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A RADON-NIKODYM THEOREM FOR VON NEUMANN ALGEBRAS

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Abstract. In this paper we present a generalization of the Radon-Nikodym theorem proved by Pedersen and Takesaki in [7]. Given a normal, semifinite and faithful (n.s.f.) weight φ on a von Neumann algebra $\mathcal{M}$ and a strictly positive operator δ, affiliated with $\mathcal{M}$ and satisfying a certain relative invariance property with respect to the modular automorphism group σ^{φ} of φ, with a strictly positive operator as the invariance factor, we construct the n.s.f. weight $\varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$. All the n.s.f. weights on $\mathcal{M}$ whose modular automorphisms commute with σ^{φ} are of this form, the invariance factor being affiliated with the centre of $\mathcal{M}$. All the n.s.f. weights which are relatively invariant under σ^{φ} are of this form, the invariance factor being a scalar.

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INTRODUCTION

In [7], G.K. Pedersen and M. Takesaki gave a construction of a normal semifinite faithful (n.s.f.) weight $\varphi(\,\cdot\,\delta)$ on a von Neumann algebra $\mathcal{M}$, starting from an n.s.f. weight φ on $\mathcal{M}$ and a strictly positive operator δ affiliated with the von Neumann algebra of elements invariant under the modular automorphisms of φ. These weights $\varphi(\,\cdot\,\delta)$ are precisely all the n.s.f. weights ψ on $\mathcal{M}$ which are invariant under the modular automorphisms of φ. In this paper we will give a construction for an n.s.f. weight $\varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$ in the case where δ satisfies the weaker hypothesis $\sigma^{\varphi}_s(\delta^{\mathrm{i}t}) = \lambda^{\mathrm{i}st}\delta^{\mathrm{i}t}$ for all $s, t \in \mathbb{R}$ and for a given strictly positive operator λ, affiliated with $\mathcal{M}$ and strongly commuting with δ. In this way we obtain precisely all n.s.f. weights ψ on $\mathcal{M}$ for which $[D\psi : D\varphi]_t = \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}$. The operators λ and δ are uniquely determined by ψ. When ψ is an n.s.f. weight on $\mathcal{M}$ we prove that σ^{ψ} and σ^{φ} commute if and only if there exist strictly positive operators λ and δ affiliated with the centre of $\mathcal{M}$ and with $\mathcal{M}$ itself, respectively, such that $\sigma^{\varphi}_s(\delta^{\mathrm{i}t}) = \lambda^{\mathrm{i}st}\delta^{\mathrm{i}t}$

for all $s,t \in \mathbb{R}$ and such that $\psi = \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$. When ψ is an n.s.f. weight on $\mathcal{M}$ and $\lambda \in \mathbb{R}_0^+$ we prove that $\psi \circ \sigma_t^\varphi = \lambda^{-t}\psi$ for all $t \in \mathbb{R}$ if and only if $\varphi \circ \sigma_t^\psi = \lambda^t \psi$ for all $t \in \mathbb{R}$ if and only if there exists a strictly positive operator δ affiliated with $\mathcal{M}$ such that $\sigma_s^\varphi(\delta^{it}) = \lambda^{ist}\delta^{it}$ for all $s,t \in \mathbb{R}$ and such that $\psi = \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$.

One important application of the Radon-Nikodym theorem of Pedersen and Takesaki arose in the theory of locally compact quantum groups. In [6], the theorem is used to obtain the modular element as the Radon-Nikodym derivative of the left and the right Haar weight. Recently, however, J. Kustermans and the author have given a new and relatively simple definition of locally compact quantum groups (see [2], [3]) and in this theory the right Haar weight is only relatively invariant under the modular automorphism group of the left Haar weight. In order still to be able to obtain the modular element, we need the more general Radon-Nikodym theorem of this paper. Further, the possibility to obtain a Radon-Nikodym derivative from the sole assumption that σ^φ and σ^ψ commute is new and could give rise to applications in von Neumann algebra theory. It is also important to notice that the most powerful tool to prove the equality of two n.s.f. weights, namely showing that the Radon-Nikodym derivative is trivial, can now be applied in much more situations.

Let us first fix some notation. Throughout the paper, a strictly positive operator will mean a positive, injective and self-adjoint operator on a Hilbert space. In Paragraphs 1 to 4 we will always assume that $\mathcal{M}$ is a von Neumann algebra, that φ is an n.s.f. weight on $\mathcal{M}$ and that λ and δ are two strictly positive, strongly commuting operators affiliated with $\mathcal{M}$. We suppose $\mathcal{M}$ acts on the GNS-space $\mathcal{H}$ of φ. We denote by J and Δ the modular operators of φ and by (σ_t) the modular automorphisms. As usual, we put $\mathfrak{N} = \{a \in \mathcal{M} \mid \varphi(a^*a) < \infty\}$ and $\mathfrak{M} = \mathfrak{N}^*\mathfrak{N}$. We denote by $\Lambda : \mathfrak{N} \to \mathcal{H}$ the map appearing in the GNS-construction of φ such that $\langle \Lambda(a), \Lambda(b)\rangle = \varphi(b^*a)$. We remark that the map Λ is closed for the weak operator topology on $\mathcal{M}$ and the weak topology on $\mathcal{H}$, and refer to Chapter 10 from [9] and Chapter I from [8] for more details about n.s.f. weights. We assume the following relative invariance :

$$\sigma_t(\delta^{is}) = \lambda^{ist}\delta^{is} \quad \text{for all } s,t \in \mathbb{R}.$$

Remark that in case $\lambda = 1$ we arrive at the premises for the construction of Pedersen and Takesaki. Because we will regularly use analytic continuations, we introduce the notation $S(z)$ for the closed strip of complex numbers with real part between 0 and $\mathrm{Re}(z)$.

Starting from all these assumptions, we will construct an n.s.f. weight φ_δ on $\mathcal{M}$ in the first paragraph. Then we will compute the modular operators and automorphisms of φ_δ and prove an explicit formula that justifies the notation $\varphi_\delta = \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$. In the fourth paragraph we compute the Connes cocycle $[D\varphi_\delta : D\varphi]$, which will enable us to prove in the last paragraph the three Radon-Nikodym type theorems mentioned above.

1. THE CONSTRUCTION OF THE WEIGHT φ_δ

Definition 1.1. For each $n \in \mathbb{N}_0$ we define an element $e_n \in \mathcal{M}$ by

$$\alpha_n = \frac{2n^2}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}, \quad e_n = \alpha_n \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(-n^2x^2 - n^4y^4)\lambda^{\mathrm{i}x}\delta^{\mathrm{i}y}\,\mathrm{d}x\,\mathrm{d}y \in \mathcal{M}.$$

The integral makes sense in the strong* topology. We remark that automatically λ satisfies $\sigma_t(\lambda^{\mathrm{i}s}) = \lambda^{\mathrm{i}s}$ for all $s, t \in \mathbb{R}$, which can be proven very easily. We also easily obtain the following lemma, using the "analytic extension techniques" of [9], Chapter 9.

Lemma 1.2. (i) *The elements $e_n \in \mathcal{M}$ are analytic with respect to σ. For all $x, y, z \in \mathbb{C}$, the operator $\delta^x\lambda^y\sigma_z(e_n)$ is bounded, with domain $\mathcal{H}$, analytic with respect to σ and satisfies $\sigma_t(\delta^x\lambda^y\sigma_z(e_n)) = \delta^x\lambda^{y+tx}\sigma_{t+z}(e_n)$ for all $t \in \mathbb{C}$.*

(ii) *For all $z \in \mathbb{C}$ we have $\sigma_z(e_n) \to 1$ strong* and bounded.*

(iii) *The function $(x, y, z) \mapsto \delta^x\lambda^y\sigma_z(e_n)$ is analytic from $\mathbb{C}^3$ to $\mathcal{M}$.*

(iv) *The elements e_n are selfadjoint.*

Inspired by the work of [1], we give the following definition:

Definition 1.3. Define a subset $\mathfrak{N}_0$ of $\mathcal{M}$ by

$$\mathfrak{N}_0 = \{a \in \mathcal{M} \mid a\delta^{\frac{1}{2}} \text{ is bounded and } \overline{a\delta^{\frac{1}{2}}} \in \mathfrak{N}\}$$

and a map

$$\Gamma : \mathfrak{N}_0 \to \mathcal{H}, \quad a \mapsto \Lambda(\overline{a\delta^{\frac{1}{2}}}),$$

where $\overline{a\delta^{\frac{1}{2}}}$ denotes the closure of $a\delta^{\frac{1}{2}}$.

