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SP-PROPERTY FOR A PAIR OF C^*-ALGEBRAS

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ABSTRACT. Recall that a C^*-algebra A has the SP-property if every non-zero hereditary C^*-subalgebra of A has a non-zero projection. Let $1 \in A \subset B$ be a pair of unital C^*-algebras. In this paper we investigate a sufficient condition for B to have the SP-property, given that A has it. In particular, if there exists a faithful conditional expectation E from B to A of index-finite type in the sense of Watatani, then B has the SP-property under the condition that A is simple with the SP-property. As an application, we have the structure theory of purely infinite simple C^*-algebras.

KEYWORDS: *C^*-index theory,* SP*-property, conditional expectation.*

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1. INTRODUCTION

A C^*-algebra A has the SP-property if every non-zero hereditary C^*-subalgebra of A has a non-zero projection. This concept has been studied by several mathematicians. For example, this concept is weaker than the real rank zero condition, which means that every hereditary C^*-subalgebra of A has an approximate identity of projections ([2], [19], and [9]). When A is a simple unital C^*-algebra, Jeong and the author ([13]) in the case of the integer group, and Kishimoto and Kumjian ([16]) in the case of a general discrete group G, proved that the reduced crossed product $A \times_{\alpha \mathrm{r}} G$ has the SP-property if A has the SP-property and α is a homomorphism from G into the set of automorphisms on A such that α_g is outer for all $g \in G$. In the case that G is finite, Jeong and the author ([14]) proved that any crossed product algebra $A \times_\alpha G$ has the SP-property when A has the SP-property. As an application, we showed that any crossed product algebra $A \times_\alpha G$ has the cancellation property under the additional condition that A has stable rank one, that is, the set of invertible elements in A is dense in it. Moreover, under the same condition if a given crossed product algebra has real rank zero, it also has stable rank one. Unfortunately, however, we do not know if a crossed product algebra of a UHF-algebra by $\mathbb{Z}_2$ has stable rank one, in general. Note that Elliott presented an example of a crossed product algebra of this type which does not have real rank zero ([8]).

In this paper, we consider a general condition for a pair of unital C^*-algebras with the same unit to have the SP-property. In particular, we consider this problem in the case of a conditional expectation from B to A of index-finite type in the sense of Watatani ([25]).

Our main theorem (Theorem 5.1) is that if there exists a faithful conditional expectation E from B to A of index-finite type, then B has the SP-property provided that A is simple with the SP-property. Before giving a proof of it, we consider the case that A is a purely infinite simple C^*-algebra in Section 4. There, we point out the existence of one pair of elements as a quasi-basis for E, and show that B is a direct sum of purely infinite simple C^*-algebras. This is a proof of an announcement by Izumi at the Fields Institute in 1995. We believe that our observation will be helpful in determining the stable rank of the crossed product algebra $A \times_\alpha G$ of a simple unital C^*-algebra A with stable rank one by a finite group G.

2. THE SP-PROPERTY

In this section we present a sufficient condition for B to have the SP-property, given that A has it.

The argument in Lemma 10 of [16] gives the following general result.

THEOREM *Let $1 \in A \subset B$ be a pair of C^*-algebras. Suppose that A has the* SP-*property and there is a faithful conditional expectation E from B to A. If for any non-zero positive element x in B and an arbitrary positive number $\varepsilon > 0$ there is an element y in A such that*

$$\|y^*(x - E(x))y\| < \varepsilon, \quad \|y^*E(x)y\| \geqslant \|E(x)\| - \varepsilon$$

then B has the SP-*property. Moreover, every non-zero hereditary C^*-subalgebra of B has a projection which is equivalent to some projection in A in the sense of Murray-von Neumann.*

Proof. Set $a = y^*E(x)y$. Consider the continuous functions f and g defined by

$$f(t) = \max(0, t - (1-\varepsilon)\|a\|), \quad g(t) = \min(t, (1-\varepsilon)\|a\|).$$

Note that $fg = (1-\varepsilon)\|a\|f$.

Since A has the SP-property, there is a non-zero projection p in $\overline{f(a)Af(a)}$. Then, there is an element $d_1 \in f(a)A$ such that $\|p - f(a)d_1\| < \frac{1}{3}$. So, $\|p - d_1^*f(a)^2d_1\| < 1$, and $\|p - pd_1^*f(a)^2d_1p\| < 1$. So, $pd_1^*f(a)^2d_1p$ is invertible in pAp. Hence, there is an element $d_2 \in pAp$ such that $p = d_2^*pd_1^*f(a)^2d_1pd_2$. Set $w = d_2^*pd_1^*f(a)^{\frac{1}{2}}$. Then, w in $\overline{f(a)Af(a)}$ such that $p = wf(a)w^*$.

