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BASES OF REPRODUCING KERNELS IN MODEL SPACES

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Abstract. This paper deals with geometric properties of sequences of reproducing kernels related to invariant subspaces of the backward shift operator in the Hardy space H^2. Let $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$, Θ be an inner function in $H^\infty(\mathcal{L}(E))$, where E is a finite dimensional Hilbert space, and $(e_n)_{n\geqslant 1}$ a sequence of vectors in E. Then we give a criterion for the vector valued reproducing kernels $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ to be a Riesz basis for $K_\Theta := H^2(E)\ominus\Theta H^2(E)$. Using this criterion, we extend to the vector valued case some basic facts that are well-known for the scalar valued reproducing kernels. Moreover, we study the stability problem, that is, given a Riesz basis $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$, we characterize its perturbations $(k_\Theta(\,\cdot\,,\mu_n))_{n\geqslant 1}$ that preserve the Riesz basis property. For the case of asymptotically orthonormal sequences, we give an effective upper bound for uniform perturbations preserving stability and compare our result with Kadeč's 1/4-theorem.

Keywords: *model subspaces of H^2, Riesz bases, reproducing kernels, stability.*

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1. INTRODUCTION

This paper is devoted to geometric properties of sequences of reproducing kernels in Hardy spaces. These properties are of interest for several reasons. First of all, if we consider a C_{00}-contraction, T, in a Hilbert space, then by Sz.-Nagy and C. Foias' theory, T is unitarily equivalent to its canonical model M_Θ,

$$M_\Theta f := P_\Theta(zf), \quad f \in K_\Theta.$$

Here K_Θ is the model space

$$K_\Theta := H^2(E) \ominus \Theta H^2(E),$$

E is an auxiliary Hilbert space, $H^2(E)$ stands for the E-valued Hardy space in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, Θ is the characteristic function of T and P_Θ

is the orthogonal projection onto K_Θ. The spectral theory of M_Θ in the language of the characteristic function Θ (see [23], [17], [20]) depends on the geometry of reproducing kernels of K_Θ, that is, on the (operator-valued) functions

$$k_\Theta(z,\lambda) := \frac{1-\Theta(z)\Theta(\lambda)^*}{1-\overline{\lambda}z}, \quad z,\lambda \in \mathbb{D},$$

satisfying

$$\langle f(\lambda), e\rangle = \langle f, k_\Theta(\,\cdot\,,\lambda)e\rangle_{H^2(E)},$$

for every $\lambda \in \mathbb{D}$, $e \in E$ and $f \in K_\Theta$. Moreover, in some approaches, the kernels $k_\Theta(z,\lambda)$ and their various analogs are the starting point for the model theory and its applications, especially to various interpolation problems; see [7], [3] and [10]. Our goal in this paper is to extend to the vector valued case some basic facts that are well-known for the scalar valued reproducing kernels ($E = \mathbb{C}$, see [17], [13]).

Recall that a scalar inner function ϑ can be written as $\vartheta = SB$, where S is the singular inner factor of ϑ,

$$S(z) = \exp\biggl(-\int\limits_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,\mathrm{d}\mu(\zeta)\biggr),$$

the measure $\mathrm{d}\mu$ being positive and singular with respect to the Lebesgue measure $\mathrm{d}m$, and B is the corresponding Blaschke product,

$$B = \prod_{n\geqslant 1} b_{\lambda_n},$$

$b_{\lambda_n} := \frac{|\lambda_n|}{\lambda_n}\frac{\lambda_n - z}{1-\overline{\lambda}_n z}$. It is known (see [18]) that $\frac{\vartheta}{z-\lambda_n} \in K_\vartheta$ are eigenvectors of M_ϑ, and $k_{\lambda_n} := \frac{1}{1-\overline{\lambda}_n z} \in K_\vartheta$ are eigenvectors of the adjoint operator M_ϑ^*:

$$M_\vartheta\Bigl(\frac{\vartheta}{z-\lambda_n}\Bigr) = \lambda_n \frac{\vartheta}{z-\lambda_n}, \quad M_\vartheta^* k_{\lambda_n} = \overline{\lambda}_n k_{\lambda_n}.$$

Moreover, it is proved in [13] that the union of eigenvectors of M_ϑ and M_ϑ^* forms an unconditional basis (a Riesz basis) in K_ϑ if and only if the reproducing kernels $(k_S(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ form the same kind of basis in K_S. The latter property is, therefore, important for some applications such as the string scattering theory, see [13] and [19] for details.

Secondly, some motivation comes from control theory and signal processing. Namely, in the special case where $\Theta = \Theta_a := \exp(a\frac{z+1}{z-1})$, the reproducing kernels $k_\Theta(\,\cdot\,,\lambda)$, with $\lambda \in \mathbb{D}$, arise as images of the exponential functions $\exp(-\mathrm{i}\overline{\mu}\omega)\chi_{(0,a)}$, with $\mu := \mathrm{i}\frac{1+\lambda}{1-\lambda}$, under a natural unitary map going from $L^2(0,a)$ to K_Θ. The property of families of exponentials (and hence, of reproducing kernels k_Θ for $\Theta = \Theta_a$) to be a Riesz basis is important for control theory. Indeed, under some further hypotheses, the theory of controllability of a dynamic system $x'(t) = Ax(t) + Bu(t)$, $t \geqslant 0$, $x(0) = x_0$, depends on the geometric properties of the system of exponentials $(\exp(-\overline{\lambda}_n t)B^*\psi_n)_{n\geqslant 1}$, where $(\psi_n)_{n\geqslant 1}$ is the sequence of eigenvectors of A^*, associated to the sequence of eigenvalues $(\overline{\lambda}_n)_{n\geqslant 1}$. For more details on the relationships between the controllability problems and the geometry of families of exponentials we refer to [1] and [18].

A third reason to study geometric properties of sequences of reproducing kernels is just to understand better non-harmonic exponentials $(\exp(\mathrm{i}\lambda_n t))_{n\geqslant 1}$ which appear frequently, say, in analysis of convolution equations (see, for instance, [13] and [29]).

To solve the problem of Riesz bases for the families $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ in scalar K_Θ, S.V. Hruscev, N.K. Nikolski and B.S. Pavlov proposed in [13] a new method (see also [17]). They gave a criterion in terms of the Carleson condition and invertibility of a certain Toeplitz operator. In the first part of this paper, using the above-mentioned operator approach, we give vector valued generalizations of several results of Hruscev, Nikolski and Pavlov.

Another subject of this paper concerns stability properties of reproducing kernel bases $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ under small perturbations of the poles λ_n. A reason to study the stability properties is that the criteria mentioned above involve, however, some properties of a given family $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ that are rather difficult to verify. On the other hand, in many cases, the given family is a slight perturbation of another family $(k_\Theta(\,\cdot\,,\mu_n)e'_n)_{n\geqslant 1}$ that is known to be a basis. This gives rise to the stability problem whose formal statement can be found in Subsection 3.1 below. Roughly speaking, given a Riesz basis $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ in K_Θ, the problem is to characterize its perturbations $(k_\Theta(\,\cdot\,,\mu_n))_{n\geqslant 1}$ that still enjoy the property of being a Riesz basis. In fact, this problem was initially raised by R.E.A.C. Paley and N. Wiener for the orthogonal basis $(\exp(\mathrm{i}nt),\ n\in\mathbb{Z})$ in $L^2(0,2\pi)$. A sufficient condition for such stability is given in [21]. For this case, the problem of uniform stability (see Subsection 3.1) was completely solved by A. Ingham and M. Kadeč (see [15] and [14]). In 1990, A. Avdonin and I. Joo gave a sufficient condition for stability of general unconditional bases of exponentials. In Section 3, we give a generalization of this result for the reproducing kernels $(k_\Theta(\,\cdot\,,\lambda_n)e_n : n\geqslant 1)$.

The paper is organized as follows. In Section 2, we deal with Riesz bases of vector valued reproducing kernels $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$. Our goal is to separate, as far as it is possible, the influence of the three parameters involved: the frequency spectrum $\Lambda=(\lambda_n)_{n\geqslant 1}$, the spatial directions e_n, $n\geqslant 1$, and the inner function Θ generating the model subspace K_Θ. In particular, if E is a finite dimensional Hilbert space, we prove, in Theorem 2.1, the vector analog of Hruscev, Nikolski and Pavlov's criterion for a family $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ to be a Riesz basis for K_Θ. If $(k_{\lambda_n}e_n)_{n\geqslant 1}$ is not a Riesz basis for K_B, then we adapt a method proposed by V. Vasyunin in the scalar case to gather "sufficiently close" eigenvectors in such a way that the rearranged family forms an unconditional basis of subspaces. If Λ is a finite union of Carleson sets and $\sup\limits_{n\geqslant 1} r(\Theta(\lambda_n))<1$, where $r(\Theta(\lambda_n))$ denotes the spectral radius of the operator $\Theta(\lambda_n)$, then we prove, in Theorem 2.4, that there exist a sequence $(e_n)_{n\geqslant 1}$ in E and $m\in\mathbb{N}$ such that $(k_{\Theta^m}(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ forms a Riesz basis in its hull. Under the assumption that $\lim\limits_{n\to\infty}\|\Theta(\lambda_n)^*e_n\|=0$, we prove in Theorem 2.8 that $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ is a Riesz basis if and only if it is uniformly minimal (see [6] for a scalar analogue of this result).

In Section 3, we consider the case where $1<p<+\infty$, $\Lambda=(\lambda_n)_{n\geqslant 1}\subset\mathbb{D}$, Θ is a (scalar) inner function such that $\sup\limits_{n\geqslant 1}|\Theta(\lambda_n)|<1$ and $K^p_\Theta := H^p\cap\Theta\overline{zH^p}$. If $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ is an unconditional basis in K^p_Θ, then we prove the existence of $\varepsilon > 0$ such that $(k_\Theta(\,\cdot\,,\mu_n))_{n\geqslant 1}$ is an unconditional basis in K^p_Θ, whenever $(\mu_n)_{n\geqslant 1} \subset \mathbb{D}$ is a sequence satisfying

$$\sup_{n\geqslant 1} |b_{\lambda_n}(\mu_n)| < \varepsilon$$

(see Theorems 3.1 and 3.3). In Theorem 3.4, we prove a vector-valued analogue of this stability result. For the case of asymptotically orthogonal sequences $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$, we give a numerical upper bound for ε (see Theorem 3.10) and compare this result with Kadeč's 1/4-theorem.

