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AN INTEGRAL REPRESENTATION FOR SEMIGROUPS OF UNBOUNDED NORMAL OPERATORS

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ABSTRACT. An integral representation for semigroups $\{U_s\}_{s\in S}$ of unbounded normal operators in a Hilbert space H is presented which admits a significantly larger class of semigroups S than usual. In particular, S need not have a topology and so the traditional assumption that the functions $s \mapsto \langle U_s x, y\rangle$, for suitable elements $x, y \in H$, are continuous is no longer a requirement. The classical spectral theorem for a single (unbounded) normal or selfadjoint operator is a *consequence* of the main result; the point is that the techniques used do not rely on the fact that a normal operator has a spectral decomposition via its resolution of the identity.

KEYWORDS: *Semigroup representation, positive definiteness, normal operator, spectral theorem.*

MSC (2000): 47B15, 47D03.

1. INTRODUCTION

The integral representation (via spectral measures) of 1-parameter semigroups of bounded selfadjoint or normal operators on a Hilbert space is well known; see [19] or [10], Chapter XXII. The extension to semigroups of unbounded selfadjoint or normal operators is also well understood, although the technical assumptions are more involved; see [5], [6], [9], [11], [14], [15], for example, and the references there in. Such results sometimes extend to semigroups S more general than $[0,\infty)$, although there are usually some sort of topological constraints on S. These topological requirements of S are often transferred to the semigroup of operators $\{U_s\}_{s\in S}$ by requiring that the $\mathbb{C}$-valued function

$$s \mapsto \langle U_s x, y\rangle\,, \quad s \in S, \tag{1.1}$$

is continuous, for suitable vectors x and y in the underlying Hilbert space H.

The aim of this note is to exhibit an integral representation for semigroups $\{U_s\}_{s\in S}$ of unbounded normal operators which admits a significantly larger class of semigroups S than is usually the case. In particular, S is not assumed to have any topology and so no continuity assumptions are required of the functions (1.1). The techniques used come from the theory of positive definite functions. It is time to be more precise.

Let S be a *commutative* semigroup with a unit element e and an involution $s \mapsto s^-$ (i.e. $(s^-)^- = s$ and $(st)^- = s^-t^-$ for all $s,t \in S$). A function $\rho : S \to \mathbb{C}$ which satisfies $\rho(e) = 1$ and $\rho(st^-) = \rho(s)\overline{\rho(t)}$ for all $s,t \in S$ is called a *character* of S. The set of all characters is denoted by S^*; it becomes a completely regular topological space when equipped with the pointwise convergence topology inherited from $\mathbb{C}^S$.

Given a Hilbert space H, let $\mathcal{L}(H)$ be the space of all continuous linear operators of H into itself and $\mathcal{N}(H)$ denote the collection of all (not necessarily bounded) normal operators in H. If $T \in \mathcal{N}(H)$, then its domain is denoted by $D(T)$; it is always dense. T^* denotes the adjoint operator of T. Each element of $\mathcal{N}(H)$ is a closed operator. As general references we use [3] and [17]. If T_1 and T_2 are linear operators in H, then $T_1 \subseteq T_2$ means $D(T_1) \subseteq D(T_2)$ and $T_1x = T_2x$ for $x \in D(T_1)$. The *closure* of a closable operator T (see [3]) is denoted by $\overline{T}$.

A function $\alpha : S \to [0,\infty)$ is an *absolute value* if α is symmetric (i.e. $\alpha(s^-) = \alpha(s)$ for $s \in S$), $\alpha(e) = 1$ and $\alpha(st) \leqslant \alpha(s)\alpha(t)$ for $s,t \in S$. The family of all absolute values on S is denoted by $\mathcal{A}(S)$. Given a map $\mathcal{U} : S \to \mathcal{N}(H)$ and $\alpha \in \mathcal{A}(S)$ define

$$D_\alpha := \Big\{x \in \bigcap_{s\in S} D(U_s) : \|U_s x\| \leqslant \alpha(s)\|x\| \text{ for all } s \in S\Big\}, \tag{1.2}$$

where U_s denotes the value of $\mathcal{U}$ at $s \in S$. Then define $D_{\mathrm{c}} := \bigcup_{\alpha\in\mathcal{A}(S)} D_\alpha$.

Definition 1.1. Let S be a commutative, unital semigroup with an involution. A map $\mathcal{U} : S \to \mathcal{N}(H)$ is called a *$*$-representation* if:

(i) $U_e = I$, the identity operator on H.
(ii) $U_{s^-} = U_s^*$, $s \in S$.
(iii) $U_tU_s \subseteq U_{st}$ with $D(U_tU_s) = D(U_{st}) \cap D(U_s)$, $s,t \in S$.
(iv) $\overline{U_tU_s} = U_{st}$, $s,t \in S$.
(v) $D_{\mathrm{c}} = \bigcup_{\alpha\in\mathcal{A}(S)} D_\alpha$ is dense in H.

We will see in Section 2 that (i)–(v) are natural if $\{U_s\}_{s\in S}$ is to have an integral representation. From (1.2) and (v) of Definition 1 we see $\bigcap_{s\in S} D(U_s)$ is necessarily a dense subspace of H (as it contains D_{c}). Our main result is the following one.

Theorem 1.2. *Let S be a commutative, unital semigroup with involution and $\mathcal{U} : S \to \mathcal{N}(H)$ be a $*$-representation. Then there exists a unique selfadjoint Radon spectral measure $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$, with* $\operatorname{supp}(Ex) \subseteq S^*$ *compact if and only if $x \in D_{\mathrm{c}}$, and*

$$U_s x = \int_{S^*} \widehat{s}(\rho)\,\mathrm{d}E(\rho)x, \quad x \in D(U_s),\ s \in S, \tag{1.3}$$

where $\widehat{s} : S^* \to \mathbb{C}$ *is defined by* $\widehat{s}(\rho) := \rho(s)$.

Some explanation is in order. By $\mathcal{B}(S^*)$ is denoted the *Borel* σ*-algebra* of S^*, i.e. the smallest σ-algebra containing all the open subsets of S^*. To say the function $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$ is a *selfadjoint spectral measure* means that $E(S^*) = I$, that each operator $E(A)$, for $A \in \mathcal{B}(S^*)$, is selfadjoint, that E is multiplicative (i.e. $E(A \cap B) = E(A)E(B)$ for $A, B \in \mathcal{B}(S^*)$), and that E is countably additive for the weak (equivalently, the strong) operator topology in $\mathcal{L}(H)$, that is, $E_{x,y} : A \mapsto \langle E(A)x, y\rangle$, for $A \in \mathcal{B}(S^*)$, is a σ-additive, $\mathbb{C}$-valued measure for $x, y \in H$. Since the values $E(A)$, for $A \in \mathcal{B}(S^*)$, are orthogonal projections we see $E_{x,x}(A) = \|E(A)x\|^2$ is actually non-negative, for $x \in H$. To say E is *Radon* means $E_{x,x}$ is a Radon measure (i.e. inner regular) for each $x \in H$; see [2], Chapter 2, for example. The selfadjointness of E implies

$$4E_{x,y} = E_{x+y,\,x+y} - E_{x-y,\,x-y} + \mathrm{i}E_{x+\mathrm{i}y,\,x+\mathrm{i}y} - \mathrm{i}E_{x-\mathrm{i}y,\,x-\mathrm{i}y}, \quad x, y \in H,$$

and so each complex measure $E_{x,y}$ is also Radon (i.e. its variation is a Radon measure). For $x \in H$, the H-valued, σ-additive measure $A \mapsto E(A)x$, for $A \in \mathcal{B}(S^*)$, is denoted by Ex. Its *support*, denoted by $\mathrm{supp}(Ex)$, is defined to be the support of $E_{x,x}$ (see [2], p. 22 for the definition of support of a non-negative Radon measure). Finally, the integral in (1.3) exists in the usual sense of integration with respect to the vector measure Ex ([12]); this is made precise in the next section.

The proof of Theorem 1.2 relies on the methods of positive definite functions, together with an earlier version of Theorem 1.2 known for $*$-representations of *bounded* normal operators ([16]). The reduction to the bounded case is possible because the closed subspaces D_α (that they have this property is not obvious from (1.2)) turn out to be invariant for each U_s, $s \in S$, and the restriction of U_s to D_α is a bounded normal operator. Moreover D_c, which also turns out to be a subspace of H, plays a crucial role because it gives a description, via *intrinsic* properties of $\mathcal{U}$, of the space of vectors $x \in \bigcap_{s\in S} D(U_s)$ for which Ex has compact support. The spectral measure E is first constructed on D_{c} and then extended to H by a continuity argument. Because the arguments are rather technical the proof is delayed until Section 3. In the following Section 2 we prefer to discuss Theorem 1.2 itself in combination with Definition 1.1 and some relevant examples. In particular, we indicate how Theorem 1.2 deviates significantly from other results in the literature which are of a "similar nature" yet, as will be seen, are actually quite different. In Section 4 we indicate how the classical spectral theorem for a single (unbounded) selfadjoint or normal operator is a *consequence* of Theorem 1.2; the point is that the proof of Theorem 1.2 *nowhere* uses the fact that a normal operator has a spectral decomposition via its resolution of the identity. This is an important feature in relation to "similar" results in the literature alluded to above, which actually *use* the spectral theorem in their proofs.

An important particular case is $S = \mathbb{N}_0^k$ (with the usual addition and the identity as involution). Theorem 1.2 then provides the answer to an operator-valued moment problem. The existence of positive definite non-moment functions on $\mathbb{N}_0^2$ ([2], Theorem 6.3.4) leads to $*$-homomorphisms $\mathcal{U} : \mathbb{N}_0^2 \to L(D)$, where $L(D)$ is the algebra of *all* endomorphisms on some appropriate pre-Hilbert space D, for which there is *no* integral formula analogous to (1.3). For the class of perfect semigroups S this phenomenon cannot occur; see the final section.

