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SPECTRAL CONDITIONS FOR THE NILPOTENCY OF LIE ALGEBRAS

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Abstract. In the paper it is proved that each of the following conditions below is sufficient for the nilpotency of a solvable Lie algebra. (They are formulated by means of the spectrum of a Banach space representation of a Lie algebra, introduced by C. Ott in [9].)

1) The validity of the projection property of the spectrum on Lie subalgebras (imposed to only one representation).

2) The existence of representations with spectra of arbitrarily small diameters (with respect to a fixed norm on the given Lie algebra).

Keywords: *Lie algebra homology, Taylor joint spectrum, projection property, spectral diameter.*

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0. INTRODUCTION

The main results of this note are two sufficient conditions, of homological nature, for certain complex finite dimensional Lie algebras to be nilpotent (see Theorem 1.3 and Theorem 2.2 below). As we shall shortly explain now, these results are suggested by the comparison between the spectral theories for solvable Lie algebras of operators introduced in [5] and [1]. (These two theories agree in the nilpotent case with the theory of [7], and in the commutative case with the several variable spectral theory of Taylor ([14]).)

One of the main results of the spectral theory of [5] (cf. also [7], [2], [3], [4], [9], [10]) is that, in an appropriate sense, the projection property of the spectrum on Lie ideals holds (see Theorem 3 in [5]). In [5] it is remarked that, in general, the projection property does not hold on any Lie subalgebra. However, [7] shows that, in the case of a nilpotent Lie algebra, the projection property on every Lie subalgebra holds (see also [10]).

On the other hand, for every finite-dimensional solvable Lie algebra of operators $\mathcal{G}$ a new spectrum $\Sigma(\mathcal{G})$ is constructed in [1]. This spectrum has the projection property on every Lie subalgebra (cf. Theorem 0.3 below). As an easy consequence of this fact, if $\Sigma(\mathcal{G}) = \{0\}$ then $\mathcal{G}$ contains only quasinilpotent operators (cf. Corollary 0.4 below) and moreover $\mathcal{G}$ is nilpotent ([15], [11]).

In the present note we study the analogues of these results of [1] in the spectral theory of [5].

Our first main result is that the nilpotent Lie algebras are characterized by the fact that the projection property of the spectrum of [5] on every Lie subalgebra holds. (Actually we prove a more general result, involving the spectrum of a representation ([9], [10]); see Theorem 1.3 below.) As a consequence we obtain that every solvable Lie algebra of operators which is not nilpotent has a subalgebra for which the spectra introduced in [5], respectively [1], are distinct. In the final of Section 1 we investigate the projection property only on hyperplane subalgebras (i.e., subalgebras of codimension 1), in the case of certain classes of Lie algebras introduced in [8] and [13].

Our second main result is that if a solvable Lie algebra endowed with a norm posseses Banach space representations with spectra (in the sense of [9]) of arbitrarily small diameters, then it must be a nilpotent Lie algebra (see Theorem 2.2 below). Consequently, if a solvable Lie algebra posseses a representation whose spectrum is $\{0\}$ then it is a nilpotent Lie algebra and the range of that representation consists only of quasinilpotent operators (see Corollary 2.3 below). This result is a strengthening of Proposition 7 of [5] and, as we remarked above, it has a variant in the framework of [1].

Next we shall introduce some conventions and notation. If $\mathcal{F}$ is a set of functions defined on a set B and A is a subset of B then we denote by $\mathcal{F}|A$ the set of restrictions to A of the functions from B. We shall work only with complex finite-dimensional Lie algebras. Consequently, by solvable (respectively nilpotent) Lie algebra we shall mean complex finite-dimensional solvable (respectively nilpotent) Lie algebra. We shall denote by $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on an arbitrary complex Banach space $\mathcal{X}$. For the definition of a Koszul complex and related matters we refer to [7], [5], [9], [10]. If $\rho : E \to \mathcal{B}(\mathcal{X})$ is a representation of the Lie algebra E then we denote (cf. [9], [10]) by $\mathrm{Kos}(\rho)$ the Koszul complex of ρ, and by $\sigma(\rho)$ the spectrum of ρ. (We note that $\mathrm{Kos}(\rho)$ is denoted in [7] by $\mathrm{Kos}(E, \rho, \mathcal{X})$.) We denote by $\widehat{E}$ the set of all characters of E (i.e., Lie algebra homomorphisms from E to the field $\mathbb{C}$ of the complex numbers). If E is a normed Lie algebra (i.e., it is endowed with a vector space norm), then $\widehat{E}$ is a vector space of bounded linear functionals on E; so $\widehat{E}$ has a natural norm and one can speak about the diameter $\mathrm{diam}(A)$ of any non-empty subset A of $\widehat{E}$. For $e, f \in E$ we denote as usual $(\mathrm{ad}\ e)f = [e, f]$.