In conclusion, let us fix some notation. E always stands for a complex Hilbert space. The symbol P_+ (respectively P_-) stands for the Riesz orthogonal projection from $L^2(E)$ onto $H^2(E)$ (respectively onto $L^2(E)\ominus H^2(E)$). The space of all bounded linear operators acting on E will be denoted by $\mathcal{L}(E)$. We write

$$L^\infty(\mathcal{L}(E)) := \{f : \mathbb{T} \to \mathcal{L}(E) \text{ such that } f \text{ is measurable and bounded}\}$$

and

$$H^\infty(\mathcal{L}(E)) := \{f : \mathbb{D} \to \mathcal{L}(E) \text{ such that } f \text{ is analytic and bounded}\}.$$

A function $\Theta \in H^\infty(\mathcal{L}(E))$ is called inner if the operator $\Theta(\zeta)$ is an isometry for almost all $\zeta \in \mathbb{T}$. For a function $\varphi \in L^\infty(\mathcal{L}(E))$, the symbol H_φ denotes the Hankel operator defined on $H^2(E)$ by

$$H_\varphi f := P_-(\varphi f),$$

and T_φ denotes the Toeplitz operator defined on $H^2(E)$ by

$$T_\varphi f := P_+(\varphi f).$$

Let (a_n) and (b_n) be real sequences. We write $(a_n) \asymp (b_n)$ if there exist two constants $c, C > 0$ such that

$$ca_n \leqslant b_n \leqslant Ca_n, \quad \forall n \geqslant 1.$$

2. UNCONDITIONAL BASES OF REPRODUCING KERNELS

Here we recall some standard facts about sequences of vectors and subspaces in a Banach space. Let $(K_n)_{n\geqslant 1}$ be a sequence of subspaces of a Banach space $\mathcal{X}$. It is called *minimal* (respectively *uniformly minimal*) if, for every n, the operators

$$\mathcal{P}_n x = \mathcal{P}_n\Big(\sum_j x_j\Big) = x_n, \tag{2.1}$$

defined on the set of finite sums $x = \sum_j x_j$, $x_j \in K_j$, are continuous (respectively, uniformly bounded, i.e. $\sup_n \|\mathcal{P}_n\| < \infty$). For the case of 1-dimensional subspaces $K_n = \mathbb{C}\cdot e_n$, $e_n \in \mathcal{X}$, one has $\mathcal{P}_n = \langle\,\cdot\,,\phi_n^*\rangle e_n$, where $(\phi_n^*)_{n\geqslant 1}$ is a biorthogonal family of $(e_n)_{n\geqslant 1}$, that is, a sequence of continuous linear functionals on $\mathcal{X}$ such that

$$\langle e_n, \phi_k^*\rangle = \delta_{nk}.$$

Hence the definition above is equivalent to the standard one. A sequence $(K_n)_{n\geqslant 1}$ is called an *unconditional basis of its hull* if, for every $x \in \mathrm{Span}\{K_n : n \geqslant 1\}$, the series $\sum_{n\geqslant 1} \mathcal{P}_n x$ unconditionally converges to x. Moreover, if $\mathrm{Span}\{K_n : n \geqslant 1\} = \mathcal{X}$, we say that $(K_n)_{n\geqslant 1}$ is *unconditionnal basis of* $\mathcal{X}$. It is well known (see, for example, [17]) that $(K_n)_{n\geqslant 1}$ is an unconditional basis of its hull if and only if it is minimal and

$$\sup_{\sigma\in\mathcal{F}} \|\mathcal{P}_\sigma\| < \infty,$$

where $\mathcal{F} := \{\sigma \subset \mathbb{N} : \sigma \text{ finite}\}$ and $\mathcal{P}_\sigma := \sum_{n\in\sigma} \mathcal{P}_n$. Let H be a Hilbert space, $(K_n)_{n\geqslant 1}$ a family of subspaces, minimal and total in H, with a total family of spectral projections $(\mathcal{P}_n)_{n\geqslant 1}$; then by Köthe-Toeplitz' theorem, $(K_n)_{n\geqslant 1}$ is an unconditional basis of H if and only if it is a *Riesz basis*, that is, there exists a constant $c > 0$ such that

$$\frac{1}{c}\sum_j \|x_j\|^2 \leqslant \left\|\sum_j x_j\right\|^2 \leqslant c\sum_j \|x_j\|^2,$$

for every finite family $x_j \in K_j$, $j \geqslant 1$. The latter is also equivalent to the existence of an isomorphism V of $\mathrm{Span}(K_j : j \geqslant 1)$ such that the subspaces VK_n, $n \geqslant 1$, are pairwise orthogonal. Note that the above definitions also work for finite sequences $(K_j)_{j\geqslant 1}$, and in particular, for a pair of subspaces K_1, K_2 of a Hilbert space H. Then the *angle*, $\alpha(K_1, K_2)$, beetween K_1 and K_2 can be defined by the conditions

$$\alpha(K_1, K_2) \in \left[0, \frac{\pi}{2}\right], \quad \cos(\alpha(K_1, K_2)) = \sup_{\substack{x\in K_1,\, y\in K_2\\ \|x\|=1,\, \|y\|=1}} |\langle x, y\rangle|.$$

It follows from the definition that

$$\cos(\alpha(K_1, K_2)) = \|P_{K_1} P_{K_2}\|, \quad \sin(\alpha(K_1, K_2)) = \|\mathcal{P}_1\|^{-1},$$

where P_{K_i} are the corresponding orthogonal projections, $i = 1, 2$, and $\mathcal{P}_1$ is defined in formula (2.1).

Let $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$, Θ be an inner function in $H^\infty(\mathcal{L}(E))$, and $(e_n)_{n\geqslant 1}$ a sequence of unit vectors in E. In this section, we are looking for necessary and sufficient conditions on Θ, Λ and $(e_n)_{n\geqslant 1}$ for the family $(k_\Theta(\,\cdot\,, \lambda_n)e_n)_{n\geqslant 1}$ to be a Riesz basis of its hull. Recall that, in the scalar case where $\dim E = 1$, the following was proved in [13]: if $\sup_{n\geqslant 1} |\Theta(\lambda_n)| < 1$ then $(k_\Theta(\,\cdot\,, \lambda_n))_{n\geqslant 1}$ is a Riesz basis of its hull if and only if $\Lambda \in (C)$ and $P_\Theta|K_B$ is an isomorphism onto its range (which is also equivalent to the fact that $T_{\overline{B}\Theta}$ is left invertible). Here $\Lambda \in (C)$ means that $\Lambda = (\lambda_n)_{n\geqslant 1}$ satisfies the Carleson condition

$$\delta(\Lambda) := \inf_{n\geqslant 1} |B_{\lambda_n}(\lambda_n)| > 0,$$

where $B_{\lambda_n} := \frac{B}{b_{\lambda_n}}$. The constant $\delta(\Lambda)$ is called the *Carleson constant* of the sequence Λ.

2.1. Some geometric properties of vector valued reproducing kernels. Here we list some general properties of vector-valued reproducing kernels. (So, below, **P** stands for "property").

(P1) *If E is a finite dimensional Hilbert space, then $(k_{\lambda_n}e_n)_{n\geqslant 1}$ is minimal in $H^2(E)$ if and only if $(\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence* (see for instance [1]).

(P2) *If $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ is minimal, then $(k_{\lambda_n}e_n)_{n\geqslant 1}$ is minimal. More generally, if T is a bounded operator in E and $(Tx_n)_{n\geqslant 1}$ is a minimal family then $(x_n)_{n\geqslant 1}$ is also minimal and its biorthogonal family is given by $(T^*\psi_n)_{n\geqslant 1}$ where $(\psi_n)_{n\geqslant 1}$ is the biorthogonal family of $(Tx_n)_{n\geqslant 1}$.*

To get a similar result for uniform minimality, we need the following property.

(P3) *Let $\Lambda\subset\mathbb{D}$, $(e_\lambda)_{\lambda\in\Lambda}$ be a family of unit vectors in E. Then*

$$\|k_\Theta(\,\cdot\,,\lambda)e_\lambda\|\asymp\|k_\lambda e_\lambda\|\iff\sup_{\lambda\in\Lambda}\|\Theta(\lambda)^*e_\lambda\|<1.$$

Indeed, since $k_\Theta(\,\cdot\,,\lambda)e_\lambda=(1-\overline{\lambda}z)^{-1}(1-\Theta\Theta(\lambda)^*)e_\lambda$, we have

$$\|k_\Theta(\,\cdot\,,\lambda)e_\lambda\|^2=\frac{1-\|\Theta(\lambda)^*e_\lambda\|^2}{1-|\lambda|^2},\quad \|k_\lambda e_\lambda\|^2=\frac{1}{1-|\lambda|^2},$$

and the result follows. ∎

The following property is a straightforward consequence of (P3) and a lemma of S. Hruscev and N. Nikolski (see [19], Lemma 120, page 173).

(P4) *If $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ is uniformly minimal and*

$$\sup_{n\geqslant 1}\|\Theta(\lambda_n)^*e_n\|<1,$$

then $(k_{\lambda_n}e_n)_{n\geqslant 1}$ is also uniformly minimal.

(P5) *If $(e_n)_{n\geqslant 1}$ is a relatively compact sequence in E, then $(k_{\lambda_n}e_n)_{n\geqslant 1}$ is a Riesz basis of its hull if and only if it is uniformly minimal* (see [24]).

(P6) *If $\dim E=N<\infty$, and $(k_{\lambda_n}e_n)_{n\geqslant 1}$ is uniformly minimal, then $\Lambda=(\lambda_n)_{n\geqslant 1}$ is the union of N-Carleson sets. In this case, we say that Λ is N-Carleson* (see, for instance, Corollary 2 on page 164 of [17]).

For N-Carleson sequences Λ, a necessary and sufficient condition for $(k_{\lambda_n}e_n)_{n\geqslant 1}$ to be a Riesz basis of its hull was found by S.A. Ivanov, see [1]. To state this condition ((P7) below), we set

$$G(\Lambda,r):=\bigcup_{\lambda\in\Lambda}\Omega(\lambda,r),$$

where $\Omega(\lambda,r)=\{z:|b_\lambda(z)|<r\}$ is the hyperbolic disc of radius r and center λ. Denote by $G_m(\Lambda,r)$, $m=1,2,\ldots$, the connected components of the set $G(\Lambda,r)$ and write $\Lambda_m(r):=\Lambda\cap G_m(\Lambda,r)$. For a sequence $(e_\lambda)_{\lambda\in\Lambda}$ in E and for $\Lambda'\subset\Lambda$, we set $\mathcal{E}(\Lambda'):=\operatorname{Span}(e_\mu:\mu\in\Lambda')$.

(P7) *Assume that $\Lambda:=(\lambda_n)_{n\geqslant 1}$ is N-Carleson. Then $(k_{\lambda_n}e_n)_{n\geqslant 1}$ is a Riesz basis of its hull if and only if there exists an $r>0$ such that*

$$\inf_{m\geqslant 1}\min_{\lambda\in\Lambda_m(r)}\alpha\big(e_\lambda,\mathcal{E}(\Lambda_m(r)\setminus\{\lambda\})\big)>0.$$

For the proof, see [1].

Now, we state some facts concerning the relationship between the properties of scalar and vector families. The idea is that the geometric properties of scalar families $(k_{\lambda_n})_{n\geqslant 1}$ are "stronger" than those of vector valued families $(k_{\lambda_n}e_n)_{n\geqslant 1}$. To make this fact clearer, some general properties of Hilbert tensor products will be useful. Let E and H be separable Hilbert spaces. We denote by $H\otimes E$ the completion of the algebraic tensor product with respect to the Hilbert-Schmidt norm $\|\cdot\|_{HS}$ defined for a finite sum $A:=\sum_k x_k\otimes y_k$ by

$$\|A\|_{HS}:=\Big(\sum_{n\geqslant 1}\|Ae_n\|^2\Big)^{1/2},$$

where $(e_n)_{n\geqslant 1}$ is an orthonormal basis of H. Here we identify A with the operator $A:H\to E$ defined by

$$Ah=\sum_k\langle h,x_k\rangle y_k,\qquad \text{for } h\in H,$$

where $\langle\cdot,\cdot\rangle$ is a bilinear pairing of H with itself. Using these definitions, one can easily identify $L^2\otimes E$ with $L^2(E)$ by

$$\sum_k x_k\otimes y_k\longrightarrow\sum_k x_k(\zeta)y_k,\quad \zeta\in\mathbb{T}$$

where $x_k\in L^2(\mathbb{T})$, $y_k\in E$.