Remark that Γ is injective and $\mathfrak{N}_0$ is a left ideal in $\mathcal{M}$. So $\Gamma(\mathfrak{N}_0 \cap \mathfrak{N}_0^*)$ becomes an involutive algebra by defining

$$\Gamma(a)\Gamma(b) = \Gamma(ab), \quad \Gamma(a)^\# = \Gamma(a^*).$$

Proposition 1.4. *When we endow the involutive algebra $\Gamma(\mathfrak{N}_0 \cap \mathfrak{N}_0^*)$ with the scalar product of $\mathcal{H}$, it becomes a left Hilbert algebra. The generated von Neumann algebra is $\mathcal{M}$.*

Proof. If $a, b \in \mathfrak{N}_0 \cap \mathfrak{N}_0^*$ we have

$$\Gamma(ab) = \Lambda(a\overline{b\delta^{\frac{1}{2}}}) = a\Gamma(b)$$

so that $\Gamma(b) \mapsto \Gamma(ab)$ is bounded. For $a, b, c \in \mathfrak{N}_0 \cap \mathfrak{N}_0^*$ we have

$$\langle\Gamma(a)\Gamma(b), \Gamma(c)\rangle = \varphi((\overline{c\delta^{\frac{1}{2}}})^* a(\overline{b\delta^{\frac{1}{2}}})) = \langle\Gamma(b), \Lambda(a^*(\overline{c\delta^{\frac{1}{2}}}))\rangle = \langle\Gamma(b), \Gamma(a)^\#\Gamma(c)\rangle.$$

If $a \in \mathfrak{N} \cap \mathfrak{N}^*$ one can easily verify that $e_n a(\delta^{-\frac{1}{2}}e_n) \in \mathfrak{N}_0 \cap \mathfrak{N}_0^*$. Moreover

$$\Gamma(e_n a(\delta^{-\frac{1}{2}}e_n)) = \Lambda(e_n a e_n) = J(\sigma_{\frac{\mathrm{i}}{2}}(e_n))^* J e_n \Lambda(a) \to \Lambda(a).$$

So $\Gamma(\mathfrak{N}_0 \cap \mathfrak{N}_0^*)$ is dense in $\mathcal{H}$. But also $e_n a e_n \in \mathfrak{N}_0 \cap \mathfrak{N}_0^*$, and this converges strongly to a. Therefore $\mathfrak{N}_0 \cap \mathfrak{N}_0^*$ is strongly dense in $\mathcal{M}$ and thus $(\Gamma(\mathfrak{N}_0 \cap \mathfrak{N}_0^*))^2$ is dense in $\mathcal{H}$.

We claim that for all $n \in \mathbb{N}_0$ and all b and b' in the Tomita algebra of φ, the element $\Lambda(e_n b b' e_n)$ belongs to the domain of the adjoint of the mapping $\Gamma(a) \mapsto \Gamma(a)^{\#}$. This will imply the closedness of that mapping, and so this will end the proof. To prove the claim, choose $a \in \mathfrak{N}_0 \cap \mathfrak{N}_0^*$. Define the element $x \in \mathfrak{N}$ by

$$x := (\delta^{\frac{1}{2}} \sigma_{-\mathrm{i}}(e_n))\sigma_{-\mathrm{i}}(b'^* b^* (\delta^{-\frac{1}{2}} e_n)).$$

Then we can make the following calculation:

$$\begin{aligned}
\langle \Lambda(x), \Gamma(a) \rangle &= \varphi((\overline{a\delta^{\frac{1}{2}}})^* (\delta^{\frac{1}{2}} \sigma_{-\mathrm{i}}(e_n))\sigma_{-\mathrm{i}}(b'^*)\sigma_{-\mathrm{i}}(b^*(\delta^{-\frac{1}{2}} e_n))) \\
&= \varphi(b^*(\delta^{-\frac{1}{2}} e_n)(\overline{a\delta^{\frac{1}{2}}})^* (\delta^{\frac{1}{2}} \sigma_{-\mathrm{i}}(e_n))\sigma_{-\mathrm{i}}(b'^*)) \\
&= \varphi(b^* e_n a^* (\delta^{\frac{1}{2}} \sigma_{-\mathrm{i}}(e_n))\sigma_{-\mathrm{i}}(b'^*)) = \varphi(b^* e_n (\overline{a^*\delta^{\frac{1}{2}}})\sigma_{-\mathrm{i}}(e_n b'^*)) \\
&= \varphi(e_n b'^* b^* e_n (\overline{a^* \delta^{\frac{1}{2}}})) = \langle \Gamma(a)^{\#}, \Lambda(e_n b b' e_n) \rangle.
\end{aligned}$$

This proves our claim. ∎

DEFINITION 1.5. We define φ_δ as the weight associated to the left Hilbert algebra $\Gamma(\mathfrak{N}_0 \cap \mathfrak{N}_0^*)$. This is an n.s.f. weight on $\mathcal{M}$.

We denote by $\mathfrak{N}'$, $\mathfrak{M}'$ and $\Lambda' : \mathfrak{N}' \to \mathcal{H}$ the evident objects associated to φ_δ. We denote by (σ'_t) the modular automorphisms of φ_δ. We remark that $\mathfrak{N}_0 \subset \mathfrak{N}'$ and $\Lambda'(a) = \Gamma(a)$ for all $a \in \mathfrak{N}_0$.

Up to now the operator λ did not appear in our formulas. We only need the relative invariance property of δ to construct the analytic elements e_n, which cut down δ properly. Further on λ will of course appear when we prove properties of φ_δ.

2. THE MODULAR OPERATORS OF φ_δ

We will now calculate the modular operators and the modular automorphisms of φ_δ. We will give explicit formulas.

LEMMA 2.1. *For all $s \in \mathbb{R}$ define*

$$u_s = J\lambda^{\frac{1}{2}\mathrm{i}s^2}\delta^{\mathrm{i}s} J \lambda^{\frac{1}{2}\mathrm{i}s^2}\delta^{\mathrm{i}s}\Delta^{\mathrm{i}s}.$$

Then (u_s) is a strongly continuous one-parameter group of unitaries on $\mathcal{H}$.

Proof. Straightforward, by using the facts that $J\mathcal{M}J = \mathcal{M}'$, $J\Delta^{\mathrm{i}s} = \Delta^{\mathrm{i}s}J$, $\Delta^{\mathrm{i}s}\delta^{\mathrm{i}t} = \lambda^{\mathrm{i}st}\delta^{\mathrm{i}t}\Delta^{\mathrm{i}s}$ and $\Delta^{\mathrm{i}s}\lambda^{\mathrm{i}t} = \lambda^{\mathrm{i}t}\Delta^{\mathrm{i}s}$ for all $s, t \in \mathbb{R}$. ∎

DEFINITION 2.2. We define Δ' as the strictly positive operator on $\mathcal{H}$ such that $u_s = \Delta'^{\mathrm{i}s}$ for all $s \in \mathbb{R}$.

Further on, in Proposition 2.4, we will give a more explicit formula for Δ'. We first need a lemma that we will use several times.