For later use we recall two well-known results concerning bicomplexes. In their statements $X_{\cdot,\cdot}$ denotes a bicomplex $(X_{i,j})_{i,j\in Z}$ consisting of complex vector spaces and linear maps; we assume that, excepting the rows $X_{j,\cdot}$, $0 \leqslant j \leqslant N$, all the other rows of $X_{\cdot,\cdot}$ consist only of zero spaces.

LEMMA 0.1. *Let j_0 be in $\{0,\ldots,N\}$ and assume that, for every index j distinct of j_0, the row $X_{j,\cdot}$ is an exact complex. Then the complex $X_{j_0,\cdot}$ is exact if and only if the totalization* $\mathrm{Tot}(X_{\cdot,\cdot})$ *is exact.*

LEMMA 0.2. *Suppose that any coloumn of $X_{\cdot,\cdot}$ contains only zero maps. Then* $\mathrm{Tot}(X_{\cdot,\cdot})$ *is exact if and only if every row of $X_{\cdot,\cdot}$ is exact.*

Finally, for the sake of completeness, we recall a few facts of the spectral theory developed in [1]. For a Lie algebra $\mathcal{H}$ we denote by $\mathrm{id}_{\mathcal{H}}$ the identity automorphism of $\mathcal{H}$ ($\mathrm{id}_{\mathcal{H}}(H) = H$ for every H in $\mathcal{H}$). If moreover $\mathcal{H}$ is a Lie subalgebra of $\mathcal{B}(\mathcal{X})$ then we can also consider $\mathrm{id}_{\mathcal{H}}$ as a representation $\mathrm{id}_{\mathcal{H}} : \mathcal{H} \to \mathcal{B}(\mathcal{X})$. Now let $\mathcal{G}$ be a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. If $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{G}$ then we denote by

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{C}_{\mathcal{H}}$$

the root decomposition determined by $\mathcal{H}$, where $\mathcal{C}_{\mathcal{H}}$ is the sum of all root spaces corresponding to non-zero roots of $\mathcal{G}$ with respect to $\mathcal{H}$ (see the remarks preceding Lemma 2 in the Chapter 8 of [12]). In this case we denote

$$\Sigma_{\mathcal{H}} := \{f \in \widehat{\mathcal{G}} : f|\mathcal{H} \in \sigma(\mathrm{id}_{\mathcal{H}}) \text{ and } f|\mathcal{C}_{\mathcal{H}} = 0\}.$$

Then one can prove that $\Sigma_{\mathcal{H}}$ is actually independent of the choice of the Cartan subalgebra $\mathcal{H}$ and that $\Sigma_{\mathcal{H}} \subset \widehat{\mathcal{G}}$ (cf. Proposition 2.1 in [1]). We denote $\Sigma_{\mathcal{H}}$ by $\Sigma(\mathcal{G})$. This subset of $\widehat{\mathcal{G}}$ was called in [1] the spectrum of $\mathcal{G}$. Unlike the spectrum from [5], the above introduced spectrum $\Sigma(\cdot)$ has the projection property on every Lie subalgebra in the following sense (see Theorem 2.6 in [1]):

THEOREM 0.3. *If $\mathcal{G}$ is a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ and $\mathcal{L}$ is a Lie subalgebra of $\mathcal{G}$, then $\Sigma(\mathcal{G})|\mathcal{L} = \Sigma(\mathcal{L})$.*

As a consequence we have (Corollary 2.7 in [1]):

COROLLARY 0.4. *If $\mathcal{G}$ is a solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ then for each $G \in \mathcal{G}$ we have*

$$\sigma(G) = \{f(G) : f \in \Sigma(\mathcal{G})\}.$$

We denote here as usual the spectrum of an operator $G \in \mathcal{B}(\mathcal{X})$ by $\sigma(G)$.

1. PROJECTION PROPERTY

We begin with an estimate of the projection of spectrum on certain hyperplane subalgebras which are not necessarily ideals.