Indeed, let A be a finite sum, $A=\sum_k x_k\otimes y_k\in H\otimes E$ and let $(b_j)_j$ be an orthonormal basis of E. Then, we have

$$\begin{aligned}
\|A\|_{HS}^2&=\sum_{n\geqslant 1}\sum_{j\geqslant 1}|\langle Ae_n,b_j\rangle|^2=\sum_{n\geqslant 1}\sum_{j\geqslant 1}\Big|\sum_k\langle e_n,x_k\rangle\langle y_k,b_j\rangle\Big|^2\\
&=\sum_{j\geqslant 1}\sum_{n\geqslant 1}\Big|\Big\langle e_n,\sum_k\langle y_k,b_j\rangle x_k\Big\rangle\Big|^2=\sum_{j\geqslant 1}\Big\|\sum_k\langle y_k,b_j\rangle x_k\Big\|_{L^2}^2\\
&=\sum_{j\geqslant 1}\int_{\mathbb{T}}\Big|\sum_k\langle y_k,b_j\rangle x_k(\zeta)\Big|^2\,\mathrm{d}m(\zeta)=\int_{\mathbb{T}}\sum_{j\geqslant 1}\Big|\Big\langle\sum_k x_k(\zeta)y_k,b_j\Big\rangle\Big|^2\,\mathrm{d}m(\zeta)\\
&=\int_{\mathbb{T}}\Big\|\sum_k x_k(\zeta)y_k\Big\|_E^2\,\mathrm{d}m(\zeta)=\Big\|\sum_k x_k(\cdot)y_k\Big\|_{L^2(E)}^2.
\end{aligned}$$

The following property is well known:

(P8) *Let $V\in\mathcal{L}(H)$ and $U\in\mathcal{L}(E)$. Then $V\otimes U:H\otimes E\to H\otimes E$ defined by*

$$(V\otimes U)\Big(\sum_k x_k\otimes y_k\Big):=\sum_k Vx_k\otimes Uy_k,$$

is a bounded operator, and we have $\|V\otimes U\|\leqslant\|V\|\,\|U\|$.

(P9) *Assume that a family $(x_n)_{n\geqslant 1}$ in H is either minimal, or uniformly minimal, or a Riesz basis. Then:*

(i) *the family $(x_n \otimes y_n)_{n\geqslant 1}$ enjoys the same property for every $y_n \in E$, $y_n \neq 0$;*

(ii) *the family of subspaces $(x_n \otimes E)_{n\geqslant 1}$ has the same property as $(x_n)_{n\geqslant 1}$;*

(iii) *if $(y_k)_{k\geqslant 1} \subset E$ possesses the same property as $(x_n)_{n\geqslant 1}$, then so does the double-indexed family $(x_n \otimes y_k)_{n,k\geqslant 1}$.*

Indeed, (i) is a consequence of (ii). To check (ii), we simply write down the corresponding (spectral) projections $\mathcal{E}_\sigma$ for $(x_n \otimes E)_{n\geqslant 1}$ in terms of the projections $\mathcal{P}_\sigma$ for the family $(x_n)_{n\geqslant 1}$. Clearly, $\mathcal{E}_\sigma = \mathcal{P}_\sigma \otimes I$ for every finite subset $\sigma \subset \mathbb{N}$. Since $\|\mathcal{E}_\sigma\| \leqslant \|\mathcal{P}_\sigma\|$ (see (P8)), claim (ii) follows from the remarks in the beginning of this section and the corresponding property of $(x_n)_{n\geqslant 1}$.

To check (iii), we observe first that $\mathcal{E}_k = \mathcal{P}_k \otimes \mathcal{Q}_k$ where $\mathcal{Q}_k$ stands for the coordinate projection for the sequence $(y_k)_{k\geqslant 1}$, i.e. $\mathcal{Q}_k = \langle \cdot, \phi_k \rangle y_k$, $(\phi_k)_{k\geqslant 1} \subset E$ being a biorthogonal sequence. Since $\|\mathcal{E}_k\| \leqslant \|\mathcal{P}_k\| \, \|\mathcal{Q}_k\|$ for every k, the minimality and uniform minimality properties follow. In the case when $(x_n)_{n\geqslant 1}$ and $(y_k)_{k\geqslant 1}$ are Riesz bases, we take isomorphisms V and U making them orthogonal. That is $(Vx_n)_{n\geqslant 1}$ and $(Uy_k)_{k\geqslant 1}$ are orthogonal sequences. Clearly $(Vx_n \otimes Uy_k)_{n,k\geqslant 1}$ is an orthogonal family in $H \otimes E$ and $V \otimes U$ is an isomorphism of $H \otimes E$. Hence, $(x_n \otimes y_k)_{n,k\geqslant 1}$ is a Riesz basis. ∎

(P10) *Let $(K_n)_{n\geqslant 1}$ be a family of subspaces in H having one of the following properties: to be minimal, uniformly minimal or a Riesz basis. Then the family of subspaces $(K_n \otimes E)_{n\geqslant 1}$ has the same property in $H \otimes E$.*

The proof is the same as for (P9) (ii) since $\mathcal{E}_\sigma = \mathcal{P}_\sigma \otimes I$ are the corresponding coordinate projections. ∎

(P11) *Let $\dim E = N < \infty$. The following assertions are equivalent:*

(i) *There exists a family of vectors $(e_\lambda)_{\lambda\in\Lambda}$ in E, $\|e_\lambda\| = 1$ such that $\mathcal{X} := (k_\lambda e_\lambda : \lambda \in \Lambda)$ is a Riesz basis of its hull.*

(ii) *Λ is N-Carleson.*

Indeed, (i)$\Rightarrow$(ii) is (P5).

To prove (ii)$\Rightarrow$(i), we set $\Lambda = \bigcup_{i=1}^{N} \Lambda_i$, $\Lambda_i \in (C)$, and write $\Lambda_i := \{\lambda_{n,i} : n \geqslant 1\}$. Consider an orthonormal basis $(e_1, e_2, \ldots, e_N)$ of E and define for $\lambda \in \Lambda$

$$e_\lambda := e_i, \quad \text{if } \lambda \in \Lambda_i.$$

As $\Lambda_i \in (C)$, the family $(k_{\lambda_{n,i}})_{n\geqslant 1}$ is a Riesz basis and (P9) implies that $(k_{\lambda_{n,i}} \otimes e_i = k_{\lambda_{n,i}} e_i : n \geqslant 1,\, 1 \leqslant i \leqslant N)$ is also a Riesz basis. ∎

2.2. SOME VECTOR VALUED GENERALIZATIONS. In this subsection, using the operator approach, we obtain the following criterion for a family $(k_\Theta(\cdot, \lambda_n)e_n)_{n\geqslant 1}$ to be a Riesz basis of its hull.

THEOREM 2.1. *Let $\Lambda \subset \mathbb{D}$, $N = \dim E < \infty$ and Θ be an inner function in $H^\infty(\mathcal{L}(E))$. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a family of vectors in E, $\|e_\lambda\| = 1$, such that*

$$\sup_{\lambda\in\Lambda} \|\Theta(\lambda)^* e_\lambda\| < 1.$$

Denote by B the inner function in $H^\infty(\mathcal{L}(E))$ satisfying

$$K_B = \operatorname{Span}\{k_\lambda e_\lambda : \lambda \in \Lambda\}.$$

Then the following assertions are equivalent:

(i) $(k_\Theta(\,\cdot\,,\lambda)e_\lambda)_{\lambda\in\Lambda}$ *is a Riesz basis of its hull.*

(ii) (a) Λ *is N-Carleson;*

(b) *there exists an $r>0$ such that* $\inf_{m\geqslant 1} \min_{\lambda\in\Lambda_m(r)} \alpha\big(e_\lambda, \mathcal{E}(\Lambda_m(r)\setminus\{\lambda\})\big) > 0$*;*

(c) $T_{B^*\Theta}$ *is left invertible in $H^2(E)$.*

We also consider the problem of rearranging a given family $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ into a basis of subspaces. Finally, under the additional assumption $\lim_{n\to\infty} \|\Theta(\lambda_n)^* e_n\| = 0$, we prove the equivalence between uniform minimality and the property of being a Riesz basis.

To prove Theorem 2.1, we need some facts which are easy generalizations of well-known results in the scalar case (see for example [19]). The proofs are similar to their scalar-valued prototypes, so we omit them.

LEMMA 2.2. *Let Θ and B be $\mathcal{L}(E)$-valued inner functions. The following statements are equivalent:*

(i) $P_\Theta|K_B$ *is an isomorphism onto its range;*

(ii) $\operatorname{dist}(B^*\Theta, H^\infty(\mathcal{L}(E))) < 1$*;*

(iii) $\|P_B\Theta\| < 1$*;*

(iv) $T_{B^*\Theta}$ *is left invertible in $H^2(E)$.*

Proof of Theorem 2.1. (i)⇒(ii): Since $(k_\Theta(\,\cdot\,,\lambda)e_\lambda : \lambda \in \Lambda)$ is a Riesz basis of its hull, it is uniformly minimal and (P4) implies that $\mathcal{X} = (k_\lambda e_\lambda : \lambda \in \Lambda)$ is also uniformly minimal. From (P6) we deduce that Λ is N-Carleson. It follows from (P7) that (ii) is satisfied. Now observe that the operator $P_\Theta|K_B$ maps the Riesz basis $\mathcal{X}$ onto the Riesz basis of its hull $(k_\Theta(\,\cdot\,,\lambda)e_\lambda : \lambda \in \Lambda)$ and thus it must be an isomorphism onto its range. This implies by Lemma 2.2 that $T_{B^*\Theta}$ is left invertible in $H^2(E)$.

(ii)⇒(i): Using (P7), we deduce from (a) and (b) that $\mathcal{X} = (k_\lambda e_\lambda : \lambda \in \Lambda)$ is a Riesz basis of K_B. Lemma 2.2 and (c) imply that $P_\Theta|K_B$ is an isomorphism onto its range. Hence $P_\Theta\mathcal{X} = (k_\Theta(\,\cdot\,,\lambda)e_\lambda : \lambda \in \Lambda)$ is a Riesz basis of its hull. ∎

REMARK 2.3. In fact, to prove (ii)⇒(i), we do not need the assumptions $\dim E < \infty$ and

$$\sup_{\lambda\in\Lambda} \|\Theta(\lambda)^* e_\lambda\| < 1.$$

In the scalar case, if a family of rootspaces $(K_\lambda)_{\lambda\in\Lambda}$ of a model operator is not a basis, V. Vasyunin proposed a method to gather "sufficiently close" rootspaces in such a way that the new family forms a Riesz basis (see, for instance, [17], Lecture IX, pp. 229–230). Using a similar method and (P10), it is easy to generalize this result to the vector case. Moreover, if we have a Riesz basis family of subspaces

$(K_{\Theta_n})_{n\geqslant 1}$ such that $\dim K_{\Theta_n} \leqslant k$, $\forall n \geqslant 1$, then it is also possible to split this family into a union of k sequences, each forming a Riesz basis of its hull.

In applications (particularly in control theory) it is useful, given a family of exponentials $(\exp(\mathrm{i}\mu_n t)u_n)_{n\geqslant 1}$, to know if one can find a real number $a > 0$ such that the family $(\exp(\mathrm{i}\mu_n t)u_n)_{n\geqslant 1}$ forms a Riesz basis of its hull in $L^2(0, a; E)$. In the language of reproducing kernels, this is equivalent to saying that there exists $a > 0$ such that $(k_{\Theta_1^a}(\,\cdot\,, \lambda_n))_{n\geqslant 1}$ is a Riesz basis of its hull, where $\Theta_1^a = \exp(a\frac{z+1}{z-1})$. For a general scalar inner function Θ, Hruscev, Nikolski and Pavlov showed that under the assumption $\sup_{n\geqslant 1} |\Theta(\lambda_n)| < 1$, the Carleson condition $\Lambda \in (C)$ is necessary and sufficient for the existence of $n \in \mathbb{N}$ such that $(k_{\Theta^n}(\,\cdot\,, \lambda) : \lambda \in \Lambda)$ is a Riesz basis of its hull. Now, we give a generalization of this result to the vector case. There are several possibilities to extend the property $\sup_{n\geqslant 1} |\Theta(\lambda_n)| < 1$ to the case $\dim E > 1$. Namely, one can think about properties $\sup_{n\geqslant 1} \|\Theta(\lambda_n)\| < 1$, $\sup_{n\geqslant 1} \|\Theta(\lambda_n)^* e_n\| < 1$ or $\sup_{n\geqslant 1} r(\Theta(\lambda_n)) < 1$, where $r(T)$ denotes the spectral radius of an operator T. All of these assumptions coincide in the scalar case ($\dim E = 1$). Now, we present a result with the assumption $\sup_{n\geqslant 1} r(\Theta(\lambda_n)) < 1$.