Lemma 2.3. *Let $z \in \mathbb{C}$ and $n, m \in \mathbb{N}_0$. If $\xi \in \mathcal{D}({\Delta'}^z)$ then $Je_nJe_m\xi \in \mathcal{D}({\Delta'}^z) \cap \mathcal{D}(\Delta^z)$ and*

$$\begin{aligned}{\Delta'}^z Je_nJe_m\xi &= J\sigma_{\mathrm{i}\bar{z}}(e_n)J\sigma_{-\mathrm{i}z}(e_m){\Delta'}^z\xi\\ \Delta^z Je_nJe_m\xi &= J\lambda^{\frac{1}{2}\mathrm{i}\bar{z}^2}\delta^{\bar{z}}\sigma_{\mathrm{i}\bar{z}}(e_n)J\lambda^{\frac{1}{2}\mathrm{i}z^2}\delta^{-z}\sigma_{-\mathrm{i}z}(e_m){\Delta'}^z\xi.\end{aligned}$$

If $\xi \in \mathcal{D}(\Delta^z)$ then $Je_nJe_m\xi \in \mathcal{D}({\Delta'}^z) \cap \mathcal{D}(\Delta^z)$ and

$$\begin{aligned}{\Delta'}^z Je_nJe_m\xi &= J\lambda^{-\frac{1}{2}\mathrm{i}\bar{z}^2}\delta^{-\bar{z}}\sigma_{\mathrm{i}\bar{z}}(e_n)J\lambda^{-\frac{1}{2}\mathrm{i}z^2}\delta^{z}\sigma_{-\mathrm{i}z}(e_m)\Delta^z\xi\\ \Delta^z Je_nJe_m\xi &= J\sigma_{\mathrm{i}\bar{z}}(e_n)J\sigma_{-\mathrm{i}z}(e_m)\Delta^z\xi.\end{aligned}$$

Proof. Let $\xi \in \mathcal{D}({\Delta'}^z)$. Recall the notation $S(z)$ from the end of the introduction. We define the function from $S(z)$ to $\mathcal{H}$ that maps α to

$$J\lambda^{\frac{1}{2}\mathrm{i}\bar{\alpha}^2}\delta^{\bar{\alpha}}\sigma_{\mathrm{i}\bar{\alpha}}(e_n)J\lambda^{\frac{1}{2}\mathrm{i}\alpha^2}\delta^{-\alpha}\sigma_{-\mathrm{i}\alpha}(e_m){\Delta'}^{\alpha}\xi.$$

This function is continuous on $S(z)$ and analytic on its interior. In is it attains the value

$$J\lambda^{-\frac{1}{2}\mathrm{i}s^2}\delta^{-\mathrm{i}s}\sigma_s(e_n)J\lambda^{-\frac{1}{2}\mathrm{i}s^2}\delta^{-\mathrm{i}s}\sigma_s(e_m){\Delta'}^{\mathrm{i}s}\xi = J\sigma_s(e_n)J\sigma_s(e_m)\Delta^{\mathrm{i}s}\xi = \Delta^{\mathrm{i}s}Je_nJe_m\xi.$$

By the results of Chapter 9 from [9] the second statement follows. The three remaining statements are proved analogously. ∎

Proposition 2.4. *Let $r \in \mathbb{R}$. The operator*

$$J\lambda^{-\frac{1}{2}\mathrm{i}r^2}J\lambda^{-\frac{1}{2}\mathrm{i}r^2}J\delta^{-r}J\delta^r\Delta^r$$

is closable and its closure equals ${\Delta'}^r$.

Proof. Let $\xi \in \mathcal{D}(J\delta^{-r}J\delta^r\Delta^r)$. Let $n, m \in \mathbb{N}_0$. By Lemma 2.3, we have $Je_nJe_m\xi \in \mathcal{D}({\Delta'}^r)$ and

$${\Delta'}^r Je_nJe_m\xi = J\lambda^{-\frac{1}{2}\mathrm{i}r^2}\sigma_{\mathrm{i}r}(e_n)J\lambda^{-\frac{1}{2}\mathrm{i}r^2}\sigma_{-\mathrm{i}r}(e_m)J\delta^{-r}J\delta^r\Delta^r\xi.$$

The operator ${\Delta'}^r$ being closed, we obtain that $\xi \in \mathcal{D}({\Delta'}^r)$ and

$${\Delta'}^r\xi = J\lambda^{-\frac{1}{2}\mathrm{i}r^2}J\lambda^{-\frac{1}{2}\mathrm{i}r^2}J\delta^{-r}J\delta^r\Delta^r\xi.$$

On the other hand, let $\xi \in \mathcal{D}({\Delta'}^r)$. Let $n, m \in \mathbb{N}_0$. By Lemma 2.3 we have that $Je_nJe_m\xi \in \mathcal{D}(J\delta^{-r}J\delta^r\Delta^r)$ and ${\Delta'}^r Je_nJe_m\xi \to {\Delta'}^r\xi$. This implies that $\mathcal{D}(J\delta^{-r}J\delta^r\Delta^r)$ is a core for ${\Delta'}^r$, and this ends our proof. ∎

Denote by S' the closure of the operator $\Gamma(a) \mapsto \Gamma(a)^{\#}$ on $\Gamma(\mathfrak{N}_0 \cap \mathfrak{N}_0^*)$. Define $J' = J\lambda^{-\frac{\mathrm{i}}{8}}J\lambda^{\frac{\mathrm{i}}{8}}J$.

Proposition 2.5.

$$S' = J'{\Delta'}^{\frac{1}{2}}.$$

So, J' and Δ' are the modular operators associated with φ_δ.

Proof. Let $a \in \mathfrak{N}_0 \cap \mathfrak{N}_0^*$ and $n, m, k, l \in \mathbb{N}_0$. Then $\Lambda(e_k a(\delta^{\frac{1}{2}}e_l)) \in \mathcal{D}(\Delta^{\frac{1}{2}})$, so by Lemma 2.3 we have

$$Je_nJe_m\Lambda(e_k a(\delta^{\frac{1}{2}}e_l)) \in \mathcal{D}({\Delta'}^{\frac{1}{2}})$$

and

$$\begin{aligned}J'\Delta'^{\frac{1}{2}}Je_nJe_m\Lambda(e_ka(\delta^{\frac{1}{2}}e_l)) &= \delta^{-\frac{1}{2}}\sigma_{\frac{\mathrm{i}}{2}}(e_n)J\lambda^{-\frac{\mathrm{i}}{4}}\delta^{\frac{1}{2}}\sigma_{-\frac{\mathrm{i}}{2}}(e_m)\Delta^{\frac{1}{2}}\Lambda(e_ka(\delta^{\frac{1}{2}}e_l))\\ &= \delta^{-\frac{1}{2}}\sigma_{\frac{\mathrm{i}}{2}}(e_n)J\sigma_{-\frac{\mathrm{i}}{2}}(\delta^{\frac{1}{2}}e_m)J\Lambda((\delta^{\frac{1}{2}}e_l)a^*e_k) = \sigma_{\frac{\mathrm{i}}{2}}(e_n)e_l\Lambda(a^*(\delta^{\frac{1}{2}}e_m)e_k)\\ &= \sigma_{\frac{\mathrm{i}}{2}}(e_n)e_lJ\sigma_{-\frac{\mathrm{i}}{2}}(e_me_k)J\Gamma(a^*).\end{aligned}$$

The last expression converges to $\Gamma(a^*) = S'\Gamma(a)$, while

$$Je_nJe_m\Lambda(e_ka(\delta^{\frac{1}{2}}e_l)) = Je_n\sigma_{-\frac{\mathrm{i}}{2}}(e_l)Je_me_k\Gamma(a)$$

converges to $\Gamma(a)$ when $n,m,k,l \to \infty$. This implies that $\Gamma(a) \in \mathcal{D}(\Delta'^{\frac{1}{2}})$ and $J'\Delta'^{\frac{1}{2}}\Gamma(a) = S'\Gamma(a)$. Thus, $S' \subset J'\Delta'^{\frac{1}{2}}$.