Let $z_0 = (1-\varepsilon)^{-\frac{1}{2}}\|a\|^{-\frac{1}{2}}wf(a)^{\frac{1}{2}}$. Then $\|z_0\| = (1-\varepsilon)^{-\frac{1}{2}}\|a\|^{-\frac{1}{2}}$ and $z_0g(a)z_0^* = p$. Since $g(a) \leqslant a$, we have $p = z_0g(a)z_0^* \leqslant z_0az_0^*$. Thus, there exists an element $z \in pA$ such that

$$zaz^* = p, \quad \|z\| \leqslant \|z_0\| = (1-\varepsilon)^{-\frac{1}{2}}\|a\|^{-\frac{1}{2}}.$$

Hence, we have

$$\|zy^*xyz^* - p\| = \|zy^*(x - E(x))yz^*\| < \frac{\varepsilon}{1-\varepsilon} \times \frac{1}{\|E(x)\| - \varepsilon}.$$

Note that the last inequality follows from the fact that

$$\|a\|^{-1} \leqslant \frac{1}{\|E(x)\| - \varepsilon}.$$

We may assume that $\|zy^*xyz^* - p\| < 1$. Since $zy^*xyz^* \in pBp$, zy^*xyz^* is invertible in pBp; that is, there exists an element $z_1 \in B$ such that $z_1y^*xyz_1^* = p$. Therefore, since $\overline{z_1y^*x^{\frac{1}{2}}Bx^{\frac{1}{2}}yz_1^*} \cong \overline{x^{\frac{1}{2}}yz_1^*Bz_1y^*x^{\frac{1}{2}}}$ by Section 1.4 of [6], $\overline{xBx}$ has a projection which is equivalent to a projection p in A. Indeed, $p = z_1y^*xyz_1^* \sim x^{\frac{1}{2}}yz_1^*z_1y^*x^{\frac{1}{2}}$ in $\overline{x^{\frac{1}{2}}yz_1^*Bz_1y^*x^{\frac{1}{2}}} \subseteq \overline{xBx}$. ∎

Next, we consider the following stronger assumption on a conditional expectation E from B to A.

Definition Let $1 \in A \subset B$ be a pair of C^*-algebras. A conditional expectation E from B to A is called *outer* if for any element $x \in B$ with $E(x) = 0$ and any non-zero hereditary C^*-subalgebra C of A,

$$\inf\{\|cxc\| : c \in C^+,\ \|c\| = 1\} = 0.$$

The following result comes from the same argument as in Lemma 3.2 of [15] and Theorem 2.1.

Corollary *Let $1 \in A \subset B$ be a pair of C^*-algebras. Suppose that A has the* SP*-property and there is a faithful conditional expectation E from B to A. If E is outer, then B has the* SP*-property.*

Proof. For the reader we write a sketch of the proof. Let x be a non-zero element in B, and let $\varepsilon > 0$ be an arbitrary positive number. Consider a continuous function $f : [0, \|x\|] \to \mathbb{R}^+$ given by

$$f(t) = \begin{cases} 1, & t \geqslant \|E(x)\|; \\ \text{linear}, & \|E(x)\| - \varepsilon \leqslant t < \|E(x)\|; \\ 0, & t < \|E(x)\| - \varepsilon. \end{cases}$$

Let C be a hereditary C^*-subalgebra of A generated by $f(E(x))$. Then, since $\big\|E(x)c - \|E(x)\|c\big\| < \varepsilon$, for any positive $c \in C$ with norm one, $\|cE(x)c\| > \|E(x)\| - \varepsilon$. Indeed, since $\big\|c(E(x)c - \|E(x)\|c)\big\| < \varepsilon$, we have $\big\|cE(x)c - \|E(x)\|c^2\big\| < \varepsilon$. Hence,

$$\|E(x)\| = \big\|\,\|E(x)\|c^2\big\| = \big\|cE(x)c + \|E(x)\|c^2 - cE(x)c\big\| < \|cE(x)c\| + \varepsilon.$$

From the outerness of E, for any arbitrary positive number $\varepsilon > 0$ there is a positive element $y \in C$ with norm one such that

$$\|y(x - E(x))y\| < \varepsilon, \quad \|yE(x)y\| > \|E(x)\| - \varepsilon.$$

Hence, B has the SP-property by Theorem 2.1. ∎

We present some examples of a pair of C^*-algebras with an outer conditional expectation.

EXAMPLE ([15]) *Let G be a discrete group and let α be a representation of G by automorphisms of a simple unital C^*-algebra A. Suppose α is outer. Then, the canonical conditional expectation from the reduced crossed product $A \times_{\alpha \mathrm{r}} G$ to A is outer.*

Proof. Let $u_g, g \in G$ be the standard unitaries in the multiplier algebra of $A \times_{\alpha \mathrm{r}} G$ implementing α. Let e be the identity of G. Let x be an element in $A \times_{\alpha \mathrm{r}} G$ with $E(x) = 0$. We approximate x by an element of the dense $*$-algebra spanned by $Au_g, g \in G$, and hence we may assume that $x = \sum_{i=1}^{n} c_i u_{g_i}$, where $c_i \in A$, and $g_1, \ldots, g_n$ are distinct elements of $G \setminus \{e\}$.