THEOREM 2.4. *Let Θ be an $\mathcal{L}(E)$-valued inner function,* $\dim E = N < \infty$ *and $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ be such that*

$$\sup_{n\geqslant 1} r(\Theta(\lambda_n)) < 1. \tag{2.2}$$

The following assertions are equivalent:

(i) *Λ is N-Carleson;*

(ii) *There exists a sequence $(e_n)_{n\geqslant 1}$ in E, $\|e_n\| = 1$, such that for all sufficiently large $m \in \mathbb{N}$, the family $(k_{\Theta^m}(\,\cdot\,, \lambda_n)e_n : n \geqslant 1)$ is a Riesz basis of its hull.*

To prove Theorem 2.4 we need two lemmas.

LEMMA 2.5. *Let $\Lambda \subset \mathbb{D}$ and $(e_\lambda)_{\lambda\in\Lambda}$ be a family of unit vectors in E such that $(k_\lambda e_\lambda : \lambda \in \Lambda)$ is a Riesz basis of its hull. Denote by B the inner function such that $K_B := \mathrm{Span}\{k_\lambda e_\lambda : \lambda \in \Lambda\}$. Then there exists a constant C such that*

$$\|P_B \Theta^*\| \leqslant C \sup_{\lambda\in\Lambda} \|\Theta(\lambda)^* e_\lambda\|,$$

for every inner function Θ in $H^\infty(\mathcal{L}(E))$.

Proof. Since $(k_\lambda e_\lambda : \lambda \in \Lambda)$ is a Riesz basis of its hull, there exist two constants $c_1, C_1 > 0$ such that

$$c_1 \sum_{\lambda\in\Lambda} |a_\lambda|^2 \|k_\lambda\|^2 \leqslant \Big\| \sum_{\lambda\in\Lambda} a_\lambda k_\lambda e_\lambda \Big\|^2 \leqslant C_1 \sum_{\lambda\in\Lambda} |a_\lambda|^2 \|k_\lambda\|^2, \tag{2.3}$$

and

$$\sum_{\lambda\in\Lambda} (1 - |\lambda|^2) \|f(\lambda)\|^2 \leqslant C_1 \|f\|^2, \tag{2.4}$$

for all finite families (a_λ) in $\mathbb{C}$ and all $f \in K_B$. Now, observing that $P_B\Theta^* k_\lambda e_\lambda = P_B k_\lambda \Theta(\lambda)^* e_\lambda$, we can write

$$\begin{aligned}\Big\| P_B\Theta^* \sum_{\lambda\in\Lambda} a_\lambda k_\lambda e_\lambda \Big\| &= \Big\| P_B \sum_{\lambda\in\Lambda} a_\lambda k_\lambda \Theta(\lambda)^* e_\lambda \Big\| \leqslant \Big\| \sum_{\lambda\in\Lambda} a_\lambda k_\lambda \Theta(\lambda)^* e_\lambda \Big\| \\ &= \sup_{\substack{f\in H^2(E)\\ \|f\|\leqslant 1}} \Big| \sum_{\lambda\in\Lambda} \overline{a}_\lambda \langle f(\lambda), \Theta(\lambda)^* e_\lambda\rangle \Big| \\ &\leqslant \sup_{\substack{f\in H^2(E)\\ \|f\|\leqslant 1}} \sum_{\lambda\in\Lambda} |a_\lambda|\, \|\Theta(\lambda)^* e_\lambda\|\, \|f(\lambda)\| \\ &\leqslant \sup_{\lambda\in\Lambda} \|\Theta(\lambda)^* e_\lambda\| \sup_{\substack{f\in H^2(E)\\ \|f\|\leqslant 1}} \sum_{\lambda\in\Lambda} |a_\lambda|\, \|f(\lambda)\|.\end{aligned}$$

On the other hand, by the Cauchy-Schwarz inequality, (2.3) and (2.4), we see that

$$\sum_{\lambda\in\Lambda} |a_\lambda|\, \|f(\lambda)\| \leqslant \frac{1}{\sqrt{c_1}} \Big\| \sum_{\lambda\in\Lambda} a_\lambda k_\lambda e_\lambda \Big\| \sqrt{C_1}\|f\|.$$

Then we obtain

$$\Big\| P_B\Theta^* \sum_{\lambda\in\Lambda} a_\lambda k_\lambda e_\lambda \Big\| \leqslant \sqrt{\frac{C_1}{c_1}} \sup_{\lambda\in\Lambda} \|\Theta(\lambda)^* e_\lambda\| \Big\| \sum_{\lambda\in\Lambda} a_\lambda k_\lambda e_\lambda \Big\|,$$

which proves the lemma with $C = \sqrt{\frac{C_1}{c_1}}$. ∎

Lemma 2.6. *Let $T \in \mathcal{L}(X)$ be a contraction in a finite dimensional Banach space, $N = \dim X < \infty$. Assume that $r := r(T) < 1$. Then, for each $\varepsilon > 0$ with $r+\varepsilon < 1$, and for each $\phi \in \mathrm{Hol}(|z| \leqslant r+\varepsilon)$, we have*

$$\|\phi(T)\| \leqslant \frac{2^{N-1}}{\varepsilon^N} \sup_{|z|=r+\varepsilon} |\phi(z)|.$$

Proof. Using the Riesz-Dunford calculus we can write

$$\|\phi(T)\| \leqslant \frac{1}{2\pi} \int\limits_{|\lambda|=r+\varepsilon} \|R_\lambda(T)\|\, |\phi(\lambda)|\, \mathrm{d}\lambda.$$

On the other hand, it is well known that

$$\|R_\lambda(T)\| \leqslant \frac{\|T-\lambda I\|^{N-1}}{|\det(T-\lambda I)|},$$

(see, for instance, [4]). Hence, we have

$$\|R_\lambda(T)\| \leqslant \frac{2^{N-1}}{\mathrm{dist}(\lambda, \sigma(T))^N},$$

which implies the result. ∎

COROLLARY 2.7. *Let* $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$, Θ *be an* $\mathcal{L}(E)$*-valued inner function,* $\dim E = N < \infty$. *Assume that*

$$r := \sup_{n\geqslant 1} r(\Theta(\lambda_n)) < 1.$$

Then, for every $0 < c < 1$, *there exists* $M \in \mathbb{N}$ *such that, for all* $m \geqslant M$, *we have*

$$\sup_{n\geqslant 1} \|\Theta(\lambda_n)^m\| < c.$$

Proof. Let $\varepsilon > 0$ be such that $r + \varepsilon < 1$. Using Lemma 2.6 with $T = \Theta(\lambda_n)$ and $\phi(z) := z^m$, we get

$$\|\Theta(\lambda_n)^m\| \leqslant \frac{2^{N-1}}{\varepsilon^N}(r+\varepsilon)^m.$$

Choose $M \in \mathbb{N}$ such that $\frac{2^{N-1}}{\varepsilon^N}(r+\varepsilon)^M < c$. Then, for all $m \geqslant M$, we have

$$\sup_{n\geqslant 1} \|\Theta(\lambda_n)^m\| < c. \quad ∎$$

Proof of Theorem 2.4. (ii)⇒(i): By Corollary 2.7, there exists $M \in \mathbb{N}$ such that, for all $m \geqslant N$, we have

$$\sup_{n\geqslant 1} \|\Theta(\lambda_n)^m\| < 1.$$

The conclusion now follows from Theorem 2.1.

(i)⇒(ii): Using (P11), we construct a family of unit vectors $(e_n)_{n\geqslant 1}$ in E such that $(k_{\lambda_n} e_n : n \geqslant 1)$ is a Riesz basis of its hull. Denote by B the inner function such that $K_B := \operatorname{Span}\{k_{\lambda_n} e_n : n \geqslant 1\}$. By Lemma 2.5, there exists a constant C such that

$$\|P_B \Theta^{p*}\| \leqslant C \sup_{n\geqslant 1} \|\Theta(\lambda_n)^{p*}\| = C \sup_{n\geqslant 1} \|\Theta(\lambda_n)^p\|.$$

Using Corollary 2.7, we now choose $M \in \mathbb{N}$ so as to ensure

$$\sup_{n\geqslant 1} \|\Theta(\lambda_n)^m\| < \frac{1}{C},$$

for all $m \geqslant M$. Hence we get

$$\|P_B \Theta^{*m}\| < 1,$$

which proves the result by Lemma 2.2 and Theorem 2.1. ∎

In general, uniform minimality is a much weaker property than that of being a Riesz basis. For the reproducing kernels $(k_\lambda e_\lambda)_{\lambda\in\Lambda}$ in $H^2(E)$, with $\dim E < \infty$, these two properties coincide. It is interesting to know whether the equivalence of these two properties remains true for the families $(k_\Theta(\,\cdot\,, \lambda_n)e_n : n \geqslant 1)$. In the scalar case, it is shown in [13] that, under the extra assumption $\lim_{n\to\infty} |\Theta(\lambda_n)| = 0$, these two properties are equivalent. This result has also been proved by I. Boricheva using different techniques based on the so-called Schur parameters of the function Θ at the zeros of the Blaschke product B (see [6]). Now, we give a generalization of this result for the vector valued case under the assumption

$$\lim_{n\to\infty} \|\Theta(\lambda_n)^* e_n\| = 0. \tag{2.5}$$

Without this assumption, the equivalence between uniform minimality and Riesz basis is still open, even in the scalar case.

