setminus \{0\}))/\mathbb{R}) \approx \mathbb{C}^{k-2} \times \mathbb{R} \times (\mathbb{C} \setminus \{0\}).$$

Hence, by using Green's result ([6], Corollary 15) again, we obtain a finite composition series $\{\mathfrak{K}_j\}_{j=1}^{n-1}$ of $C_0(\mathbb{C} \times (\mathbb{C}^{n-1} \setminus \{0_{n-1}\})) \rtimes \mathbb{R}$ with $\mathfrak{K}_0 = \{0\}$ such that

$$\mathfrak{K}_j/\mathfrak{K}_{j-1} \cong C_0(\mathbb{C}^{n-j-1} \times \mathbb{R} \times (\mathbb{C} \setminus \{0\})) \otimes \mathbb{K}.$$

THE GENERAL CASE. Summing up we obtain the main result for the structure of group C^*-algebras for the Lie semi-direct products $\mathbb{C}^n \rtimes \mathbb{R}$.

THEOREM 2.1. *If G is a Lie semi-direct product $\mathbb{C}^n \rtimes \mathbb{R}$ $(n \geqslant 1)$, then $C^*(G)$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^{K}$ with $\mathfrak{I}_K = C^*(G)$, $\mathfrak{I}_0 = \{0\}$ such that*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0(\mathbb{C}^{n_0+u} \times \mathbb{R}) = C_0(\widehat{G}_1) & \text{for } j = K, \\ C_0(\mathbb{C}^{n_0+s_j} \times (\mathbb{C} \setminus \{0\})^{t_j} \times \mathbb{T}) \otimes \mathbb{K} & \text{or,} \\ C_0(\mathbb{C}^{n_0+s_j} \times (\mathbb{C} \setminus \{0\})^{t_j} \times \mathbb{R}) \otimes \mathbb{K} & \text{or,} \\ C_0(\mathbb{C}^{n_0+s_j} \times \mathbb{R}_+^{u_j}) \otimes \mathfrak{A}_{\Theta(i_2,\ldots,i_{u_j})} \otimes \mathbb{K} & \text{for } 1 \leqslant j \leqslant K-1; \end{cases}$$

with $0 \leqslant n_0 \leqslant n$ and $0 \leqslant s_j, t_j \leqslant n - n_0$ and $2 \leqslant u_j \leqslant n - n_0$,
$s_j + t_j + 1 \leqslant n - n_0$, and $s_j + u_j \leqslant n - n_0$, and
n_0 the number of zero Jordan blocks of $\mathrm{d}\alpha$, and u the number of nonzero Jordan blocks of $\mathrm{d}\alpha$ with zero on the diagonal,

where the second (respectively third) case occurs when some (respectively every) eigenvalue of $\mathrm{d}\alpha$ on an $\widehat{\alpha}$-invariant subspace of $\mathbb{C}^n$ is nonzero and not purely imaginary (respectively zero or purely imaginary but $\widehat{\alpha}$ is not diagonal), and the fourth case occurs when $\widehat{\alpha}$ is rotational on an $\widehat{\alpha}$-invariant subspace of $\mathbb{C}^n$, and $\mathfrak{A}_{\Theta(i_2,\ldots,i_{u_j})}$ is simple if and only if elements of $\Theta(i_1,\ldots,i_{u_j})$ are not rationally dependent.

Proof. In the above setting we may have the direct sum $\mathbb{C}^n = \prod_{i=0}^{l} \mathbb{C}^{n_i}$ with each $\mathbb{C}^{n_i}$ $\widehat{\alpha}$-invariant, where $\widehat{\alpha}$ is trivial on $\mathbb{C}^{n_0}$, diagonal on $\mathbb{C}^{n_1}$ and non trivial on each direct factor of $\mathbb{C}^{n_1}$, and non-diagonal on each $\mathbb{C}^{n_i}$ $(2 \leqslant i \leqslant l)$ such that

$$\widehat{\alpha}_t = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \oplus \begin{pmatrix} e^{t\overline{\lambda_{11}}} & & 0 \\ & \ddots & \\ 0 & & e^{t\overline{\lambda_{1n_1}}} \end{pmatrix} \oplus \left(\bigoplus_{i=2}^{l} e^{t\overline{\lambda_i}} \begin{pmatrix} 1 & t & & * \\ & \ddots & \ddots & \\ & & \ddots & t \\ 0 & & & 1 \end{pmatrix} \right)$$