2. THEOREM 1.2: DISCUSSION

Consider a selfadjoint spectral measure $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$. Every bounded Borel function $f : S^* \to \mathbb{C}$ is integrable with respect to the $\mathcal{L}(H)$-valued measure E in the sense of vector measures ([7], [12]). This integral, denoted by $\int_{S^*} f \,\mathrm{d}E$, is an element of $\mathcal{L}(H)$. Also, f is integrable with respect to each H-valued measure Ex, $x \in H$, and

$$\left\langle \Big(\int_{S^*} f \,\mathrm{d}E\Big)x, y \right\rangle = \left\langle \int_{S^*} f \,\mathrm{d}(Ex), y \right\rangle = \int_{S^*} f \,\mathrm{d}E_{x,y}, \quad x, y \in H.$$

The map $f \mapsto \int_{S^*} f \,\mathrm{d}E$ is linear, multiplicative, i.e. $\int_{S^*} fg \,\mathrm{d}E = \big(\int_{S^*} f \,\mathrm{d}E\big) \cdot \big(\int_{S^*} g \,\mathrm{d}E\big)$, and satisfies $\big\|\int_{S^*} f \,\mathrm{d}E\big\| \leqslant |f|_E$ where

$$|f|_E = \inf\Big\{\sup_{\rho \in A} |\, f(\rho)| : A \in \mathcal{B}(S^*), E(A) = I\Big\}.$$

The operator $\int_{S^*} f \,\mathrm{d}E$ is normal with resolution of the identity $P_f : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(H)$ given by $P_f(A) = E(f^{-1}(A))$. For these facts see [8], Chapter XVII and [17], Chapter 12.

More generally, if $f : S^* \to \mathbb{C}$ is merely Borel measurable, then "$\int_{S^*} f \,\mathrm{d}E$" still exists, but now as an unbounded operator. Its domain $D(E, f)$ is given by

$$D(E, f) = \Big\{x \in H : \lim_{n\to\infty} \Big(\int_{S^*} f\chi_f^{(n)} \,\mathrm{d}E\Big)x \text{ exists in } H\Big\},$$

where $\chi_f^{(n)}$ is the characteristic function of $|f|^{-1}([0, n])$, for $n \in \mathbb{N}$, and the operator $\int_{S^*} f \,\mathrm{d}E$ is defined by $x \mapsto \lim_{n\to\infty} \Big(\int_{S^*} f\chi_f^{(n)} \,\mathrm{d}E\Big)x$, for $x \in D(E, f)$. The existence of $\lim_{n\to\infty} \Big(\int_{S^*} f\chi_f^{(n)} \,\mathrm{d}E\Big)x$ is *equivalent* to f being integrable with respect to the H-valued vector measure Ex in the sense of [12], i.e. $f \in L^1(E_{x,y})$, for all $y \in H$, and for each $A \in \mathcal{B}(S^*)$ there is a vector in H, denoted by $\int_A f \,\mathrm{d}(Ex)$, satisfying $\langle \int_A f \,\mathrm{d}(Ex), y\rangle = \int_A f \,\mathrm{d}E_{x,y}$, for $y \in H$. The unbounded operator $\int_{S^*} f \,\mathrm{d}E$ so defined is closed, densely defined and satisfies $\Big(\int_{S^*} f \,\mathrm{d}E\Big)x = \int_{S^*} f \,\mathrm{d}(Ex)$, for $x \in D(E, f)$. For these facts we refer to [7], Section 1; [8], Chapter XVIII. Moreover, $\int_{S^*} f \,\mathrm{d}E$ is *normal* and its adjoint is $\int_{S^*} \overline{f} \,\mathrm{d}E$. This follows from standard results in [3], Chapter 5; [8], p. 2268; [10], Section 22.2, Chapter XXII or [17], Chapter 13, once we verify that

$$D(E, f) = \Big\{x \in H : \int_{S^*} |f|^2 \,\mathrm{d}E_{x,x} < \infty\Big\}. \tag{2.1}$$

To see this, let $x \in H$ satisfy $\int_{S^*} |f|^2 \,\mathrm{d}E_{x,x} < \infty$. Then $f \in L^1(E_{x,y})$, for $y \in H$, since $\int_{S^*} |f| \,\mathrm{d}|E_{x,y}| \leqslant \|y\| \Big(\int_{S^*} |f|^2 \,\mathrm{d}E_{x,x} \Big)^{1/2} < \infty$ ([17], Lemma 13.23). Since Hilbert spaces have the BP-property, f is Ex-integrable ([12], p. 31), i.e. $x \in D(E, f)$. Conversely, if $x \in D(E, f)$, then f is Ex-integrable. Let $F_n = \{\rho \in S^* : |f(\rho)|^2 \leqslant n\}$, for $n \in \mathbb{N}$. Then both $f\chi_{F_n}$ and $\overline{f}\chi_{F_n}$, $n \in \mathbb{N}$, are bounded Borel functions and so

$$\int_{F_n} |f|^2 \,\mathrm{d}E_{x,x} = \Big\langle \Big(\int_{S^*} \overline{f}\chi_{F_n} \cdot f\chi_{F_n} \,\mathrm{d}E \Big) x, x \Big\rangle = \Big\langle \Big(\int_{S^*} \overline{f}\chi_{F_n} \,\mathrm{d}E \Big) \cdot \Big(\int_{S^*} f\chi_{F_n} \,\mathrm{d}E \Big) x, x \Big\rangle,$$

where the last equality uses the multiplicativity of $\varphi \mapsto \int_{S^*} \varphi \,\mathrm{d}E$ on the space of all bounded Borel functions φ. Since $\Big(\int_{S^*} \overline{f}\chi_{F_n} \,\mathrm{d}E \Big)^* = \int_{S^*} f\chi_{F_n} \,\mathrm{d}E$ ([17], Theorem 12.21), it follows that $\int_{F_n} |f|^2 \,\mathrm{d}E_{x,x} = \Big\| \Big(\int_{S^*} f\chi_{F_n} \,\mathrm{d}E \Big) x \Big\|^2$, for $n \in \mathbb{N}$. But, $\{f\chi_{F_n}\}_{n=1}^{\infty}$ converges pointwise to f on S^* and $|f\chi_{F_n}| \leqslant |f|$, $n \in \mathbb{N}$, with $|f|$ an Ex-integrable function ([12], p. 27). By the dominated convergence theorem for vector measures ([12], p. 30),

$$\lim_{n\to\infty} \Big(\int_{S^*} f\chi_{F_n} \,\mathrm{d}E \Big) x = \lim_{n\to\infty} \int_{S^*} f\chi_{F_n} \,\mathrm{d}(Ex) = \int_{S^*} f \,\mathrm{d}(Ex).$$

Hence, $\int_{S^*} |f|^2 \,\mathrm{d}E_{x,x} = \sup_{n\in\mathbb{N}} \int_{F_n} |f|^2 \,\mathrm{d}E_{x,x} = \sup_{n\in\mathbb{N}} \Big\| \Big(\int_{S^*} f\chi_{F_n} \,\mathrm{d}E \Big) x \Big\|^2 < \infty$ showing that x belongs to the right-hand-side of (2.1). So, (2.1) is indeed valid.

For each $s \in S$ the function $\widehat{s} : S^* \to \mathbb{C}$ is continuous and hence, Borel measurable. Denoting $D(E, \widehat{s})$ simply by $D(\widehat{E}(s))$ and $\int_{S^*} \widehat{s} \,\mathrm{d}E$ simply by $\widehat{E}(s)$ we have the *canonical representation* $\widehat{E} : S \to \mathcal{N}(H)$ given by

$$\widehat{E}(s)x = \int_{S^*} \widehat{s} \,\mathrm{d}Ex, \quad s \in S,\ x \in D(\widehat{E}(s)); \tag{2.2}$$

moreover $\widehat{E}$ is called the *generalized Laplace transform* of E (cf. Remark 2.3 below). We now summarize the basic properties of $\{\widehat{E}(s)\}_{s\in S}$. If E is also a Radon measure it follows that $\widehat{E} : S \to \mathcal{N}(H)$ as defined by (2.2) is a $*$-representation as in Definition 1.1. So, the requirements of Definition 1.1 are quite natural since they are *necessarily* satisfied by the canonical representation corresponding to any given selfadjoint, Radon spectral measure.

Proposition 2.1. *Let S be a commutative, unital semigroup with an involution. Let $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$ be a selfadjoint spectral measure and let $\widehat{E} : S \to \mathcal{N}(H)$ be its associated canonical representation as given by* (2.2).

(i) $E(A)D(\widehat{E}(s)) \subseteq D(\widehat{E}(s))$, $A \in \mathcal{B}(S^*)$, $s \in S$.

(ii) $E(A)\widehat{E}(s) \subseteq \widehat{E}(s)E(A)$, $A \in \mathcal{B}(S^*)$, $s \in S$.

(iii) *If* $x \in D(\widehat{E}(s))$ *and* $y = \widehat{E}(s)x$, *then* $\mathrm{d}E_{y,y} = |\widehat{s}|^2 \,\mathrm{d}E_{x,x}$.

(iv) $D(\widehat{E}(s)^*) = D(\widehat{E}(s^-) = D(\widehat{E}(s))$ *and* $\widehat{E}(s)^* = \widehat{E}(s^-)$, *for each* $s \in S$.

(v) $D(\widehat{E}(t)\widehat{E}(s)) = D(\widehat{E}(st)) \cap D(\widehat{E}(s))$ *and* $\widehat{E}(t)\widehat{E}(s) \subseteq \widehat{E}(ts)$, *for all* $s, t \in S$.

(vi) $\overline{\widehat{E}(t)\widehat{E}(s)} = \widehat{E}(ts)$, $t, s \in S$.

(vii) $\widehat{E}(e) = I$.

If, in addition, E is a Radon measure, then

(viii) $D_{\mathrm{c}} = \bigcup_{\alpha \in \mathcal{A}(s)} D_\alpha$ *is dense in H and so, in particular, $\widehat{E}$ is a $*$-representation.*

(ix) $D_{\mathrm{c}} = \{x \in H : \operatorname{supp}(Ex)$ *is compact*$\}$.

Before proving Proposition 2.1 we require some preliminaries. A function $\varphi : S \to \mathbb{C}$ is *positive definite* if $\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(s_j s_k^-) \geqslant 0$ for all choices of $n \in \mathbb{N}$, $\{s_1, \ldots, s_n\} \subseteq S$ and $\{c_1, \ldots, c_n\} \subseteq \mathbb{C}$. In particular, every character $\rho \in S^*$ is positive definite. Let $\alpha \in \mathcal{A}(S)$. A function $f : S \to \mathbb{C}$ is *α-bounded* if there exists $C > 0$ such that $|f(s)| \leqslant C\alpha(s)$ for $s \in S$. If f is positive definite and α-bounded, then $C = \varphi(e)$. A character $\rho \in S^*$ is α-bounded iff $|\rho| \leqslant \alpha$. Hence, the set S^α of all α-bounded characters is a compact subset of S^*. For all these notions we refer to [2], Chapter 4. The space of all non-negative Radon measures on S^* is denoted by $M_+(S^*)$. Given $\alpha \in \mathcal{A}(S)$ the subspace of $M_+(S^*)$ consisting of all Radon measures supported in $S^\alpha \subseteq S^*$ is denoted by $M_+(S^\alpha)$. The following result is the Berg-Maserick theorem ([1], Theorem 2.1).