THEOREM 2.8. *Let E be a finite dimensional Hilbert space,* $\dim E = N$. *Let* $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ *and* Θ *be an inner function in* $H^\infty(\mathcal{L}(E))$. *Let* $(e_n)_{n\geqslant 1}$ *be a family of unit vectors in* E. *Assume that* (2.5) *is satisfied. Then the following assertions are equivalent:*

(i) $(k_\Theta(\,\cdot\,,\lambda_n)e_n : n \geqslant 1)$ *is a Riesz basis of its hull;*

(ii) $(k_\Theta(\,\cdot\,,\lambda_n)e_n : n \geqslant 1)$ *is uniformly minimal.*

Proof. (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(i): Without loss of generality, we can assume that Θ is a purely contractive function, that is $\|\Theta(z)e\| < \|e\|$ for every $z \in \mathbb{D}$ and every $e \neq 0$ (see [23]). Thus (2.5) implies that

$$\sup_{\lambda\in\Lambda} \|\Theta(\lambda)^* e_\lambda\| < 1,$$

and (P4) implies that $\mathcal{X} := (k_{\lambda_n} e_n : n \geqslant 1)$ is uniformly minimal. Hence it is a Riesz basis by (P5). For $M \geqslant 1$, denote by B_M the inner function generating the subspace

$$K_{B_M} := \mathrm{Span}(k_{\lambda_n} e_n : n \geqslant M).$$

By Lemma 2.5, there exists a constant C, depending on the family $\mathcal{X}$ only, such that

$$\|P_B \Theta^* | K_{B_M}\| \leqslant C \sup_{n\geqslant M} \|\Theta(\lambda_n)^* e_n\|.$$

As $\lim\limits_{n\to\infty} \|\Theta(\lambda_n)^* e_n\| = 0$, we can find $M \geqslant 1$ such that

$$\sup_{n\geqslant M} \|\Theta(\lambda_n)^* e_n\| < \frac{1}{C}.$$

So we get

$$\|P_B \Theta^* | K_{B_M}\| < 1,$$

which implies by Lemma 2.2 and Theorem 2.1 that $(k_\Theta(\,\cdot\,,\lambda_n)e_n : n \geqslant M)$ forms a Riesz basis of its hull. Thanks to minimality of the whole family, it is easy to see that $(k_\Theta(\,\cdot\,,\lambda_n)e_n : n \geqslant 1)$ is a Riesz basis of its hull. ∎

3. STABILITY OF UNCONDITIONAL BASES OF REPRODUCING KERNELS

3.1. THE STABILITY PROBLEM. Now, we are going to consider the stability problem mentioned in the introduction. Namely, let $\Lambda = (\lambda_n)_{n\geqslant 1}$ be a sequence in $\mathbb{D}$, $\mathcal{E} = (e_n)_{n\geqslant 1}$ a sequence in E, and Θ be an $\mathcal{L}(E)$-valued inner function. Suppose that the sequence of reproducing kernels $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ is an unconditional basis of its hull and let $(\varepsilon_n)_{n\geqslant 1}$ be a sequence of positive numbers. We say that $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ is (ε_n)-*stable* if each sequence $(k_\Theta(\,\cdot\,,\lambda_n')e_n')_{n\geqslant 1}$ satisfying

$$|b_{\lambda_n}(\lambda_n')| \leqslant \varepsilon_n \quad \text{and} \quad \|e_n - e_n'\| \leqslant \varepsilon_n, \quad n \geqslant 1,$$

is a Riesz basis of its hull. Given an unconditional basis of its hull $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$, the *stability problem* is to describe $(\varepsilon_n)_{n\geqslant 1}$ such that $(k_\Theta(\,\cdot\,,\lambda_n)e_n)_{n\geqslant 1}$ is (ε_n)-stable. We will consider the case of k_{λ_n} as a limit case of $k_\Theta(\,\cdot\,,\lambda_n)$, with $\Theta \equiv 0$, and therefore, use the same language for the families of reproducing kernels $(k_{\lambda_n}e_n)_{n\geqslant 1}$ in $H^2(E)$. We will also study the case of scalar reproducing kernels $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ in Banach spaces K_Θ^p, $1 < p < +\infty$, and use the same definition of stability. First, we recall some well-known facts.

Setting $H_-^p := \{f \in L^p : \widehat{f}(n) = 0,\ n \geqslant 0\}$, for $p \in (1,\infty)$, we know, from the M. Riesz theorem on conjugate functions, that L^p is the direct sum of H^p and H_-^p, and hence using the duality

$$\langle f, g\rangle := \int_{\mathbb{T}} f\overline{g}\,\mathrm{d}m,$$

we may identify (in the sesquilinear manner) the dual space $(H^p)^*$ with H^q, where q is the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. For a scalar inner function Θ, K_Θ^p denotes the subspace of H^p defined by

$$K_\Theta^p := H^p \cap \Theta\overline{H_0^p},$$

where $H_0^p := zH^p$ and

$$k_\Theta(\,\cdot\,,\lambda) := \frac{1 - \overline{\Theta(\lambda)}\Theta}{1 - \overline{\lambda}z} \in K_\Theta^p.$$

Then

$$f(\lambda) = \langle f, k_\Theta(\,\cdot\,,\lambda)\rangle, \quad \lambda \in \mathbb{D},\ f \in K_\Theta^q.$$

If $\Lambda = (\lambda_n)_{n\geqslant 1}$ is a sequence in $\mathbb{D}$ and Θ is an inner function, we denote by $J_{\Theta,\Lambda}$ the operator defined on K_Θ^q by

$$J_{\Theta,\Lambda} f = (f(\lambda_n))_{n\geqslant 1}, \quad f \in K_\Theta^q.$$

The following characterization of unconditional bases is shown in [13], Part II, Theorem 6.3. Assume that $\sup\limits_{n\geqslant 1} |\Theta(\lambda_n)| < 1$. The family $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ is an unconditional basis of K_Θ^p if and only if the operator $J_{\Theta,\Lambda}$ is an isomorphism from K_Θ^q onto $\ell^q((1 - |\lambda_n|^2)^{1/q})$.

As mentioned in the introduction, the story of (ε_n)-stability started with R.E. Paley and N. Wiener in 1934. Namely, Paley and Wiener looked for the better constant $\delta > 0$ such that $(\exp(\mathrm{i}\mu_n t))_{n\geqslant 1}$ is a Riesz basis of $L^2(0, 2\pi)$, whenever

$(\mu_n)_{n\geqslant 1}$ is a sequence satisfying $|\mu_n - n| \leqslant \delta$. The Paley-Wiener result, with $\delta = \frac{1}{\pi^2}$, was later improved by R.J. Duffin and J.J. Eachus ([9]) up to $\delta = \frac{\log 2}{\pi}$. In 1936, A. Ingham ([14]) gave an example showing that such a δ should be less than $\frac{1}{4}$. And, finally, in 1964, M. Kadeč ([15]) showed that $\delta < \frac{1}{4}$ is sufficient (see [17], Lectures 8 and 11, for an exposition of relations between exponentials and reproducing kernels and for an alternative proof of M. Kadeč' result). This result was the object of several generalizations by V. Katsnelson ([16]), A. Avdonin ([2]), and Hruscev-Nikolski-Pavlov ([13]). In particular, in [13] (Corollary 2.5, p. 295), the following generalization of an earlier result by R. Duffin and A. Schaeffer is proved: given a Riesz basis $(\exp(\mathrm{i}\lambda_n t))_{n\geqslant 1}$ of $L^2(0,a)$ such that $\inf_{n\geqslant 1} \operatorname{Im} \lambda_n > 0$, then there exists $\varepsilon > 0$ such that $(\exp(\mathrm{i}\mu_n t))_{n\geqslant 1}$ is a Riesz basis of $L^2(0,a)$ whenever the μ_n's satisfy $|\mu_n - \lambda_n| \leqslant \varepsilon$. Later on, this result was generalized to the case of vector valued exponentials by Avdonin, Ivanov and Joo ([1]). We now give generalizations of these results to reproducing kernels. The following three theorems are the main results of this section. In Subsection 3.2, we compare these results with Kadeč' 1/4-theorem quoted above. Subsection 3.3 contains the proofs of Theorems 3.1, 3.3, 3.10, and a remark for the proof of Theorem 3.4.

In this section, if Λ is a Blaschke sequence in $\mathbb{D}$, we denote by B_Λ the Blaschke product associated to Λ.

THEOREM 3.1. *Let $1 < p < \infty$, $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$, and Θ be an inner function in H^∞ such that $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ is an unconditional basis of K^p_Θ. Assume that*

$$\sup_{n\geqslant 1} |\Theta(\lambda_n)| < 1.$$

Then there exists $\varepsilon = \varepsilon(\Lambda,\Theta,p)$ such that $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ is (ε_n)-stable in K^p_Θ for every $(\varepsilon_n)_{n\geqslant 1}$ satisfying $\sup_{n\geqslant 1} \varepsilon_n < \varepsilon$.

REMARK 3.2. It follows from our proof that the constant $\varepsilon = \varepsilon(\Lambda,\Theta,p)$ can be taken to be any number satisfying $0 < \varepsilon < \frac{\delta}{2}$ and

$$\sup_{n\geqslant 1} |\Theta(\lambda_n)| + 2\varepsilon < 1, \tag{3.1}$$

$$\frac{2\varepsilon}{\delta/2-\varepsilon}\|J^{-1}_{\Theta,\Lambda}\|\left(128\frac{1+\varepsilon}{1-\varepsilon}(1+6\log 1/\delta)\frac{1+\varepsilon+\delta/2}{1-\varepsilon-\delta/2}\right)^{1/q} < 1 \tag{3.2}$$

where $\delta = \delta(\Lambda)$ is the Carleson constant of the sequence Λ and q is the conjugate exponent of p.

In fact, for $p = 2$, we can improve the upper bound for ε as follows.

THEOREM 3.3. *Let $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$, Θ be an inner function such that*

$$\sup_{n\geqslant 1} |\Theta(\lambda_n)| < 1.$$

Assume that $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ is a Riesz basis of its hull. Then $(k_\Theta(\,\cdot\,,\lambda_n))_{n\geqslant 1}$ is (ε_n)-stable for every $(\varepsilon_n)_{n\geqslant 1}$ satisfying

$$\sup_{n\geqslant 1} \varepsilon_n < \frac{\delta^6}{8}\frac{1-\gamma}{1+\gamma},$$

where $\delta = \delta(\lambda)$ *and* $\gamma := \operatorname{dist}(\Theta\overline{B}_\Lambda, H^\infty)$.

The vector valued analogue of Theorem 3.1 is as follows.

THEOREM 3.4. *Let* $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$, *E be a finite dimensional Hilbert space,* $(e_n)_{n\geqslant 1} \subset E$, $\|e_n\| = 1$, *and* $\Theta \in H^\infty(\mathcal{L}(E))$ *be an inner function such that* $(k_\Theta(\,\cdot\,, \lambda_n)e_n)_{n\geqslant 1}$ *is a Riesz basis of* $K_\Theta = H^2(E) \ominus \Theta H^2(E)$. *Assume that*

$$\sup_{n\geqslant 1} \|\Theta(\lambda_n)^* e_n\| < 1.$$

Then there exists $\varepsilon = \varepsilon(\Lambda, \Theta, (e_n))$ *such that* $(k_\Theta(\,\cdot\,, \lambda_n)e_n)_{n\geqslant 1}$ *is* (ε_n)*-stable for every* $(\varepsilon_n)_{n\geqslant 1}$ *satisfying* $\sup\limits_{n\geqslant 1} \varepsilon_n < \varepsilon$.

REMARK 3.5. (i) Since $N = \dim E < \infty$, we know from Theorem 2.1 that Λ is N-Carleson, $\Lambda = \bigcup\limits_{i=1}^{N} \Lambda_i$, $\Lambda_i \in (C)$. Let $\delta := \inf\limits_i \delta(\Lambda_i) > 0$. Moreover, since $(k_\Theta(\,\cdot\,, \lambda_n)e_n)_{n\geqslant 1}$ is an unconditional basis of K_Θ, the operator $J_{\Theta,\Lambda}$ defined by

$$J_{\Theta,\Lambda} f := (\langle f(\lambda_n), e_n\rangle)_{n\geqslant 1}, \quad f \in K_\Theta,$$

is an isomorphism from K_Θ onto $\ell^2((1-|\lambda_n|^2)^{1/2})$. Then it follows from our proof that the constant $\varepsilon = \varepsilon(\Lambda, \Theta, (e_n))$ can be taken to be any number $\varepsilon > 0$ satisfying $\varepsilon < \frac{\delta}{2}$ and

$$\sup_{n\geqslant 1} \|\Theta(\lambda_n)^* e_n\| + 2\varepsilon(\varepsilon + 1) + \varepsilon < 1, \tag{3.3}$$

$$\frac{2\varepsilon}{\delta/2 - \varepsilon} \|J_{\Theta,\Lambda}^{-1}\| \left(128N \frac{1+\varepsilon}{1-\varepsilon}(1 + 6\log 1/\delta) \frac{1+\varepsilon+\delta/2}{1-\varepsilon-\delta/2}\right)^{1/2} < 1. \tag{3.4}$$

(ii) In fact, the assumption $\dim E < \infty$ of Theorem 3.4 can be dropped if we assume that Λ is N-Carleson.