PROPOSITION 1.1. *Let $\rho : E \to \mathcal{B}(\mathcal{X})$ be a representation of the Lie algebra E. Let B be a hyperplane subalgebra of E such that there exists an ideal I of E with* $\dim I = 1$ *and* $E = I + B$. *Take $\widetilde{\gamma} \in \widehat{B}$ such that*

$$(\forall)\, b \in B, \quad (\mathrm{ad}\, b)|I = \widetilde{\gamma}(b)\mathrm{id}_I. \tag{1.1}$$

Then the following estimate of the "projection" $\sigma(\rho)|B$ of $\sigma(\rho)$ on B holds

$$\sigma(\rho|B) \bigtriangleup (\widetilde{\gamma} + \sigma(\rho|B)) \subseteq \sigma(\rho)|B \subseteq \sigma(\rho|B) \cup (\widetilde{\gamma} + \sigma(\rho|B)), \tag{1.2}$$

where $\bigtriangleup$ denotes the symmetric difference of sets. If moreover $I \subseteq \ker \rho$, then

$$\sigma(\rho)|B = \sigma(\rho|B) \cup (\widetilde{\gamma} + \sigma(\rho|B)). \tag{1.3}$$

Proof. We begin with a preliminary observation. Take $a \in I \setminus \{0\}$ and for $\widetilde{\lambda} \in \widehat{E}$ denote by $X_{\rho-\tilde{\lambda}}$ the following diagram

$$\begin{array}{ccccccccccc}
0 & \longleftarrow & \mathcal{X} & \xleftarrow{\beta'} & \mathcal{X} \otimes B & \xleftarrow{\beta'} & \cdots & \xleftarrow{\beta'} & \mathcal{X} \otimes \wedge^p B & \xleftarrow{\beta'} & \cdots \\
 & & \Big\downarrow{\scriptstyle \rho(a)} & & \Big\downarrow{\scriptstyle \rho(a)\otimes \mathrm{id}_B} & & & & \Big\downarrow{\scriptstyle \rho(a)\otimes \mathrm{id}_{\wedge^p B}} & & \\
0 & \longleftarrow & \mathcal{X} & \xleftarrow{\beta} & \mathcal{X} \otimes B & \xleftarrow{\beta} & \cdots & \xleftarrow{\beta} & \mathcal{X} \otimes \wedge^p B & \xleftarrow{\beta} & \cdots
\end{array} \tag{1.4}$$

whose upper row is $\mathrm{Kos}(\rho|B - \widetilde{\lambda}|B + \widetilde{\gamma})$ and whose lower row is $\mathrm{Kos}(\rho|B - \widetilde{\lambda}|B)$. By (1.1) we get

$$(\forall) b \in B, \quad \rho(a)(\rho(b) - \widetilde{\lambda}(b) + \widetilde{\gamma}(b)) = (\rho(b) - \widetilde{\lambda}(b))\rho(a).$$

This fact easily implies that (1.4) is a commutative diagram. Then, remarking that $\wedge^p E = \wedge^p B \oplus (a \wedge (\wedge^{p-1} B)) \simeq \wedge^p B \oplus \wedge^{p-1} B$, $p \in \mathbb{N}$, a straightforward computation shows that $\mathrm{Tot}(X_{\rho-\tilde{\lambda}})$ is isomorphic to $\mathrm{Kos}(\rho - \widetilde{\lambda})$.

We now pass to the proof of (1.2). Take $\widetilde{\lambda}_0 \in \sigma(\rho|B) \bigtriangleup (\widetilde{\gamma} + \sigma(\rho|B))$ and define $\widetilde{\lambda} : E \to \mathbb{C}$ by $\widetilde{\lambda}|B = \widetilde{\lambda}_0$ and $\widetilde{\lambda}|I = 0$. Then $\widetilde{\lambda} \in \widehat{E}$ and $0 \in \sigma(\rho|B - \widetilde{\lambda}|B) \bigtriangleup (\widetilde{\gamma} + \sigma(\rho|B - \widetilde{\lambda}|B))$. Consequently, one row of (1.4) is exact and the other is not exact. Hence, by Lemma 0.1, $\mathrm{Tot}(X_{\rho-\tilde{\lambda}})$ is not exact. Then, by the preliminary observation, $\mathrm{Kos}(\rho - \widetilde{\lambda})$ is not exact. We get $\widetilde{\lambda} \in \sigma(\rho)$ and the first inclusion of (1.2) is proved. Next take $\widetilde{\lambda} \in \sigma(\rho)$. Then $\mathrm{Kos}(\rho - \widetilde{\lambda})$ is not exact, so $\mathrm{Tot}(\rho - \widetilde{\lambda})$ is not exact. Hence, by Lemma 0.1, at least one row of (1.4) is not exact. This implies $\widetilde{\lambda}|B \in \sigma(\rho|B) \cup (\widetilde{\gamma} + \sigma(\rho|B))$ and (1.2) is completely proved.