where λ_{1k} $(1 \leqslant k \leqslant n_1)$, λ_i $(2 \leqslant i \leqslant l)$ are eigenvalues of $\mathrm{d}\alpha$. Moreover, we may assume that the eigenvalue of $\mathrm{d}\alpha$ is zero on $\mathbb{C}^{n_i}$ for any $2 \leqslant i \leqslant i_0$ with some $i_0 \leqslant l$. Then since $\mathbb{C}^{n_0}$ and the direct factor $\mathbb{C} \times \{0_{n_i-1}\}$ of $\mathbb{C}^{n_i}$ for $2 \leqslant i \leqslant i_0$ are fixed under $\widehat{\alpha}$, we obtain that $\widehat{G}_1$ is homeomorphic to $\mathbb{C}^{n_0} \times \mathbb{C}^{i_0-1} \times \mathbb{R}$, say $u = i_0 - 1$. Then it is clear that n_0 is equal to the number of zero Jordan blocks of $\mathrm{d}\alpha$, and u is equal to the number of nonzero Jordan blocks of $\mathrm{d}\alpha$ with zero on the diagonal.

From the analysis in this section, we have a finite composition series of $C^*(G)$ such that its each subquotient has the following form:

$$C_0(\mathbb{C}^{n_0} \times (\mathbb{C} \setminus \{0\})^{k_1} \times (\mathbb{C} \setminus \{0\})^{k_2} \times \prod_{m=p_1}^{p_t} (\mathbb{C}^{h_m} \times (\mathbb{C} \setminus \{0\}))) \rtimes \mathbb{R}$$

where $k_1 + k_2 \leqslant n_1$, and $0 \leqslant h_m \leqslant n_m - 1$, and $2 \leqslant p_1 < \cdots < p_t \leqslant l$. Denote by $C_0(X) \rtimes \mathbb{R}$ the above C^*-algebra. From construction, we notice that dimension of X increases when k_1 or k_2 or t increase, but the indices of corresponding subquotients decrease.

If the eigenvalue of $\mathrm{d}\alpha$ on some $\mathbb{C}^{h_m} \times (\mathbb{C} \setminus \{0\})$ is nonzero and not purely imaginary, or $k_1 \neq 0$, then $\widehat{\alpha}$ is free and wandering on X. It follows from [6], Corollary 15 that $C_0(X) \rtimes \mathbb{R} \cong C_0(X/\mathbb{R}) \otimes \mathbb{K}$. In the first case, the quotient space $X/\mathbb{R}$ by $\widehat{\alpha}$ has the fiber structure with $(\mathbb{C} \times (\mathbb{C} \setminus \{0\}))/\mathbb{R}$ the base space and fibers given by

$$\mathbb{C}^{n_0} \times (\mathbb{C} \setminus \{0\})^{k_1+k_2} \times \mathbb{C}^{h_m-1} \times \prod_{q=p_1, q\neq m}^{p_t} (\mathbb{C}^{h_q} \times (\mathbb{C} \setminus \{0\})).$$

In the second case, $X/\mathbb{R}$ has the fiber structure with $\mathbb{T}$ the base space and fibers

$$\mathbb{C}^{n_0} \times (\mathbb{C} \setminus \{0\})^{k_1-1} \times (\mathbb{C} \setminus \{0\})^{k_2} \times \prod_{m=p_1}^{p_t} (\mathbb{C}^{h_m} \times (\mathbb{C} \setminus \{0\})).$$

In both cases, each fiber structure splits into the product space since any orbit under $\widehat{\alpha}$ in X has the same type when $t \in \mathbb{R}$ is large enough. Therefore, we have that the quotient space $X/\mathbb{R}$ is homeomorphic to $\mathbb{C}^{n_0 + \sum_{m=p_1}^{p_t} h_m} \times (\mathbb{C} \setminus \{0\})^{t-1+k_1+k_2} \times \mathbb{T}$.