On the other hand, let $\xi \in \mathcal{D}(J\delta^{-\frac{1}{2}}J\delta^{\frac{1}{2}}\Delta^{\frac{1}{2}})$. Take a sequence (ξ_k) in $\Lambda(\mathfrak{N}\cap\mathfrak{N}^*)$ such that $\xi_k \to \xi$ and $\Delta^{\frac{1}{2}}\xi_k \to \Delta^{\frac{1}{2}}\xi$. Let $n,m,k \in \mathbb{N}_0$. Then $Je_nJe_m\xi_k \in \mathcal{D}(\Delta'^{\frac{1}{2}})$ and

$$\Delta'^{\frac{1}{2}}Je_nJe_m\xi_k = J\lambda^{-\frac{\mathrm{i}}{8}}\delta^{-\frac{1}{2}}\sigma_{\frac{\mathrm{i}}{2}}(e_n)J\lambda^{-\frac{\mathrm{i}}{8}}\delta^{\frac{1}{2}}\sigma_{-\frac{\mathrm{i}}{2}}(e_m)\Delta^{\frac{1}{2}}\xi_k.$$

If $k \to \infty$ this converges to

$$J\lambda^{-\frac{\mathrm{i}}{8}}\sigma_{\frac{\mathrm{i}}{2}}(e_n)J\lambda^{-\frac{\mathrm{i}}{8}}\sigma_{-\frac{\mathrm{i}}{2}}(e_m)J\delta^{-\frac{1}{2}}J\delta^{\frac{1}{2}}\Delta^{\frac{1}{2}}\xi = J\sigma_{\frac{\mathrm{i}}{2}}(e_n)J\sigma_{-\frac{\mathrm{i}}{2}}(e_m)\Delta'^{\frac{1}{2}}\xi.$$

If $n,m \to \infty$ this converges to $\Delta'^{\frac{1}{2}}\xi$. Because $Je_nJe_m\xi_k \in \mathcal{D}(S')$ for all $n,m,k \in \mathbb{N}$ and because of the previous proposition, we have finally proved that $\mathcal{D}(S')$ is a core for $\Delta'^{\frac{1}{2}}$. ∎

COROLLARY 2.6. *We have the formula*

$$\sigma'_s(x) = \lambda^{\frac{1}{2}\mathrm{i}s^2}\delta^{\mathrm{i}s}\sigma_s(x)\delta^{-\mathrm{i}s}\lambda^{-\frac{1}{2}\mathrm{i}s^2}$$

for all $s \in \mathbb{R}$ and all $x \in \mathcal{M}$.

COROLLARY 2.7. *For all $s \in \mathbb{R}$, $x,y,z \in \mathbb{C}$ and $n \in \mathbb{N}_0$ we have*

$$\sigma_s(\lambda^x\delta^y\sigma_z(e_n)) = \sigma'_s(\lambda^x\delta^y\sigma_z(e_n)).$$

Note that these formulas become easier when λ is affiliated with the centre of $\mathcal{M}$, in particular when λ is a positive real number. In that case Δ' is the closure of $J\delta^{-1}J\delta\Delta$ and J' equals $\lambda^{\frac{\mathrm{i}}{4}}J$, because $Jx = x^*J$ for all x belonging to the centre of $\mathcal{M}$. Moreover, we have $\sigma'_s(x) = \delta^{\mathrm{i}s}\sigma_s(x)\delta^{-\mathrm{i}s}$ in that case.

3. A FORMULA FOR φ_δ

Before proving an explicit formula for φ_δ we need two lemmas. The second one will also be used in the next section.

Lemma 3.1. *There exists a net $(x_l)_{l\in L}$ in $\mathfrak{N}_0 \cap \mathfrak{N}_0^*$ such that x_l is analytic with respect to σ' for all l and $\sigma'_z(x_l) \to 1$ strong* and bounded for all $z \in \mathbb{C}$.*

Proof. Because $\mathfrak{N}_0 \cap \mathfrak{N}_0^*$ is a strongly dense $*$-subalgebra of $\mathcal{M}$, we can take a net $(a_k)_{k\in K}$ in $\mathfrak{N}_0 \cap \mathfrak{N}_0^*$ such that $a_k^* = a_k$, $\|a_k\| \leqslant 1$ for all k and $a_k \to 1$ strongly. Define $q_k \in \mathcal{M}$ by

$$q_k = \frac{1}{\sqrt{\pi}} \int \exp(-t^2)\sigma'_t(a_k)\,\mathrm{d}t.$$

Clearly q_k is analytic with respect to σ' and

$$\sigma'_z(q_k) = \frac{1}{\sqrt{\pi}} \int \exp(-(t-z)^2)\sigma'_t(a_k)\,\mathrm{d}t.$$

Also $\sigma'_z(q_k) \to 1$ strong* and bounded.

Define $L = \mathbb{N}_0 \times K \times \mathbb{N}_0$ with the product order, and $x_{(n,k,m)} = e_n q_k e_m$. Then x_l is analytic with respect to σ' for all l and $\sigma'_z(x_l) \to 1$ strong* and bounded for all $z \in \mathbb{C}$. Let $n, m \in \mathbb{N}_0$ and $k \in K$. The operator $e_n q_k e_m \delta^{\frac{1}{2}}$ is bounded, with closure

$$e_n q_k(\delta^{\frac{1}{2}} e_m) = e_n \frac{1}{\sqrt{\pi}} \int \exp(-t^2)\sigma'_t(a_k)(\delta^{\frac{1}{2}} e_m)\,\mathrm{d}t.$$

For all $t \in \mathbb{R}$ the integrand of this expression equals

$$\exp(-t^2)\delta^{\mathrm{i}t}\lambda^{\frac{1}{2}\mathrm{i}t^2}\sigma_t(\overline{a_k\delta^{\frac{1}{2}}}(\lambda^{\frac{1}{2}(\mathrm{i}t^2-t)}\delta^{-\mathrm{i}t}\sigma_{-t}(e_m))).$$

This belongs to $\mathfrak{N}$. When we apply Λ on it, we obtain

$$\exp(-t^2)\delta^{\mathrm{i}t}\lambda^{\frac{1}{2}\mathrm{i}t^2}\Delta^{\mathrm{i}t}J\lambda^{-\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}\sigma_{-\frac{\mathrm{i}}{2}-t}(e_m)J\Lambda(\overline{a_k\delta^{\frac{1}{2}}}).$$

As a function of t this is weakly integrable. Because the mapping Λ is closed for the weak operator topology on $\mathcal{M}$ and the weak topology on $\mathcal{H}$, we can conclude that $e_n q_k(\delta^{\frac{1}{2}} e_m) \in \mathfrak{N}$. This means that $e_n q_k e_m \in \mathfrak{N}_0$. Analogously, we obtain that $e_n q_k e_m \in \mathfrak{N}_0^*$. ∎

Lemma 3.2. *If $a \in \mathfrak{N}'$, then $a(\delta^z e_n)$ belongs to $\mathfrak{N}$ for all $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$. We have $\Lambda(a(\delta^z e_n)) = \Lambda'(a(\delta^{z-\frac{1}{2}} e_n))$.*

Proof. Take a net $(x_l)_{l\in L}$ as in the previous lemma. Then $a x_l(\delta^z e_n) \to a(\delta^z e_n)$ strongly. Because $x_l \in \mathfrak{N}_0$ we have $a x_l(\delta^z e_n) \in \mathfrak{N}$ and

$$\begin{aligned}\Lambda(a x_l(\delta^z e_n)) = \Lambda'(a x_l(\delta^{z-\frac{1}{2}} e_n)) &= J'(\sigma'_{\frac{\mathrm{i}}{2}}(x_l(\delta^{z-\frac{1}{2}} e_n)))^* J'\Lambda'(a) \\ &\to J'(\sigma'_{\frac{\mathrm{i}}{2}}(\delta^{z-\frac{1}{2}} e_n))^* J'\Lambda'(a) = \Lambda'(a(\delta^{z-\frac{1}{2}} e_n)).\end{aligned}$$