By Lemma 3.2 of [15], for any $\varepsilon > 0$ there is a positive element $c \in C$ such that $\|c\| = 1$,

$$\|c c_i u_{g_i} c\| < \frac{\varepsilon}{n}, \quad i = 1, \ldots, n.$$

Hence,

$$\|cxc\| = \Big\| c \Big(\sum_{i=1}^{n} c_i u_{g_i} \Big) c \Big\| \leqslant \sum_{i=1}^{n} \|c c_i u_{g_i} c\| < \varepsilon.$$

This completes the proof. ∎

EXAMPLE *Let ρ be a corner endomorphism of a unital C^*-algebra A, and let E be the canonical conditional expectation from the crossed product $A \times_\rho \mathbb{N}$ by ρ to A. Suppose that*

$$\widetilde{\mathbb{T}}(\rho) = \{\lambda \in \mathbb{T} : \widehat{\rho}(I) = I \ \textit{for} \ \forall I \in \mathrm{Prime}(A \times_\rho \mathbb{N})\} = \mathbb{T}.$$

Then E is outer.

Proof. This comes from the same argument as in Example 2.4 modifying Proposition 2.2 of [13]. ∎

We now have the structure theorem for the pure infiniteness of a simple crossed product algebra of a purely infinite simple C^*-algebra by a discrete group.

COROLLARY ([12], [16]) *Let A be a purely infinite simple C^*-algebra, G a discrete group, and α an action of G on A. Suppose that α is outer. Then the reduced crossed product $A \times_{\alpha \mathrm{r}} G$ is a purely infinite simple C^*-algebra.*

Proof. In the case of a countable abelian group G see [12], Corollary 3.3. In the case of a general discrete group G see [16], Lemma 10. ∎

3. C^*-INDEX THEORY

In this section, we summarize the C^*-index theory of Watatani ([25]).

Let $1 \in A \subseteq B$ be a pair of C^*-algebras, and let $E : B \to A$ be a faithful conditional expectation from B to A.

A finite family $\{(u_1, v_1), \ldots, (u_n, v_n)\}$ in $B \times B$ is called a *quasi-basis* for E if

$$\sum_{i=1}^{n} u_i E(v_i b) = \sum_{i=1}^{n} E(b u_i) v_i = b \quad \text{for } b \in B.$$

We say that a conditional expectation E is of *index-finite type* if there exists a quasi-basis for E. In this case the index of E is defined by

$$\mathrm{Index}(E) = \sum_{i=1}^{n} u_i v_i.$$

Note that $\mathrm{Index}(E)$ does not depend on the choice of a quasi-basis and every conditional expectation E of index-finite type on a C^*-algebra has a quasi-basis of the form $\{(u_1, u_1^*), \ldots, (u_n, u_n^*)\}$ (Lemma 2.1.6, [25]). Moreover, $\mathrm{Index}(E)$ is always contained in the centre of B, so that it is a scalar whenever B has a trivial centre, in particular when B is simple.

Let $E : B \to A$ be a faithful conditional expectation. Then $B_A (= B)$ is a pre-Hilbert module over A with an A-valued inner product

$$\langle x, y \rangle = E(x^* y), \quad x, y \in B_A.$$

Let $\mathcal{E}$ be the completion of B_A with respect to the norm on B_A defined by

$$\|x\|_{B_A} = \|E(x^* x)\|_A^{\frac{1}{2}}, \quad x \in B_A.$$

Then $\mathcal{E}$ is a Hilbert C^*-module over A. Since E is faithful, the canonical map $B \to \mathcal{E}$ is injective. Let $L_A(\mathcal{E})$ be the set of all (right) A-module homomorphisms $T : \mathcal{E} \to \mathcal{E}$ with an adjoint A-module homomorphism $T^* : \mathcal{E} \to \mathcal{E}$ such that

$$\langle T\xi, \zeta \rangle = \langle \xi, T^* \zeta \rangle \quad \xi, \zeta \in \mathcal{E}.$$

Then $L_A(\mathcal{E})$ is a C^*-algebra with the operator norm $\|T\| = \sup\{\|T\xi\| : \|\xi\| = 1\}$. There is an injective $*$-homomorphism $\lambda : B \to L_A(\mathcal{E})$ defined by

$$\lambda(b)x = bx$$

for $x \in B_A$, $b \in B$, so that B can be viewed as a C^*-subalgebra of $L_A(\mathcal{E})$. Note that the map $e_A : B_A \to B_A$ defined by

$$e_A x = E(x), \quad x \in B_A$$

is bounded and thus it can be extended to a bounded linear operator, denoted by e_A again, on $\mathcal{E}$. Then $e_A \in L_A(\mathcal{E})$ and $e_A = e_A^2 = e_A^*$; that is, e_A is a projection in $L_A(\mathcal{E})$.