In the cases left, if the eigenvalue of $d\alpha$ on every $\mathbb{C}^{h_m} \times (\mathbb{C} \setminus \{0\})$ is zero or purely imaginary, but $\widehat{\alpha}$ is not rotational on $\prod_{m=p_1}^{p_t} (\mathbb{C}^{h_m} \times (\mathbb{C} \setminus \{0\}))$, then

$$X = \mathbb{C}^{n_0} \times (\mathbb{C} \setminus \{0\})^{k_2} \times \prod_{m=p_1}^{p_t} (\mathbb{C}^{h_m} \times (\mathbb{C} \setminus \{0\}))$$

on which $\widehat{\alpha}$ is also free and wandering. Then $X/\mathbb{R}$ has the fiber structure with the base space given by $(\mathbb{C} \times (\mathbb{C} \setminus \{0\}))/\mathbb{R}$ and fibers by

$$\mathbb{C}^{n_0} \times (\mathbb{C} \setminus \{0\})^{k_2} \times \mathbb{C}^{h_{p_1}-1} \times \prod_{m=p_2}^{p_t} (\mathbb{C}^{h_m} \times (\mathbb{C} \setminus \{0\})),$$

and $(\mathbb{C} \times (\mathbb{C} \setminus \{0\}))/\mathbb{R} \approx \mathbb{R} \times (\mathbb{C} \setminus \{0\})$. Since every orbit under $\widehat{\alpha}$ in X has the same type when $t \in \mathbb{R}$ is large enough, $X/\mathbb{R} \approx \mathbb{C}^{n_0 - 1 + \sum_{m=p_1}^{p_t} h_m} \times (\mathbb{C} \setminus \{0\})^{t+k_2} \times \mathbb{R}$.

In the cases left in the end, we have $X = \mathbb{C}^{n_0} \times (\mathbb{C} \setminus \{0\})^{k_2+t}$ where the action of $\mathbb{R}$ on $(\mathbb{C} \setminus \{0\})^{k_2+t}$ is diagonally rotational. ∎

REMARK 2.2. The above theorem will be the first result for the fine structure of group C^*-algebras for a class of simply connected solvable Lie groups including the Mautner group. The expressions of subquotients in some cases are reasonable by Green or Poguntke's results ([7], [12]). As a note, the construction of composition series in this theorem is a realization for Pedersen's result ([11]) in the case of those Lie semi-direct products.

COROLLARY 2.3. *Under the same situation as in Theorem* 2.1*, we obtain that*

$$2 \vee \dim_{\mathbb{C}} \widehat{G}_1 \leqslant \mathrm{sr}(C^*(G)) \leqslant \dim_{\mathbb{C}} \widehat{G}_1 + 1,$$
$$2 \leqslant \mathrm{csr}(C^*(G)) \leqslant \dim_{\mathbb{C}} \widehat{G}_1 + 1.$$

Proof. By repeatedly using (F2) and (F3) in Section 1, we obtain that

$$\mathrm{sr}(\mathfrak{I}_j) \leqslant 2, \quad \mathrm{csr}(\mathfrak{I}_j) \leqslant 2$$

for $1 \leqslant j \leqslant K-1$. We use the fact of [17] that $\mathrm{csr}(C_0(\mathbb{R}^n)) = 2$ for $n = 1$, 1 for $n = 2$ and $[(n+1)/2] + 1$ for $n \geqslant 3$. Since $\dim \widehat{G}_1 = 2k + 1$ $(k \geqslant 0)$, we have that

$$\dim_{\mathbb{C}} \widehat{G}_1 + 1 = k + 2, \quad 2 \vee \mathrm{csr}(C_0(\widehat{G}_1)) = [(\dim \widehat{G}_1 + 1)/2] + 1 = k + 2.$$

Moreover, for any simply connected, solvable Lie group G, we know that $\mathrm{sr}(C^*(G)) = 1$ if and only if $G \cong \mathbb{R}$ ([21], Lemma 3.7). On the other hand, by Connes' Thom isomorphism, we get the following calculation of K-groups of C^*-algebras:

$$\mathrm{K}_1(C^*(G)) \cong \mathrm{K}_1(C_0(\mathbb{C}^n) \rtimes \mathbb{R}) \cong \mathrm{K}_0(C_0(\mathbb{C}^n)) \cong \mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}.$$

Hence, by Elhage Hassan's result ([4], Corollary 1.6), we obtain $\mathrm{csr}(C^*(G)) \geqslant 2$. ∎

REMARK 2.4. It would be true that $\mathrm{sr}(C^*(G)) = 2 \vee \dim_{\mathbb{C}} \widehat{G}_1$ for any $G = \mathbb{C}^n \rtimes \mathbb{R}$ $(n \geqslant 1)$. This formula is true for any simply connected, solvable Lie group of type I with its dimension $\geqslant 2$ (cf. [21]). As a note, we get that $\mathrm{sr}(C_0(\mathbb{R}) \otimes C^*(G)) = \dim_{\mathbb{C}}(\mathbb{R} \times \widehat{G}_1)$.