Proposition 2.2. *Let $\alpha \in \mathcal{A}(S)$ and $\varphi : S \to \mathbb{C}$ be an α-bounded, positive definite function. Then there exists a compactly supported Radon measure $\mu \in M_+(S^\alpha)$ such that $\varphi(s) = \int_{S^*} \widehat{s}\,\mathrm{d}\mu$, for $s \in S$, and μ is unique within $M_+(S^*)$.*

Remark 2.3. The map $\widehat{\mu} : s \mapsto \int_{S^*} \widehat{s}\,\mathrm{d}\mu$ is the *generalized Laplace transform* of μ. If $M^{\mathrm{c}}(S^*)$ the space of all complex Radon measures on S^* having compact support, then $\mu \mapsto \widehat{\mu}$ is injective on $M^{\mathrm{c}}(S^*)$ ([2], p. 96).

Proof of Proposition 2.1. Parts (i)–(iii) follow from a combination of Theorem 13.24 and the Remark on p. 345 of [17] together with the identity

$$D(\widehat{E}(s)E(A)) = \left\{x \in H : \int_A |\widehat{s}|^2 \,\mathrm{d}E_{x,x} < \infty\right\}, \quad A \in \mathcal{B}(S^*),\ s \in S,$$

which follows from (2.1) with $\widehat{s}$ in place of f. Property (iv) is immediate from [17], Theorem 13.24 (c). Theorem 13.24 (b) of [17] yields (v), and (vi) is immediate from

[3], p. 136, Theorem 7. Property (vii) is a simple consequence of the definition of $\widehat{E}(e)$ after noting that $\widehat{e}$ is the (bounded) function $\mathbb{1}$ constantly equal to 1 on S^*.

Suppose now, in addition, that E is Radon. Fix $s \in S$. Given a compact set $K \subseteq S^*$ let $z \in E(K)H$, i.e. $z = E(K)x$ for some $x \in H$. It is routine to check that $\int_{S^*} |\widehat{s}|^2 \,\mathrm{d}E_{z,z} = \int_K |\widehat{s}|^2 \,\mathrm{d}E_{z,z} = \int_K |\widehat{s}|^2 \,\mathrm{d}E_{x,x}$, which is finite as K is compact, $\widehat{s}$ is continuous and $E_{z,z}$ is a finite measure. This shows $z \in D(\widehat{E}(s))$ and hence, that $E(K)H \subseteq D(\widehat{E}(s))$. Since K and s are arbitrary we have shown $\bigcup_{K\in\mathcal{K}(S^*)} E(K)H \subseteq \bigcap_{s\in S} D(\widehat{E}(s))$, where $\mathcal{K}(S^*)$ is the family of all compact subsets of S^*. For $K \in \mathcal{K}(S^*)$ define $\alpha_K : S \to [0,\infty)$ by

$$\alpha_K(s) := \sup\{|\widehat{s}(\rho)| : \rho \in K\}, \quad s \in S. \tag{2.3}$$

Then $\alpha_K \in \mathcal{A}(S)$. If $x \in H$, it was noted above that $z = E(K)x \in \bigcap_{s\in S} D(\widehat{E}(s))$. From (iii) with x and A replaced by z and S^*, respectively, we have $\|\widehat{E}(s)z\|^2 = \int_{S^*} |\widehat{s}|^2 \,\mathrm{d}E_{z,z} = \int_K |\widehat{s}|^2 \,\mathrm{d}E_{z,z}$, for $s \in S$. But, for $s \in S$,

$$\int_K |\widehat{s}|^2 \,\mathrm{d}E_{z,z} \leqslant \int_K (\alpha_K(s))^2 \,\mathrm{d}E_{z,z} = (\alpha_K(s))^2 \int_K \mathbb{1} \,\mathrm{d}E_{z,z} \leqslant (\alpha_K(s))^2 \|z\|^2.$$

Accordingly, $\|\widehat{E}(s)z\| \leqslant \alpha_K(s)\|z\|$ for $s \in S$. It follows from (1.2) that $z \in D_{\alpha_K}$. This establishes that $\bigcup_{K\in\mathcal{K}(S^*)} E(K)H \subseteq \bigcup_{K\in\mathcal{K}(S^*)} D_{\alpha_K} \subseteq D_{\mathrm{c}}$. So, to prove (viii) it suffices to verify that $\bigcup_{K\in\mathcal{K}(S^*)} E(K)H$ is dense. Fix $x \in H$. Since $E_{x,x}$ is Radon and the net of compact subsets of S^* (directed by inclusion) is upwards filtering to S^*, it follows that $\lim_{K\in\mathcal{K}(S^*)} E_{x,x}(K) = E_{x,x}(S^*) = \|x\|^2$. Then the equalities

$$\begin{aligned}\|x - E(K)x\| &= \|E(K^{\mathrm{c}})x\| = [E_{x,x}(K^{\mathrm{c}})]^{1/2} = [E_{x,x}(S^*) - E_{x,x}(K)]^{1/2}\\ &= [\|x\|^2 - E_{x,x}(K)]^{1/2},\end{aligned}$$

valid for each $K \in \mathcal{K}(S^*)$, show that $\lim_{K\in\mathcal{K}(S^*)} E(K)x = x$ in the norm of H. In particular, x belongs to the closure of $\bigcup_{K\in\mathcal{K}(S^*)} E(K)H$ and (viii) is verified.

(ix) Suppose $x \in H$ satisfies $K = \mathrm{supp}(Ex) = \mathrm{supp}(E_{x,x})$ is compact. Since $\widehat{s}$ is continuous, $\int_{S^*} |\widehat{s}|^2 \,\mathrm{d}E_{x,x} = \int_K |\widehat{s}|^2 \,\mathrm{d}E_{x,x} < \infty$ and so $x \in D(\widehat{E}(s))$, for $s \in S$. By [17], Theorem 13.24 (a) we have $\langle \widehat{E}(s)x, x\rangle = \int_{S^*} \widehat{s} \,\mathrm{d}E_{x,x} = \int_K \widehat{s} \,\mathrm{d}E_{x,x}$, for $s \in S$. If α_K is as in (2.3), then it follows that

$$|\langle \widehat{E}(s)x, x\rangle| \leqslant \int_K |\widehat{s}| \,\mathrm{d}E_{x,x} \leqslant \alpha_K(s) \int_K \mathbb{1} \,\mathrm{d}E_{x,x} \leqslant \alpha_K(s)\|x\|^2, \quad s \in S,$$

and so Proposition 3.2 (iv) of Section 3 implies that $x \in D_{\alpha_K} \subseteq D_{\mathrm{c}}$.

Conversely, suppose $x \in D_\alpha \subseteq \bigcap_{s\in S} D(\widehat{E}(s))$ for some $\alpha \in \mathcal{A}(S)$. By the Cauchy-Schwarz inequality $|\langle \widehat{E}(s)x, x\rangle| \leqslant \alpha(s)\|x\|^2$, for $s \in S$, which shows that $\varphi_x : s \mapsto \langle \widehat{E}(s)x, x\rangle$ is α-bounded on S. Moreover, φ_x is also positive definite since, for any finite subsets $\{c_1, \ldots, c_n\} \in \mathbb{C}$ and $\{s_1, \ldots, s_n\} \subseteq S$ we have (by (iv) and (v)) that

$$\sum_{j,k} c_j\bar{c}_k\varphi_x(s_k^- s_j) = \sum_{j,k} c_j\bar{c}_k\langle \widehat{E}(s_k^- s_j)x, x\rangle = \sum_{j,k} c_j\bar{c}_k\langle \widehat{E}(s_k^-)\widehat{E}(s_j)x, x\rangle$$

$$= \sum_{j,k} c_j\bar{c}_k\langle \widehat{E}(s_j)x, \widehat{E}(s_k)x\rangle = \left\|\sum_{j=1}^{n} c_j\widehat{E}(s_j)x\right\|^2 \geqslant 0.$$

By the Berg-Maserick theorem there is $\mu_x \in M_+(S^\alpha)$ such that $\langle \widehat{E}(s)x, x\rangle = \int_{S^*} \widehat{s}\,\mathrm{d}\mu_x = \int_{S^\alpha} \widehat{s}\,\mathrm{d}\mu_x$, for $s \in S$, and μ_x is unique in $M_+(S^*)$. But, also $\langle \widehat{E}(s)x, x\rangle = \int_{S^*} \widehat{s}\,\mathrm{d}E_{x,x}$ ([17], Theorem 13.24 (a)), with $E_{x,x} \in M_+(S^*)$ and so, by uniqueness, we conclude $E_{x,x} = \mu_x$. In particular, $\operatorname{supp}(Ex) = \operatorname{supp}(\mu_x)$ is compact. This completes the proof of Proposition 2.1. ∎

Remark 2.4. If $\sigma(\widehat{E}(s))$ is the spectrum of $\widehat{E}(s)$ ([17], p. 346), then $\sigma(\widehat{E}(s))$ coincides with the *E-essential range* of $\widehat{s}$ ([17], Theorem 13.27 (c)); see [17], p. 303 for the definition of E-essential range. Since $\widehat{s} : S^* \to \mathbb{C}$ is continuous, it follows that the E-essential range of $\widehat{s}$ coincides with $\overline{\widehat{s}(\operatorname{supp}(E))}$; the bar denotes closure in $\mathbb{C}$ and $\operatorname{supp}(E) := \overline{\bigcup_{x\in H} \operatorname{supp}(E_{x,x})}$. Accordingly, the canonical $*$-representation $\widehat{E}$ associated to E has the property, that $\sigma(\widehat{E}(s)) = \overline{\widehat{s}(\operatorname{supp}(E))}$, for $s \in S$.

We now show property (v) in Definition 1.1 does *not* follow from (i)–(iv) in general.