Because Λ, as above, is closed, we conclude that $a(\delta^z e_n) \in \mathfrak{N}$ and $\Lambda(a(\delta^z e_n)) = \Lambda'(a(\delta^{z-\frac{1}{2}} e_n))$. ∎

PROPOSITION 3.3. *For all $x \in \mathcal{M}^+$ we have*
$$\varphi_\delta(x) = \lim_n \varphi((\delta^{\frac{1}{2}} e_n) x (\delta^{\frac{1}{2}} e_n)).$$

Proof. If $x \in \mathfrak{N}'$ we have $x(\delta^{\frac{1}{2}} e_n) \in \mathfrak{N}$ for all n and
$$\begin{aligned}\varphi((\delta^{\frac{1}{2}} e_n) x^* x (\delta^{\frac{1}{2}} e_n)) &= \|\Lambda(x(\delta^{\frac{1}{2}} e_n))\|^2 = \|\Lambda'(x e_n)\|^2 \\ &= \|J' \sigma_{-\frac{\mathrm{i}}{2}}(e_n) J' \Lambda'(x)\|^2 \to \|\Lambda'(x)\|^2 = \varphi_\delta(x^* x).\end{aligned}$$
This gives the proof for all $x \in \mathfrak{M}'^{+}$. Now let $x \in \mathcal{M}^+$ and $\varphi_\delta(x) = +\infty$. Suppose $\varphi((\delta^{\frac{1}{2}} e_n) x (\delta^{\frac{1}{2}} e_n))$ does not converge to $+\infty$. Then there exists an $M > 0$ and a subsequence $(e_{n_k})_k$ such that $\varphi((\delta^{\frac{1}{2}} e_{n_k}) x (\delta^{\frac{1}{2}} e_{n_k})) \leqslant M$ for all k. Thus $x^{\frac{1}{2}}(\delta^{\frac{1}{2}} e_{n_k}) \in \mathfrak{N}$ for all k, so $x^{\frac{1}{2}} e_{n_k} \in \mathfrak{N}'$ and $\varphi_\delta(e_{n_k} x e_{n_k}) \leqslant M$ for all k. Because $e_{n_k} x e_{n_k} \to x$ strong* and bounded, this contradicts with $\varphi_\delta(x) = +\infty$ and the σ-weak lower semicontinuity of φ_δ. ∎

4. THE CONNES COCYCLE $[D\varphi_\delta : D\varphi]$

We first state the following lemma, which can be proved in exactly the same way as Lemma 2.1.

LEMMA 4.1. *For all $s \in \mathbb{R}$ define $v_s = \lambda^{\frac{1}{2}\mathrm{i}s^2} \delta^{\mathrm{i}s} \Delta^{\mathrm{i}s}$. Then (v_s) is a strongly continuous one-parameter group of unitaries on $\mathcal{H}$.*

DEFINITION 4.2. We define ρ as the strictly positive operator on $\mathcal{H}$ such that $v_s = \rho^{\mathrm{i}s}$ for all $s \in \mathbb{R}$.

Recall the notation $S(z)$ from the end of the introduction.

LEMMA 4.3. *If $x \in \mathfrak{N} \cap \mathfrak{N}'^*$, then $\Lambda(x) \in \mathcal{D}(\rho^{\frac{1}{2}})$ and*
$$J \lambda^{-\frac{\mathrm{i}}{8}} \rho^{\frac{1}{2}} \Lambda(x) = \Lambda'(x^*).$$

Proof. Let $x \in \mathfrak{N} \cap \mathfrak{N}'^*$ and $n, m \in \mathbb{N}$. Then $e_m x \in \mathfrak{N} \cap \mathfrak{N}^*$ because of Lemma 3.2. We can define a function from $S(\frac{1}{2})$ to $\mathcal{H}$ mapping α to
$$\lambda^{-\frac{1}{2}\mathrm{i}\alpha^2} \delta^\alpha \sigma_{-\mathrm{i}\alpha}(e_n) \Delta^\alpha \Lambda(e_m x).$$
This function is continuous on $S(\frac{1}{2})$ and analytic on its interior. It attains the value
$$\lambda^{\frac{1}{2}\mathrm{i}t^2} \delta^{\mathrm{i}t} \sigma_t(e_n) \Delta^{\mathrm{i}t} \Lambda(e_m x) = \rho^{\mathrm{i}t} \Lambda(e_n e_m x)$$
in $\mathrm{i}t$. So $\Lambda(e_n e_m x) \in \mathcal{D}(\rho^{\frac{1}{2}})$ and
$$\begin{aligned}J \lambda^{-\frac{\mathrm{i}}{8}} \rho^{\frac{1}{2}} \Lambda(e_n e_m x) &= J \sigma_{-\frac{\mathrm{i}}{2}}(\delta^{\frac{1}{2}} e_n) \Delta^{\frac{1}{2}} \Lambda(e_m x) = J \sigma_{-\frac{\mathrm{i}}{2}}(\delta^{\frac{1}{2}} e_n) J \Lambda(x^* e_m) \\ &= \Lambda(x^*(\delta^{\frac{1}{2}} e_n) e_m) = \Lambda'(x^* e_n e_m) = J' \sigma'_{-\frac{\mathrm{i}}{2}}(e_n e_m) J' \Lambda'(x^*).\end{aligned}$$
Because $\rho^{\frac{1}{2}}$ is closed, we can conclude that $\Lambda(x) \in \mathcal{D}(\rho^{\frac{1}{2}})$ and $J \lambda^{-\frac{\mathrm{i}}{8}} \rho^{\frac{1}{2}} \Lambda(x) = \Lambda'(x^*)$. ∎

Proposition 4.4. *The Connes cocycle* $[D\varphi_\delta : D\varphi]_t$ *equals* $\lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}$ *for all* $t \in \mathbb{R}$.

Proof. Let $x \in \mathfrak{N}^* \cap \mathfrak{N}'$ and $y \in \mathfrak{N} \cap \mathfrak{N}'^*$. Denote $u_t = \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}$. Define $F(\alpha) = \langle \rho^\alpha \Lambda(y), \Lambda(x^*)\rangle$ when $\alpha \in \mathbb{C}$ and $0 \leqslant \mathrm{Re}(\alpha) \leqslant \frac{1}{2}$. Define $G(\alpha) = \langle \lambda^{\frac{\mathrm{i}}{8}} J\Lambda'(y^*), \rho^{\bar{\alpha}-1}\lambda^{\frac{\mathrm{i}}{8}} J\Lambda'(x)\rangle$ when $\alpha \in \mathbb{C}$ and $\frac{1}{2} \leqslant \mathrm{Re}(\alpha) \leqslant 1$. Because of the previous lemma, F and G are both well defined, continuous on their domain and analytic in the interior. For any $t \in \mathbb{R}$ we have

$$\begin{aligned}
F(\mathrm{i}t) &= \langle \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}\Delta^{\mathrm{i}t}\Lambda(y), \Lambda(x^*)\rangle = \varphi(xu_t\sigma_t(y))\\
F\Big(\mathrm{i}t + \frac{1}{2}\Big) &= \langle \rho^{\mathrm{i}t}\lambda^{\frac{\mathrm{i}}{8}} J\Lambda'(y^*), \Lambda(x^*)\rangle\\
G\Big(\mathrm{i}t + \frac{1}{2}\Big) &= \langle \lambda^{\frac{\mathrm{i}}{8}} J\Lambda'(y^*), \rho^{-\mathrm{i}t}\Lambda(x^*)\rangle = F\Big(\mathrm{i}t + \frac{1}{2}\Big)\\
G(\mathrm{i}t+1) &= \langle J\Lambda'(y^*), \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{-\mathrm{i}t}\Delta^{-\mathrm{i}t} J\Lambda'(x)\rangle = \langle \Lambda'(x), J\lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t} J\Delta^{\mathrm{i}t}\Lambda'(y^*)\rangle\\
&= \langle \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}\Lambda'(x), \Delta'^{\mathrm{i}t}\Lambda'(y^*)\rangle = \varphi_\delta(\sigma'_t(y)u_t x).
\end{aligned}$$