The *(reduced) C^*-basic construction* is a C^*-subalgebra of $L_A(\mathcal{E})$ defined to be

$$C^*(B, e_A) = \overline{\mathrm{span}\{\lambda(x) e_A \lambda(y) \in L_A(\mathcal{E}) : x, y \in B\}}^{\|\cdot\|}$$

([25], Definition 2.1.2).

Then,

LEMMA ([25], Lemma 2.1.4) (i) $e_A C^*(B, e_A) e_A = \lambda(A) e_A$.
(ii) $\psi : A \to e_A C^*(B, e_A) e_A$, $\psi(a) = \lambda(a) e_A$, *is a* $*$*-isomorphism (onto).*

LEMMA ([25], Lemma 2.1.5) *The following are equivalent:*
(i) $E : B \to A$ *is of index-finite type.*
(ii) $C^*(B, e_A)$ *has an identity and there exists a number* c *with* $0 < c < 1$ *such that*

$$E(x^* x) \geqslant c(x^* x), \quad x \in B.$$

The above inequality was shown first in [20] by Pimsner and Popa for the conditional expectation $E_N : M \to N$ from a type II_1 factor M onto its subfactor N (c can be taken as the inverse of the Jones index $[M : N]$).

The conditional expectation $E_B : C^*(B, e_A) \to B$ defined by

$$E_B(\lambda(x) e_A \lambda(y)) = (\mathrm{Index}(E))^{-1} xy, \quad x, y \in B$$

is called *the dual conditional expectation* of $E : B \to A$. If E is of index-finite type, so is E_B with a quasi-basis $\{(w_i, w_i^*)\}$, where $w_i = \sqrt{\mathrm{Index}(E)} u_i e_A$, and $\{(u_i, u_i^*)\}$ is a quasi-basis for E ([25], Proposition 2.3.4).

Even if $\mathrm{Index}(E)$ is scalar, we do not know the relation between the number of pairs in a quasi-basis and $\mathrm{Index}(E)$. Izumi, however, showed recently that if we extend a conditional expectation E from σ-unital C^*-algebra D to a stable simple C^*-algebra C with $\overline{DC} = D$ to the multiplier algebra $M(D)$, then it has only one pair as a quasi-basis. In the case that C and D are stable, we have the following result.

THEOREM ([11]) *Let* $1 \in A \subseteq B$ *be a pair of unital* C^**-algebras, and let* E *be a faithful conditional expectation from* B *on* A *of index-finite type. Suppose that* A *is simple. Let* $\widetilde{E}$ *be the restriction of* $(E \otimes \mathrm{id})^{**}$ *to the multiplier algebra* $M(B \otimes \mathbf{K})$ *of* $B \otimes \mathbf{K}$*, where* $\mathbf{K}$ *denotes a* C^**-algebra of compact operators on some separable infinite-dimensional Hilbert space. Then,* $\widetilde{E}$ *is a conditional expectation from* $M(B \otimes \mathbf{K})$ *to* $M(A \otimes \mathbf{K})$*. Moreover, there exists an isometry* W *in* $M(B \otimes \mathbf{K})$ *such that* $\big\{\big(\sqrt{\mathrm{Index}(E)} W^*, \sqrt{\mathrm{Index}(E)} W\big)\big\}$ *is a quasi-basis for* $\widetilde{E}$.

Proof. For completeness, we will give a sketch of the proof.

Let e_A be the projection on the right A-Hilbert module B_A defined by $e_A x = E(x)$. Then, $e_A \otimes \mathrm{id}$ is a projection in the multiplier algebra $M(C^*(B, e_A) \otimes \mathbf{K})$ of $C^*(B, e_A) \otimes \mathbf{K}$. Since $C^*(B, e_A)$ is isomorphic to a full hereditary algebra of some matrix algebra over A (Lemma 3.1 (ii)), $C^*(B, e_A) \otimes \mathbf{K}$ is simple, so $e_A \otimes \mathrm{id}$ is a full projection in $M(C^*(B, e_A) \otimes \mathbf{K})$. By Lemma 2.5 of [3] there is an isometry V in $M(C^*(B, e_A) \otimes \mathbf{K})$ such that $V^* V = 1$ and $V V^* = e_A \otimes \mathrm{id}$.

Let E_B be the dual conditional expectation of E from $C^*(B, e_A)$ to B, and $\widetilde{E}_B$ be the restriction of $(E_B \otimes \mathrm{id})^{**}$ to $M(C^*(B, e_A) \otimes \mathbf{K})$. Then, for any $x \in M(C^*(B, e_A) \otimes \mathbf{K})$ we have

$$(e_A \otimes \mathrm{id}) \widetilde{E}_B((e_A \otimes \mathrm{id}) x) = \frac{1}{\mathrm{Index}(E)}((e_A \otimes \mathrm{id}) x).$$