Example 2.5. Let S be the set of all $\mathbb{C}$-valued, Borel measurable functions on [0,1]. Under pointwise multiplication and pointwise (complex) conjugation of functions, S is a commutative, unital (with $e = \mathbb{1}$) semigroup with involution. Let $H = L^2([0,1])$. For $\varphi \in S$ define the operator U_φ on $D(U_\varphi) = \{h \in H : \varphi h \in H\}$ by $U_\varphi : h \mapsto \varphi h$. It is routine to verify the map $\mathcal{U} : \varphi \mapsto U_\varphi$, for $\varphi \in S$, has range in $\mathcal{N}(H)$ and satisfies (i)–(iv) of Definition 1.1. However, it fails (v) since $D_\mathrm{c} = \{0\}$. To see this it suffices to show $\bigcap_{\varphi\in S} D(U_\varphi) = \{0\}$; see (1.2). So, let $g \in \bigcap_{\varphi\in S} D(U_\varphi)$ and suppose $g \neq 0$. Since also $|g| \in \bigcap_{\varphi\in S} D(U_\varphi)$, as $h \in D(U_\varphi)$ iff $|h| \in D(U_\varphi)$, for all $\varphi \in S$, we may assume $g \geqslant 0$. Hence, $A = g^{-1}((0,\infty))$ has positive measure. Since Lebesgue measure λ on $\mathcal{B}([0,1])$ is non-atomic there are pairwise disjoint Borel sets $A_n \subseteq A$, each with $\lambda(A_n) > 0$. Then $\varphi_0 = (1/g)\sum_{n=1}^{\infty}(\lambda(A_n))^{-1/2}\chi_{A_n}$ belongs to S and $\{\varphi_0 > 0\} \subseteq A$. A direct calculation shows $g\varphi_0 \notin H$, that is, $g \notin D(U_{\varphi_0})$ which is a contradiction. Hence, $\{U_s\}_{s\in S}$ fails to be a $*$-representation because it does not satisfy (v) of Definition 1.1. ∎

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the additive semigroup of non-negative integers. As involution on $\mathbb{N}_0$ we take the identity function. Let $\mathbb{N}_0 \times \mathbb{N}_0$ be the commutative, unital (with $e = (0,0)$) semigroup whose semigroup operation is given by $(m,n) + (u,v) = (m+u, n+v)$ and whose involution is given by $(m,n)^- = (n,m)$.

Lemma 2.6. (i) *If* $\alpha \in \mathcal{A}(\mathbb{N}_0)$, *there is* $\lambda > 0$ *with* $\alpha(n) \leqslant \lambda^n$, *for* $n \in \mathbb{N}_0$.
(ii) *If* $\alpha \in \mathcal{A}(\mathbb{N}_0 \times \mathbb{N}_0)$, *there is* $\lambda > 0$ *with* $\alpha(m,n) \leqslant \lambda^{m+n}$, *for* $m, n \in \mathbb{N}_0$.

Proof. (i) is obvious, with $\lambda := \alpha(1)$. For (ii) put $\lambda := \max\{\alpha(1,0), \alpha(0,1)\}$. Then, for $m, n \in \mathbb{N}_0$, we have $\alpha(m,n) \leqslant \alpha(m,0)\alpha(0,n) \leqslant \alpha(1,0)^m \alpha(0,1)^n \leqslant \lambda^{m+n}$. ∎

For the rest of this section we discuss Theorem 1.2 in relation to some related results in the literature. Suppose S is a locally compact full semigroup (see [11], [15]) and assume that S is also commutative and unital; this is not a requirement in [11], [15]. In [15], A.E. Nussbaum considers representations $\mathcal{U} : S \to \mathcal{N}(H)$ which take their values in the *selfadjoint* operators and satisfy:

(N1) $U_{st} \subseteq U_s U_t$, $s, t \in S$.
(N2) $s \mapsto \langle U_s x, y\rangle$ is continuous for each $x \in \bigcap_{s\in S} D(U_s)$ and $y \in H$.
(N3) There is a countable set $S' \subseteq S$ with $S' \cap \{ts : t \in S\} \neq \emptyset$, for $s \in S$.

Since each U_s is selfadjoint we may take the identity function on S as involution, in which case $U_{s^-} = U_s^*$, for $s \in S$. Condition (N3) implies $\bigcap_{s\in S} D(U_s)$ is dense ([15], Theorem 5), and (N1) implies that $U_s U_t = U_{st}$ for $s, t \in S$ ([15], p. 134). A real character of S is a *continuous* homomorphism of S into $\mathbb{R}$ and $\widehat{S}$ is the space of all real characters with the topology of uniform convergence on compact subsets of S ([15], pp. 136–137). The main result in [15], Theorem 6 states if $\mathcal{U} : S \to \mathcal{N}(H)$ takes its values in the selfadjoint operators and satisfies (N1)–(N3), then there exists a spectral measure $E : \mathcal{B}(\widehat{S}) \to \mathcal{L}(H)$ such that $U_s x = \int_{\widehat{S}} \widehat{s}\,\mathrm{d}Ex$, for $s \in S$ and $x \in D(U_s)$.

In [11], C. Ionescu Tulcea considers semigroup representations $\mathcal{U} : S \to \mathcal{N}(H)$, with S a locally compact full semigroup, which satisfy the following conditions:

(IT1) $U_{st} \subseteq U_s U_t$, $s, t \in S$.
(IT2) U_s, U_t commute, for $s, t \in S$, i.e. their resolutions of the identity commute.
(IT3) $s \mapsto \langle U_s x, y\rangle$ is continuous for each $x \in \bigcap_{s\in S} D(U_s)$ and $y \in H$.
(IT4) $\bigcap_{s\in S} D(U_s) = \bigcap_{s\in S'} D(U_s)$ for some countable set $S' \subseteq S$.

It is known that (IT2) and (N3) together imply (IT4) ([11], p. 106). Any *continuous* homomorphism $\rho \not\equiv 0$ of S into $\mathbb{C}$ is called a character ([11], p. 97), and if S happens to have a *continuous* involution (which we assume henceforth), then ρ is also required to satisfy $\rho(s^-) = \overline{\rho(s)}$ for all $s \in S$ ([11], p. 98). Let $\widetilde{S}$ be the space of all characters, equipped with a suitable topology ([11], Section 2), and $\beta(\widetilde{S})$ be the Stone-Čech compactification of $\widetilde{S}$. The main result in [11], Theorem 3 states if $\mathcal{U} : S \to \mathcal{N}(H)$ satisfies (IT1)–(IT4), then there is a unique selfadjoint, Radon spectral measure $E : \mathcal{B}(\beta(\widetilde{S})) \to \mathcal{L}(H)$ which is concentrated on $\widetilde{S}$ (see [11], p. 97) such that $U_s x = \int_{\widetilde{S}} \widehat{s}\, \mathrm{d}Ex$, for $x \in D(U_s)$ and $s \in S$. A similar result occurs in [14], Theorem 1, where S is only required to be a locally compact space (i.e. not necessarily a semigroup).

The hypotheses of Definition 1.1 are different to those of Nussbaum and Ionescu Tulcea. To begin with, S need not have a topology in our setting and so (N2) and (IT3) are not available. Moreover, the arguments in [11] and [15] rely on the topology of S, the continuity requirements (N2) and (IT3) (and of the involution in the case of [11]) and the representation techniques of abelian C^*-algebras of operators via their structure space. It is clear from Definition 1.1 and our assumptions on S that such methods are not available in our setting; they will be replaced by techniques from the theory of positive definite functions. Moreover, (iii) and (iv) of Definition 1.1, which are analogues of (N1)/(IT1), are typically weaker than (N1)/(IT1). Indeed, as seen by the next example, some rather simple semigroups of normal operators (which satisfy Definition 1.1) are excluded in [11] and [15] as they do not satisfy (N1)/(IT1).

Example 2.7. Let $S = \mathbb{R}$ with addition as the semigroup operation and the identity function as involution. Then S is a locally compact full semigroup with involution. Let $H = L^2([0,\infty))$. For $s \in \mathbb{R}$ let $\rho_s(t) = \mathrm{e}^{-st}$, for $t \in [0,\infty)$. Define U_s to be the operator with domain $D(U_s) = \{h \in H : \rho_s h \in H\}$ given by $U_s : h \mapsto \rho_s h$ for $h \in D(U_s)$. Then $D(U_s) = H$ if $s \geqslant 0$; otherwise $D(U_s)$ is a proper dense subspace. For $s \in S$, the operator U_s is selfadjoint. Moreover, $U_{-s}U_s = I$ whereas U_sU_{-s} is the restriction of I to $D(U_{-s})$, for $s > 0$. Of course, $U_{-s+s} = I$ for $s \in S$. So, (N1)/(IT1) is *not* satisfied. However, $\mathcal{U}$ is a $*$-representation as in Definition 1.1; conditions (i)–(iv) are easily checked. To check (v) of Definition 1.1 let $U_s^{(n)}$ be the restriction of U_s to the closed, U_s-invariant subspace $H^{(n)} = \{h \in H : \chi_{(n,\infty)} h = 0 \text{ a.e.}\}$, for $n \in \mathbb{N}$. Define $\alpha_n : S \to [0,\infty)$ by $\alpha_n(s) = \|U_s^{(n)}\|_{\mathcal{L}(H^{(n)})}$, for $s \in S$, after noting $U_s^{(n)} \in \mathcal{L}(H^{(n)})$. Then $\alpha_n \in \mathcal{A}(S)$ for $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. If $h \in H^{(n)}$, then $\|U_s h\| = \|U_s^{(n)} h\| \leqslant \|U_s^{(n)}\|_{\mathcal{L}(H^{(n)})}\|h\| = \alpha_n(s)\|h\|$, for $s \in S$, which shows $h \in D_{\alpha_n}$, i.e. $H^{(n)} \subseteq D_{\alpha_n}$. Since $\bigcup_{n=1}^{\infty} H^{(n)}$ is dense and $\bigcup_{n=1}^{\infty} H^{(n)} \subseteq \bigcup_{n=1}^{\infty} D_{\alpha_n} \subseteq D_{\mathrm{c}}$ we have established (v). ∎

The following $*$-representation (in our sense) fails both (N3) and (IT4).