Next let us suppose that $I \subseteq \ker \rho$, i.e., $\rho(a) = 0$. In view of (1.2) we must prove only the inclusion $\supseteq$ of (1.3). Let $\widetilde{\lambda}_0$ be arbitrary in the right hand side of (1.3) and define the linear functional $\widetilde{\lambda} : E \to \mathbb{C}$ by $\widetilde{\lambda}|B = \widetilde{\lambda}_0$ and $\widetilde{\lambda}|I = 0$. Then $\widetilde{\lambda} \in \widehat{E}$ and at least one row of (1.4) is not exact. Consequently, by Lemma 0.2, the complex $\mathrm{Tot}(X_{\rho-\tilde{\lambda}})$ is not exact. In view of the preliminary observation we get $\widetilde{\lambda} \in \sigma(\rho)$ and (1.3) is completely proved. ∎

REMARKS 1.2. (a) The example of Section 3 of [5] (see also Example 3 of [10]) can be justified by means of Proposition 1.1 above.

(b) The relation (1.3) from Proposition 1.1 above generalizes Remarks 2.9(1^o) of [7].

Now we apply Proposition 1.1 to establish a converse to the fact that, for the spectrum of a representation of a nilpotent Lie algebra, the projection property on every Lie subalgebra holds (cf. [7], [10]).

THEOREM 1.3. *Let* $\rho : L \to \mathcal{B}(\mathcal{X})$ *be a representation of the Lie algebra* L. *If for every Lie subalgebra* E *of* L *with* $\dim E \leqslant 2$ *we have*

$$\sigma(\rho)|E = \sigma(\rho|E)$$

then L *is a nilpotent Lie algebra.*

Proof. Assume that the Lie algebra L is not nilpotent. Then, in view of the Engel Theorem, one can find $b, a \in L \setminus \{0\}$ and $\gamma \in \mathbb{C}^*$ such that $[b, a] = \gamma a$. Denote $E = \mathbb{C}a + \mathbb{C}b$, $B = \mathbb{C}b$, $I = \mathbb{C}a$. In view of the hypothesis we get

$$\sigma(\rho|B) = \sigma(\rho)|B = (\sigma(\rho)|E)|B = \sigma(\rho|E)|B.$$

But this is a contradiction with Proposition 1.1 above (see the first inclusion in (1.2)) because in this case $\widetilde{\gamma} \neq 0$. ∎

REMARK 1.4. In Theorem 1.3 above we do not assume that L is a solvable Lie algebra.

COROLLARY 1.5. *Let* $\mathcal{G}$ *be a solvable Lie subalgebra of* $\mathcal{B}(\mathcal{X})$ *which is not nilpotent. Then there exists a Lie subalgebra* $\mathcal{L}$ *of* $\mathcal{G}$ *such that*

$$\sigma(\mathrm{id}_{\mathcal{L}}) \neq \Sigma(\mathcal{L}).$$

Proof. Here we use notation introduced in [1]. If $\Sigma(\mathcal{L}) = \sigma(\mathrm{id}_{\mathcal{L}})$ for every Lie subalgebra $\mathcal{L}$ of $\mathcal{G}$ then, by Theorem 2.6 in [1], the spectrum of the identic representation $\mathrm{id}_{\mathcal{G}} : \mathcal{G} \to \mathcal{B}(\mathcal{X})$ has the projection property on every Lie subalgebra of $\mathcal{G}$. Hence $\mathcal{G}$ is a nilpotent Lie algebra by Theorem 1.3 above, a contradiction with the hypothesis. ∎

To conclude this section, we show that, for certain types of solvable Lie algebras, it suffices to impose the condition from Theorem 1.3 only for hyperplane subalgebras to get the nilpotency of the algebra. First we give the definitions, suggested by [13] and [8].

DEFINITION 1.6. A Lie algebra L is called *strongly hypersolvable* if for each Lie subalgebra E with $\dim E \geqslant 3$ the following condition holds:

($*$) for every $e \in E$ one can find a proper ideal $\mathcal{I}(e)$ of E with $e \in \mathcal{I}(e)$.

REMARKS 1.7. (a) In [13] a Lie algebra is called hypersolvable if it is solvable and verifies the condition ($*$) for every Lie subalgebra E with $\dim E \geqslant 4$.

(b) Using the classical Levi-Malcev theorem ([12]) one easily checks that each strongly hypersolvable Lie algebra (in the sense of Definition 1.6 above) is solvable, hence it is hypersolvable in the sense of [13].

(c) Every nilpotent Lie algebra is strongly hypersolvable but not conversely (cf. the example of K. Teleman in [13]).

DEFINITION 1.8. ([8]) A Lie algebra is called Δ-*reduced* if the intersection of its hyperplane subalgebras is $\{0\}$.

REMARK 1.9. Unlike [8] we use here complex Lie algebras.