So we can glue together the functions F and G and define $H(\alpha) = F(\alpha)$ when α belongs to the domain of F and $H(\alpha) = G(\alpha)$ when α belongs to the domain of G. Then H is continuous on $S(1)$ and analytic on its interior. We have

$$H(\mathrm{i}t) = \varphi(xu_t\sigma_t(y)) \quad \text{and} \quad H(\mathrm{i}t+1) = \varphi_\delta(\sigma'_t(y)u_t x)$$

for all $t \in \mathbb{R}$. Because it is easily verified that

$$u_{t+s} = u_t\sigma_t(u_s), \quad u_{-t} = \sigma_{-t}(u_t^*), \quad \sigma'_t(x) = u_t\sigma_t(x)u_t^*$$

for all $s, t \in \mathbb{R}$ and $x \in \mathcal{M}$, we conclude that $u_t = [D\varphi_\delta : D\varphi]_t$ for all t. ∎

The previous proposition also implies that the operators λ and δ are uniquely determined by φ_δ. If we put $u_t = [D\varphi_\delta : D\varphi]_t$, we have $\lambda^{\mathrm{i}t} = u_t^* u_1^* u_{t+1}$ and $\delta^{\mathrm{i}t} = u_t\lambda^{-\frac{1}{2}\mathrm{i}t^2}$ for all $t \in \mathbb{R}$, which proves our claim.

5. THREE RADON-NIKODYM THEOREMS

In this paragraph we denote by (σ_t^φ) the modular automorphism group of an n.s.f. weight φ on a von Neumann algebra. We denote by $\mathfrak{N}_\varphi, \mathfrak{M}_\varphi, \Lambda_\varphi, J_\varphi$ and Δ_φ the same objects as defined in the introduction but we add a subscript φ for the sake of clarity.

Proposition 5.1. *Let ψ and φ be two n.s.f. weights on a von Neumann algebra $\mathcal{M}$. Let λ and δ be two strongly commuting, strictly positive operators affiliated with $\mathcal{M}$. Then the following are equivalent:*

(i) $[D\psi : D\varphi]_t = \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}$ *for all* $t \in \mathbb{R}$;

(ii) $\sigma_t^\varphi(\delta^{\mathrm{i}s}) = \lambda^{\mathrm{i}st}\delta^{\mathrm{i}s}$ *for all* $s, t \in \mathbb{R}$ *and* $\psi = \varphi_\delta$.

Proof. The implication (ii) ⇒ (i) follows from Proposition 4.4.

To prove (i) ⇒ (ii), denote $u_t = [D\psi : D\varphi]_t$. Let $s, t \in \mathbb{R}$. Then

$$\lambda^{\frac{1}{2}\mathrm{i}t^2}\lambda^{\frac{1}{2}\mathrm{i}s^2}\lambda^{\mathrm{i}st}\delta^{\mathrm{i}t}\delta^{\mathrm{i}s} = u_{t+s} = u_t\sigma_t^\varphi(u_s) = \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}\sigma_t^\varphi(\lambda^{\frac{1}{2}\mathrm{i}s^2}\delta^{\mathrm{i}s}).$$

This implies that

$$\lambda^{\frac{1}{2}\mathrm{i}s^2}\lambda^{\mathrm{i}st}\delta^{\mathrm{i}s} = \sigma_t^\varphi(\lambda^{\frac{1}{2}\mathrm{i}s^2}\delta^{\mathrm{i}s}) \quad \text{for all } s,t \in \mathbb{R}. \tag{1}$$

It follows that for all $r,s,t \in \mathbb{R}$

$$\sigma_r^\varphi(\lambda^{\frac{1}{2}\mathrm{i}s^2+\mathrm{i}st})\sigma_r^\varphi(\delta^{\mathrm{i}s}) = \sigma_{r+t}^\varphi(\lambda^{\frac{1}{2}\mathrm{i}s^2}\delta^{\mathrm{i}s}) = \lambda^{\frac{1}{2}\mathrm{i}s^2}\lambda^{\mathrm{i}s(r+t)}\delta^{\mathrm{i}s}. \tag{2}$$

But equation (1) implies that $\sigma_r^\varphi(\delta^{\mathrm{i}s}) = \sigma_r^\varphi(\lambda^{-\frac{1}{2}\mathrm{i}s^2})\lambda^{\frac{1}{2}\mathrm{i}s^2}\lambda^{\mathrm{i}sr}\delta^{\mathrm{i}s}$, so by equation (2) we get

$$\sigma_r^\varphi(\lambda^{\mathrm{i}st})\lambda^{\frac{1}{2}\mathrm{i}s^2}\lambda^{\mathrm{i}sr}\delta^{\mathrm{i}s} = \lambda^{\frac{1}{2}\mathrm{i}s^2}\lambda^{\mathrm{i}s(r+t)}\delta^{\mathrm{i}s}.$$

This gives us $\sigma_r^\varphi(\lambda^{\mathrm{i}st}) = \lambda^{\mathrm{i}st}$ for all $r,s,t \in \mathbb{R}$. Then equation (1) implies that $\sigma_t^\varphi(\delta^{\mathrm{i}s}) = \lambda^{\mathrm{i}st}\delta^{\mathrm{i}s}$ for all $s,t \in \mathbb{R}$. So we can construct the weight φ_δ such that $[D\varphi_\delta : D\varphi]_t = \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}$. But then $\varphi_\delta = \psi$. ∎

We will now consider the more specific case in which λ is affiliated to the centre of $\mathcal{M}$. We will prove that we obtain exactly all the weights whose automorphism group commutes with that of φ.

PROPOSITION 5.2. *Let φ and ψ be two n.s.f. weights on a von Neumann algebra $\mathcal{M}$. Then the following are equivalent:*

(i) *The modular automorphism groups σ^ψ and σ^φ commute.*

(ii) *There exist a strictly positive operator δ affiliated with $\mathcal{M}$ and a strictly positive operator λ affiliated with the centre of $\mathcal{M}$ such that $\sigma_s^\varphi(\delta^{\mathrm{i}t}) = \lambda^{\mathrm{i}st}\delta^{\mathrm{i}t}$ for all $s,t \in \mathbb{R}$ and such that $\psi = \varphi_\delta$.*

(iii) *There exist a strictly positive operator δ affiliated with $\mathcal{M}$ and a strictly positive operator λ affiliated with the centre of $\mathcal{M}$ such that $[D\psi : D\varphi]_t = \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}$ for all $t \in \mathbb{R}$.*

Proof. The equivalence of (ii) and (iii) follows from Proposition 5.1. The implication (ii) ⇒ (i) follows from Corollary 2.6 by a direct computation. We will prove the implication (i) ⇒ (iii). Denote $u_t = [D\psi : D\varphi]_t$ for all $t \in \mathbb{R}$ and denote by $\mathcal{Z}$ the centre of $\mathcal{M}$. For all $x \in \mathcal{M}$ and $s,t \in \mathbb{R}$ we have

$$\sigma_t^\psi(\sigma_s^\varphi(x)) = u_t\sigma_{t+s}^\varphi(x)u_t^* \quad \text{and} \quad \sigma_s^\varphi(\sigma_t^\psi(x)) = \sigma_s^\varphi(u_t)\sigma_{t+s}^\varphi(x)\sigma_s^\varphi(u_t^*).$$