Example 2.8. Let $S = \mathbb{C}^{\mathbb{N}_0}$ be the semigroup of all functions $\varphi : \mathbb{N}_0 \to \mathbb{C}$ with multiplication of functions (defined pointwise) as the semigroup operation, in which case $e = \mathbb{1}$ is the unit, and with complex conjugation $\varphi \mapsto \overline{\varphi}$ (defined pointwise) as involution. Let $H = \ell^2(\mathbb{N}_0)$. For $\varphi \in S$, define U_φ to be the operator with domain $D(U_\varphi) = \{\xi \in H : \varphi\xi \in H\}$ given by $U_\varphi : \xi \mapsto \varphi\xi$ for $\xi \in D(U_\varphi)$. Then $\mathcal{U} : S \to \mathcal{N}(H)$ defined by $\varphi \mapsto U_\varphi$ has properties (i)–(iv) of Definition 1.1 and

$$\bigcap_{\varphi \in S} D(U_\varphi) = \{\xi \in H : \mathrm{supp}(\xi) \text{ is a finite subset of } \mathbb{N}_0\}\,. \tag{2.4}$$

To verify (v) of Definition 1.1 define, for $n \in \mathbb{N}_0$, an element $\alpha_n \in \mathcal{A}(S)$ by $\alpha_n(\varphi) = \max\{|\varphi(k)| : 0 \leqslant k \leqslant n\}$, for $\varphi \in S$. Fix $n \in \mathbb{N}_0$ and let $H^{(n)}$ be the closed, U_φ-invariant subspace of H consisting of all $\xi \in H$ with $\operatorname{supp}(\xi) \subseteq \{0, 1, \dots, n\}$. Then, for $\xi \in H^{(n)}$, we have $\|U_\varphi \xi\|^2 = \|\varphi\xi\|^2 = \sum_{j=0}^{n} |\varphi(j)|^2 |\xi(j)|^2 \leqslant (\alpha_n(\varphi))^2 \|\xi\|^2$, for $\varphi \in S$, which shows $\xi \in D_{\alpha_n}$, i.e. $H^{(n)} \subseteq D_{\alpha_n}$. Since $\bigcup_{n=0}^{\infty} H^{(n)}$ is dense and $\bigcup_{n=0}^{\infty} H^{(n)} \subseteq \bigcup_{n=0}^{\infty} D_{\alpha_n} \subseteq D_{\mathrm{c}}$ we have established (v). So, $\mathcal{U} : S \to \mathcal{N}(H)$ is a $*$-representation.

It is routine to verify (IT2). We noted above that (IT2) and (N3) together imply (IT4). So, if we show (IT4) fails, then also (N3) fails. Proceeding by contradiction, let $D = \bigcap_{\varphi \in S} D(U_\varphi)$ and suppose there is a sequence $\{\varphi_n\}_{n=1}^{\infty}$ with $D = \bigcap_{n=1}^{\infty} D(U_{\varphi_n})$. Since $D(U_{\varphi_n}) = D(U_{|\varphi_n|})$ we may suppose $\varphi_n \geqslant 0$, for $n \in \mathbb{N}$. Also, if $0 \leqslant \varphi \leqslant \psi$, then $D(U_\psi) \subseteq D(U_\varphi)$. Accordingly, $\psi_n = \sum_{j=1}^{n} \varphi_j$ satisfies $D(U_{\psi_n}) \subseteq D(U_{\varphi_n})$, for $n \in \mathbb{N}$, and so $D \subseteq \bigcap_{n=1}^{\infty} D(U_{\psi_n}) \subseteq \bigcap_{n=1}^{\infty} D(U_{\varphi_n}) = D$. Hence, we may suppose $0 \leqslant \varphi_1 \leqslant \varphi_2 \leqslant \cdots$. Also, $\varphi_n \leqslant (\mathbb{1} \vee \varphi_n)$ and so $D \subseteq D(U_{\mathbb{1} \vee \varphi_n}) \subseteq D(U_{\varphi_n})$ for $n \in \mathbb{N}$ which yields $D \subseteq \bigcap_{n=1}^{\infty} D(U_{\mathbb{1} \vee \varphi_n}) \subseteq \bigcap_{n=1}^{\infty} D(U_{\varphi_n}) = D$. So, we may suppose $\mathbb{1} \leqslant \varphi_1 \leqslant \varphi_2 \leqslant \cdots$ and $D = \bigcap_{n=1}^{\infty} D(U_{\varphi_n})$. Define $\xi : \mathbb{N}_0 \to \mathbb{C}$ by $\xi(n) = 1/((n+1)\varphi_{n+1}(n))$, for $n \in \mathbb{N}_0$, in which case $\sum_{n=0}^{\infty} |\xi(n)|^2 \leqslant \sum_{n=1}^{\infty} n^{-2} < \infty$. Hence, $\xi \in H$. Also, $(\varphi_n \xi)(k) = \varphi_n(k)/((k+1)\varphi_{k+1}(k))$ for $k \in \mathbb{N}_0$ and $(\varphi_n \xi)(k) \leqslant (k+1)^{-1}$ whenever $k \geqslant (n-1)$. Accordingly, $\varphi_n \xi \in H$, for $n \in \mathbb{N}$. That is, $\xi \in \bigcap_{n=1}^{\infty} D(U_{\varphi_n}) = D$ which contradicts (2.4) as ξ does not have finite support. Hence, no such sequence $\{\varphi_n\}_{n=1}^{\infty} \subseteq S$ can exist and so (IT4) is *not* satisfied. ∎

Remark 2.9. (a) In the notation of Example 2.8 define $\varphi, \psi \in S$ by $\varphi(n) = n$, for $n \in \mathbb{N}_0$, and $\psi = \chi_F$ where $F = \{0, 2, 4, \dots\}$. Define $\xi \in H$ by $\xi(0) = 0$ and $\xi(n) = n^{-1} \chi_{F^c}(n)$ for $n \geqslant 1$. Then $\xi \in D(U_\varphi U_\psi)$, but $\xi \notin D(U_\psi U_\varphi)$. This shows (iii) and (iv) of Definition 1.1 need *not* imply $U_t U_s = U_s U_t$ for $s, t \in S$, but only that $\overline{U_t U_s} = \overline{U_s U_t}$.

(b) It was noted earlier that (N3) necessarily implies the density of $\bigcap_{s \in S} D(U_s)$ in H. Example 2.8 shows that $\bigcap_{s \in S} D(U_s)$ can be dense without (N3) being satisfied.

(c) If we take S to be $\mathbb{R}^{\mathbb{N}_0}$ rather than $\mathbb{C}^{\mathbb{N}_0}$ in Example 2.8, then the $*$-representation $\{U_s\}_{s \in S}$ consists of *selfadjoint* operators. In this setting (a) above implies that (N1) cannot hold since, as noted earlier, (N1) always implies that $U_s U_t = U_t U_s$ for all $s, t \in S$.

(d) In Example 2.8 it is clear from (2.4) and the definition of D_{α_n}, for $n \in \mathbb{N}$, that $D_{\mathrm{c}} = \bigcap_{\varphi\in S} D(U_\varphi)$. This is special to this example. To see this, let $S = \mathbb{N}_0$ and $H = L^2([0,\infty))$. Let T be the selfadjoint operator with $D(T) = \{h \in H : \varphi h \in H\}$ given by $T : h \mapsto \varphi h$, for $h \in D(T)$, where φ is the identity function on $[0,\infty)$. Then $\mathcal{U} : S \to \mathcal{N}(H)$ given by $U_n = T^n$, for $n \in S$, has properties (i)–(iv) of Definition 1.1. For each $\lambda > 0$ let $\alpha_\lambda(n) = \lambda^n$, for $n \in S$, in which case $\alpha_\lambda \in \mathcal{A}(S)$. Lemma 2.6 (i) implies $D_{\mathrm{c}} = \bigcup_{\lambda>0} D_{\alpha_\lambda}$. For $\lambda > 0$, let $H_\lambda = \big\{h \in H : \chi_{(\lambda,\infty)} h = 0 \text{ a.e.}\big\}$. Then H_λ is a closed subspace contained in $\bigcap_{n\in S} D(U_n)$ and, if $h \in H_\lambda$, then $\|U_n h\| = \|\varphi^n h\| \leqslant \lambda^n\|h\|$, for $s \in S$. This shows $H_\lambda \subseteq D_{\alpha_\lambda}$ and, since $\bigcup_{\lambda>0} H_\lambda$ is dense, it follows that D_{c} is dense. So, $\mathcal{U}$ is a $*$-representation. To show the inclusion $D_{\mathrm{c}} \subseteq \bigcap_{n\in S} D(U_n)$ is strict consider $h(t) = \mathrm{e}^{-t/2}$, for $t \in [0,\infty)$. Then $\|h\| = 1$ and $h \in \bigcap_{n\in S} D(U_n)$. Fix $\lambda > 0$ and suppose $h \in D_{\alpha_\lambda}$. Then $\|U_n h\| = \|\varphi^n h\| \leqslant \alpha_\lambda(n)\|h\| = \lambda^n$, for $n \in \mathbb{N}_0$. But, $\|\varphi^n h\| = [(2n)!]^{1/2}$ and it follows that $(2n)! \leqslant \lambda^{2n}$, for $n \in \mathbb{N}_0$, which is nonsense. Hence, no such $\lambda > 0$ exists and so $h \notin D_{\mathrm{c}}$.

3. PROOF OF THEOREM 1.2

The main aim of this section is to establish Theorem 1.2. This will be achieved via a series of lemmata. The following fact is straight-forward to verify.

Lemma 3.1. *Let X be a Hausdorff topological space. Then the space $M(X)$ of all $\mathbb{C}$-valued Radon measures on X is complete with respect to the total variation norm $|||\cdot|||$.*

We begin by recording some basic properties of the collection $\{D_\alpha\}_{\alpha\in\mathcal{A}(S)}$ given via (1.2). At this stage it is not even known that the D_α are vector spaces!

Proposition 3.2. *Let S be a commutative, unital semigroup with involution and $\mathcal{U} : S \to \mathcal{N}(H)$ be a $*$-representation. Denote the subspace $\bigcap_{s\in S} D(U_s)$, of H, briefly by $D_{\mathcal{U}}$.*

(i) *$U_s(D_{\mathcal{U}}) \subseteq D_{\mathcal{U}}$, $s \in S$.*

(ii) *For each $s, t \in S$ and $z \in D_{\mathcal{U}}$ we have*

$$\langle U_{st}z, z\rangle = \langle U_s U_t z, z\rangle = \langle U_t z, U_s^* z\rangle = \langle U_t z, U_{s^-} z\rangle \,. \tag{3.1}$$

(iii) *The function $\varphi_z(s) := \langle U_s z, z\rangle$, $s \in S$, is positive definite, for each $z \in D_{\mathcal{U}}$.*

(iv) *For each $\alpha \in \mathcal{A}(S)$ we have*

$$D_\alpha = \{x \in D_{\mathcal{U}} : |\langle U_s x, x\rangle| \leqslant \alpha(s)\|x\|^2,\ \textit{for all } s \in S\}. \tag{3.2}$$

(v) *D_α is a closed linear subspace of H, for each $\alpha \in \mathcal{A}(S)$.*

(vi) *For each $\alpha \in \mathcal{A}(S)$ and $s \in S$ we have $D_\alpha \subseteq D(U_s)$ and $U_s(D_\alpha) \subseteq D_\alpha$. In particular, $D_{\mathrm{c}} \subseteq D(U_s)$ and $U_s(D_{\mathrm{c}}) \subseteq D_{\mathrm{c}}$, for every $s \in S$.*

Proof. (i) Fix $s \in S$. Given $x \in D_{\mathcal{U}}$ we have to show $y := U_s x \in D_{\mathcal{U}}$, i.e. $y \in D(U_t)$ for $t \in S$. Since $D_{\mathcal{U}} \subseteq D(U_{st}) \cap D(U_s) = D(U_t U_s)$, (see Definition 1.1 (iii)), we have

$$x \in D(U_t U_s) = \{h \in H : h \in D(U_s) \text{ and } U_s h \in D(U_t)\}, \tag{3.3}$$

from which it follows that $y \in D(U_t)$. Since $t \in S$ is arbitrary we conclude that $y \in D_{\mathcal{U}}$.