PROPOSITION 1.10. *Let $\rho : L \to \mathcal{B}(\mathcal{X})$ be a representation of the Lie algebra L and assume that for every hyperplane subalgebra E of L we have*

$$\sigma(\rho)|E \subseteq \sigma(\rho|E).$$

(i) *If L is strongly hypersolvable then it is nilpotent.*
(ii) *If L is metabelian (i.e., $[[L,L],[L,L]] = \{0\}$) then it is nilpotent.*
(iii) *If L is Δ-reduced solvable then it is even abelian.*

Proof. We begin with a preliminary observation: the hypothesis implies that every one-dimensional ideal of L commutes with L. Indeed, assume that there exists an ideal I of L such that $\dim I = 1$ and $[I, L] \neq \{0\}$. As a by-product of the proof of Theorem 1.6.9 from [6] it then follows that one can find a hyperplane Lie subalgebra B of L such that $L = B + I$. Let $\widetilde{\gamma}$ be as in Proposition 1.1. By Proposition 1.1 we have $(\widetilde{\gamma} + \sigma(\rho|B)) \setminus \sigma(\rho|B) \subset \sigma(\rho)|B$. But $\widetilde{\gamma} \neq 0$ (since $[I, L] \neq \{0\}$) and $\sigma(\rho|B)$ is a compact set (see Theorem 2 in [5] and also Corollary 11 in [10]). Hence we have a contradiction with the hypothesis $\sigma(\rho)|B \subseteq \sigma(\rho|B)$. The above observation is proved.

We now come back to the proof of the assertions (i)–(iii).

(i) In view of the previous observation it suffices to prove by induction on $n := \dim L$ that if L is strongly hypersolvable and every one-dimensional ideal of L commutes with L then L is nilpotent. This is obvious if $\dim L = 2$. Now assume that this fact holds for Lie algebras of dimension strictly less than n. Let J be a proper Lie ideal of L. If J has an one-dimensional ideal which does not commute with J then consider a chain of ideals of L,

$$J = J_0 \subset J_1 \subset \cdots \subset J_k = L$$

such that $\dim(J_{i+1}/J_i) = 1$ for $i = 0, \ldots, k-1$. (Such a chain exists since the Lie algebra L is solvable by Remark 1.7 (b).) Now if we apply step by step Lemma 7 from [13] we deduce that L has an one-dimensional ideal which does not commute with L. But this contradicts our assumption. Thus every one-dimensional ideal of J commutes with J. Since J is in turn hypersolvable, the induction hypothesis then implies that J is nilpotent. Consequently every proper ideal of L is nilpotent. Since L is solvable (see Remark 1.7 (b)) it then follows that L is even nilpotent.

(ii) Suppose that L is not nilpotent. Let H be a Cartan subalgebra of L, R be the corresponding set of non-zero roots and $\{L^\alpha\}_{\alpha \in R}$ be the corresponding set of root spaces. Then we have $\bigoplus\limits_{\alpha \in R} L^\alpha \subseteq [L, L]$ (see e.g. Proposition 1.2 (3)) in [6]). Hence the hypothesis implies that $[L^\alpha, L^\beta] = \{0\}$ for all $\alpha, \beta \in R$.

Now let us choose $\alpha \in R$ and $l \in L^\alpha \setminus \{0\}$ such that for every $h \in H$ we have $[h, l] = \alpha(h) l$. Since $[L^\beta, l] = \{0\}$ for every $\beta \in R$ and $L = H \oplus \bigoplus\limits_{\beta \in R} L^\beta$, it follows that $I := \mathbb{C}\, l$ is a one-dimensional ideal of L which is not central (i.e., $[I, L] \neq \{0\}$). But this contradicts the observation from the beginning of the proof.

(iii) By Proposition 11 of [8] (which holds also for complex Lie algebras) one easily checks that, if $[L, L] \neq \{0\}$, then one can find a one-dimensional Lie ideal I of L such that $[I, L] \neq \{0\}$. But this contradicts the observation from the beginning of the proof. Hence the Lie algebra L is abelian. ∎

2. DIAMETERS OF SPECTRA

In this section we extend Proposition 7 of [5] to the case of solvable Lie algebras. To this end we need the following elementary result.

LEMMA 2.1. *Let $\gamma_1, \dots, \gamma_s$ be non-zero complex numbers. For any non-empty compact subset K of $\mathbb{C}$ we denote $K_0 = K$ and, for $1 \leqslant p \leqslant s$,*

$$K_p = \bigcup_{1 \leqslant i_1 < \dots < i_p \leqslant s} (K + \gamma_{i_1} + \dots + \gamma_{i_p}).$$

Then there exists a positive number ω depending on $\gamma_1, \dots, \gamma_s$ and independent of K, such that the following set

$$M = \{z \in \mathbb{C} : \text{there exists a unique } p \text{ in } \{0, \dots, s\} \text{ such that } z \in K_p\}$$

has the diameter at least ω. In particular, M has at least two distinct elements.