(iii) Theorem 3.4 allows the following asymptotic form. Under the same hypotheses, let $(\mu_n)_{n\geqslant 1} \subset \mathbb{D}$ and $(a_n)_{n\geqslant 1} \subset E$ be such that

$$\lim_{n\to+\infty} |b_{\lambda_n}(\mu_n)| = 0, \quad \text{and} \quad \lim_{n\to+\infty} \|a_n - e_n\| = 0.$$

Then there exists $N \in \mathbb{N}$ such that $(k_\Theta(\,\cdot\,, \mu_n)a_n)_{n\geqslant N}$ is a Riesz basis of its hull.

Indeed, let $\varepsilon > 0$ be a constant defined by Theorem 3.4. Choose $N \in \mathbb{N}$ such that

$$\sup_{n\geqslant N} |b_{\lambda_n}(\mu_n)| < \varepsilon, \quad \text{and} \quad \sup_{n\geqslant N} \|a_n - e_n\| < \varepsilon.$$

It follows from the proof of Theorem 3.4 that $(k_\Theta(\,\cdot\,, \mu_n)a_n)_{n\geqslant N}$ is a Riesz basis of its hull.

Let us comment on these results. First of all, they strengthen the Hruscev-Nikolski-Pavlov result cited above, the Euclidean neighborhoods $|\lambda_n - \mu_n| \leqslant \varepsilon$ being replaced by the hyperbolic ones $|\frac{\lambda_n-\mu_n}{\lambda_n-\overline{\mu}_n}| \leqslant \varepsilon$. These two neighborhoods are comparable when $|\lambda_n - \mu_n| \leqslant A\varepsilon \operatorname{Im}\lambda_n$, where A is an absolute constant; this means that for the case $\sup\limits_n(\operatorname{Im}\lambda_n) = \infty$ our result is sharper.

As another commentary, we compare these results with known general stability criteria for Riesz bases in Hilbert spaces. In fact, L. Dovbysh, N. Nikolski and V. Sudakov proved that general Riesz bases are (ε_n)-stable only if $\sum_n \varepsilon_n^2 < \infty$ (see [8]). Theorems 3.1 and 3.4 show that, for bases of reproducing kernels in K_Θ, the situation is much better, and one can guarantee stability with respect to uniform perturbations $\sup_{n\geqslant 1} \varepsilon_n < \varepsilon$.

3.2. ASYMPTOTICALLY ORTHONORMAL BASES OF REPRODUCING KERNELS. In this subsection, we introduce the notion of asymptotically orthonormal bases. Then we give a method to construct asymptotically orthonormal bases consisting of reproducing kernels. In this case, we compare our constant in Theorem 3.3 with the constant in Kadeč's 1/4-theorem.

DEFINITION 3.6. Let $(f_n)_{n\geqslant 1}$ be a sequence in a Hilbert space H. We say that $(f_n)_{n\geqslant 1}$ is an *asymptotically orthonormal basis* of its hull (and write $(f_n)_{n\geqslant 1} \in$ (AOS)) if, for every sufficiently large N, there exist $c_N, C_N > 0$ such that

$$c_N \sum_{n\geqslant N} |a_n|^2 \leqslant \left\| \sum_{n\geqslant N} a_n f_n \right\|^2 \leqslant C_N \sum_{n\geqslant N} |a_n|^2, \tag{3.5}$$

for every finite sum $\sum_{n\geqslant N} a_n f_n$, and

$$\lim_{N\to\infty} c_N = 1, \quad \lim_{N\to\infty} C_N = 1.$$

The following lemma shows that this definition is equivalent to those of A.L. Volberg (see [28]). But first, we need the notion of orthogonalizer. If $(x_n)_{n\geqslant 1}$ is a Riesz basis of its hull then there exists an isomorphism V defined on $X :=$ Span$(x_n : n \geqslant 1)$ which transforms $(x_n)_{n\geqslant 1}$ onto an orthonormal basis. Such an operator V is called an orthogonalizer of $(x_n)_{n\geqslant 1}$.

LEMMA 3.7. *Let $(f_n)_{n\geqslant 1}$ be a Riesz basis of its hull and let V be an orthogonalizer of $(f_n)_{n\geqslant 1}$. The following assertions are equivalent:*

(i) $(f_n)_{n\geqslant 1} \in$ (AOS).

(ii) *There exist a unitary operator U and a compact operator K such that $V = U + K$.*

(iii) *There exist a compact operator K such that the Gram matrix $G = (\langle f_n, f_k\rangle)_{n,k}$ can be written as*

$$G = I + K,$$

with I the identity mapping.

Proof. (ii)$\Rightarrow$(iii): Let V be an isomorphism from $\mathcal{X} :=$ Span$\{f_n : n \geqslant 1\}$ onto ℓ^2 and $Vf_n = e_n$, $n \geqslant 1$, with $(e_n)_{n\geqslant 1}$ the standard orthonormal basis of ℓ^2. Denote $V_1 := V^{-1}$, and let $a = (a_n)_{n\geqslant 1} \in \ell^2$. Then

$$\left\| V_1\left(\sum_{n\geqslant 1} a_n e_n\right)\right\|^2 = \sum_{n,j} a_n \overline{a}_j \langle f_n, f_j\rangle = \langle Ga, a\rangle.$$

Hence $G = V_1^* V_1$. If $V = U + K$ then it is clear that $V_1 = V^{-1} = U_1 + K_1$, with U_1 a unitary operator and K_1 a compact operator. It follows that $G = (U_1^* + K_1^*)(U_1 + K_1) = I + K_2$, with K_2 a compact operator.

(iii)⇒(ii): Assume that $G = V_1^* V_1 = I + K$ with K a compact operator. Since $(f_n)_{n\geqslant 1}$ is a Riesz basis of its hull, V_1 is an isomorphism from ℓ^2 onto $\mathcal{X}$. Consider the polar decomposition of $V_1 = JR$ with R a positive operator on ℓ^2 and J a unitary operator (see for instance [22], Theorem 12.35, page 332). Then $V_1^* V_1 = RJ^*JR = R^2 = I + K$. So $R^2 - I = K$. Since R is positive, $I + R$ is invertible and it follows that $R - I = (R + I)^{-1}K$ is compact. Hence, $V_1 = J + JK_1 = J + K_2$.

(ii)⇒(i): Let $V = U + K$, $V^{-1} = U_1 + K_1$, with U, U_1 unitary operators and K, K_1 compact operators. Let $\varepsilon_N := \|K|\mathrm{Span}(f_n : n \geqslant N)\|$ and $\widetilde{\varepsilon}_N := \|K_1|\mathrm{Span}(f_n : n \geqslant N)\|$. Since K is compact, we have $\lim\limits_{N\to+\infty} \varepsilon_N = 0$, and for all $f \in \mathrm{Span}(f_n : n \geqslant N)$

$$\|Vf\| = \|Uf + Kf\| \leqslant \|f\| + \varepsilon_N\|f\| = (1 + \varepsilon_N)\|f\|.$$

Similarly, for all $g \in \mathrm{Span}(e_n : n \geqslant N)$, we have

$$\|V^{-1}g\| \leqslant (1 + \widetilde{\varepsilon}_N)\|g\|,$$

which implies that $(f_n)_{n\geqslant 1} \in$ (AOS).

(i)⇒(iii): Using the same computations, we show that

$$\|P_N(I - G)P_N\| = \sup_{a\in\ell^2,\, \|a\|\leqslant 1} \left| \sum_{n\geqslant N} |a_n|^2 - \left\| \sum_{n\geqslant N} a_n f_n \right\|^2 \right|.$$

Hence $\|P_N(I-G)P_N\| \leqslant 1 - c_N$, which implies $\lim\limits_{N\to+\infty} \|P_N(I-G)P_N\| = 0$. Hence, to conclude, it suffices to note that

$$I - G = P_N(I - G) + (I - P_N)(I - G) = P_N(I - G)P_N + T_N,$$

with a finite rank operator T_N . ∎

Now we recall Volberg's necessary and sufficient condition for a family $R(\Lambda) := \left(\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}\right)_{n\geqslant 1}$ of H^2 reproducing kernels to be an asymptotically orthonormal basis of its hull (see [28]).

THEOREM 3.8. (Volberg) *Let $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$. The following statements are equivalent:*

(i) *$R(\Lambda)$ is an asymptotically orthonormal basis of its hull.*

(ii) $\lim\limits_{n\to+\infty} |B_{\lambda_n}(\lambda_n)| = 1$, *where* $B_{\lambda_n} := \prod\limits_{k\neq n} b_{\lambda_k}$.

Next, we need the following lemma.

LEMMA 3.9. *Let* $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ *and* Θ *be an inner function in* H^∞.

(i) *Assume that*

$$\lim_{n\to+\infty} |B_{\lambda_n}(\lambda_n)| = 1, \quad \text{and} \quad \lim_{n\to+\infty} |\Theta(\lambda_n)| = 0.$$

Then $R_\Theta(\Lambda) := \left(\frac{k_\Theta(\,\cdot\,,\lambda_n)}{\|k_\Theta(\,\cdot\,,\lambda_n)\|}\right) \in (\mathrm{AOS})$.

(ii) *Assume that* $R_\Theta(\Lambda) \in (\mathrm{AOS})$. *Then* $R(\Lambda) \in (\mathrm{AOS})$.

(iii) *Let* $M = (\mu_n)_{n\geqslant 1} \subset \mathbb{D}$ *be such that*

$$|b_{\lambda_n}(\mu_n)| \leqslant \varepsilon < 1.$$

If $R(\Lambda) \in (\mathrm{AOS})$ *then* $R(M) \in (\mathrm{AOS})$.

Proof. (i) Since $\lim\limits_{n\to+\infty} |B_{\lambda_n}(\lambda_n)| = 1$, we get from Volberg's theorem that $R(\Lambda) \in (\mathrm{AOS})$. Therefore there exist $c_N, C_N \to 1$ such that

$$c_N \sum_{n\geqslant N} |a_n|^2 \leqslant \Big\| \sum_{n\geqslant N} a_n \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}\Big\|^2 \leqslant C_N \sum_{n\geqslant N} |a_n|^2. \tag{3.6}$$

Since $\lim\limits_{n\to+\infty} |\Theta(\lambda_n)| = 0$, we have

$$\lim_{n\to+\infty} \frac{\|k_\Theta(\,\cdot\,,\lambda_n)\|^2}{\|k_{\lambda_n}\|^2} = \lim_{n\to+\infty} (1 - |\Theta(\lambda_n)|^2) = 1.$$

Hence $R_\Theta(\Lambda) \in (\mathrm{AOS})$ if and only if $\left(\frac{k_\Theta(\,\cdot\,,\lambda_n)}{\|k_{\lambda_n}\|}\right)_{n\geqslant 1} \in (\mathrm{AOS})$. Moreover

$$\Big\| \sum_{n\geqslant N} a_n \frac{k_\Theta(\,\cdot\,,\lambda_n)}{\|k_{\lambda_n}\|}\Big\|^2 = \Big\| \sum_{n\geqslant N} a_n \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}\Big\|^2 - \Big\| \sum_{n\geqslant N} a_n \overline{\Theta(\lambda_n)} \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}\Big\|^2,$$

and

$$\begin{aligned} \inf_{n\geqslant N} |\Theta(\lambda_n)|^2 c_N \sum_{n\geqslant N} |a_n|^2 &\leqslant \Big\| \sum_{n\geqslant N} a_n \overline{\Theta(\lambda_n)} \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}\Big\|^2 \\ &\leqslant \sup_{n\geqslant N} |\Theta(\lambda_n)|^2 C_N \sum_{n\geqslant N} |a_n|^2. \end{aligned} \tag{3.7}$$

Comparing (3.6) and (3.7), we find that $R_\Theta(\Lambda) \in (\mathrm{AOS})$.