(ii) Since $D_{\mathcal{U}} \subseteq D(U_{st}) \cap D(U_t) = D(U_s U_t)$ with $U_s U_t \subseteq U_{st}$ (see Definition 1.1 (iii)) we have (as $z \in D_{\mathcal{U}}$) that

$$\langle U_{st} z, z\rangle = \langle U_s U_t z, z\rangle. \tag{3.4}$$

But, also $z \in D_{\mathcal{U}} \subseteq D(U_{s^-}) = D(U_s^*)$ and so, by Definition 1.1 (ii) and the definition of adjoints, we have $\langle U_s w, z\rangle = \langle w, U_{(s^-)} z\rangle$, for $w \in D(U_s)$. By part (i) we see $w = U_t z \in D_{\mathcal{U}} \subseteq D(U_s)$ which, upon substitution into the previous identity, yields $\langle U_s U_t z, z\rangle = \langle U_t z, U_s^* z\rangle = \langle U_t z, U_{(s^-)} z\rangle$. Combining this identity with (3.4) gives (3.1).

(iii) Now that (3.1) is known to be valid we can repeat the calculations (for φ_x) in the proof of Proposition 2.1 (ix) to deduce φ_z is positive definite.

(iv) Fix $\alpha \in \mathcal{A}(S)$. If $x \in D_\alpha$, then it follows from (1.2), via the Cauchy-Schwarz inequality, that $|\langle U_s x, x,\rangle| \leqslant \|U_s x\| \cdot \|x\| \leqslant \alpha(s)\|x\|^2$, for $s \in S$. So, x belongs to the right-hand-side of (3.2). For $x \in D_{\mathcal{U}}$, (3.1) and Definition 1.1 (ii) imply that

$$\langle U_s U_{(s^-)} x, x\rangle = \langle U_{(s^-)} x, U_{(s^-)} x\rangle = \|U_{(s^-)} x\|^2 = \|U_s^* x\|^2 = \|U_s x\|^2, \tag{3.5}$$

for each $s \in S$, where the last equality in (3.5) follows from the fact that U_s is normal ([17], Theorem 13.32). If, in addition, x belongs to the right-hand-side of (3.2), then

$$|\langle U_s U_{(s^-)} x, x\rangle| \leqslant \alpha(ss^-)\|x\|^2 \leqslant \alpha(s)\alpha(s^-)\|x\|^2 = (\alpha(s))^2\|x\|^2, \quad s \in S.$$

Combining this inequality with (3.5) shows that $\|U_s x\| \leqslant \alpha(s)\|x\|$, for $s \in S$, i.e. $x \in D_\alpha$.

(v) Fix $\alpha \in \mathcal{A}(S)$. We first show that D_α is a vector subspace of H. It follows easily from (1.2) that $\lambda x \in D_\alpha$ whenever $\lambda \in \mathbb{C}$ and $x \in D_\alpha$. So, let $x, y \in D_\alpha$. Using both (1.2) and (3.2) it is straightforward to check that

$$\begin{aligned}|\langle U_s(x+y), (x+y)\rangle| &\leqslant |\langle U_s x, x\rangle| + |\langle U_s x, y\rangle| + |\langle U_s y, x\rangle| + |\langle U_s y, y\rangle| \\ &\leqslant \alpha(s)(\|x\| + \|y\|)^2,\end{aligned}$$

for $s \in S$. This shows, with $z = x + y$, that φ_z in part (iii) is α-bounded. By (iii) the function φ_z is also positive definite and hence, $|\varphi_z(s)| \leqslant \varphi_z(e)\alpha(s)$ for $s \in S$ ([2], Chapter 4, Proposition 1.12). But, Definition 1.1 (i) implies $\varphi_z(e) = \|x+y\|^2$ and it follows $|\langle U_s(x+y), (x+y)\rangle| \leqslant \alpha(s)\|x+y\|^2$, for $s \in S$, i.e. $(x+y) \in D_\alpha$. So, D_α is indeed a subspace of H.

To see D_α is closed, let $\{x_n\}_{n=1}^{\infty} \subseteq D_\alpha$ converge to some $x \in H$. Since D_α is a subspace the differences $(x_n - x_m) \in D_\alpha$, for all $m, n \in \mathbb{N}$, and so (1.2) implies

$$\|U_s x_n - U_s x_m\| = \|U_s(x_n - x_m)\| \leqslant \alpha(s)\|x_n - x_m\|, \quad s \in S. \tag{3.6}$$

Fix $s_0 \in S$. Then (3.6) guarantees $y(s_0) \in H$ such that $\{U_{s_0} x_n\}_{n=1}^\infty$ converges to $y(s_0)$. Since $D_\alpha \subseteq D_{\mathcal{U}} \subseteq D(U_{s_0})$ and $U_{s_0} \in \mathcal{N}(H)$ is a closed operator, it follows $x \in D(U_{s_0})$ and $U_{s_0} x = y(s_0)$. But, $s_0 \in S$ is arbitrary and so $x \in D_{\mathcal{U}}$. Moreover, since $\{x_n\}_{n=1}^\infty \subseteq D_\alpha$ it follows from (1.2) that $\|U_{s_0} x_n\| \leqslant \alpha(s_0)\|x_n\|$, for $n \in \mathbb{N}$. Letting $n \to \infty$ shows that $\|U_{s_0} x\| \leqslant \alpha(s_0)\|x\|$. Since $s_0 \in S$ is arbitrary it follows (from (1.2)) that $x \in D_\alpha$.

(vi) Fix $\alpha \in \mathcal{A}(S)$. From the definition of D_α it is clear that $D_\alpha \subseteq D_{\mathcal{U}} \subseteq D(U_s)$, for $s \in S$. Fix $s \in S$ and $x \in D_\alpha$. For each $t \in S$, it follows from (3.1) that

$$\begin{aligned} |\langle U_t(U_s x), U_s x\rangle| &= |\langle U_{ts} x, U_s x\rangle| \leqslant \|U_{ts} x\| \cdot \|U_s x\| \\ &\leqslant \alpha(st)\|x\| \cdot \alpha(s)\|x\| \leqslant M\alpha(t), \end{aligned}$$

where $M = (\alpha(s))^2\|x\|^2$. This shows, with $z = U_s x$, that the function φ_z of part (iii) is α-bounded. By (iii) it is also positive definite which, as noted above, implies the inequality $|\langle U_t z, z\rangle| = |\varphi_z(t)| \leqslant \varphi_z(e)\alpha(t) = \|z\|^2\alpha(t)$, for $t \in S$. Hence, $z = U_s x \in D_\alpha$. This establishes that $U_s(D_\alpha) \subseteq D_\alpha$. ∎

Remark 3.3. If there is at least one $s_0 \in S$ such that $D(U_{s_0}) \neq H$, then $D_\alpha \neq D_{\mathcal{U}}$ for *every* $\alpha \in \mathcal{A}(S)$. For, suppose there is $\alpha \in \mathcal{A}(S)$ with $D_\alpha = D_{\mathcal{U}}$. Then $D_{\mathcal{U}}$ is closed in H (cf. Proposition 3.2 (v)) and dense (as $D_{\mathrm{c}} \subseteq D_{\mathcal{U}}$), which forces $D_{\mathcal{U}} = H$. Since $D_{\mathcal{U}} \subseteq D(U_{s_0})$ it follows that $D(U_{s_0}) = H$, contrary to the assumption on U_{s_0}.

Proposition 3.4. ([16], Theorem 2) *Let S be a commutative, unital semigroup with involution and $\mathcal{U} : S \to \mathcal{L}(H)$ be a map satisfying $U_e = I$ and $U_{st^-} = U_s U_t^*$, for all $s, t \in S$. Then there exists a unique selfadjoint, Radon spectral measure $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$ which has compact support and satisfies $U_s = \int_{S^*} \widehat{s}\,\mathrm{d}E$, for $s \in S$.*

To begin the proof of Theorem 1.2 let $\mathcal{U} : S \to \mathcal{N}(H)$ be a $*$-representation. Fix $\alpha \in \mathcal{A}(S)$. By Proposition 3.2 (vi) the restriction $U_s^{(\alpha)} := U_s|D_\alpha$ of U_s to $D_\alpha \subseteq D(U_s)$ is defined, for $s \in S$, as an operator from D_α into D_α. Moreover, (1.2) implies that $U_s^{(\alpha)} \in \mathcal{L}(D_\alpha)$ since the inequalities $\|U_s^{(\alpha)} x\| = \|U_s x\| \leqslant \alpha(s)\|x\|$, for $x \in D_\alpha$ and $s \in S$, show that

$$\|U_s^{(\alpha)}\|_{\mathcal{L}(D_\alpha)} \leqslant \alpha(s) < \infty, \quad s \in S. \tag{3.7}$$

Using the properties listed in Proposition 3.2 it is routine to verify that $\mathcal{U}^{(\alpha)} : S \to \mathcal{L}(D_\alpha)$ defined by $s \mapsto U_s^{(\alpha)}$ satisfies the assumptions of Proposition 3.4 in the Hilbert space D_α. Accordingly, by Proposition 3.4 there exists a unique selfadjoint, Radon spectral measure $E^{(\alpha)} : \mathcal{B}(S^*) \to \mathcal{L}(D_\alpha)$ which is compactly supported and satisfies

$$U_s^{(\alpha)} = \int_{S^*} \widehat{s}\,\mathrm{d}E^{(\alpha)}, \quad s \in S. \tag{3.8}$$

An examination of the proof of Theorem 2 in [16] (i.e. of Proposition 3.4 above) shows $\mathrm{supp}(E^{(\alpha)}) \subseteq S^\delta$, where $\delta : S \to [0,\infty)$ is the absolute value $\delta(s) := \|U_s^{(\alpha)}\|_{\mathcal{L}(D_\alpha)}$, for $s \in S$. But, (3.7) implies $\delta(s) \leqslant \alpha(s)$, for $s \in S$, and so $\mathrm{supp}(E^{(\alpha)}) \subseteq S^\alpha$.