Proof. After a suitable rotation of the complex plane we may suppose that $x_i = \operatorname{Re} \gamma_i \neq 0$ for $1 \leqslant i \leqslant s$. The following situations can occur:

(i) Not all the numbers $x_1, \dots, x_s$ have the same sign.
(ii) All the numbers $x_1, \dots, x_s$ have the same sign.

We shall study only the first of these possibilities since the second possibility can be treated similarly. In the case (i) we necessarily have $s \geqslant 2$. After a suitable renumbering we may suppose that there exists $q \in \{1, \dots, s-1\}$ such that all the numbers $x_1, \dots, x_q$ are negative and all the numbers $x_{q+1}, \dots, x_s$ are positive. We define

$$\omega := -x_1 - \dots - x_q + x_{q+1} + \dots + x_s.$$

Let us choose $\zeta^-, \zeta^+ \in K$ such that

$$\operatorname{Re} \zeta^- = \inf\{\operatorname{Re} \lambda : \lambda \in K\}, \ \operatorname{Re} \zeta^+ = \sup\{\operatorname{Re} \lambda : \lambda \in K\}.$$

We shall verify that the following complex numbers

$$\lambda^- := \zeta^- + \sum_{j=1}^{q} \gamma_j, \quad \lambda^+ := \zeta^+ + \sum_{j=q+1}^{s} \gamma_j$$

are two elements of M with $|\lambda^- - \lambda^+| \geqslant \omega$. First we note that $|\lambda^- - \lambda^+| \geqslant \operatorname{Re}(\lambda^+ - \lambda^-) \geqslant \omega$. Moreover, if $p \in \{1, \dots, s\} \setminus \{q\}$ and $1 \leqslant i_1 < \dots < i_p \leqslant s$ then $x_1 + \dots + x_q < x_{i_1} + \dots + x_{i_p}$ (because $x_1, \dots, x_q$ are *all* the negative terms of the sequence $x_1, \dots, x_s$) and $x_1, \dots, x_q < 0$. Hence for every $\lambda \in K$ we have

$$\operatorname{Re} \lambda^- \leqslant \operatorname{Re}(\lambda + \gamma_1 + \dots + \gamma_q) < \operatorname{Re}(\lambda + \gamma_{i_1} + \dots + \gamma_{i_p})$$

and

$$\operatorname{Re} \lambda^- < \operatorname{Re} \zeta^- \leqslant \operatorname{Re} \lambda.$$

Consequently, $\lambda^- \notin K_p$ for $p \in \{0, \dots, s\} \setminus \{q\}$; but $\lambda^- \in K_q$, so $\lambda^- \in M$. Similarly, one can check that $\lambda^+ \notin K_p$ for $p \in \{0, \dots, s\} \setminus \{s-q\}$; but $\lambda^+ \in K_{s-q}$, so we have also $\lambda^+ \in M$. ∎

THEOREM 2.2. *Let E be a normed solvable Lie algebra. If E posseses Banach space representations with spectra of arbitrarily small diameters then it is nilpotent.*

Proof. Actually we shall prove the following statement equivalent with that of Theorem 2.2: if the normed solvable Lie algebra E is not nilpotent then there exists a positive number ω such that for every Banach space representation $\rho : E \to \mathcal{B}(\mathcal{X})$ we have $\operatorname{diam}(\sigma(\rho)) \geqslant \omega$. To this end we proceed by induction on $n = \dim E$.

For $n = 1$ the statement is obvious. Next let us assume that $n \geqslant 2$ and that the above statement holds for Lie algebras of dimension strictly less than n. We consider each Lie subalgebra endowed with the norm inherited from E.

If E has a proper Lie ideal J such that J is not a nilpotent Lie algebra then in view of the induction hypothesis we can choose a positive number ω which is not bigger than the diameter of the spectrum of every Banach space representation of J. In particular, using the projection property on Lie ideals (cf. Theorem 3 of [5]; see also Theorem 3.4 of [10]), we get

$$\operatorname{diam}(\sigma(\rho)) \geqslant \operatorname{diam}(\sigma(\rho)|J) = \operatorname{diam}(\sigma(\rho|J)) \geqslant \omega$$

for every Banach space representation $\rho : E \to \mathcal{B}(\mathcal{X})$.

