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ON BAND ALGEBRAS

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Abstract. It is shown that a nest in a Hilbert space $\mathcal{H}$ is the lattice of closed invariant subspaces of a band algebra in $\mathcal{B}(\mathcal{H})$ (i.e. an algebra generated by a semigroup of idempotent operators) if and only if all finite-dimensional atoms of the nest have dimension 1.

A canonical operator matrix form for operator bands, developed by the authors, is used to demonstrate that the set of algebraic operators in $\mathcal{B}(\mathcal{H})$ coincides with the union of all band subalgebras of $\mathcal{B}(\mathcal{H})$.

Several sufficient conditions for an operator band to be reducible and triangularizable are presented, and a new proof is given for a theorem on algebraic triangularizability of arbitrary operator bands.

Keywords: *Semigroups, idempotents, invariant subspaces, bands, reducible, irreducible representations.*

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1. INTRODUCTION

By an *operator* we mean a bounded linear transformation (in a Hilbert space setting) or a linear transformation (in a vector space setting), depending on the context. The set of all operators on a Hilbert space $\mathcal{H}$ (respectively vector space $\mathcal{V}$) is denoted by $\mathcal{B}(\mathcal{H})$ (respectively $\mathcal{L}(\mathcal{V})$). We write I and 0 for the identity and the zero operator respectively. All Hilbert spaces ($\mathcal{H}$) in this paper are assumed to be over the field $\mathbb{C}$ of complex numbers.

We consider semigroups of idempotent operators under the operation of composition (i.e. operator multiplication). We refer to these bands as *operator bands.* A linear span of an operator band is an algebra called a *band algebra.* Since every abstract band can be represented (via the left regular representation) as a band of linear transformations on a vector space over an arbitrary field, the study of operator bands is essential for understanding all bands.

An operator band in $\mathcal{B}(\mathcal{H})$ is said to be *reducible* if it has a proper closed non-trivial invariant subspace. An operator band $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ is said to be *triangularizable* if there exists a maximal chain of closed subspaces of $\mathcal{H}$ each of which is invariant under $\mathcal{S}$.

We begin by demonstrating that a nest in a Hilbert space $\mathcal{H}$ is the lattice of closed invariant subspaces of a band (equivalently, of a band algebra) in $\mathcal{B}(\mathcal{H})$ if and only if all finite-dimensional atoms of the nest have dimension 1. This is a step towards characterizing closed subspace lattices that are attainable as $\mathrm{Lat}(\mathcal{S})$, where $\mathcal{S}$ is an operator band, a general question that remains open.

We proceed to use the canonical operator matrix form for bands, developed in [9], to demonstrate that an operator is algebraic in $\mathcal{B}(\mathcal{H})$ if and only if it belongs to a band subalgebra of $\mathcal{B}(\mathcal{H})$. It is shown that a similar result holds in a vector space setting, provided that the minimal polynomial for the operator splits over the underlying field. (The same canonical form is later used to give an operator-theoretic proof for a theorem on algebraic triangularizability of arbitrary operator bands.)

We will also present two sufficient conditions for triangularizability of an operator band in $\mathcal{B}(\mathcal{H})$; the first in terms of the set of commutators, and the second via semigroup structure. These are to be contrasted with the result of R. Drnovšek ([2]) who has constructed a weakly dense (and thus irreducible) band in $\mathcal{B}(l^2)$.

Even though we mostly deal with a Hilbert space setting, our Hilbert space results have Banach space generalizations (with appropriate quotients used in place of orthogonal complements as required). In several cases the results have obvious algebraic counterparts as well. We try, where possible, to present proofs in a way which makes the necessary modifications apparent.

The following is standard in semigroup theory. A book by A.H. Clifford and G.B. Preston ([1]) is a classic. For a more recent account see J.M. Howie ([6]).

An element $a \in \mathcal{S}$ is said to be *idempotent* with respect to an operation $\circ : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, if $a \circ a = a$. A *band* is a semigroup of idempotents (the reference to some fixed operation is implicit). A *sub-band* of a band is a subset that is closed under the operation. A sub-band $\mathcal{J}$ of a band $\mathcal{S}$ is a *band ideal* in $\mathcal{S}$, if $a \circ b, b \circ a, a \circ b \circ a \in \mathcal{S}$ whenever $a \in \mathcal{S}$, $b \in \mathcal{J}$.

Relation '$\sim$', defined on a band $(\mathcal{S}, \circ)$ by

$$a \sim b \Leftrightarrow \begin{cases} a \circ b \circ a = a, \\ b \circ a \circ b = b, \end{cases}$$

is an equivalence relation.

For each element a of a band $\mathcal{S}$, write $\mathcal{C}_a$ for the $\sim$-equivalence class of a. Then $\mathcal{C}_a$ is a sub-band of $\mathcal{S}$. We refer to $\mathcal{C}_a$ as the *component* of $\mathcal{S}$ containing a. A band that has only one component is said to be *rectangular.*

A semigroup $\mathcal{Q}$ is said to be a *left zero semigroup* if $a \circ b = a$ for all $a, b \in \mathcal{Q}$. *Right zero semigroups* are defined similarly. Every rectangular band is isomorphic to a direct product of a left zero semigroup and a right zero semigroup.

The relation '$\sim$' is a semigroup congruence. An operation '$\odot$' is (well-)defined on the set $\mathcal{S}_{/\sim}$ of components of a band $\mathcal{S}$ by

$$\mathcal{C}_a \odot \mathcal{C}_b = \mathcal{C}_{a \circ b}.$$

We refer to the abelian band $(\mathcal{S}_{/\sim}, \odot)$ as *the band of components of* $\mathcal{S}$.

One simple consequence of $\mathcal{S}_{/\sim}$ being abelian is that

$$a \circ b \circ c \sim b \circ a \circ c \sim c \circ b \circ a \sim \cdots$$

for all $a, b, c \in \mathcal{S}$.

Relation '$\precsim$', defined on a band $\mathcal{S}$ by

$$a \precsim b \Leftrightarrow a \circ b \circ a = a,$$

is a pre-order (i.e. reflexive and transitive). We refer to '$\precsim$' as *the band pre-order on* $\mathcal{S}$. Clearly

$$a \sim b \Leftrightarrow \begin{cases} a \precsim b, \\ b \precsim a. \end{cases}$$

It follows that '$\precsim$' is a partial order exactly when all components of $\mathcal{S}$ are singletons. Therefore the band pre-order on $\mathcal{S}_{/\sim}$ is a partial order. We denote it by '$\preceq$' and refer to it as *the band order on* $\mathcal{S}_{/\sim}$. It is easy to see that

$$\mathcal{C}_a \preceq \mathcal{C}_b \Leftrightarrow a \precsim b.$$

It is worth observing that if $\mathcal{S}$ is an operator band containing I and 0, then $0 \precsim T \precsim I$ for every $T \in \mathcal{S}$.