We note that $\alpha \vee \beta \in \mathcal{A}(S)$ whenever $\alpha, \beta \in \mathcal{A}(S)$, where $(\alpha \vee \beta)(s) := \max\{\alpha(s), \beta(s)\}$, for $s \in S$. So, defining $\leqslant$ in $\mathcal{A}(S)$ by $\alpha \leqslant \beta$ iff $\alpha(s) \leqslant \beta(s)$ for all $s \in S$, we have $(\mathcal{A}(S), \leqslant)$ is a partially ordered set. It is also a directed set, since $\alpha \leqslant \alpha \vee \beta$ and $\beta \leqslant \alpha \vee \beta$ whenever $\alpha, \beta \in \mathcal{A}(S)$. It is a consequence of (1.2) that $D_\alpha \subseteq D_\beta$ whenever $\alpha \leqslant \beta$ in $\mathcal{A}(S)$. Of course, D_α is a closed subspace of D_β. It is routine to check D_α is invariant for each operator $U_s^{(\beta)}$, $s \in S$, and that $U_s^{(\beta)}|D_\alpha = U_s^{(\alpha)}$ for each $s \in S$. Since $\{D_\alpha\}_{\alpha \in \mathcal{A}(S)}$ is an upwards directed family of subspaces it is clear that $D_{\mathrm{c}} = \bigcup\limits_{\alpha \in \mathcal{A}(S)} D_\alpha$ is a vector subspace of H (contained in $D_{\mathcal{U}}$, of course).

Lemma 3.5. *Let S be a commutative, unital semigroup with an involution and $\mathcal{U} : S \to \mathcal{N}(H)$ be a $*$-representation.*

(i) *Let $\alpha \leqslant \beta$ in $\mathcal{A}(S)$. Then, for each $x \in D_\alpha \subseteq D_\beta$, we have $E_{x,x}^{(\alpha)} = E_{x,x}^{(\beta)}$ as elements of $M_+^{\mathrm{c}}(S^*) := M_+(S^*) \cap M^{\mathrm{c}}(S^*)$.*

(ii) *Given any $\alpha, \beta \in \mathcal{A}(S)$, the equality $E_{x,x}^{(\alpha)} = E_{x,x}^{(\beta)}$ is valid as elements of $M_+^{\mathrm{c}}(S^*)$, for every $x \in D_\alpha \cap D_\beta$.*

Proof. (i) For $x \in D_\alpha$ fixed, both $E_{x,x}^{(\alpha)}, E_{x,x}^{(\beta)} \in M_+(S^\beta)$. So, it suffices to show their generalized Laplace transforms $(E_{x,x}^{(\alpha)})^\wedge$ and $(E_{x,x}^{(\beta)})^\wedge$ coincide. But, from (3.8) we see

$$(E_{x,x}^{(\alpha)})^\wedge(s) = \int\limits_{S^*} \widehat{s}\,\mathrm{d}E_{x,x}^{(\alpha)} = \Big\langle \Big(\int\limits_{S^*} \widehat{s}\,\mathrm{d}E^{(\alpha)} \Big) x, x \Big\rangle = \langle U_s^{(\alpha)} x, x \rangle, \tag{3.9}$$

for each $s \in S$. A similar calculation shows that $(E_{x,x}^{(\beta)})^\wedge(s) = \langle U_s^{(\beta)} x, x \rangle$, for each $s \in S$. Since $U_s^{(\beta)}|D_\alpha = U_s^{(\alpha)}$ it follows that $(E_{x,x}^{(\alpha)})^\wedge = (E_{x,x}^{(\beta)})^\wedge$.

(ii) Since $\alpha \leqslant (\alpha \vee \beta)$ and $\beta \leqslant (\alpha \vee \beta)$, part (i) implies that $E_{x,x}^{(\alpha)} = E_{x,x}^{(\alpha \vee \beta)}$ and $E_{x,x}^{(\beta)} = E_{x,x}^{(\alpha \vee \beta)}$, whenever $x \in D_\alpha \cap D_\beta$. In particular, $E_{x,x}^{(\alpha)} = E_{x,x}^{(\beta)}$. ∎

Let $x \in D_{\mathrm{c}}$. Then $x \in D_\alpha$ for some $\alpha \in \mathcal{A}(S)$. Lemma 3.5 ensures the compactly supported Radon measure $\mu_x : \mathcal{B}(S^*) \to [0, \infty)$ given by $\mu_x(A) := E_{x,x}^{(\alpha)}(A)$, for $A \in \mathcal{B}(S^*)$, is well defined. If $x, y \in D_{\mathrm{c}}$, we define $\mu_{x,y} \in M^{\mathrm{c}}(S^*)$ by $\mu_{x,y} = \frac{1}{4}(\mu_{x+y} - \mu_{x-y} + \mathrm{i}\mu_{x+\mathrm{i}y} - \mathrm{i}\mu_{x-\mathrm{i}y})$. Choose $\alpha \in \mathcal{A}(S)$ with both $x, y \in D_\alpha$. Then $(x \pm y) \in D_\alpha$ and $(x \pm \mathrm{i}y) \in D_\alpha$. It follows from the selfadjointness of $E^{(\alpha)}(A) \in \mathcal{L}(D_\alpha)$ and the polarization identity that $\mu_{x,y}(A) = \langle E^{(\alpha)}(A)x, y \rangle$, for $A \in \mathcal{B}(S^*)$. A calculation as in (3.9) shows $\widehat{\mu}_{x,y}(s) = \langle U_s^{(\alpha)} x, y \rangle$, for $s \in S$. So, we have the following result.

Lemma 3.6. *Let $x, y \in D_{\mathrm{c}}$. Then $\mu_{x,y}(A) = \langle E^{(\alpha)}(A)x, y \rangle$, for $A \in \mathcal{B}(S^*)$, and*

$$\widehat{\mu}_{x,y}(s) = \langle U_s^{(\alpha)} x, y \rangle, \quad s \in S, \tag{3.10}$$

for every $\alpha \in \mathcal{A}(S)$ with the property that both $x, y \in D_\alpha$.

An immediate consequence is the following fact.

Lemma 3.7. *The map* $(x,y) \mapsto \mu_{x,y}$ *is sesquilinear from* $D_{\mathrm{c}} \times D_{\mathrm{c}}$ *into* $M^{\mathrm{c}}(S^*)$.

Proof. Fix $x,y \in D_{\mathrm{c}}$ and $\lambda \in \mathbb{C}$. Choose $\alpha \in \mathcal{A}(S)$ with $x,y \in D_\alpha$. It follows from (3.10) that $\widehat{\mu}_{\lambda x,y}(s) = \langle U_s^{(\alpha)}\lambda x, y\rangle = \lambda\langle U_s^{(\alpha)}x,y\rangle = (\lambda\mu_{x,y})^\wedge(s)$, for $s \in S$. Uniqueness of generalized Laplace transforms gives $\mu_{\lambda x,y} = \lambda\mu_{x,y}$. A similar calculation shows $\mu_{x,\lambda y} = \overline{\lambda}\mu_{x,y}$.

Fix $x_1,x_2,y \in D_{\mathrm{c}}$. Choose $\alpha \in \mathcal{A}(S)$ with $x_1,x_2,y \in D_\alpha$. Again by (3.10) we have $\widehat{\mu}_{x_1+x_2,y}(s) = \langle U_s^{(\alpha)}(x_1+x_2),y\rangle = \langle U_s^{(\alpha)}x_1,y\rangle + \langle U_s^{(\alpha)}x_2,y\rangle = \widehat{\mu}_{x_1,y}(s) + \widehat{\mu}_{x_2,y}(s) = (\mu_{x_1,y}+\mu_{x_2,y})^\wedge(s)$, for $s \in S$. It follows that $\mu_{x_1+x_2,y} = \mu_{x_1,y} + \mu_{x_2,y}$. A similar calculation shows $\mu_{x,y_1+y_2} = \mu_{x,y_1} + \mu_{x,y_2}$, whenever $x,y_1,y_2 \in D_{\mathrm{c}}$. ∎

Fix $B \in \mathcal{B}(S^*)$ and define $\Psi_B : D_{\mathrm{c}} \times D_{\mathrm{c}} \to \mathbb{C}$ by $(x,y) \mapsto \mu_{x,y}(B)$. Lemma 3.7 shows Ψ_B is sesquilinear. Given $\{x_1,\dots,x_n\} \subseteq D_{\mathrm{c}}$ and $\{c_1,\dots,c_n\} \subseteq \mathbb{C}$ define $\nu \in M^{\mathrm{c}}(S^*)$ by $\nu := \sum_{j,k} c_j\overline{c}_k\mu_{x_j,x_k}$. Choose $\alpha \in \mathcal{A}(S)$ with $\{x_j\}_{j=1}^n \subseteq D_\alpha$. Using (3.10) it follows from the relevant calculation concerning ν in the proof of Theorem 2 in [16] (performed now for $\mathcal{U}^{(\alpha)} : S \to \mathcal{L}(D_\alpha)$) that $\nu \geqslant 0$. Again the relevant calculation concerning Ψ_B in the proof of Theorem 2 in [16] shows Ψ_B is a positive definite kernel and

$$|\Psi_B(x,y)| \leqslant \|x\| \cdot \|y\|, \quad (x,y) \in D_{\mathrm{c}} \times D_{\mathrm{c}}. \tag{3.11}$$

Hence, Ψ_B has a unique extension to a sesquilinear form $\overline{\Psi}_B : H \times H \to \mathbb{C}$. So, there is a unique operator $E(B) \in \mathcal{L}(H)$ with $\|E(B)\| \leqslant 1$ and $\langle E(B)x,y\rangle = \overline{\Psi}_B(x,y)$, for $(x,y) \in H \times H$. Of course, if $(x,y) \in D_{\mathrm{c}} \times D_{\mathrm{c}}$, then $\langle E(B)x,y\rangle = \mu_{x,y}(B)$. Moreover,

$$E(B)x = E^{(\alpha)}(B)x, \quad \text{if } x \in D_\alpha \text{ and } \alpha \in \mathcal{A}(S). \tag{3.12}$$

The claim is that $E : B \mapsto E(B)$, for $B \in \mathcal{B}(S^*)$, is the required spectral measure. If $x,y \in D_{\mathrm{c}}$, then $\langle E(S^*)x,y\rangle = \mu_{x,y}(S^*) = \widehat{\mu}_{x,y}(e) = \langle U_e^{(\alpha)}x,y\rangle = \langle x,y\rangle$, for any $\alpha \in \mathcal{A}(S)$ with $x,y \in D_\alpha$. Since D_{c} is dense it follows $E(S^*) = I$. The relevant calculation in the proof of Theorem 2 in [16] shows, for fixed $B \in \mathcal{B}(S^*)$, that $\langle E(B)^*x,y\rangle = \langle E(B)x,y\rangle$, for $x,y \in D_{\mathrm{c}}$, and again density of D_{c} implies $E(B)^* = E(B)$.