Thus we can conclude that $u_t^*\sigma_s^\varphi(u_t) \in \mathcal{Z}$ for all $s,t \in \mathbb{R}$. But then $u_t^*u_s^*u_{s+t} \in \mathcal{Z}$ for all $s,t \in \mathbb{R}$. Because σ^φ acts trivially on $\mathcal{Z}$, we get

$$u_t^*\sigma_s^\varphi(u_t) = \sigma_{-t}^\varphi(u_t^*\sigma_s^\varphi(u_t)) = u_{-t}\sigma_{s-t}^\varphi(u_t) = u_{-t}u_{s-t}^*u_s \in \mathcal{Z}.$$

We can conclude that $u_tu_{s+t}^*u_s \in \mathcal{Z}$ for all $s,t \in \mathbb{R}$. Then we define for $(s,t) \in \mathbb{R}^2$, $w(s,t) = u_t^*u_s^*u_{s+t}$. The function w is strong* continuous from $\mathbb{R}^2$ to the unitaries of $\mathcal{Z}$. Let $s,s',t \in \mathbb{R}$. Because of the previous remarks we can make the following calculation:

$$\begin{aligned} w(s+s',t) &= u_t^*u_{s+s'}^*u_{s+s'+t} = u_t^*\sigma_{s'}^\varphi(u_s^*)u_{s'}^*u_{s'+t}u_s(u_s^*\sigma_{s'+t}^\varphi(u_s)) \\ &= u_t^*u_s^*\sigma_{s'}^\varphi(\sigma_t^\varphi(u_s)u_s^*)u_{s'}^*u_{s'+t}u_s = u_t^*u_s^*\sigma_{s'}^\varphi(u_t^*u_{s+t}u_s^*)u_{s'}^*u_{s'+t}u_s \\ &= u_t^*u_s^*(u_{s'}^*u_{s'+t}u_t^*)u_{s+t}(u_t^*u_t) = u_t^*u_s^*u_{s+t}u_t^*u_{s'}^*u_{s'+t} = w(s,t)w(s',t). \end{aligned}$$

Next let $s, t, t' \in \mathbb{R}$. We have

$$\begin{aligned} w(s, t+t') &= u_{t+t'}^* u_s^* u_{s+t+t'} = (\sigma_{t'}^{\varphi}(u_t^*) u_t) u_t^* u_{t'}^* u_s^* u_{t'} \sigma_{t'}^{\varphi}(u_{s+t}) \\ &= u_t^* u_{t'}^* u_s^* u_{t'} \sigma_{t'}^{\varphi}(u_{s+t} u_t^*) u_t = u_t^* u_{t'}^* u_s^* u_{t'} \sigma_{t'}^{\varphi}(u_s) \sigma_{t'}^{\varphi}(u_s^* u_{s+t} u_t^*) u_t \\ &= u_t^* u_{t'}^* u_s^* u_{t'} \sigma_{t'}^{\varphi}(u_s) u_s^* u_{s+t} = u_t^* (u_{t'}^* u_s^* u_{s+t'}) u_s^* u_{s+t} \\ &= u_t^* u_s^* u_{s+t} u_{t'}^* u_s^* u_{s+t'} = w(s,t) w(s,t'). \end{aligned}$$

For each $t \in \mathbb{R}$ we can now take a strictly positive operator λ_t affiliated with $\mathcal{Z}$ such that $\lambda_t^{\mathrm{i}s} = w(s,t)$ for all $s, t \in \mathbb{R}$. Let $t, t' \in \mathbb{R}$. Because λ_t and $\lambda_{t'}$ are strongly commuting we can write

$$(\lambda_t \hat{\cdot} \lambda_{t'})^{\mathrm{i}s} = \lambda_t^{\mathrm{i}s} \lambda_{t'}^{\mathrm{i}s} = w(s,t) w(s,t') = \lambda_{t+t'}^{\mathrm{i}s}$$

for all $s \in \mathbb{R}$, where $\lambda_t \hat{\cdot} \lambda_{t'}$ denotes the closure of $\lambda_t \lambda_{t'}$. It follows that $\lambda_{t+t'} = \lambda_t \hat{\cdot} \lambda_{t'}$. Put $\lambda = \lambda_1$. It follows from functional calculus that $\lambda^q = \lambda_q$ for all $q \in \mathbb{Q}$. Then we have

$$\lambda^{\mathrm{i}sq} = (\lambda^q)^{\mathrm{i}s} = \lambda_q^{\mathrm{i}s} = w(s,q)$$

for all $s \in \mathbb{R}$ and $q \in \mathbb{Q}$. Because of strong* continuity we have $\lambda^{\mathrm{i}st} = w(s,t)$ and thus $u_{s+t} = \lambda^{\mathrm{i}st} u_t u_s$ for all $s, t \in \mathbb{R}$. Now we can easily verify that $v_t = \lambda^{-\frac{1}{2}\mathrm{i}t^2} u_t$ defines a strong* continuous one-parameter group of unitaries in $\mathcal{M}$. So we can take a strictly positive operator δ affiliated with $\mathcal{M}$ such that $[D\psi : D\varphi]_t = u_t = \lambda^{\frac{1}{2}\mathrm{i}t^2} \delta^{\mathrm{i}t}$ for all $t \in \mathbb{R}$. This gives us (iii). ∎

Now we will look at the even more specific case $\lambda \in \mathbb{R}_0^+$. So the following proposition becomes meaningful.

Proposition 5.3. *Let φ be an n.s.f. weight on a von Neumann algebra $\mathcal{M}$. Let δ be a strictly positive operator affiliated with $\mathcal{M}$ and $\lambda \in \mathbb{R}_0^+$ such that $\sigma_t^{\varphi}(\delta^{\mathrm{i}s}) = \lambda^{\mathrm{i}st} \delta^{\mathrm{i}s}$ for all $s, t \in \mathbb{R}$. Then we have*

$$\varphi_\delta \circ \sigma_t^{\varphi} = \lambda^{-t} \varphi_\delta \quad \textit{and} \quad \varphi \circ \sigma_t^{\varphi_\delta} = \lambda^t \varphi \quad \textit{for all } t \in \mathbb{R}.$$

Proof. Let $a \in \mathfrak{N}_{\varphi_\delta}$ and $t \in \mathbb{R}$. Then $\sigma_t^{\varphi}(a) = \delta^{-\mathrm{i}t} \sigma_t^{\varphi_\delta}(a) \delta^{\mathrm{i}t}$. This belongs to $\mathfrak{N}_{\varphi_\delta}$ because $\delta^{\mathrm{i}t}$ is analytic with respect to σ^{φ_δ}, and we have

$$\Lambda_{\varphi_\delta}(\sigma_t^{\varphi}(a)) = \delta^{-\mathrm{i}t} J_{\varphi_\delta} \lambda^{-\frac{1}{2}t} \delta^{-\mathrm{i}t} J_{\varphi_\delta} \Delta_{\varphi_\delta}^{\mathrm{i}t} \Lambda_{\varphi_\delta}(a).$$

So we get

$$\varphi_\delta(\sigma_t^{\varphi}(a^* a)) = \lambda^{-t} \varphi_\delta(a^* a) \quad \text{for all } t \in \mathbb{R}.$$

Now, the conclusion follows easily. The second statement is proved analogously. ∎

After stating a lemma, we will prove our third Radon-Nikodym theorem.