One more property deserves mention. (The use of $\circ$ and $\odot$ is henceforth abandoned. We rely on the context for the appropriate interpretation.) Clearly $\mathcal{C}_a\mathcal{C}_b\mathcal{C}_a = \mathcal{C}_a$ whenever $a \precsim b$. Restated without reference to components this means that $abd = ad$ whenever $a \sim d \precsim b$. It turns out, in general, that

$$ab_1b_2\cdots b_md = ad \quad \text{whenever} \quad a \sim d \precsim b_1, b_2, \ldots, b_m.$$

We refer to this property as *the sandwich property.*

2. Nests attainable as a lattice of a band

Definition 2.1. A *nest* $(\mathcal{M}, \subset)$ is a chain of closed subspaces of a Hilbert space $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$ and closed under intersection and closed span; i.e.

$$\bigcap_{W \in \mathcal{L}} W \in \mathcal{M} \quad \text{and} \quad \overline{\bigcup_{W \in \mathcal{L}} W} \in \mathcal{M}$$

for every subset $\mathcal{L}$ of $\mathcal{M}$; (the overline indicates norm closure). Given $\emptyset \neq W \in \mathcal{L}$ we write

$$W_- = \overline{\bigcup_{W \supsetneq U \in \mathcal{L}} U}.$$

When $\dim(W \ominus W_-) = \alpha > 0$ we say that (W_-, W) is *a gap of dimension* α in $(\mathcal{M}, \subset)$ and call the space $W \ominus W_-$ an α*-dimensional atom* of $\mathcal{M}$.

Given a subset $\mathcal{D}$ in $\mathcal{B}(\mathcal{H})$, the lattice of all closed subspaces of $\mathcal{H}$ invariant under $\mathcal{D}$ is denoted by $\mathrm{Lat}(\mathcal{D})$. $\mathcal{D}$ is called *unicellular* if $\mathrm{Lat}(\mathcal{D})$ is totally ordered by inclusion, in which case it is automatically a nest.

Remark 2.2. In [2] R. Drnovšek gave a procedure for constructing a weakly dense (and thus irreducible) band in $\mathcal{B}(l^2)$, and hence on any separable Hilbert space. Every infinite-dimensional Hilbert space $\mathcal{H}$ can be expressed as a countably infinite orthogonal direct sum of equi-dimensional subspaces of $\mathcal{H}$. A close examination of [2] shows that Drnovšek's procedure can be carried out on any infinite-dimensional $\mathcal{H}$, by replacing scalar entries in the original matrices with appropriate operator entries. In particular, every infinite-dimensional Hilbert space $\mathcal{H}$ admits an irreducible Drnovšek-type band, which we shall denote by $\mathcal{R}_H$ henceforth. By Theorem 5.3 of [9], $\mathcal{R}_H$ has no non-trivial zero-divisors, and consequently we may assume that $0 \notin \mathcal{R}_H$ and $I \in \mathcal{R}_H$.

If $\mathcal{S}$ is a unicellular band and (W_1, W_2) is a finite-dimensional gap of $\mathrm{Lat}(\mathcal{S})$, then the compression of $\mathcal{S}$ to the atom $W_2 \ominus W_1$ is an operator band which must be irreducible. Since operator bands on finite-dimensional vector spaces are triangularizable ([12]), it follows that $\dim(W_2 \ominus W_1) = 1$. Hence a necessary condition for a nest to be the lattice of closed invariant subspaces of an operator band (i.e. to be "attainable as a lattice of a band") is that all finite-dimensional atoms have dimension 1. Our next theorem shows that this condition is sufficient as well.

THEOREM 2.3. *The following assertions are equivalent for a nest $\mathcal{M}$ in a Hilbert space $\mathcal{H}$:*

(i) *All finite-dimensional atoms of $\mathcal{M}$ have dimension* 1.

(ii) *There exists an operator band $\mathcal{S}$ in $\mathcal{B}(\mathcal{H})$ such that the components of $\mathcal{S}$ form a chain with respect to the usual order and $\mathcal{M} = \mathrm{Lat}(\mathcal{S})$.*

(iii) *There exists an operator band $\mathcal{S}$ in $\mathcal{B}(\mathcal{H})$ such that $\mathcal{M} = \mathrm{Lat}(\mathcal{S})$.*

Proof. Only (i) $\Rightarrow$ (ii) remains to be shown.

If $W \in \mathcal{M}$ and $W \ominus W_-$ is not an infinite-dimensional atom in $\mathcal{M}$, we write

$$\mathcal{K}_W = \{E \in \mathcal{B}(\mathcal{H}) \mid E^2 = E, \,\mathrm{Range}(E) = W\}.$$

If $W \ominus W_-$ is an infinite-dimensional atom, we shall denote by $[T]_{W \ominus W_-}$ the compression of $T \in \mathcal{B}(\mathcal{H})$ to $W \ominus W_-$, and let

$$\mathcal{K}_W = \big\{E \in \mathcal{B}(\mathcal{H}) \mid E^2 = E, W_- \subset \mathrm{Range}(E) \subset W, \\ E(W \ominus W_-) \subset W \ominus W_-, [E]_{W \ominus W_-} \in \mathcal{R}_{W \ominus W_-}\big\}.$$

CLAIM. $\mathcal{K}_W$ is a band for every $W \in \mathcal{M}$.

If $W \in \mathcal{M}$ and $W \ominus W_-$ is not an infinite-dimensional atom in $\mathcal{M}$ then $\mathcal{K}_W$ is a right zero band.

If $W \ominus W_-$ is an infinite-dimensional atom, and $E, F \in \mathcal{K}_W$, then E and F can be (simultaneously) written as block-upper-triangular matrices

$$E = \begin{pmatrix} I & 0 & * \\ 0 & R_1 & * \\ 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} I & 0 & * \\ 0 & R_2 & * \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the orthogonal decomposition $H = W_- \oplus (W \ominus W_-) \oplus (H \ominus W)$, where $R_1, R_2 \in \mathcal{R}_{W \ominus W_-}$. Thus $EF, FE \in \mathcal{K}_W$, so that $\mathcal{K}_W$ is a band, as claimed.