Next, using K-theory of C^*-algebras, we give an answer to a fundamental question for extension of C^*-algebras about whether a short exact sequence is split or not.

PROPOSITION 2.5. *Under the same situation as in Theorem* 2.1*, we have the following exact sequence non split: with* $m = n_0 + u$,

$$0 \to \mathfrak{I}_{K-1} = C_0(\mathbb{C}^n \setminus \mathbb{C}^m \times \{0_{n-m}\}) \rtimes \mathbb{R} \to C^*(G) \to C_0(\mathbb{C}^m \times \mathbb{R}) = C_0(\widehat{G}_1) \to 0.$$

Proof. We calculate K-groups of C^*-algebras in the above exact sequence by using Connes' Thom isomorphism and Bott periodicity (cf. [2], [23]) as follows:

$$\begin{aligned}
\mathrm{K}_0(C_0(\mathbb{C}^n \setminus (\mathbb{C}^m \times \{0_{n-m}\})) \rtimes \mathbb{R}) &\cong \mathrm{K}_1(C_0(\mathbb{C}^m \times (\mathbb{C}^{n-m} \setminus \{0_{n-m}\}))) \\
&\cong \mathrm{K}_1(C_0(\mathbb{C}^{n-m} \setminus \{0_{n-m}\})) \cong \mathrm{K}_1(C_0(\mathbb{R}_+ \times S^{2(n-m)})) \\
&\cong \mathrm{K}_0(C_0(\mathbb{R}^{2(n-m)}) \oplus \mathbb{C}) \cong \mathbb{Z}^2, \\
\mathrm{K}_0(C^*(G)) \cong \mathrm{K}_1(C_0(\mathbb{C}^n)) \cong 0, &\quad \mathrm{K}_0(C_0(\mathbb{C}^m \times \mathbb{R})) \cong \mathrm{K}_1(\mathbb{C}^m) \cong 0
\end{aligned}$$

where S^{2n} denotes the $2n$-dimensional sphere. If the exact sequence in the statement is split, then so is the following exact sequence of K-groups (cf. [23], Corollary 8.2.2):

$$0 \to \mathrm{K}_0(\mathfrak{I}_{K-1}) \to \mathrm{K}_0(C^*(G)) \to \mathrm{K}_0(C_0(\widehat{G}_1)) \to 0,$$

that is, $0 \cong \mathbb{Z}^2 \oplus 0$, which is false. ∎

REMARK 2.6. The non-splitting of the exact sequence associated with the group C^*-algebra of the real generalized Heisenberg group of the form $\mathbb{R}^{n+1} \rtimes_\alpha \mathbb{R}^n$ $(n \geqslant 1)$ is obtained by Kasparov (cf. [16]). The above proposition extends the case of $n = 1$.

3. STRUCTURE OF GROUP C^*-ALGEBRAS OF $\mathbb{R}^n \rtimes \mathbb{R}$

In this section, we first give an application of our main result Theorem 2.1 as follows:

THEOREM 3.1. *If* G *is a Lie semi-direct product* $\mathbb{R}^n \rtimes \mathbb{R}$ $(n \geqslant 1)$*, then* $C^*(G)$ *has a finite composition series* $\{\mathfrak{D}_j\}_{j=1}^K$ *with* $\mathfrak{D}_K = C^*(G)$, $\mathfrak{D}_0 = \{0\}$ *such that*

$$\mathfrak{D}_j/\mathfrak{D}_{j-1} \cong \begin{cases} C_0(\mathbb{R}^{n'+1}) = C_0(\widehat{G}_1) & \text{for } j = K, \\ C_0(\Omega_j) \otimes \mathbb{K} & \text{or}, \\ C_0(\Omega_j') \otimes \mathfrak{A}_{\Theta(i_2,\dots,i_{u_j})} \otimes \mathbb{K} & \text{or}, \\ C_0(\Omega_j') \otimes \mathfrak{B}_j \otimes \mathbb{K} & \text{for } 1 \leqslant j \leqslant K-1; \end{cases}$$