Set $W = \sqrt{\mathrm{Index}(E)} \widetilde{E}_B(V)$. Then, $W \in M(B \otimes \mathbf{K})$, and $V = \sqrt{\mathrm{index}(E)}(e_A \otimes \mathrm{id}) W$. Hence, we have

$$1 = V^* V = \mathrm{Index}(E) W^* (e_A \otimes \mathrm{id}) W.$$

Therefore $\big\{\sqrt{\mathrm{Index}(E)} W^*, \sqrt{\mathrm{Index}(E)} W\big\}$ is a quasi-basis for $\widetilde{E}$. ∎

However, in the case that A is purely infinite simple, we can find only one pair as a quasi-basis for E as follows:

LEMMA *Let $1 \in A \subset B$ be a pair of C^*-algebras, and let E be a conditional expectation from B to A of index-finite type. Suppose that A is a purely infinite simple C^*-algebra. Then there is a co-isometry $\frac{1}{\sqrt{\mathrm{Index}(E)}}v$ such that the pair $\{v, v^*\}$ is a quasi-basis for E in B. That is, for any element x in B*

$$x = E(xv)v^*.$$

Proof. Let e_A be the projection on the right A-Hilbert module B_A defined by $e_A(x) = E(x)$. Then the basic extension $C^*(B, e_A)$ of B is isomorphic to a hereditary subalgebra of some matrix algebra over A. Since A is a purely infinite simple C^*-algebra, so is $C^*(B, e_A)$. Thus, there is a co-isometry w in $C^*(B, e_A)$ such that $w^*w \leqslant e_A$ and $ww^* = 1$ ([7]). Note that $we_A = w$.

Using a similar argument as in Lemma 1.2 of [20], there exists a non-zero element $v \in B$ such that $we_A = ve_A$. Then we have $ve_Av^* = 1$. Since $E_B(ve_Av^*) = \frac{1}{\mathrm{Index}(E)}vv^* = 1$, $\frac{1}{\sqrt{\mathrm{Index}(E)}}v$ is a co-isometry.

Moreover, we know that $\{v, v^*\}$ is a quasi-basis for E. Indeed, for any $b \in B$ $b = (ve_Av^*)(b) = vE(v^*b) = E(bv)v^*$. ∎

4. THE STRUCTURE THEORY FOR PURELY INFINITE SIMPLE C^*-ALGEBRAS

At the end of Section 2, we claimed that if A is a purely infinite simple C^*-algebra and α is an outer action from a discrete group G on A, then the reduced crossed product $A \times_{\alpha\mathrm{r}} G$ is also purely infinite simple. In this section, we consider the case of a pair of unital simple C^*-algebras $1 \in A \subseteq B$ with a finite C^*-index, and deduce the pure infiniteness of B under the condition that A is purely infinite, which was announced by Izumi at the Fields Institute in 1995. To conclude this result, we use the characterization of the simplicity of the corona algebra of a stable C^*-algebra by Rørdam ([24]).

First, we discuss a special case of the SP-property for a pair of C^*-algebras, which will help the reader to understand the general case.

PROPOSITION *Let $1 \in A \subset B$ be a pair of C^*-algebras, and let E be a faithful conditional expectation from B to A of index-finite type. Suppose that A is a purely infinite simple C^*-algebra. Then B has the* SP*-property.*

We need several lemmas.

LEMMA *Let A be a unital C^*-algebra with the* SP*-property. Let a be a non-zero, non-invertible positive element in A. Then for any positive number $\varepsilon > 0$ there is a projection e in A such that $\|ea\| < \varepsilon$.*

Proof. Choose a continuous function $f : [0, \|a\|] \to \mathbf{R}^+$ so that $f(t) = 0$ if $t \geqslant \varepsilon$ and $f(a) \neq 0$. Since A has the SP-property, there is a non-zero projection e in the hereditary subalgebra $\overline{f(a)Af(a)}$. Then we have $\|ea\| < \varepsilon$. ∎

LEMMA ([18], [21]) *Let $C^*(p,q)$ be the universal unital C^*-algebra generated by two projections.*

(i) *There is an isomorphism from $C^*(p,q)$ onto*

$$D = \{f \in C([0,1], M_2(\mathbf{C})) : f(0), f(1) \text{ are diagonal}\}$$

which carries the generating projections into the functions

$$p_D(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q_D(t) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

(ii) *The spectrum of $C^*(p,q)$ is homeomorphic to the quotient of two copies of $[0,1]$ in which the corresponding points in $(0,1)$ have been identified.*

Proof of Proposition 4.1. Let x be a non-zero positive element in B with $\|x\| = 1$. Since A is purely infinite simple, there is an element $z \in A$ such that $E(z^*xz) = 1$. Set $y = z^*(x - E(x))z$.

From Lemma 3.4 there is a quasi-basis $\{v, v^*\}$ for E so that $b = E(bv)v^*$ for $b \in B$. So, there is $a \in A$ such that $y = av^*$, $a \in A$. Then a is not invertible in A. Indeed, if a is invertible, $v^* = a^{-1}y$, hence $E(v^*) = E(a^{-1}y) = 0$. On the contrary, $1 = E(v)v^* = 0$. This is a contradiction. Hence, either $|a|$ or $|a^*|$ is not invertible.