If $W_1, W_2 \in \mathcal{M}$, $W_1 \subsetneq W_2$, $E_1 \in \mathcal{K}_{W_1}$ and $E_2 \in \mathcal{K}_{W_2}$, then E_2 acts as an identity on W_1 (which contains the range of E_1). Thus

$$E_2 E_1 = E_1.$$

Moreover

$$E_1 E_2 = E_1 E_1 E_2 = E_1(E_2 E_1)E_2 = (E_1 E_2)^2,$$

so that E_1E_2 is idempotent as well. Obviously, $\mathrm{Range}(E_1E_2) \subset \mathrm{Range}(E_1) \subset W_1$ and $[E_1E_2]_{W_1} = [E_1]_{W_1}$ (because E_2 acts as an identity on W_1). Therefore $\mathrm{Range}(E_1E_2) = \mathrm{Range}(E_1)$ and $E_1E_2 \in \mathcal{K}_{W_1}$, by the definition of $\mathcal{K}_{W_1}$. We have shown that

$$(\mathcal{K}_{W_1}\mathcal{K}_{W_2}) \cup (\mathcal{K}_{W_2}\mathcal{K}_{W_1}) = \mathcal{K}_{W_1}.$$

It follows that

$$\bigcup_{W \in \mathcal{M}} \mathcal{K}_W$$

is a band, which will be denoted by $\mathcal{S}$ henceforth.

Since E_2 acts as an identity on W_1 and $\mathrm{Range}(E_2) \subset W_2$, every closed subspace Z of $\mathcal{H}$, with $Z \subset W_1$ or $W_2 \subset Z$, is invariant under E_2 and hence under $\mathcal{K}_{W_2}$. This shows that each $Z \in \mathcal{M}$ is invariant under

$$\bigcup_{Z \subsetneq W \in \mathcal{M}} \mathcal{K}_W$$

and under

$$\bigcup_{W \subset Z \in \mathcal{M}} \mathcal{K}_W.$$

Hence $\mathcal{M} \subset \mathrm{Lat}(\mathcal{S})$.

The rest of the argument is devoted to demonstrating that $\mathcal{M} = \mathrm{Lat}(\mathcal{S})$. Since $\mathrm{Span}(\mathcal{S} \cup \{I\})$ is a unital algebra and $\mathrm{Lat}(\mathcal{S}) = \mathrm{Lat}(\mathrm{Span}(\mathcal{S} \cup \{I\}))$, it is enough to show that the set of closed cyclic invariant subspaces of $\mathrm{Span}(\mathcal{S} \cup \{I\})$ is a subset of $\mathcal{M}$.

For non-triviality, suppose $x \in \mathcal{H}\backslash\{0\}$. Our goal is to show

$$\overline{\mathrm{Span}(\mathcal{S} \cup \{I\})(x)} \in \mathcal{M}.$$

Suppose $W \in \mathcal{M}$ and $x \notin W$. Then the orthogonal projection (z_x) of x onto $W^\perp$ is non-zero. From the definition of $\mathcal{K}_W$ it follows that

$$\mathcal{K}_W(x) = W;$$

(obviously, $\mathcal{K}_W(x) \subset W$; for the reverse inclusion note that, by the definition of $\mathcal{K}_W$, for each $y \in W$ there exists $E \in \mathcal{K}_W$ such that $E(z_x) = y$.) Hence $W \subset \mathcal{S}(x)$. This shows that the subspace W_x defined by

$$W_x = \overline{\bigcup_{x \notin W \in \mathcal{M}} W},$$

is an element of $\mathcal{M}$ and is contained in $\overline{\mathcal{S}(x)}$.

Case 1. If $x \in W_x$ then

$$(\mathcal{S} \cup \{I\})(x) \subset (\mathcal{S} \cup \{I\})(W_x) \subset W_x,$$

because $W_x \in \mathcal{M} \subset \mathrm{Lat}(\mathcal{S})$. Therefore

$$\overline{\mathrm{Span}(\mathcal{S} \cup \{I\})(x)} = W_x \in \mathcal{M},$$

as a direct consequence of

$$(\mathcal{S} \cup \{I\})(x) \subset W_x \subset \overline{\mathcal{S}(x)}.$$

Case 2. If $x \notin W_x$ then $W_x \subset \overline{\mathcal{S}(x)}$ and W_x is the largest element in M which does not contain x. Let

$$U_x = \bigcap_{W_x \subsetneq U \in \mathcal{M}} U.$$

It is clear that $x \in U_x \in \mathcal{M} \subset \mathrm{Lat}(\mathcal{S})$, so that

$$W_x \subsetneq \overline{\mathrm{Span}(\mathcal{S} \cup \{I\})(x)} \subset U_x.$$

Obviously, $U_x \ominus W_x$ is an atom of $\mathcal{M}$.

If $\dim(U_x \ominus W_x) = 1$ then $\overline{\mathrm{Span}(\mathcal{S} \cup \{I\})(x)} = U_x \in \mathcal{M}$, which is the desired conclusion.

If $U_x \ominus W_x$ is infinite-dimensional and $P_{U_x \ominus W_x}$ is the orthogonal projection onto $U_x \ominus W_x$, then $P_{U_x \ominus W_x}(x) \neq 0$. It follows from the irreducibility of $\mathcal{R}_{U_x \ominus W_x}$ and the definition of $\mathcal{K}_{U_x}$ that

$$\begin{aligned} U_x \ominus W_x &= \overline{\mathrm{Span}(\mathcal{R}_{U_x \ominus W_x})(P_{U_x \ominus W_x}(x))} \\ &\subset P_{U_x \ominus W_x}(\overline{\mathrm{Span}(\mathcal{K}_{U_x})(x)}) \subset P_{U_x \ominus W_x}(\overline{\mathrm{Span}(\mathcal{S} \cup \{I\})(x)}). \end{aligned}$$

Since $W_x \subset \overline{\mathcal{S}(x)}$ we also see that

$$U_x = W_x \oplus (U_x \ominus W_x) \subset \overline{\mathrm{Span}(\mathcal{S} \cup \{I\})(x)} \subset U_x \in \mathcal{M},$$

which leads to the desired conclusion and completes the proof. ∎

3. BAND ALGEBRAS AND ALGEBRAIC OPERATORS

The goal of this section is to demonstrate that an operator is algebraic in $\mathcal{B}(\mathcal{H})$ if and only if it belongs to a band subalgebra of $\mathcal{B}(\mathcal{H})$. (This is the converse to Theorem 3.4. It can be seen that a similar result holds in a vector space setting, provided that the minimal polynomial for the operator splits over the underlying field.) We need some preliminary results.