$$\text{with } 0 \leqslant n' \leqslant n \text{ and } 0 \leqslant s_j, t_j \leqslant n - n',$$
$$1 \leqslant u_j \leqslant n - n' \text{ and } s_j + t_j + 1, s_j + u_j \leqslant n - n'$$

where Ω_j *is a closed subspace of* $\mathbb{C}^{n'+s_j} \times (\mathbb{C} \setminus \{0\})^{t_j} \times \mathbb{T}$ *or* $\mathbb{C}^{n'+s_j} \times (\mathbb{C} \setminus \{0\})^{t_j} \times \mathbb{R}$ *with* $0 \leqslant s_j, t_j \leqslant n - n'$ *and* $s_j + t_j + 1 \leqslant n - n'$, Ω'_j *is a closed subspace of* $\mathbb{C}^{n'+s_j} \times \mathbb{R}_+^{u_j}$ *with* $2 \leqslant u_j \leqslant n - n'$ *and* $s_j + u_j \leqslant n - n'$, *and* $\mathfrak{A}_{\Theta(i_2,\ldots,i_{u_j})}$ *is simple, and when it is not simple,* $\mathfrak{B}_j$ *is its quotient or itself.*

Proof. One can embed $G = \mathbb{R}^n \rtimes_\alpha \mathbb{R}$ into $\widetilde{G} = \mathbb{C}^n \rtimes_{\widetilde{\alpha}} \mathbb{R}$ with the action $\widetilde{\alpha}$ of $\mathbb{R}$ on $\mathbb{C}^n$ defined by $\widetilde{\alpha}_t(x + \mathrm{i}y) = \alpha_t(x) + \mathrm{i}\alpha_t(y)$ for $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. By definition of $\widetilde{\alpha}$, since $\mathbb{R}^n$ is invariant under $\widetilde{\alpha}$ and closed in $\mathbb{C}^n$, we have the following exact sequence:

$$0 \to C_0(\mathbb{C}^n \setminus \mathbb{R}^n) \rtimes_{\widetilde{\alpha}^\wedge} \mathbb{R} \to C^*(\widetilde{G}) \to C^*(G) \to 0$$

with $\widetilde{\alpha}^\wedge$ the dual action of $\widetilde{\alpha}$ on $\mathbb{C}^n$. Put $\mathfrak{I} = C_0(\mathbb{C}^n \setminus \mathbb{R}^n) \rtimes_{\widetilde{\alpha}^\wedge} \mathbb{R}$. By Theorem 2.1, we find a finite composition series $\{\mathfrak{I}_k\}_{k=1}^N$ of $C^*(\widetilde{G})$ such that each subquotient $\mathfrak{I}_k/\mathfrak{I}_{k-1}$ has the form as stated in it. Set $\mathfrak{D}_k = \mathfrak{I}_k + \mathfrak{I}/\mathfrak{I}$. Then, note that $\{\mathfrak{D}_k\}_{k=1}^K$ is a finite composition series of $C^*(G)$ with each subquotient satisfying the following:

$$\begin{aligned}\mathfrak{D}_k/\mathfrak{D}_{k-1} &= (\mathfrak{I}_k + \mathfrak{I}/\mathfrak{I})/(\mathfrak{I}_{k-1} + \mathfrak{I}/\mathfrak{I}) \cong (\mathfrak{I}_k + \mathfrak{I})/(\mathfrak{I}_{k-1} + \mathfrak{I}) \\ &= (\mathfrak{I}_k + \mathfrak{I}_{k-1} + \mathfrak{I})/(\mathfrak{I}_{k-1} + \mathfrak{I}) \cong \mathfrak{I}_k/(\mathfrak{I}_k \cap (\mathfrak{I}_{k-1} + \mathfrak{I}))\end{aligned}$$