Suppose that $|a^*|$ is not invertible. Let ε be an arbitrary positive number. Then, from Lemma 4.2, there is a projection e such that $\|e|a^*|\| < \frac{\varepsilon}{\|v\|}$. Write $a = u|a|$ by a polar decomposition in A^{**}. Then

$$\|ea\| = \|e|a^*|u\| < \frac{\varepsilon}{\|v\|}.$$

So we have $\|ey\| < \varepsilon$, hence $\|ez^*(x - E(x))ze\| = \|eye\| < \varepsilon$. Since $\|ez^*E(x)ze\| = \|e\| = 1 > \|E(x)\| - \varepsilon$, $\overline{xBx}$ has a non-zero projection from the proof of Theorem 2.1.

Suppose that $|a^*|$ is invertible. Set $u^* = a^*|a^*|^{-1}$. Then, u is a co-isometry in A, and $a = |a^*|u$. Then, $y = |a^*|uv^*$. Set $p = u^*u$ and $q = \frac{1}{\mathrm{Index}(E)}v^*v$. Then p and q are non-zero projections. Let π be a homomorphism from the C^*-algebra D in Lemma 4.3 (i) to a C^*-algebra $C^*(p,q)$ generated by p,q such that $\pi(p_D) = p$ and $\pi(q_D) = q$. From Lemma 4.3 (ii) we can write the spectrum $\widehat{D}$ of D as follows:

$$\begin{array}{ccc} t_0 & & t_1 \\ \cdot & & \cdot \\ \circ & - - - - - - - - - - - - - - & \circ \\ \cdot & & \cdot \\ s_0 & & s_1, \end{array}$$

where t_0, s_0, t_1, and s_1 are end points of $\widehat{D}$.

We may assume that

$$\widehat{p_D D p_D} \subset \begin{array}{ccc} t_0 & & t_1 \\ \cdot & & \cdot \\ \circ & - - - - - - - - - - - - - - & \circ \end{array}$$

Then, we consider two cases.

Case 1. $\widehat{\ker(\pi)} \not\ni t_0$.

Let $\eta > 0$ be an arbitrary positive number. Consider a continuous function $g : [0,1] \to [0,1]$ such that $g(0) = 1$, $g(t) = \eta$ for $t \geqslant \eta$, and linear on $[0,\eta]$. Define

$$T(t) = \begin{pmatrix} g(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad t \in \widehat{p_D D p_D}.$$

Then $T \in p_D D p_D$ and $\|T\| = 1$. Note that

$$p_D q_D p_D(t) = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \quad t \in \widehat{p_D D p_D}.$$

Hence,

$$T(t)(p_D q_D p_D)(t) = \begin{pmatrix} g(t)t & 0 \\ 0 & 0 \end{pmatrix}$$

and $\|T p_D q_D p_D\| \leqslant \eta$.

Set $c = \pi(T)$. Since $t_0 \notin \widehat{\ker(\pi)}$,

$$\|c\| = \|\pi(T)\| = 1.$$

Set $b = \alpha^{-1/2} c u^* |a^*|^{-1}$, where α is a positive number and $(aa^*)^{-1} \geqslant \alpha$. Then,

$$bb^* \geqslant cu^*uc = cpc = \pi(T^2),$$

so $\|b\| \geqslant 1$. Note that $by = \alpha^{-1/2} cu^*|a^*|^{-1} y = \alpha^{-1/2} cu^*uv^* = \alpha^{-1/2} cpv^*$. Hence,

$$\|by\|^2 = \|byy^*b^*\| = \|\alpha^{-1}\mathrm{Index}(E) cpqpc\| \leqslant \alpha^{-1}\mathrm{Index}(E)\eta.$$

So, if we take η sufficiently small, then $\|byb^*\| < \varepsilon$. Hence, $\|bz^*(x - E(x))zb\| = \|byb^*\| < \varepsilon$. Since $\|bz^*E(x)zb^*\| = \|bb^*\| \geqslant 1 > \|E(x)\| - \varepsilon$, from the proof of Theorem 2.1 $\overline{xBx}$ has a non-zero projection.

(Set $a = bz^*E(x)zb^* = bb^*$ in Theorem 2.1. Then, since bb^* is invertible in pBp and $\overline{f(p)Af(p)}$ has a non-zero projection, we know that $\overline{f(a)Bf(a)}$ has a non-zero projection. Note that $f(a) \geqslant \lambda f(p)$ for some $\lambda > 0$.)

Case 2. $\widehat{\ker(\pi)} \ni t_0$.

We may assume that z^*xz is not invertible.

Since $t_0 \in \widehat{\ker(\pi)}$, $pqp = \pi(p_D q_D p_D)$ is invertible in $pC^*(p,q)p$. So, there is a positive number λ such that $pqp \geqslant \lambda p$. Since $y = y^*$, we have

$$y^2 = yy^* = |a^*|upqpu^*|a^*| \geqslant \lambda|a^*|upu^*|a^*| = \lambda|a^*|(uu^*)^2|a^*| = \lambda|a^*|^2.$$

Hence y is invertible, since $|a^*|$ is invertible. Note that $\|z^*xz\| \geqslant \|E(z^*xz)\| = 1$.