Theorem 3.1. ([4]) *Finitely generated bands are finite.*

Theorem 3.2. ([9]) *Suppose $\mathcal{S}$ is a non-zero rectangular band of operators on a Hilbert space $\mathcal{H}$. Then there exist an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ ($\mathcal{H}_2 \neq \{0\}$) and sets $\Omega \subset \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$, $\Lambda \subset \mathcal{L}(\mathcal{H}_3, \mathcal{H}_2)$, such that S has matrix form*

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 & X & XY \\ 0 & I & Y \\ 0 & 0 & 0 \end{pmatrix} \middle| \; X \in \Omega, \, Y \in \Lambda \right\}$$

with respect to this decomposition.

A similar result holds in a vector space setting.

Theorem 3.3. ([9]) *If a band $\mathcal{S}$ of operators on a Hilbert space $\mathcal{H}$ has finitely many components, then there exist an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_m$ (m is usually greater than the number of components) with respect to which all elements of $\mathcal{S}$ have a block-upper-triangular matrix form, with each block* *on the diagonal being either* 0 *or* I*. Components of* $\mathcal{S}$ *are subsets that consist of all elements with the same block-diagonal.*

A similar result holds in a vector space setting.

Theorem 3.4. ([9]) *If* $\mathcal{S}$ *is an operator band on a vector space* $\mathcal{V}$ *over a field* $\mathbb{F}$*, then every operator in the linear span of* $\mathcal{S}$ *is algebraic with a minimal polynomial that splits over* $\mathbb{F}$*.*

Remark. The claim about minimal polynomials is not explicit in [9]. It is shown that every operator in the linear span of $\mathcal{S}$ can be block-upper-triangularized with finitely many blocks, in such a way that the diagonal blocks are scalar multiples of the identity matrix. This clearly implies the result stated here.

Theorem 3.5. *If* T *is a bounded algebraic operator on a Hilbert space* $\mathcal{H}$*, then* T *belongs to a span of an operator band in* $\mathcal{B}(\mathcal{H})$*.*

If T *is an algebraic operator on a vector space* $\mathcal{V}$ *over a field* $\mathbb{F}$*, such that the minimal polynomial of* T *splits over* $\mathbb{F}$*, then* T *belongs to a span of an operator band on* $\mathcal{V}$*.*

Proof. We restrict attention to a Hilbert space setting. The proof of the vector space result is analogous, with Primary Decomposition Theorem replacing Riesz idempotents.

The proof is in two steps. The first shows that the theorem is true for all bounded nilpotent operators T. The second demonstrates the general case by reducing it to the case considered in the first step.

Suppose $T \in \mathcal{B}(\mathcal{H})$ is nilpotent of order n. Then there exists an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \cdots \oplus \mathcal{H}_n$, such that with respect to this decomposition T is represented by a block-upper-triangular operator matrix:

$$T = \begin{pmatrix} 0 & T_{12} & T_{13} & T_{14} & \cdots & T_{1n-1} & T_{1n} \\ 0 & 0 & T_{23} & T_{24} & \cdots & T_{2n-1} & T_{2n} \\ 0 & 0 & 0 & T_{34} & \cdots & T_{3n-1} & T_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & T_{n-1n} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For each $k = 1, 2, \ldots, n$, consider the set

$$\mathcal{C}_k = \{E \in \mathcal{B}(\mathcal{H}) \mid E^2 = E \text{ and } \mathrm{Range}(E) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k\}.$$

Each $E \in \mathcal{C}_k$ acts as an identity on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k$. Hence $EF = F$ for all $E, F \in \mathcal{C}_k$, so that $\mathcal{C}_k$ is a right zero band.

If $E \in \mathcal{C}_k$ and $F \in \mathcal{C}_m$ with $k \leqslant m$, then $FE = E$. Consequently, $(EF)(EF) = E(FE)F = EEF = EF$. It follows that both EF and FE are idempotent. Since $\mathrm{Range}(EF) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k = \mathrm{Range}(E) = \mathrm{Range}(FE)$, both FE and EF are elements of $\mathcal{C}_k$. Therefore the set $\mathcal{S} = \bigcup_{k=1}^{n} \mathcal{C}_k$ is a band with components $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$. For each k, operators

$$A_k = \begin{pmatrix} I_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & I_2 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_k & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$B_k = \begin{pmatrix} I_1 & 0 & 0 & 0 & T_{1k+1} & \cdots & 0 \\ 0 & I_2 & 0 & 0 & T_{2k+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_k & T_{kk+1} & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

are elements of $\mathcal{C}_k$. ('I_j' denotes the identity operator on $\mathcal{H}_j$.) Hence

$$T = \sum_{k=1}^{n-1} (B_k - A_k) \in \operatorname{Span}(\mathcal{S}),$$

which completes the first step of the proof.

Suppose now that T is an algebraic operator in $\mathcal{B}(\mathcal{H})$ with spectrum $\sigma(T) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. For each $k = 1, 2, \dots, n$, the Riesz idempotent R_k corresponding to λ_k commutes with T. Consequently, the ranges of $R_1, R_2, \dots, R_n$ are closed complementary subspaces of $\mathcal{H}$ invariant under T. An operator matrix representation of T with respect to the (not necessarily orthogonal) decomposition $\mathcal{H} = \operatorname{Range}(R_1) \oplus \operatorname{Range}(R_2) \oplus \cdots \oplus \operatorname{Range}(R_n)$ is of the form

$$T = \begin{pmatrix} T_1 & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & T_n \end{pmatrix},$$

where T_k is algebraic and $\sigma(T_k) = \{\lambda_k\}$. It follows that $T_k - \lambda_k I_k$ is an algebraic quasinilpotent operator on $\operatorname{Range}(R_k)$ and consequently must be nilpotent.

According to the first step of the proof, there exist operator bands $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$ in $\mathcal{B}(\operatorname{Range}(R_1)), \mathcal{B}(\operatorname{Range}(R_2)), \dots, \mathcal{B}(\operatorname{Range}(R_n))$ respectively, such that $T_k - \lambda_k I_k \in \operatorname{Span}(\mathcal{S}_k)$, $k = 1, 2, \dots, n$. Clearly $\mathcal{S}_k \cup \{I_k\}$ is still a band and $T_k \in \operatorname{Span}(\mathcal{S}_k \cup \{I_k\})$.