(ii) Let $(\psi_n)_{n\geqslant 1}$ be the biorthogonal family of $R_\Theta(\Lambda)$. Then it is clear (for instance, from Lemma 3.7) that $(\psi_n)_{n\geqslant 1}$ is also an asymptotically orthonormal basis of its hull. Moreover, we have

$$\lim_{n\to+\infty} \mathrm{dist}^2\left(\frac{k_\Theta(\,\cdot\,,\lambda_n)}{\|k_\Theta(\,\cdot\,,\lambda_n)\|}, \mathrm{Span}(k_\Theta(\,\cdot\,,\lambda_k) : k \neq n)\right) = \lim_{n\to+\infty} \frac{1}{\|\psi_n\|^2} = 1.$$

Then we use a formula proved by I.A. Boricheva:

$$\begin{aligned} &\mathrm{dist}^2\left(\frac{k_\Theta(\,\cdot\,,\lambda_n)}{\|k_\Theta(\,\cdot\,,\lambda_n)\|}, \mathrm{Span}(k_\Theta(\,\cdot\,,\lambda_k)\, : k \neq n)\right) \\ &\qquad = |B_{\lambda_n}(\lambda_n)|^2 \prod_{i\geqslant 1} \frac{1 - |\Theta_{i+1}(\lambda_n)|^2}{1 - |\Theta_{i+1}(\lambda_n)|^2 |b_{\lambda_i}(\lambda_n)|^2}, \end{aligned}$$

where Θ_i are the Schur-Nevanlinna functions associated to (Θ, Λ) (see [6]). Since

$$\prod_{i\geqslant 1} \frac{1-|\Theta_{i+1}(\lambda_n)|^2}{1-|\Theta_{i+1}(\lambda_n)|^2|b_{\lambda_i}(\lambda_n)|^2} \leqslant 1,$$

we get

$$\operatorname{dist}^2\left(\frac{k_\Theta(\,\cdot\,,\lambda_n)}{\|k_\Theta(\,\cdot\,,\lambda_n)\|}, \operatorname{Span}(k_\Theta(\,\cdot\,,\lambda_k) : k\neq n)\right) \leqslant |B_{\lambda_n}(\lambda_n)|^2,$$

so $\lim\limits_{n\to+\infty} |B_{\lambda_n}(\lambda_n)| = 1$, which implies by Volberg's theorem that $R(\Lambda)$ is an asymptotically orthonormal basis of its hull.

(iii) Set $\delta_n := |B_{\lambda_n}(\lambda_n)|$. Since $\lim\limits_{n\to+\infty} \delta_n = 1$, there exists $N \in \mathbb{N}$ such that

$$\lambda := \frac{2\varepsilon}{1+\varepsilon^2} < \inf_{n\geqslant N} \delta_n.$$

Using Lemma 3.13 below, we get, for $k \geqslant N$,

$$\prod_{j\neq k} |b_{\mu_j}(\mu_k)| \geqslant \frac{\prod\limits_{j\neq k} |b_{\lambda_j}(\lambda_k)| - \lambda}{1-\lambda \prod\limits_{j\neq k} |b_{\lambda_j}(\lambda_k)|} = \frac{\delta_k - \lambda}{1-\lambda\delta_k}.$$

Hence

$$\lim_{k\to+\infty} \prod_{j\neq k} |b_{\mu_j}(\mu_k)| = 1,$$

which implies by Volberg's theorem that $R(M)$ is an asymptotically orthonormal basis of its hull. ∎

The following stability result will be derived from Theorem 3.3 and Lemma 3.9.

THEOREM 3.10. *Let $R_\Theta(\Lambda) \in$ (AOS) be such that*

$$\lim_{n\to+\infty} |\Theta(\lambda_n)| = 0.$$

Let $M := (\mu_n)_{n\geqslant 1} \subset \mathbb{D}$ such that

$$|b_{\lambda_n}(\mu_n)| \leqslant \varepsilon < 1, \quad n \geqslant 1.$$

Then there exists $N \in \mathbb{N}$ such that $\left(\frac{k_\Theta(\,\cdot\,,\mu_n)}{\|k_\Theta(\,\cdot\,,\mu_n)\|}\right)_{n\geqslant N}$ is a Riesz basis of its hull.

REMARK 3.11. When compared to Kadeč's theorem, our result seems to be surprising. But, in fact, as it was mentioned above, we use hyperbolic distances whereas Kadeč's theorem uses the Euclidean ones. So our result is sharper for the case $\sup\limits_{n\geqslant 1} \operatorname{Im}(\lambda_n) = +\infty$ and worse for the case $\inf\limits_{n\geqslant 1} \operatorname{Im}(\lambda_n) = 0$.

3.3. PROOFS OF THEOREMS 3.1, 3.3 AND 3.10.

First of all, we give auxiliary results which will be useful in the sequel.

Lemma 3.12. *Let $\lambda, \mu \in \mathbb{D}$ and $0 < \varepsilon < 1$ be such that $|b_\lambda(\mu)| \leqslant \varepsilon$. Then we have*

$$\frac{1}{2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right) \leqslant \frac{1-|\lambda|^2}{1-|\mu|^2} \leqslant 2\left(\frac{1+\varepsilon}{1-\varepsilon}\right).$$

Proof. Thanks to a lemma of Vinogradov-Havin (see [12]), we have

$$\frac{1-\varepsilon}{1+\varepsilon} \leqslant \frac{1-|\lambda|}{1-|\mu|} \leqslant \frac{1+\varepsilon}{1-\varepsilon}.$$

Then it suffices to notice that $1-|\lambda| \leqslant 1-|\lambda|^2 \leqslant 2(1-|\lambda|)$. ∎

Lemma 3.13. *Let $\Lambda(\lambda_n)_{n\geqslant 1}$ be a sequence satisfying the Carleson condition with constant $\delta = \delta(\Lambda)$. Let $0 < \lambda < 1$. If $\frac{2\lambda}{1+\lambda^2} < \delta$ and*

$$|b_{\lambda_n}(\mu_n)| \leqslant \lambda, \quad n \geqslant 1,$$

then $M := (\mu_n)_{n\geqslant 1} \in (C)$ and, more precisely, we have

$$\delta(M) \geqslant \frac{\delta - 2\lambda/(1+\lambda^2)}{1-2\lambda\delta/(1+\lambda^2)}.$$

Proof. See e.g. [11], page 310. ∎

Remark 3.14. This lemma means that $(k_{\lambda_n})_{n\geqslant 1}$ is (ε_n)-stable as soon as $\sup\limits_{n\geqslant 1} \varepsilon_n < \lambda$.

If $\lambda = \delta/3$, we have

$$\delta(M) \geqslant \delta/3.$$

If $\lambda = \delta/2$, then we have

$$\delta(M) \geqslant \delta^3.$$

Lemma 3.15. *Let $\Lambda = (\lambda_n)_{n\geqslant 1} \in (C)$ so that $\delta(\Lambda) \geqslant \delta > 0$. Then for any function $f \in H^q$, we have*

$$\sum_{n\geqslant 1} |f(\lambda_n)|^q (1-|\lambda_n|^2) \leqslant 32(1+2\log 1/\delta)\|f\|_q^q.$$

Proof. For $q=2$, see [17]. For $q \neq 2$, using the Riesz-Smirnov factorization, we get the result from the case $q=2$. ∎

Lemma 3.16. *Let $\Lambda = (\lambda_n)_{n\geqslant 1} \in (C)$, and let $\delta = \delta(\Lambda)$ be its Carleson constant. Suppose $(\mu_n)_{n\geqslant 1}$ is a sequence in $\mathbb{D}$ satisfying*

$$|b_{\lambda_n}(\mu_n)| \leqslant \varepsilon < \frac{\delta}{2}, \quad \forall n \geqslant 1.$$

Then for any function $f \in H^q$, we have

$$\sum_{n\geqslant 1} |f(\lambda_n)-f(\mu_n)|^q (1-|\mu_n|^2) \leqslant 64(1+6\log 1/\delta)\left(\frac{1+\varepsilon+\delta/2}{1-\varepsilon-\delta/2}\right)\left(\frac{2\varepsilon}{\delta/2-\varepsilon}\right)^q \|f\|_q^q.$$

Proof. Set $g(z) := \frac{f(z)-f(\mu_n)}{b_{\mu_n}(z)}$. Then it is easy to see that $g \in H^q$. As $\overline{\Omega(\lambda_n, \delta/2)} \subset \mathbb{D}$, we can apply the maximum principle. Hence

$$|g(\lambda_n)| \leqslant \sup_{\xi \in \partial\Omega(\lambda_n,\delta/2)} |g(\xi)|.$$

If $\xi \in \partial\Omega(\lambda_n, \delta/2)$, we have $|b_{\mu_n}(\xi)| \geqslant ||b_{\mu_n}(\lambda_n)| - |b_{\lambda_n}(\xi)|| \geqslant \delta/2 - \varepsilon$. Therefore,

$$\left|\frac{f(\lambda_n) - f(\mu_n)}{b_{\mu_n}(\lambda_n)}\right| \leqslant \frac{1}{\delta/2 - \varepsilon} \sup_{\xi \in \partial\Omega(\lambda_n,\delta/2)} |f(\xi) - f(\mu_n)|.$$

Then, consider $u_n \in \partial\Omega(\lambda_n, \delta/2)$ such that

$$|f(u_n)| = \sup_{\xi \in \partial\Omega(\lambda_n,\delta/2)} |f(\xi)|.$$

We get

$$\left|\frac{f(\lambda_n) - f(\mu_n)}{b_{\mu_n}(\lambda_n)}\right| \leqslant \frac{2}{\delta/2 - \varepsilon}|f(u_n)|.$$

Consequently,

$$|f(\lambda_n) - f(\mu_n)|^q(1 - |\mu_n|^2) \leqslant \left(\frac{2\varepsilon}{\delta/2 - \varepsilon}\right)^q (1 - |\mu_n|^2)|f(u_n)|^q.$$

On the other hand, $|b_{\mu_n}(u_n)| \leqslant |b_{\mu_n}(\lambda_n)| + |b_{\lambda_n}(u_n)| \leqslant \varepsilon + \delta/2$. From Lemma 3.12, we get

$$1 - |\mu_n|^2 \leqslant 2\left(\frac{1 + \varepsilon + \delta/2}{1 - \varepsilon - \delta/2}\right)(1 - |u_n|^2).$$

Thus

$$|f(\lambda_n) - f(\mu_n)|^q(1 - |\mu_n|^2) \leqslant 2\left(\frac{2\varepsilon}{\delta/2 - \varepsilon}\right)^q \left(\frac{1 + \varepsilon + \delta/2}{1 - \varepsilon - \delta/2}\right)(1 - |u_n|^2)|f(u_n)|^q.$$

Moreover, as $|b_{\lambda_n}(u_n)| \leqslant \delta/2$, it follows from Lemma 3.13 that $(u_n)_{n\geqslant 1}$ satisfies the Carleson condition and we have

$$\inf_{k\geqslant 1} \prod_{j\neq k} \left|\frac{u_k - u_j}{1 - \overline{u}_k u_j}\right| \geqslant \delta^3.$$