Claim. $1 \notin \sigma(z^*xz)$ (= the set of spectrum of z^*xz).

Suppose that $1 \in \sigma(z^*xz)$. Then, $z^*xz - 1$ is not invertible. So, $y = z^*(x - E(x))z = z^*xz - 1$ is not invertible. This is a contradiction to the fact that y is invertible.

So, since $\|z^*xz\| \geqslant 1$ and $0 \in \sigma(z^*xz)$, there is a non-zero spectral projection $\chi(z^*xz)$ of z^*xz which is not equal to one. Then, $\chi(z^*xz) \in \overline{z^*xBxz}$. Since $\overline{z^*xBxz} \cong \overline{xzBz^*x}$ by [6], 1.4, $\overline{xBx}$ has a non-zero projection. ∎

REMARK In Proposition 4.1, since A is simple and E is of index-finite type, we know that B is a direct sum of finitely simple C^*-algebras B_i by the next lemma. So, if we conclude that the last projection $\chi(z^*xz)$ is infinite, then we could conclude that each simple C^*-algebra B_i is purely infinite. In fact, in the case that B is a crossed product algebra of A by a finite group, we can conclude that any non-zero hereditary C^*-subalgebra of B has a non-zero projection that is equivalent to some projection in $M_n(A)$ for some n, using the same method as in Theorem 1.1 ([13], Theorem 4.2).

LEMMA ([11]) *Let $1 \in A \subset B$ be a pair of C^*-algebras, and let E be a faithful conditional expectation from B to A of index-finite type.*

(i) *If A is simple, then B can be written as a finite direct sum of simple C^*-algebras.*

(ii) *If B is simple, then A can be written as a finite direct sum of simple C^*-algebras.*

Proof. We have only to show case (i). Case (ii) can be deduced from the fact that A is isomorphic to a corner hereditary C^*-subalgebra of $C^*(B, e_A)$, by Lemma 3.1 (ii).

Let J be a non-zero closed two-sided ideal of B, and $\{u_\lambda\}$ be an approximate identity for J. Then $\{u_\lambda\}$ converges strongly to a positive element z in the centre of B^{**}. We claim that $z \in B$. Let E^{**} be a faithful conditional expectation from B^{**} to A^{**}, which is derived from E. Note that $\mathrm{Index}(E^{**}) = \mathrm{Index}(E)$. Then $E(u_\lambda)$ converges strongly to $E^{**}(z)$ in the centre of A^{**}. Set $c = \sup\{\|E(u_\lambda)\| : \lambda\}$. Since A is simple, $c = E^{**}(z)$. In fact, assume that $c \neq E^{**}(z)$. Then, for an arbitrary positive number ε there exists a non-zero projection r in the centre of A^{**} such that

$$rE^{**}(z) < (c - \varepsilon)r.$$

Since A is simple, $\|E(u_\lambda)r\| = \|E(u_\lambda)\|$, hence $\|rE^{**}(z)\| = c$, which is a contradiction.

Set $Q(A) = \{\varphi \in A_+^* : \|\varphi\| = 1\}$. Then $Q(A)$ is weak$*$-compact. Since $\varphi(c - E(u_\lambda))$ converges to 0 for $\varphi \in Q(A)$, $c - E(u_\lambda)$ converges uniformly to 0 by Dini's Theorem. Hence,

$$\|E(u_\lambda) - E^{**}(z)\| \longrightarrow 0, \quad \lambda \nearrow .$$

On the contrary, since E^{**} is of index-finite type, there is a positive number d such that

$$E^{**}(x) \geqslant dx, \quad x \in B_+^{**}$$

from Lemma 3.2 (ii). Since $\|E^{**}(u_\lambda - z)\|$ converges to 0 uniformly, $\|u_\lambda - z\|$ converges uniformly to 0. Hence z is in the centre of B, and $J = zB$. Since the centre of B is finite-dimensional by Proposition 2.7.3 of [25], B can be written as a direct sum of finitely simple C^*-algebras. ∎

A proof of the following theorem is given more directly by Theorem 4.5 of [13] in the case that B is a crossed product algebra of A by a finite group.

THEOREM ([10]) *Let $1 \in A \subseteq B$ be a pair of separable C^*-algebras, and let E be a faithful conditional expectation from B to A of index-finite type. Suppose that A is a purely infinite simple C^*-algebra. Then B is a finite direct sum of purely infinite simple C^*-algebras.*

Proof. By Lemma 4.5 (i), B is a finite direct sum of simple C^*-algebras B_i. Take central projections p_i in B such that $p_iB = B_i$. Then there are conditional expectations F_i of index-finite type : $B_i \to p_iAp_i$. Since each p_iAp_i is purely infinite simple, we may assume that B is simple.