Let $\mathcal{S}$ denote the set

$$\left\{ \begin{pmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & D_n \end{pmatrix} \,\middle|\, D_k \in \mathcal{S}_k \cup \{I_k\} \right\}.$$

Then $\mathcal{S}$ is a band in $\mathcal{B}(\mathcal{H})$ and $T \in \operatorname{Span}(\mathcal{S})$. ∎

Theorem 3.3 has a number of consequences, some of which were addressed in [9] and shall be strengthened here. We shall say that the spectrum (σ) is *operationally subordinate* on an algebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ if

$$\sigma(AB + C) \subset \sigma(A)\sigma(B) + \sigma(C) = \{ab + c \mid a \in \sigma(A),\, b \in \sigma(B),\, c \in \sigma(C)\},$$

for all $A, B, C \in \mathcal{A}$. Clearly, the spectrum is operationally subordinate on a unital algebra $\mathcal{A}$ exactly when it is both sublinear and submultiplicative.

The spectrum is said to be *permutable* ([8]) on a semigroup $\mathcal{S}$ of operators in $\mathcal{B}(\mathcal{H})$ if $\sigma(ABC) = \sigma(BAC), \forall A, B, C \in \mathcal{S}$.

We refer the reader to [13] and [14] for an account of the connection between these properties and reducibility (or triangularizability) of the semigroup. Theorem 4.2 of the next section is a sample result. Our present goal is to demonstrate that every band algebra in $\mathcal{B}(\mathcal{H})$ has an operationally subordinate spectrum which is permutable. In particular, in view of R. Drnovšek's example ([2]), the combination of these conditions does not imply reducibility even for band algebras.

Theorem 3.6. *Every band algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ has an operationally subordinate spectrum which is permutable.*

Proof. The proof proceeds along the lines of Lemma 5.9 in [9] and is presented here for completeness.

Let $\mathcal{A}$ be generated by a band $\mathcal{S}$. If $A_1, \dots, A_n \in \mathcal{A}$ then there exist $m \in \mathbb{N}$, scalars $\{c_{ij}\}_{i=1,\, j=1}^{i=n,\, j=m}$ and elements $\{F_{ij}\}_{i=1,\, j=1}^{i=n,\, j=m}$ of $\mathcal{S}$, such that

$$A_i = \sum_{j=1}^{m} c_{ij} F_{ij}, \quad i = 1, \dots, n.$$

The sub-band $\mathcal{T}$ of $\mathcal{S}$ generated by $\{F_{ij}\}_{i=1,\, j=1}^{i=n,\, j=m}$ is finite by Green-Rees Theorem 3.1. Apply Theorem 3.3 to decompose $\mathcal{H}$ as an orthogonal sum $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k$, with respect to which all elements of $\mathcal{T}$ have a block-upper-triangular matrix form, with each block on the diagonal being either 0 or I. It follows that $A_1, \dots, A_n$ are block-upper-triangular with respect to this decomposition, with each diagonal block being some scalar multiple of identity. The spectrum of each A_i (including the multiplicities) can be read off the block-diagonal. The desired conclusion follows. ∎

4. SUFFICIENT CONDITIONS FOR REDUCIBILITY AND TRIANGULARIZABILITY

Next we present several sufficient conditions for reducibility and triangularizability of an operator band in $\mathcal{B}(\mathcal{H})$. These should be considered in contrast to Drnovšek's example.

Theorem 4.1. ([12]) *If a non-zero band ideal $\mathcal{J}$ in an operator band $\mathcal{S}$ is reducible, then so is $\mathcal{S}$. A similar result is true for all operator algebras.*

Theorem 4.2. ([13]) *For a semigroup $\mathcal{S}$ of compact operators in $\mathcal{B}(\mathcal{H})$ the following are equivalent:*
(i) *$\mathcal{S}$ is triangularizable;*
(ii) *the spectrum is strongly permutable on $\mathcal{S}$;*
(iii) *the spectrum is sublinear on $\mathcal{S}$.*

Theorem 4.3. *If the norm closure of a band algebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ contains a non-zero compact operator, then $\mathcal{A}$ is reducible.*

Proof. If K is a non-zero compact operator in the norm closure $\overline{\mathcal{A}}$ of $\mathcal{A}$, then the ideal $\mathcal{I}_K$ (of $\overline{\mathcal{A}}$) generated by K consists of compact operators. Since the spectrum is operationally subordinate on $\mathcal{A}$ by Theorem 3.6, and every compact operator is a point of continuity of the spectrum ([11]), it follows easily that the spectrum is operationally subordinate on $\mathcal{I}_K$. By Theorem 4.2, $\mathcal{I}_K$ is triangularizable, and consequently $\overline{\mathcal{A}}$ is reducible by Theorem 4.1. ∎

Corollary 4.4. *If for every E in a band $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ either E or $I - E$ has finite rank then $\mathcal{S}$ is triangularizable.*

Proof. A compression of $\mathcal{S}$ to an invariant subspace still satisfies the hypothesis (on the new Hilbert space). Theorem 4.3 gives reducibility of $\mathcal{S}$, and an application of Zorn's Lemma does the rest. ∎

Theorem 4.5. ([3]) *A subset $\mathcal{J}$ of a band $\mathcal{S}$ is a band ideal in $\mathcal{S}$ if and only if:*
(a) *$\mathcal{J}$ is a union of some components of $\mathcal{S}$.*
(b) *If $\mathcal{C}$ and $\mathcal{D}$ are components of $\mathcal{S}$ such that $\mathcal{C} \prec \mathcal{D}$ and $\mathcal{D} \subset \mathcal{J}$, then $\mathcal{C} \subset \mathcal{J}$.*

Given two operators A and B on the same space, *the commutator* $[A, B]$ is the operator $AB - BA$. If $\mathcal{W}$ is a set of operators, we write:

$$[\mathcal{W}, \mathcal{W}] = \{[A, B] \mid A, B \in \mathcal{W}\}.$$

Theorem 4.6. *If $\mathcal{S}$ is an operator band then*

$$[\mathcal{S}, \mathcal{S}] = \{C - D \mid C \sim D \text{ in } \mathcal{S}\}.$$

Proof. $[\subset]$: Immediate, since $AB \sim BA$ for all $A, B \in \mathcal{S}$.
$[\supset]$: If $C \sim D$ in $\mathcal{S}$ then

$$C - D = (CD)(DC) - (DC)(CD) \in [\mathcal{S}, \mathcal{S}]. \quad ∎$$

The following corollary can be proved by direct computation.