Next let us assume that every proper Lie ideal of E is nilpotent. Since the Lie algebra E is solvable, we can choose a basis $\{e_1, \ldots, e_n\}$ of E such that $\|e_k\| = 1\,(1 \leqslant k \leqslant n)$,

$$[e_j, e_i] = \sum_{k=1}^{i} c_{ji}^k e_k \quad 1 \leqslant i < j \leqslant n \tag{2.1}$$

and, for certain m, $\{e_k : 1 \leqslant k \leqslant m\}$ is a basis of $[E, E]$. Let I be the Lie ideal of E spanned by $e_1, \ldots, e_{n-1}$. In view of our assumption, I is nilpotent. So $c_{ji}^i = 0$ for $1 \leqslant i < j \leqslant n-1$. Next we study the two possible cases:

(a) There exists an i_0 in $\{1, \ldots, n-1\}$ such that $c_{n\,i_0}^{i_0} = 0$. Then the determinant

$$\begin{vmatrix} c_{n\,n-1}^{n-1} & \cdots & c_{n\,n-1}^{1} \\ 0 & \ddots & \vdots \\ 0 & 0 & c_{n\,1}^{1} \end{vmatrix}$$

vanishes hence, by (2.1), the vectors $[e_n, e_{n-1}], \ldots, [e_n, e_1] \in [E, E]$ are linearly dependent. Then, since we already know that $c_{ji}^i = 0$ for $1 \leqslant i < j \leqslant n-1$, one easily deduces as above that every $n-1$ vectors from the set $\{[e_j, e_i] : 1 \leqslant i < j \leqslant n\}$ are linearly dependent. Consequently we have

$$m = \dim[E, E] \leqslant n - 2.$$

Then $J_1 := \mathbb{C}\, e_n + [E, E]$ and $J_2 := \mathbb{C}\, e_{n-1} + \cdots + \mathbb{C}\, e_{m+1} + [E, E]$ are proper Lie ideals of E. Hence J_1, J_2 are nilpotent in view of our assumption. Since $E = J_1 + J_2$ we get that E is a nilpotent Lie algebra, thus contradicting the hypothesis. Hence the present case cannot actually appear.

(b) Let us suppose now that $\gamma_i := c_{ni}^i \neq 0$ for each i in $\{1, \ldots, n-1\}$. Then we can apply Lemma 2.1 for $\gamma_1, \ldots, \gamma_{n-1}$; denote by ω the corresponding positive number. We show that ω has the desired property. To this end let us consider an arbitrary Banach space representation $\rho : E \to \mathcal{B}(\mathcal{X})$. We consider in $\wedge^p I$ the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \leqslant i_1 < \cdots < i_p \leqslant n-1\}$ and endow the set of multi-indices $\{(i_1, \ldots, i_p) : 1 \leqslant i_1 < \cdots < i_p \leqslant n-1\}$ with the lexicographic ordering (see e.g. [3], page 89). Let us consider the operator, where $0 \leqslant p \leqslant n-1$) $\theta^p_{e_n} : \mathcal{X} \otimes \wedge^p I \to \mathcal{X} \otimes \wedge^p I$:

$$\theta^p_{e_n}(x \otimes \underline{e}) = \rho(e_n)x \otimes \underline{e} + \sum_{j=1}^{p} (-1)^{j-1} x \otimes [e_n, e_{i_j}] \wedge \overset{j}{\overset{\wedge}{\underline{e}}}$$

defined for $x \in \mathcal{X}$, $\underline{e} = e_{i_1} \wedge \cdots \wedge e_{i_p}$, where the notation $\overset{j}{\wedge}$ means the omission of e_{i_j} (cf. page 6 in [7]). As in [3] we compute now the spectrum of $\theta^p_{e_n}$. The operator $\theta^p_{e_n}$ is represented by a lower-triangular matrix with entries from $\mathcal{B}(\mathcal{X})$. If we denote $b = \rho(e_n) \in \mathcal{B}(\mathcal{X})$ then by (2.1) we get

$$\theta^p_{e_n}(x \otimes e_{i_1} \wedge \cdots \wedge e_{i_p}) = (b + \gamma_{i_1} + \cdots + \gamma_{i_p})x \otimes e_{i_1} \wedge \cdots \wedge e_{i_p} + S$$

where S is a sum of terms of the form $c_{k_1 \cdots k_p} x \otimes e_{k_1} \wedge \cdots \wedge e_{k_p}$ with $c_{k_1 \cdots k_p} \in \mathbb{C}$ and $(k_1, \ldots, k_p)$ being a multi-index lexicographically strictly less than $(i_1, \ldots, i_p)$. It follows that, on the diagonal of the lower-triangular matrix representing $\theta^p_{e_n}$, one finds the operators

$$b + \gamma_{i_1} + \cdots + \gamma_{i_p}, \quad 1 \leqslant i_1 < \cdots < i_p \leqslant n-1,$$

and below the diagonal the entries are scalar multiples of $\mathrm{id}_{\mathcal{X}}$. Consequently for $\lambda \in \mathbb{C}$ we have

$$\text{(2.2)} \qquad \theta^p_{e_n} - \lambda \,\text{is not invertible} \iff \lambda \in \bigcup_{1 \leqslant i_1 < \cdots < i_p \leqslant n-1} (\sigma(b) + \gamma_{i_1} + \cdots + \gamma_{i_p}).$$