Consider a conditional expectation $\widetilde{E}$ of index-finite type from $M(B \otimes \mathbf{K})$ to $M(A \otimes \mathbf{K})$. Then there is a conditional expectation F of index-finite type from $M(B \otimes \mathbf{K})/(B \otimes \mathbf{K})$ to a C^*-algebra $\{x + B \otimes \mathbf{K} : x \in M(A \otimes \mathbf{K})\}$ $(= D)$. Since A is purely infinite simple, $M(A \otimes \mathbf{K})/(A \otimes \mathbf{K})$ is simple by Theorem 3.2 of [24]. So, $M(A \otimes \mathbf{K})/(A \otimes \mathbf{K})$ is isomorphic to D. Since F is of index-finite type, $M(B \otimes \mathbf{K})/(B \otimes \mathbf{K})$ is a direct sum of some simple C^*-algebras by Lemma 4.5 (i).

We claim that $M(B \otimes \mathbf{K})/(B \otimes \mathbf{K})$ is simple. Indeed, since B is a separable simple C^*-algebra with the SP-property by Proposition 4.1, this corona algebra is prime by Theorem 2.7 of [17]. So it should be simple. Again, by Theorem 3.2 of [24], B is purely infinite. ∎

5. MAIN THEOREM

In this section we present the main theorem. The proof is almost the same as in Proposition 4.1 using Theorem 3.3.

THEOREM *Let $1 \in A \subset B$ be a pair of unital C^*-algebras, and let E be a faithful conditional expectation from B to A of index-finite type. Suppose that A is simple and has the* SP*-property. Then B has the* SP*-property.*

Note that in the case that $B = A \times_\alpha G$ is a crossed product algebra of A by a finite group G, Jeong and Osaka concluded the statement more directly ([13], Theorem 4.2).

Proof. We show that $B \otimes \mathbf{K}$ has the SP-property, where $\mathbf{K}$ denotes a C^*-algebra of compact operators on some separable infinite-dimensional Hilbert spaces. From Theorem 3.3 there is a conditional expectation $\widetilde{E}$ from $M(B \otimes \mathbf{K})$ to $M(A \otimes \mathbf{K})$, and an isometry W in $M(B \otimes \mathbf{K})$ such that $\{(\sqrt{\mathrm{Index}(E)}W^*, \sqrt{\mathrm{Index}(E)}W)\}$ is a quasi-basis for $\widetilde{E}$. Set $v = \sqrt{\mathrm{Index}(E)}W^*$.

Take $x \in (B \otimes \mathbf{K})_+$ with $\|x\| = 1$. As in the proof of Theorem 2.1 there is a continuous function $f : [0,1] \to [0,1]$ such that $\overline{f(\widetilde{E}(x))(A \otimes \mathbf{K})f(\widetilde{E}(x))}$ has a non-zero projection r and $z \in \overline{r(A \otimes \mathbf{K})f(\widetilde{E}(x))}$ such that $z\widetilde{E}(x)z^* = r$. Set $y = z(x - \widetilde{E}(x))z^*$. Then, $\widetilde{E}(y) = 0$ and $ry = y = yr$. So, $y \in r(B \otimes \mathbf{K})r$. Write $y = \widetilde{E}(yv)v^* = av^*$. Since

$$ra = r\widetilde{E}(yv) = \widetilde{E}(ryv) = \widetilde{E}(yv) = a,$$

$|a^*| \in r(A \otimes \mathbf{K})r$. Note that $r(A \otimes \mathbf{K})r$ has the SP-property.

Let ε be an arbitrary positive number.

Suppose that $|a^*|$ is not invertible. Then, since $r(A \otimes \mathbf{K})r$ is a unital C^*-algebra with the SP-property, from Lemma 4.2 there is a projection e in $r(A \otimes \mathbf{K})r$ such that $\| e|a^*| \| < \frac{\varepsilon}{\|v\|}$. So we can conclude that $\overline{x(B \otimes \mathbf{K})x}$ has a non-zero projection, by the same argument as in Proposition 4.1.

Suppose that $|a^*|$ is invertible. Then using Lemma 4.3 we can conclude that $\overline{x(B \otimes \mathbf{K})x}$ has a non-zero projection, by the same steps in the proof of Proposition 4.1.

Therefore, $B \otimes \mathbf{K}$ has the SP-property, and so has B. ∎

COROLLARY *Let $1 \in A \subset B$ be a pair of simple unital C^*-algebras, and let E be a conditional expectation from B to A of index-finite type. Then A has the* SP*-property if and only if B has the* SP*-property.*

Proof. Suppose that B has the SP-property. Consider the basic construction:

$$1 \in A \subset B \subset C^*(B, e_A).$$

Since A is simple, $C^*(B, e_A)$ is simple from Corollary 2.2.14 in [25]. So we know that $C^*(B, e_A)$ has the SP-property from Theorem 5.1. Hence from Lemma 3.1 (ii) we know that A has the SP-property. ∎

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