Corollary 4.7. *If $\mathcal{S}$ is an operator band, then $[A,B]^3 = 0$ for every $A, B \in \mathcal{S}$.*

Theorem 4.8. *If $\mathcal{S}$ is an operator band containing a non-zero operator A, distinct from the identity, such that $[A,B]^2 = 0$ for every $B \in \mathcal{S}$, then $\mathcal{S}$ is reducible.*

Proof. Suppose $\mathcal{S}$ is a band in $\mathcal{B}(\mathcal{H})$. Without loss of generality assume $\{0, I\} \subsetneq \mathcal{S}$. Let $A \in \mathcal{S}$ be the operator described in the hypothesis. According to Theorem 4.1, it is sufficient to show that the band ideal $\{B \in \mathcal{S} \mid B \precsim A\}$ is reducible. Denote this ideal by $\mathcal{J}$ and consider an arbitrary element B of $\mathcal{J}$.

Observe that A has matrix form

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = \text{Range}(A) \dot{+} \text{Kernel}(A)$; (Range($A$) and Kernel($A$) are (non-trivial, proper) closed complementary (not necessarily orthogonal) subspaces). A calculation shows that (with respect to the same decomposition) B has matrix form

$$B = \begin{pmatrix} E & X \\ Y & YX \end{pmatrix},$$

with some X, Y. Consequently:

$$[A,B] = \begin{pmatrix} 0 & X \\ -Y & 0 \end{pmatrix}.$$

Since $[A,B]^2 = 0$ by the hypothesis, it follows that $YX = XY = 0$. Therefore

$$B = \begin{pmatrix} E & X \\ Y & 0 \end{pmatrix},$$

so that $B(\text{Kernel}(A)) \subset \text{Range}(A)$. Set

$$\mathcal{J}(\text{Kernel}(A)) = \{B(x) \mid B \in \mathcal{J},\, x \in \text{Kernel}(A)\}.$$

Since it has been shown that $\mathcal{J}(\text{Kernel}(A)) \subset \text{Range}(A)$, $\overline{\mathcal{J}(\text{Kernel}(A))}$ is a proper closed invariant subspace of $\mathcal{J}$. If $\mathcal{J}(\text{Kernel}(A)) \neq \{0\}$ then $\mathcal{J}$ is reducible.

If $\mathcal{J}(\text{Kernel}(A)) = \{0\}$ then $\mathcal{J}$ is reducible again, because Kernel(A) is a proper non-trivial closed invariant subspace of $\mathcal{J}$. ∎

The converse of Theorem 4.8 is false. The rectangular band

$$\left\{ \begin{pmatrix} 0 & I & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & I \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I & I \\ 0 & I & I \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

serves as a counterexample.

Corollary 4.9. *If $\mathcal{S}$ is an operator band and $T^2 = 0$ for every $T \in [\mathcal{S}, \mathcal{S}]$, then $\mathcal{S}$ is triangularizable.*

Proof. Since a compression of $\mathcal{S}$ to an invariant subspace satisfies the hypothesis of Theorem 4.8 (on the subspace), a standard Zorn's Lemma argument gives the desired result. ∎

Theorem 4.10. *If every component of an operator band $\mathcal{S}$ is either a left zero semigroup or a right zero semigroup, then the band is triangularizable.*

Proof. If $\mathcal{S}$ satisfies the hypothesis then so does every compression of $\mathcal{S}$ to an invariant subspace. Therefore, by Zorn's Lemma, it is sufficient to show that $\mathcal{S}$ has a proper non-trivial closed invariant subspace. Assume for non-triviality that $\mathcal{S} \neq \{0, I\}$. Let A be a non-zero element of $\mathcal{S}$ different from the identity. Then $\{B \in \mathcal{S} \mid B \precsim A\}$ (henceforth denoted by $\mathcal{J}$) is a non-zero band ideal in $\mathcal{S}$. According to Theorem 4.1, it is enough to demonstrate that $\mathcal{J}$ is reducible.

If $\mathcal{C}$ is a component of $\mathcal{J}$ (i.e. a component of $\mathcal{S}$ contained in $\mathcal{J}$) and $\mathcal{C}$ is a left zero semigroup, then $BA = B$ for every $B \in \mathcal{C}$. (This is because $BA = B(BA)$ and both B and BA are elements of the left zero semigroup $\mathcal{C}$.) Similarly, if $\mathcal{C}$ is a component of $\mathcal{J}$ which is a right zero semigroup, then $AB = B$ for every $B \in \mathcal{C}$. Hence $AB = B$ or $BA = B$ for every $B \in \mathcal{J}$.

A has operator matrix form

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathrm{Kernel}(A) \dot{+} \mathrm{Range}(A)$.

It follows that every $B \in \mathcal{J}$ is of the form

$$\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}.$$

Consequently, $B(\mathrm{Kernel}(A)) \subset \mathrm{Range}(A)$ for all $B \in \mathcal{J}$. Therefore the same is true for all B in the linear span of $\mathcal{J}$.

If $B(\mathrm{Kernel}(A)) = \{0\}$ for every $B \in \mathcal{J}$, then $\mathrm{Kernel}(A)$ is a proper closed non-trivial invariant subspace of $\mathcal{J}$. If $B(\mathrm{Kernel}(A)) \neq \{0\}$ for some $B \in \mathcal{J}$, then the norm closure of $\{B(x) \mid B \in \mathrm{Span}(\mathcal{J}),\, x \in \mathrm{Kernel}(A)\}$ is a proper closed non-trivial invariant subspace of $\mathcal{J}$. ∎

5. A VIEW TO ALGEBRAIC REDUCIBILITY

In this section we use operator-theoretic methods to show how one can use a canonical operator representation of an operator band with finitely many components, given in Theorem 3.3, to derive a simple description for the set $\mathcal{N}_{\mathcal{S}}$ of all nilpotent operators in the band algebra generated by $\mathcal{S}$. Consequently we obtain a number of algebraic results, including an operator-theoretic proof for the fact that every operator band is algebraically triangularizable. This should be compared to [10] and contrasted with [2].