Next we consider the commutative diagram

$$\text{(2.3)} \qquad \begin{array}{ccccccccccccc} & & 0 & & & & 0 & & & & 0 & & \\ & & \downarrow & & & & \downarrow & & & & \downarrow & & \\ 0 & \leftarrow & \mathcal{X} & \leftarrow & \cdots & \leftarrow & \mathcal{X} \otimes \wedge^p I & \leftarrow & \cdots & \leftarrow & \mathcal{X} \otimes \wedge^{n-1} I & \leftarrow & 0 \\ & & \downarrow & & & & \downarrow & & & & \downarrow & & \\ 0 & \leftarrow & \mathcal{X} & \leftarrow & \cdots & \leftarrow & \mathcal{X} \otimes \wedge^p I & \leftarrow & \cdots & \leftarrow & \mathcal{X} \otimes \wedge^{n-1} I & \leftarrow & 0 \\ & & \downarrow & & & & \downarrow & & & & \downarrow & & \\ & & 0 & & & & 0 & & & & 0 & & \end{array}$$

whose rows coincide with $\mathrm{Kos}(\rho|I)$ and whose p-th column is $\theta^p_{e_n} - \lambda$ (see Lemma 1.1 in [7]). Then, in view of the choice of ω (see (2.2) and Lemma 2.1 applied for $K = \sigma(b)$ and $s = n-1$) we obtain two complex numbers λ_1, λ_2 with $|\lambda_1 - \lambda_2| \geqslant \omega \; (> 0)$ such that, for $\lambda \in \{\lambda_1, \lambda_2\}$, precisely $n-1$ columns of (2.3) are not exact. Then, for such λ, the totalization of (2.3) is not exact (by Lemma 0.1). If we define $\widetilde{\lambda}_j : E \to \mathbb{C}$ by $\widetilde{\lambda}_j | I = 0$ and $\widetilde{\lambda}_j(e_n) = \lambda_j$, then $\widetilde{\lambda}_j \in \widehat{E}$ and, by Lemma 1.5 of [7], $\mathrm{Kos}(\rho - \widetilde{\lambda}_j)$ is not exact ($j \in \{1, 2\}$). Consequently we get $\widetilde{\lambda}_1, \widetilde{\lambda}_2 \in \sigma(\rho)$. But $\|\widetilde{\lambda}_1 - \widetilde{\lambda}_2\| \geqslant |\widetilde{\lambda}_1(e_n) - \widetilde{\lambda}_2(e_n)| = |\lambda_1 - \lambda_2| \geqslant \omega$. So $\mathrm{diam}(\sigma(\rho)) \geqslant \omega$, as desired. ∎

Now we can formulate the following generalization of Proposition 7 of [5].

Corollary 2.3. *Let ρ be a Banach space representation of the solvable Lie algebra E. If $\sigma(\rho)$ contains only one element $\widetilde{\lambda}$ then E is a nilpotent Lie algebra and for every $e \in E$ the spectrum of the operator $\rho(e)$ is $\{\widetilde{\lambda}(e)\}$.*

Proof. We endow E with a norm and then apply Theorem 2.2 to get the nilpotency of E. Then we apply the projection property of the spectrum on one-dimensional Lie subalgebras (see e.g. Corollary 0.2 in [1]). ∎

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Note added in proof. A notion of essential spectrum for Banach space representations of Lie algebras was introduced by A. Dosiev in the paper "The algebra of power series of elements of a nilpotent Lie algebra and Słodkowski spectra" (to appear in the journal *Algebra i Analiz*), extending the notion of essential spectrum of operators. We note that the main results of the present paper (notably Theorem 1.3 and Theorem 2.2) still hold when $\sigma(\rho)$ stands for the essential spectrum of the representation ρ.

On the other hand, we note two references where self-contained introductions to the spectral theory of representations of Lie algebras can be found:

[1] C. Ott, *Gemeinsame Spektren auflösbarer Operator-Liealgebren*, Ph.D. Dissertation, Kiel 1997.

[2] D. Beltiţă, M. Şabac, *Lie Algebras of Bounded Operators*, Oper. Theory Adv. Appl., vol. 120, Birkhäuser, Basel 2001.