(cf. [9], Remark 3.1.3). Since $\mathfrak{I}_{k-1} \subset \mathfrak{I}_k \cap (\mathfrak{I}_{k-1} + \mathfrak{I})$, we see that $\mathfrak{D}_k/\mathfrak{D}_{k-1}$ is isomorphic to a quotient C^*-algebra of $\mathfrak{I}_k/\mathfrak{I}_{k-1}$. In particular, $\mathfrak{D}_K/\mathfrak{D}_{K-1}$ is isomorphic to a quotient of $\mathfrak{I}_K/\mathfrak{I}_{K-1} \cong C_0((\widetilde{G})_1^\wedge)$, and thus isomorphic to $C_0(\widehat{G}_1)$. In other cases, when a noncommutative torus $\mathfrak{A}_\Theta$ is a tensor factor of $\mathfrak{I}_k/\mathfrak{I}_{k-1}$ and simple, from the form of $\mathfrak{I}_k/\mathfrak{I}_{k-1}$ and that $\mathbb{K}$, $\mathfrak{A}_\Theta \otimes \mathbb{K}$ are simple, we know that $\mathfrak{D}_k/\mathfrak{D}_{k-1}$ is isomorphic to the form as in this statement. And if $\mathfrak{A}_\Theta$ is not simple, $\mathfrak{D}_k/\mathfrak{D}_{k-1}$ is isomorphic to the form in the fourth case having a tensor factor which is a quotient of $\mathfrak{A}_\Theta$. ∎

REMARK 3.2. As a note, the similar result can be easily obtained in the cases for Lie semi-direct products of connected, commutative Lie groups by $\mathbb{R}$, i.e, $(\mathbb{R}^n \times \mathbb{T}^s) \rtimes \mathbb{R}$ for some $n, s \geqslant 0$ by applying the same methods to their universal covering groups $\mathbb{R}^{n+s} \rtimes \mathbb{R}$.

COROLLARY 3.3. *Under the assumption of Theorem* 3.1, *we obtain that*

$$\begin{cases} \mathrm{sr}(C^*(G)) = 2 \vee \dim_{\mathbb{C}} \widehat{G}_1 & \textit{if } \dim \widehat{G}_1 \textit{ is even,} \\ 2 \vee \dim_{\mathbb{C}} \widehat{G}_1 \leqslant \mathrm{sr}(C^*(G)) \leqslant \dim_{\mathbb{C}} \widehat{G}_1 + 1 & \textit{if } \dim \widehat{G}_1 \textit{ is odd;} \end{cases}$$

$$\begin{cases} \mathrm{csr}(C^*(G)) \leqslant 2 \vee \mathrm{csr}(C_0(\widehat{G}_1)) = [(\dim \widehat{G}_1 + 1)/2] + 1, \quad \textit{and} \\ \mathrm{csr}(C^*(G)) \geqslant 2 \quad \textit{if } \dim G \textit{ is odd.} \end{cases}$$

Proof. The claim follows from the same argument as in Corollary 2.3. ∎

REMARK 3.4. It would be true that $\mathrm{sr}(C^*(G)) = 2 \vee \dim_{\mathbb{C}} \widehat{G}_1$ for any $G = \mathbb{R}^n \rtimes \mathbb{R}$ $(n \geqslant 1)$. If $\dim \widehat{G}_1$ is odd, we know that $\mathrm{sr}(C_0(\mathbb{R}) \otimes C^*(G)) = \dim_{\mathbb{C}}(\mathbb{R} \times \widehat{G}_1)$.

Finally, we give an example to illustrate Theorem 2.1 and 3.1 as follows:

Example 3.5. Let $G = \mathbb{R}^2 \rtimes_\alpha \mathbb{R}$ with $\alpha_t(x,y) = (\mathrm{e}^t x, \mathrm{e}^{-t} y)$ for $t \in \mathbb{R}$, $x, y \in \mathbb{R}$ (cf. [15], [22]). Set $\widetilde{G} = \mathbb{C}^2 \rtimes_{\widetilde{\alpha}} \mathbb{R}$ with $\widetilde{\alpha}_t(z,w) = (\mathrm{e}^t z, \mathrm{e}^{-t} w)$ for $t \in \mathbb{R}$, $z, w \in \mathbb{C}$. Then we have the structure of $C^*(\widetilde{G})$ as follows:

$$0 \to C_0(\mathbb{C}^2 \setminus \{0_2\}) \rtimes_{\widetilde{\alpha}^\wedge} \mathbb{R} \to C^*(\widetilde{G}) \to C_0(\mathbb{R}) \to 0,$$

$$0 \to C_0((\mathbb{C} \setminus \{0\})^2) \rtimes \mathbb{R} \to C_0(\mathbb{C}^2 \setminus \{0_2\}) \rtimes_{\widetilde{\alpha}^\wedge} \mathbb{R} \to \bigoplus_{k=1}^{2} C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R} \to 0,$$

$$C_0((\mathbb{C} \setminus \{0\})^2) \rtimes \mathbb{R} \cong C_0((\mathbb{C} \setminus \{0\}) \times \mathbb{T}) \otimes \mathbb{K}, \quad C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R} \cong C(\mathbb{T}) \otimes \mathbb{K}.$$