Lemma 5.1. *If $A_1, \ldots, A_n$ belong to the same component $\mathcal{C}$ of an operator band $\mathcal{S}$ and $t_1, t_2, \ldots, t_n \in \mathbb{F}$, then there exist $C_2, \ldots, C_n \in [\mathcal{C}, \mathcal{C}]$ such that*

$$(t_1 A_1 + \cdots + t_n A_n) + (t_2 C_2 + \cdots + t_n C_n) = (t_1 + \cdots + t_n) A_1.$$

Proof. This is a trivial consequence of Theorem 4.6. ∎

THEOREM 5.2. *If $\mathcal{S}$ is an operator band then* $\mathcal{N}_{\mathcal{S}} = \mathrm{Span}([\mathcal{S}, \mathcal{S}])$.

Proof. [$\supset$]: Suppose $T = \sum_{i=1}^{n} t_i[A_i, B_i]$ for some $t_i \in \mathbb{F}$ and $A_i, B_i \in \mathcal{S}$. Let $\mathcal{S}_T$ be the sub-band of $\mathcal{S}$ generated by the set $\{A_1, B_1, \ldots, A_n, B_n\}$. Since $\mathcal{S}_T$ is finitely generated, it is finite by Green-Rees Theorem 3.1, and so has finitely many components. Apply Theorem 3.3 to deduce the existence of complementary subspaces $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ of $\mathcal{V}$, such that with respect to the decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_m$ all elements of $\mathcal{S}_T$ have a block-upper-triangular matrix form, with each block on the diagonal being either 0 or I.

It follows from Theorem 4.6 that elements of $[\mathcal{S}_T, \mathcal{S}_T]$ also have the block-upper-triangular matrix form described above (with respect to $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_m$) and all of the block-diagonal entries in their matrices are zero. The same is true for elements of the linear span of $[\mathcal{S}_T, \mathcal{S}_T]$, and therefore every operator in this linear span is nilpotent. Since T is one such operator the proof is complete.

Note that the reasoning used here can be easily modified to yield a proof of the fact that $\mathcal{N}_{\mathcal{S}}$ is always a linear space.

[$\subset$]: Let T be a non-zero nilpotent operator in the linear span of $\mathcal{S}$. Then $T = \sum_{i=1}^{n} t_i A_i$ for some $0 \neq t_1, t_2, \ldots, t_n \in \mathbb{F}$ and $A_1, A_2, \ldots, A_n \in \mathcal{S}$. According to Green-Rees Theorem 3.1, the sub-band $\mathcal{S}_{\{A_1,\ldots,A_n\}}$ of $\mathcal{S}$ generated by the set $\{A_1, \ldots, A_n\}$ is finite and has finitely many (say: m) components. The proof is by induction on m.

(*Base case*) $m = 1$: Let $A_1, \ldots, A_n$ belong to the same component $\mathcal{C}$ of $\mathcal{S}$. According to Theorem 3.2 there exist complementary subspaces $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ of $\mathcal{V}$ ($\mathcal{V}_2 \neq \{0\}$), such that with respect to the decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$ every element of $\mathcal{C}$ has matrix form

$$\begin{pmatrix} 0 & L & LM \\ 0 & I & M \\ 0 & 0 & 0 \end{pmatrix},$$

for some L, M. The matrix for T (with respect to the same decomposition) must be of the form

$$T = \begin{pmatrix} 0 & * & * \\ 0 & (t_1 + \cdots + t_n) & * \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that $t_1 + \cdots + t_n = 0$ because T is nilpotent. According to Lemma 5.1 there exist $C_2, \ldots, C_n \in [\mathcal{C}, \mathcal{C}]$ such that

$$(t_1 A_1 + \cdots + t_n A_n) + (t_2 C_2 + \cdots + t_n C_n) = (t_1 + \cdots + t_n) A_1 = 0.$$

Therefore

$$T = (t_1 A_1 + \cdots + t_n A_n) = -(t_2 C_2 + \cdots + t_n C_n) \in \mathrm{Span}([\mathcal{C}, \mathcal{C}]) \subset \mathrm{Span}([\mathcal{S}, \mathcal{S}]).$$

(*Inductive step*): Suppose the result is true for $m = 1, 2, \ldots, k$ and $\mathcal{S}_{\{A_1,\ldots,A_n\}}$ has $k + 1$ components.

We may assume without loss of generality that no two operators among $A_1, A_2, \ldots, A_n$ belong to the same component. (According to Lemma 5.1, if A_1, A_2 belong to the same component $\mathcal{C}$ then $t_1 A_1 + t_2 A_2 = (t_1 + t_2) A_1 - t_2 C_2$, for some $C_2 \in [\mathcal{C}, \mathcal{C}] \subset [\mathcal{S}, \mathcal{S}]$. Consequently $T + t_2 C_2 = (t_1 + t_2) A_1 + A_3 + \cdots + A_n$.

Repeat the procedure to obtain $T + Q = \sum_{r=1}^{\hat{n}} s_{i_r} A_{i_r}$, for some $A_{i_1}, A_{i_2}, \ldots, A_{i_{\hat{n}}}$ in $\mathcal{S}$, no two of which lie in the same component, some $Q \in \mathrm{Span}([\mathcal{S},\mathcal{S}])$ and $s_{i_1}, s_{i_2}, \ldots, s_{i_{\hat{n}}} \in \mathbb{F}$. Clearly $T \in \mathrm{Span}([\mathcal{S},\mathcal{S}]) \Leftrightarrow T + Q \in \mathrm{Span}([\mathcal{S},\mathcal{S}])$ and we have observed that in the first part of the proof that $T + Q$ is still nilpotent.)

We may also assume that $t_1 = 1$ (replace T by $(1/t_1)T$).

Re-index $A_1, A_2, \ldots, A_n$ if necessary, so that A_1 belongs to a component $\mathcal{C}_{A_1}$ of $\mathcal{S}_{\{A_1,\ldots,A_n\}}$, maximal with respect to '$\preceq$'; (in other words, $A_1 \not\precsim A_j$ for $j = 2, 3, \ldots, n$). Then $\mathcal{G} = \mathcal{S}_{\{A_1,\ldots,A_n\}} \setminus \mathcal{C}_{A_1}$ is a band (!), containing $A_2, \ldots, A_n$. Clearly $\mathcal{G}$ has at most k components.