Using Lemma 3.15, we obtain

$$\sum_{n\geqslant 1}(1 - |u_n|^2)|f(u_n)|^q \leqslant 32(1 + 6\log 1/\delta)\|f\|_q^q,$$

which completes the proof. ∎

Proof of Theorem 3.1. Let $\varepsilon < \delta/2$ be a number satisfying (3.1) and (3.2) and let $M = (\mu_n)_{n\geqslant 1} \subset \mathbb{D}$ be such that

$$\sup_{n\geqslant 1} |b_{\lambda_n}(\mu_n)| < \varepsilon.$$

From Lemma 3.12, it follows that

$$\frac{1}{2}\left(\frac{1 - \varepsilon}{1 + \varepsilon}\right) \leqslant \frac{1 - |\lambda_n|^2}{1 - |\mu_n|^2} \leqslant 2\left(\frac{1 + \varepsilon}{1 - \varepsilon}\right), \quad \forall n \geqslant 1. \tag{3.8}$$

On the other hand, we have

$$\left|\frac{\Theta(\lambda_n)-\Theta(\mu_n)}{b_{\lambda_n}(\mu_n)}\right| \leqslant \left\|\frac{\Theta-\Theta(\mu_n)}{b_{\mu_n}}\right\|_\infty = \|\Theta-\Theta(\mu_n)\|_\infty \leqslant 2.$$

Hence $|\Theta(\lambda_n)-\Theta(\mu_n)| \leqslant 2\varepsilon$ and then

$$\sup_{n\geqslant 1}|\Theta(\mu_n)| \leqslant \sup_{n\geqslant 1}|\Theta(\lambda_n)| + 2\varepsilon < 1.$$

Consequently, thanks to the characterization of unconditional bases recalled in Subsection 3.1, to prove that $(k_\Theta(\,\cdot\,,\mu_n))_{n\geqslant 1}$ is an unconditional basis in K^p_Θ, it suffices to check that $J_{\Theta,M}$ is an isomorphism from K^q_Θ onto $\ell^q((1-|\mu_n|^2)^{1/q})$. In view of Lemma 3.6, we have, for all $f\in H^q$,

$$\left(\sum_{n\geqslant 1}|f(\lambda_n)-f(\mu_n)|^q(1-|\mu_n|^2)\right)^{1/q} \leqslant C(\delta,\varepsilon)\|f\|_q, \tag{3.9}$$

where

$$C(\delta,\varepsilon) := \left(64(1+6\log 1/\delta)\left(\frac{1+\varepsilon+\delta/2}{1-\varepsilon-\delta/2}\right)\right)^{1/q}\left(\frac{2\varepsilon}{\delta/2-\varepsilon}\right).$$

This implies that $J_{\Theta,M}$ is a continuous operator from K^q_Θ into $\ell^q((1-|\mu_n|^2)^{1/q})$. Denote by U the operator defined from $\ell^q((1-|\lambda_n|^2)^{1/q})$ onto $\ell^q((1-|\mu_n|^2)^{1/q})$ by $Ua := a$, $a\in\ell^q((1-|\lambda_n|^2)^{1/q})$. From (3.8), it follows that U is an isomorphism and

$$\|U^{-1}\| \leqslant \left(2\frac{1+\varepsilon}{1-\varepsilon}\right)^{1/q}.$$

Further, we have

$$J_{\Theta,M} = UJ_{\Theta,\Lambda} + J_{\Theta,M} - UJ_{\Theta,\Lambda} = UJ_{\Theta,\Lambda}(I + J^{-1}_{\Theta,\Lambda}U^{-1}(J_{\Theta,M}-UJ_{\Theta,\Lambda})).$$

Hence to prove that $J_{\Theta,M}$ is an isomorphism, it suffices to check that

$$\|J^{-1}_{\Theta,\Lambda}\|\,\|U^{-1}\|\|J_{\Theta,M}-UJ_{\Theta,\Lambda}\| < 1.$$

But inequality (3.9) means that

$$\|J_{\Theta,M}-UJ_{\Theta,\Lambda}\| \leqslant C(\delta,\varepsilon),$$

and the result follows from (3.2). ∎

To prove Theorem 3.3, we use an operator approach based on the criterion of Hruscev, Nikolski and Pavlov for bases of reproducing kernels.

Proof. Let $(\mu_n)_{n\geqslant 1}\subset\mathbb{D}$ be such that

$$|b_{\lambda_n}(\mu_n)| \leqslant \varepsilon := \sup_{n\geqslant 1}\varepsilon_n < \frac{\delta^6}{8}\frac{1-\gamma}{1+\gamma}.$$

Since $\dfrac{\delta^6}{8}\dfrac{1-\gamma}{1+\gamma} < \dfrac{\delta}{2}$, we deduce from Lemma 3.13 that $M := (\mu_n)_{n\geqslant 1}\in(C)$ and $\delta(M)\geqslant\delta^3$. Using the criterion of Hruscev, Nikolski and Pavlov, it remains to prove that

$$\operatorname{dist}(\Theta\overline{B}_M, H^\infty) < 1.$$

It follows from a theorem of P. Jones and S.A. Vinogradov (see, for instance, [17], Lecture VIII, Section 4) that there exists $f \in H^\infty$ satisfying

$$f(\mu_n) = B_\Lambda(\mu_n), \quad \text{and} \quad \|f\|_\infty \leqslant \frac{8}{\delta^6} \sup_{n \geqslant 1} |B_\Lambda(\mu_n)|.$$

Therefore $B_\Lambda - f \in B_M H^\infty$ and $\|f\|_\infty \leqslant \frac{8}{\delta^6}\varepsilon$. So

$$\operatorname{dist}(B_\Lambda \overline{B}_M, H^\infty) \leqslant \|f\|_\infty \leqslant \frac{8}{\delta^6}\varepsilon.$$

Let $g, h \in H^\infty$ be such that

$$\|\Theta\overline{B}_\Lambda - h\|_\infty = \operatorname{dist}(\Theta\overline{B}_\Lambda, H^\infty) = \gamma, \tag{3.10}$$

and

$$\|B_\Lambda\overline{B}_M - g\|_\infty = \operatorname{dist}(B_\Lambda\overline{B}_M, H^\infty) \leqslant \frac{8}{\delta^6}\varepsilon \tag{3.11}$$

which implies that $\|g\|_\infty \leqslant \frac{8}{\delta^6}\varepsilon + 1$. Moreover since $\Theta\overline{B}_M - gh = \Theta\overline{B}_\Lambda(B_\Lambda\overline{B}_M - g) + (\Theta\overline{B}_\Lambda - h)g$, we have

$$\operatorname{dist}(\Theta\overline{B}_M, H^\infty) \leqslant \|B_\Lambda\overline{B}_M - g\|_\infty + \|g\|_\infty \|\Theta\overline{B}_\Lambda - h\|_\infty.$$

Hence we get from (3.11)

$$\operatorname{dist}(\Theta\overline{B}_M, H^\infty) \leqslant \frac{8}{\delta^6}\varepsilon + \left(\frac{8}{\delta^6}\varepsilon + 1\right)\gamma < 1. \quad \blacksquare$$

Proof of Theorem 3.10. Using Lemma 3.9, we see that $R(\Lambda)$ and $R(M)$ are (AOS). Hence there exist constants $c_N, \widetilde{c}_N, C_N, \widetilde{C}_N$ tending to 1 and such that, for every sufficiently large N, we have

$$c_N \sum_{n \geqslant N} |a_n|^2 \|k_{\lambda_n}\|^2 \leqslant \Big\| \sum_{n \geqslant N} a_n k_{\lambda_n} \Big\|^2 \leqslant C_N \sum_{n \geqslant N} |a_n|^2 \|k_{\lambda_n}\|^2, \tag{3.12}$$

and

$$\widetilde{c}_N \sum_{n \geqslant N} |a_n|^2 \|k_{\mu_n}\|^2 \leqslant \Big\| \sum_{n \geqslant N} a_n k_{\mu_n} \Big\|^2 \leqslant \widetilde{C}_N \sum_{n \geqslant N} |a_n|^2 \|k_{\mu_n}\|^2. \tag{3.13}$$

Write $\Lambda_N = (\lambda_n)_{n \geqslant N}$ and $M_N = (\mu_n)_{n \geqslant N}$. Then, we have

$$\operatorname{dist}(\Theta\overline{B}_{\Lambda_N}, H^\infty) = \|H_{\Theta\overline{B}_{\Lambda_N}}\| = \|P_{B_{\Lambda_N}} | \Theta H^2\| = \|P_{\Theta H^2} | K_{B_{\Lambda_N}}\|,$$

where $P_{\Theta H^2}$ is the orthogonal projection onto ΘH^2. Moreover,

$$\begin{aligned}
\Big\| P_{\Theta H^2}\Big(\sum_{n \geqslant N} a_n k_{\lambda_n} \Big) \Big\|^2 = \Big\| \sum_{n \geqslant N} a_n \overline{\Theta}(\lambda_n) k_{\lambda_n} \Big\|^2 &\leqslant C_N \sum_{n \geqslant N} |a_n|^2 |\Theta(\lambda_n)|^2 \|k_{\lambda_n}\|^2 \\
&\leqslant C_N \sup_{n \geqslant N} |\Theta(\lambda_n)|^2 \sum_{n \geqslant N} |a_n|^2 \|k_{\lambda_n}\|^2 \\
&\leqslant \frac{C_N}{c_N} \sup_{n \geqslant N} |\Theta(\lambda_n)|^2 \Big\| \sum_{n \geqslant 1} a_n k_{\lambda_n} \Big\|^2,
\end{aligned}$$

by (3.12). Hence $\|P_{\Theta H^2}|K_{B_{\Lambda_N}}\| \leqslant \sqrt{\frac{C_N}{c_N}} \sup\limits_{n\geqslant N} |\Theta(\lambda_n)|$, which implies that

$$\lim_{N\to\infty} \operatorname{dist}(\Theta\overline{B}_{\Lambda_N}, H^\infty) = 0.$$

Moreover, as previously, we show that

$$\operatorname{dist}(B_{\Lambda_N}\overline{B}_{M_N}, H^\infty) \leqslant \sqrt{\frac{\widetilde{C}_N}{\widetilde{c}_N}}\varepsilon.$$

Therefore, arguing as in the proof of Theorem 3.3, we get

$$\operatorname{dist}(\Theta\overline{B}_{M_N}, H^\infty) \leqslant \sqrt{\frac{\widetilde{C}_N}{\widetilde{c}_N}}\varepsilon + \left(\sqrt{\frac{\widetilde{C}_N}{\widetilde{c}_N}}\varepsilon + 1\right)\operatorname{dist}(\Theta\overline{B}_{\Lambda_N}, H^\infty),$$

and we can choose N sufficiently large to make the last quantity strictly less than 1, whenever $\varepsilon < 1$. ∎

The proof of Theorem 3.4 is similar to that of Theorem 3.1, with the following lemma in place of 3.15.

LEMMA 3.17. *Let E be a complex separable Hilbert space. Let $\Lambda = (\lambda_n)_{n\geqslant 1}$ be an N-Carleson subset of $\mathbb{D}$, so that*

$$\Lambda = \bigcup_{i=1}^{N} \Lambda_i, \quad \Lambda_i \in (C),$$

and denote by δ_i the Carleson constant of Λ_i. Then for any function $f \in H^2(E)$, we have

$$\sum_{\lambda\in\sigma}(1-|\lambda|^2)\|f(\lambda)\|_E^2 \leqslant 32\sum_{i=1}^{N}(1+2\log 1/\delta_i)\|f\|^2_{H^2(E)}.$$

Using an orthonormal basis of E, we prove this lemma in the same way as Lemma 3.15.

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