Since $\mathbb{R}^2$ is invariant under $\widetilde{\alpha}^\wedge$ and closed in $\mathbb{C}^2$, we have the following:

$$0 \to \mathfrak{I} = C_0(\mathbb{C}^2 \setminus \mathbb{R}^2) \rtimes_{\widetilde{\alpha}^\wedge} \mathbb{R} \to C^*(\widetilde{G}) \to C^*(G) \to 0.$$

Then we construct a composition series $\{\mathfrak{D}_j\}_{j=1}^3$ of $C^*(G)$ with $\mathfrak{D}_3 = C^*(G)$ as follows:

$$\begin{aligned}
\mathfrak{D}_2 &= (C_0(\mathbb{C}^2 \setminus \{0_2\}) \rtimes_{\widetilde{\alpha}^\wedge} \mathbb{R} + \mathfrak{I})/\mathfrak{I} \\
&\cong C_0(\mathbb{C}^2 \setminus \{0_2\}) \rtimes_{\widetilde{\alpha}^\wedge} \mathbb{R}/(C_0(\mathbb{C}^2 \setminus \{0_2\}) \rtimes_{\widetilde{\alpha}^\wedge} \mathbb{R} \cap \mathfrak{I}) \cong C_0(\mathbb{R}^2 \setminus \{0_2\}) \rtimes \mathbb{R}, \\
\mathfrak{D}_1 &= C_0((\mathbb{C} \setminus \{0\})^2) \rtimes \mathbb{R} + \mathfrak{I})/\mathfrak{I} \cong C_0((\mathbb{C} \setminus \{0\})^2) \rtimes \mathbb{R}/(C_0((\mathbb{C} \setminus \{0\})^2) \rtimes \mathbb{R} \cap \mathfrak{I}) \\
&\cong C_0((\mathbb{R} \setminus \{0\})^2) \rtimes \mathbb{R} \cong \bigoplus_{k=1}^{4} C_0(\mathbb{R}_+^2) \rtimes \mathbb{R} \cong \bigoplus_{k=1}^{4} C_0(\mathbb{R}) \otimes \mathbb{K}.
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
\mathfrak{D}_3/\mathfrak{D}_2 &= C^*(G)/((C_0(\mathbb{C}^2 \setminus \{0_2\}) \rtimes_{\widetilde{\alpha}^\wedge} \mathbb{R} + \mathfrak{I})/\mathfrak{I}) \\
&\cong C^*(\widetilde{G})/(C_0(\mathbb{C}^2 \setminus \{0_2\}) \rtimes_{\widetilde{\alpha}^\wedge} \mathbb{R} + \mathfrak{I}) \cong C_0(\mathbb{R}), \\
\mathfrak{D}_2/\mathfrak{D}_1 &\cong C_0(\mathbb{R}^2 \setminus \{0_2\}) \rtimes \mathbb{R}/C_0((\mathbb{R} \setminus \{0\})^2) \rtimes \mathbb{R} \cong \bigoplus_{k=1}^{4} C_0(\mathbb{R}_+) \rtimes \mathbb{R} \cong \bigoplus_{k=1}^{4} \mathbb{K}
\end{aligned}$$

where the last isomorphism is a well known fact and is also obtained by [6], Corollary 15. Hence, the structure of $C^*(G)$ is given by

$$0 \to C_0(\mathbb{R}^2 \setminus \{0_2\}) \rtimes \mathbb{R} \to C^*(G) \to C_0(\mathbb{R}) \to 0,$$

$$0 \to \bigoplus_{k=1}^{4} C_0(\mathbb{R}) \otimes \mathbb{K} \to C_0(\mathbb{R}^2 \setminus \{0_2\}) \rtimes \mathbb{R} \to \bigoplus_{k=1}^{4} \mathbb{K} \to 0.$$

Remark 3.6. This example suggests that it is difficult to give the structure of $C^*(\mathbb{R}^n \rtimes_\alpha \mathbb{R})$ explicitly from that of $C^*(\mathbb{C}^n \rtimes_{\widetilde{\alpha}} \mathbb{R})$ in general, and it is easier to analyze it directly for specified actions α. But no systematic results more than Theorem 3.1 are known.

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