We aim to show that T belongs to the linear span of $\mathcal{G}$. Since $T^p = 0$ for some $p \in \mathbb{N}$, and $T = \sum_{i=1}^{n} t_i A_i = A_1 + D$, for some $D \in \mathrm{Span}(\mathcal{G})$, it follows (from the non-commutative binomial expansion) that

$$(A_1)^p + R + D^p = 0$$

for some $R \in \mathrm{Span}(\mathcal{G})$; (note that the linear span of $\mathcal{G}$ is an algebra containing $A_1 D$, DA_1 and $A_1 D A_1$). Therefore

$$T = A_1 + D = -R - D^p + D \in \mathrm{Span}(\mathcal{G})$$

and consequently $T \in \mathrm{Span}(\mathcal{G})$.

Hence $T = \sum_{i=1}^{N} s_i B_i$ for some $s_1, s_2, \ldots, s_n \in \mathbb{F}$ and $B_1, B_2, , \ldots, B_n \in \mathcal{G}$, and therefore $T \in \mathrm{Span}([\mathcal{S},\mathcal{S}])$ by the inductive assumption (the band generated by $B_1, B_2, \ldots, B_N$ is a sub-band of $\mathcal{S}$ with at most k components). ∎

Denote the Jacobson Radical of an algebra $\mathcal{A}$ by $\mathcal{R}_{\mathcal{A}}$. In the case when $\mathcal{A}$ is a linear span of some operator band $\mathcal{S}$, we write $\mathcal{R}_{\mathcal{S}}$ instead of $\mathcal{R}_{\mathrm{Span}(\mathcal{S})}$.

We will use the following theorem of D. Hadwin:

Theorem 5.3. ([5]) *Suppose $\mathcal{A}$ is an algebra of algebraic operators on a vector space $\mathcal{V}$ (over a field $\mathbb{F}$) such that the minimal polynomial of each element of $\mathcal{A}$ splits over $\mathbb{F}$. Then the following are equivalent:*

(i) *$\mathcal{A}$ is algebraically triangularizable;*

(ii) *$\mathcal{R}_{\mathcal{A}} = \{T \in \mathcal{A} \mid T \text{is nilpotent}\}$;*

(iii) *$\mathcal{A}/\mathcal{R}_{\mathcal{A}}$ is abelian.*

Theorem 5.4. *If $\mathcal{S}$ is an operator band then $\mathcal{N}_{\mathcal{S}}$ is an ideal in the linear span of $\mathcal{S}$.*

Proof. In view of Theorem 5.2 it is sufficient to show that $A[B,C]$ and $[B,C]A$ belong to $[\mathcal{S},\mathcal{S}]$ for $A, B, C \in \mathcal{S}$. Since $ABC \sim ACB \sim BCA \sim CBA$, it follows by Theorem 4.6 that $A[B,C] = ABC - ACB \in [\mathcal{S},\mathcal{S}]$ and $[B,C]A = BCA - CBA \in [\mathcal{S},\mathcal{S}]$. ∎

Corollary 5.5. *If $\mathcal{S}$ is an operator band then $\mathcal{N}_{\mathcal{S}}$ is the Jacobson Radical $\mathcal{R}_{\mathcal{S}}$ of the linear span of $\mathcal{S}$.*

Proof. According to Theorem 3.4 every element of the linear span of $\mathcal{S}$ is algebraic. Thus by Theorem 14 (and the Remark following Theorem 12) in Part 2 of [7], $\mathcal{R}_{\mathcal{S}}$ is the largest nil ideal (i.e. the largest ideal containing only nilpotents) in the linear span of $\mathcal{S}$. Since $\mathcal{N}_{\mathcal{S}}$ is an ideal in $\mathcal{S}$, it must be that $\mathcal{R}_{\mathcal{S}} = \mathcal{N}_{\mathcal{S}}$. ∎

COROLLARY 5.6. *If $\mathcal{S}$ is an operator band then* $\mathrm{Span}(\mathcal{S})/\mathcal{R}_{\mathcal{S}}$ *is abelian and $\mathcal{S}$ is algebraically triangularizable.*

Equivalently: the only (algebraically) irreducible representations of a band are trivial representations.

Proof. Theorem 3.4, Corollary 5.5, and Theorem 5.3. ∎

The close relationship between the algebra $\mathrm{Span}(\mathcal{S})/\mathcal{R}_{\mathcal{S}}$ and the band $\mathcal{S}_{/\sim}$ of components of $\mathcal{S}$ is now apparent. Let $\mathcal{S}/\mathcal{R}_{\mathcal{S}}$ denote the image of $\mathcal{S}$ under the quotient map $\pi : \mathrm{Span}(\mathcal{S}) \to \mathrm{Span}(\mathcal{S})/\mathcal{R}_{\mathcal{S}}$. Clearly $\mathcal{S}/\mathcal{R}_{\mathcal{S}}$ is a band because π is an algebra homomorphism.

COROLLARY 5.7. *If $\mathcal{S}$ is an operator band then $\mathcal{S}/\mathcal{R}_{\mathcal{S}}$ is isomorphic to the band $\mathcal{S}_{/\sim}$ of components of $\mathcal{S}$.*

Proof. If $A, B \in \mathcal{S}$ satisfy $\pi(A) = \pi(B)$, then $A - B \in \mathcal{R}_{\mathcal{S}}$ and consequently $A - B$ is nilpotent. Let $\mathcal{S}_{A,B}$ be the sub-band of $\mathcal{S}$ generated by A and B. Then $\mathcal{S}_{A,B}$ is finite with at most 6 elements. According to Theorem 3.3 there exist complementary subspaces $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ of $\mathcal{V}$, such that with respect to the decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_m$ all elements of $\mathcal{S}_{A,B}$ have block-upper-triangular matrix form with each block on the diagonal being either 0 or I. Components of $\mathcal{S}_{A,B}$ are subsets that consist of all elements with the same block-diagonal. Since $A - B$ is nilpotent, A and B have the same block diagonal with respect to the above decomposition of $\mathcal{V}$. Hence $A \sim B$. Therefore $\mathcal{C}_A = \mathcal{C}_B$ whenever $\pi(A) = \pi(B)$.

Since the converse is also true (reverse the direction of the argument and use Theorem 4.6, Corollary 4.7 (b) and Corollary 5.5), it follows that

$$\mathcal{C}_A = \mathcal{C}_B \Leftrightarrow \pi(A) = \pi(B).$$

Hence, the map $\Psi : \mathcal{S}/\mathcal{R}_{\mathcal{S}} \to \mathcal{S}_{/\sim}$, specified by

$$\Psi(\pi(A)) = \mathcal{C}_A \quad \text{for all } A \in \mathcal{S},$$

is well-defined and bijective. Since π is a homomorphism, so is Ψ. ∎

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