Lemma 5.4. *Let φ be an n.s.f. weight on a von Neumann algebra $\mathcal{M}$ and $a \in \mathcal{M}$. If $\mathfrak{N}_\varphi a \subset \mathfrak{N}_\varphi$, $\mathfrak{N}_\varphi a^* \subset \mathfrak{N}_\varphi$ and if there exists a $\lambda \in \mathbb{R}_0^+$ such that $\varphi(ax) = \lambda \varphi(xa)$ for all $x \in \mathfrak{M}_\varphi$, then $\sigma_t^{\varphi}(a) = \lambda^{\mathrm{i}t} a$ for all $t \in \mathbb{R}$.*

Proof. The proof of Result 6.29 in [1] can be taken over literally. Also a slight adaptation of the proof of Theorem 3.6 in [7] yields the result. ∎

PROPOSITION 5.5. *Let ψ and φ be two n.s.f. weights on a von Neumann algebra $\mathcal{M}$. Let $\lambda \in \mathbb{R}_0^+$. The following statements are equivalent:*

(i) *For all $t \in \mathbb{R}$ we have $\varphi \circ \sigma_t^\psi = \lambda^t \varphi$.*

(ii) *For all $t \in \mathbb{R}$ we have $\psi \circ \sigma_t^\varphi = \lambda^{-t} \psi$.*

(iii) *There exists a strictly positive operator δ affiliated with $\mathcal{M}$ such that $\sigma_t^\varphi(\delta^{\mathrm{i}s}) = \lambda^{\mathrm{i}st}\delta^{\mathrm{i}s}$ for all $s, t \in \mathbb{R}$ and such that $\psi = \varphi_\delta$.*

(iv) *There exists a strictly positive operator δ affiliated with $\mathcal{M}$ such that $[D\psi : D\varphi]_t = \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}$ for all $t \in \mathbb{R}$.*

Proof. We have already proven the equivalence of (iii) and (iv) and the implications (iii) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i). Suppose now that (i) is valid. Put $u_t = [D\psi : D\varphi]_t$ and let $x \in \mathcal{M}^+$. Then we have

$$\begin{aligned}\varphi(u_t^* x u_t) &= \varphi(\sigma_{-t}^\varphi(u_t^*)\sigma_{-t}^\varphi(x)\sigma_{-t}^\varphi(u_t)) \\ &= \varphi(u_{-t}\sigma_{-t}^\varphi(x)u_{-t}^*) = \varphi(\sigma_{-t}^\psi(x)) = \lambda^{-t}\varphi(x).\end{aligned}$$

So we have $\mathfrak{N}_\varphi u_t \subset \mathfrak{N}_\varphi$ for all $t \in \mathbb{R}$, and thus $\mathfrak{N}_\varphi u_t^* = \sigma_t^\varphi(\mathfrak{N}_\varphi u_{-t}) \subset \mathfrak{N}_\varphi$ for all $t \in \mathbb{R}$. Then we get for every $x \in \mathfrak{M}_\varphi$ that

$$\varphi(x u_t) = \varphi(u_t^* u_t x u_t) = \lambda^{-t}\varphi(u_t x).$$

From the previous lemma we can conclude that $\sigma_s^\varphi(u_t) = \lambda^{\mathrm{i}st}u_t$ for all $s, t \in \mathbb{R}$. Put $v_t = \lambda^{-\frac{1}{2}\mathrm{i}t^2}u_t \in \mathcal{M}$. Then we have that $t \mapsto v_t$ is a strongly continuous one-parameter group of unitaries. Define δ such that $\delta^{\mathrm{i}t} = v_t$ for all $t \in \mathbb{R}$. So δ is affiliated with $\mathcal{M}$ and $[D\psi : D\varphi]_t = \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}$ and this gives us (iv). Finally suppose (ii) is valid. From the proven implication (i) $\Rightarrow$ (iv) we get the existence of a strictly positive operator δ affiliated with $\mathcal{M}$ such that $[D\varphi : D\psi]_t = \lambda^{-\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}$ for all $t \in \mathbb{R}$. Changing δ to δ^{-1} we get $[D\psi : D\varphi]_t = \lambda^{\frac{1}{2}\mathrm{i}t^2}\delta^{\mathrm{i}t}$ for all $t \in \mathbb{R}$. This gives us again (iv). ∎

We conclude this paper by giving an example which shows that all situations can really occur : we can have $\sigma_t^\varphi(\delta^{\mathrm{i}s}) = \lambda^{\mathrm{i}st}\delta^{\mathrm{i}s}$ with λ and δ strongly commuting but λ not central, with λ central but not scalar, and with λ scalar. Indeed, define $\mathcal{M}_1 = B(L^2(\mathbb{R}))$ and define the selfadjoint operators P and Q on the obvious domains by

$$(P\xi)(\gamma) = \gamma\xi(\gamma) \quad \text{and} \quad (Q\xi)(\gamma) = -\mathrm{i}\xi'(\gamma).$$

Put $H = \exp(P)$ and $K_1 = \exp(Q)$ and denote by Tr the canonical trace on $\mathcal{M}_1$. Remark that Tr has a trivial modular automorphism group so that we can define $\varphi_1 = \mathrm{Tr}_H$ as in Definition 1.5. An easy calculation yields that $\sigma_t^{\varphi_1}(K_1^{\mathrm{i}s}) = H^{\mathrm{i}t}K_1^{\mathrm{i}s}H^{-\mathrm{i}t} = \mathrm{e}^{-\mathrm{i}ts}K_1^{\mathrm{i}s}$, where e denotes the well known real number e. This gives an example of our third case. Define $\mathcal{M}_2$ as the von Neumann algebra of two by two matrices over $\mathcal{M}_1$ and φ_2 as the balanced weight $\theta(\varphi_1, \varphi_1)$ (see [8]). Define $K_2 = \begin{pmatrix} K_1 & 0 \\ 0 & K_1^{-1} \end{pmatrix}$. We easily have $\sigma_t^{\varphi_2}(K_2^{\mathrm{i}s}) = \begin{pmatrix} \mathrm{e}^{-1} & 0 \\ 0 & \mathrm{e} \end{pmatrix}^{its} K_2^{\mathrm{i}s}$, which gives an example of our first case because $\mathcal{M}_2$ is a factor. Define $\mathcal{M}_3$ as the diagonal matrices in $\mathcal{M}_2$. We can restrict φ_2 to $\mathcal{M}_3$ and keep K_2. We have the same formula as above, and in this way an example of our second case, $\begin{pmatrix} \mathrm{e}^{-1} & 0 \\ 0 & \mathrm{e} \end{pmatrix}$ being central now.

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REFERENCES

1. J. KUSTERMANS, KMS-weights on C^*-algebras, Odense Universitet, preprint #funct-an/9704008, 1997.
2. J. KUSTERMANS, S. VAES, A simple definition for locally compact quantum groups, *C.R. Acad. Sci. Paris Sér. I* **328**(1999), 871–876.
3. J. KUSTERMANS, S. VAES, Locally compact quantum groups, *Ann. Sci. École Norm. Sup.* **33**(2000), 837–934.
4. J. KUSTERMANS, S. VAES, The operator algebra approach to quantum groups, *Proc. Nat. Acad. Sci. USA* **97**(2000), 547–552.
5. J. KUSTERMANS, A. VAN DAELE, C^*-algebraic quantum groups arising from algebraic quantum groups, *Int. J. Math.* **8**(1997), 1067–1139.
6. T. MASUDA, Y. NAKAGAMI, A von Neumann algebra framework for the duality of the quantum groups, *Publ. Res. Inst. Math. Sci.* **30**(1994), 799–850.
7. G.K. PEDERSEN, M. TAKESAKI, The Radon-Nikodym theorem for von Neumann algebras, *Acta Math.* **130**(1973), 53–87.
8. Ş. STRĂTILĂ, *Modular Theory in Operator Algebras*, Abacus Press, Tunbridge Wells 1981.
9. Ş. STRĂTILĂ, L. ZSIDÓ, *Lectures on von Neumann Algebras*, Abacus Press, Tunbridge Wells 1979.

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