Fix $B \in \mathcal{B}(S^*)$. Let $x \in H$ and $y \in D_{\mathrm{c}}$. Choose a sequence $\{x_n\}_{n=1}^\infty \subseteq D_{\mathrm{c}}$ such that $\|x - x_n\| \to 0$ as $n \to \infty$. Then it follows from (3.11) that

$$|\mu_{x_n,y}(B) - \mu_{x_m,y}(B)| = |\Psi_B(x_n - x_m, y)| \leqslant \|x_n - x_m\| \cdot \|y\|, \quad m,n \in \mathbb{N}.$$

Since $B \in \mathcal{B}(S^*)$ is arbitrary, we have $|||\mu_{x_n,y} - \mu_{x_m,y}||| \leqslant 4\|x_n - x_m\| \cdot \|y\|$, for $m,n \in \mathbb{N}$. By Lemma 3.1 there is a Radon measure $\widetilde{\mu}_{x,y} : \mathcal{B}(S^*) \to \mathbb{C}$ such that $|||\mu_{x_n,y} - \widetilde{\mu}_{x,y}||| \to 0$. By continuity of each $E(B) \in \mathcal{L}(H)$ and the identities $\mu_{x_n,y}(B) = \Psi_B(x_n,y) = \langle E(B)x_n,y\rangle$, for $B \in \mathcal{B}(S^*)$ and $n \in \mathbb{N}$, we see $\langle E(B)x_n,y\rangle \to \langle E(B)x,y\rangle$. But, also $\mu_{x_n,y}(B) \to \widetilde{\mu}_{x,y}(B)$ and so $\langle E(B)x,y\rangle = \widetilde{\mu}_{x,y}(B)$, for $B \in \mathcal{B}(S^*)$. This shows $\langle E(\cdot)x,y\rangle$ is σ-additive whenever $x \in H$ and $y \in D_{\mathrm{c}}$. Now fix $x,y \in H$. Choose $\{y_n\}_{n=1}^\infty \subseteq D_{\mathrm{c}}$ such that $\|y_n - y\| \to 0$. For $B \in \mathcal{B}(S^*)$ we have

$$|\widetilde{\mu}_{x,y_m}(B) - \widetilde{\mu}_{x,y_n}(B)| = |\overline{\Psi}_B(x, y_m - y_n)| \leqslant \|x\| \cdot \|y_m - y_n\|, \quad m,n \in \mathbb{N}.$$

The above argument can be repeated to deduce $\langle E(\cdot)x, y\rangle$ is σ-additive. Hence, $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$ is σ-additive for the weak (and so also strong) operator topology.

Fix $x \in D_c$. Choose $\alpha \in \mathcal{A}(S)$ with $x \in D_\alpha$. By (3.12) and multiplicativity of $E^{(\alpha)}$ we have $E(A\cap B)x = E^{(\alpha)}(A\cap B)x = E^{(\alpha)}(A)E^{(\alpha)}(B)x = E(A)E(B)x$, for all sets $A, B \in \mathcal{B}(S^*)$. The usual density argument yields $E(A\cap B) = E(A)E(B)$ in $\mathcal{L}(H)$. Accordingly, E is multiplicative.

Hence, we have established that E is a selfadjoint spectral measure. It is Radon because $E_{x,x} = \widetilde{\mu}_{x,x}$ is Radon, for $x \in H$. Moreover, if $x \in D_c$, then $E_{x,x} = E^{(\alpha)}_{x,x}$, for any $\alpha \in \mathcal{A}(S)$ with $x \in D_\alpha$, and so $\operatorname{supp}(E_{x,x}) = \operatorname{supp}(Ex) = \operatorname{supp}(E^{(\alpha)}_{x,x})$ is compact. Fix $s \in S$. Since $\widehat{s}$ is continuous it is Ex-integrable (still assuming $x \in D_c$) and

$$\Big\langle \int_{S^*} \widehat{s}\, \mathrm{d}(Ex), y \Big\rangle = \int_{S^*} \widehat{s}\, \mathrm{d}E_{x,y}, \quad y \in H. \tag{3.13}$$

But, if also $y \in D_c$, then it follows from Lemma 3.6 (after choosing another $\alpha \in \mathcal{A}(S)$, if necessary, such that both $x, y \in D_\alpha$) and the identity $E_{x,y} = E^{(\alpha)}_{x,y} = \mu_{x,y}$ that

$$\int_{S^*} \widehat{s}\, \mathrm{d}E_{x,y} = \widehat{\mu}_{x,y}(s) = \langle U^{(\alpha)}_s x, y\rangle = \langle U_s x, y\rangle. \tag{3.14}$$

Since D_c is dense in H we conclude from (3.13) and (3.14) that $U_s x = \int_{S^*} \widehat{s}\, \mathrm{d}(Ex)$. So, (1.3) holds for all $x \in D_c$ and $s \in S$.

Suppose that $F : \mathcal{B}(S^*) \to \mathcal{L}(H)$ is another selfadjoint, Radon spectral measure such that $\operatorname{supp}(Fx)$ is compact for each $x \in D_c$ and

$$U_s x = \int_{S^*} \widehat{s}\, \mathrm{d}Fx, \quad x \in D_c,\ s \in S. \tag{1.3$'$}$$

Then, for all $x, y \in D_c$, it follows from (1.3) and (1.3)$'$ that $(E_{x,y})^\wedge = (F_{x,y})^\wedge$. Hence, $E_{x,y} = F_{x,y}$ as elements of $M^c(S^*)$. So, for any $A \in \mathcal{B}(S^*)$, we have $\langle E(A)x, y\rangle = \langle F(A)x, y\rangle$ for all $x, y \in D_c$. By the usual density argument it follows that $E(A) = F(A)$.

Hence, we have established there exists a unique, selfadjoint Radon spectral measure $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$ such that $\operatorname{supp}(Ex) \subseteq S^*$ is compact, for $x \in D_c$, and

$$U_s x = \int_{S^*} \widehat{s}\, \mathrm{d}(Ex), \quad s \in S,\ x \in D_c. \tag{3.15}$$

LEMMA 3.8. *Let $\mathcal{U} : S \to \mathcal{N}(H)$ be a $*$-representation and $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$ be the unique selfadjoint, Radon spectral measure which satisfies* (3.15). *For each $\alpha \in \mathcal{A}(S)$, let $Q_\alpha \in \mathcal{L}(H)$ be the orthogonal projection of H onto the closed subspace D_α.*

(i) $\lim_{a\in\mathcal{A}(S)} Q_\alpha = I$ *in the strong operator topology of* $\mathcal{L}(H)$.

(ii) *For each* $\alpha \in \mathcal{A}(S)$ *we have*

$$Q_\alpha E(A) = E(A) Q_\alpha, \quad A \in \mathcal{B}(S^*). \tag{3.16}$$

(iii) *For each* $\alpha \in \mathcal{A}(S)$ *and* $s \in S$ *we have*

$$U_s Q_\alpha y = Q_\alpha U_s y, \quad y \in D(U_s). \tag{3.17}$$

Proof. (i) Fix $x \in D_{\mathrm{c}}$. Choose $\alpha_0 \in \mathcal{A}(S)$ with $x \in D_{\alpha_0}$, in which case $Q_{\alpha_0} x = x$. For $\alpha \in \mathcal{A}(S)$ with $\alpha \geqslant \alpha_0$ we have $D_{\alpha_0} \subseteq D_\alpha$ and so $Q_\alpha x = Q_{\alpha_0} x = x = Ix$. This implies $\lim_{\alpha \in \mathcal{A}(S)} Q_\alpha x = Ix$. Since D_{c} is dense and $\{Q_\alpha\}_{\alpha \in \mathcal{A}(S)}$ is uniformly bounded it follows that $\lim_{\alpha \in \mathcal{A}(S)} Q_\alpha = I$ for the strong operator topology.

(ii) Fix $\alpha \in \mathcal{A}(S)$ and $A \in \mathcal{B}(S^*)$. By (3.12) we have $E(A) D_\alpha \subseteq D_\alpha$. So, given any $y \in H$ we see that $Q_\alpha y \in D_\alpha$ and hence, $E(A) Q_\alpha y \in D_\alpha$. Accordingly, $Q_\alpha E(A) Q_\alpha y = E(A) Q_\alpha y$ and, since $y \in H$ is arbitrary, it follows that $E(A) Q_\alpha = Q_\alpha E(A) Q_\alpha$. Taking adjoints gives $Q_\alpha E(A) = Q_\alpha E(A) Q_\alpha$ and (3.16) follows.

(iii) Fix $\alpha \in \mathcal{A}(S)$ and $s \in S$. Suppose first that $x \in D_{\mathrm{c}}$. Since $Q_\alpha x \in D_\alpha \subseteq D_{\mathrm{c}}$ it follows from (3.15) and part (ii) that

$$U_s Q_\alpha x = \int_{S^*} \widehat{s}\, \mathrm{d}(E[Q_\alpha x]) = Q_\alpha \int_{S^*} \widehat{s}\, \mathrm{d}(Ex) = Q_\alpha U_s x, \quad x \in D_{\mathrm{c}}. \tag{3.18}$$

Fix $y \in D(U_s)$. If $x \in D_{\mathrm{c}}$, then $\langle U_s Q_\alpha y, x \rangle = \langle Q_\alpha y, U_s^* x \rangle = \langle Q_\alpha y, U_{(s^-)} x \rangle$; the first equality holds since $Q_\alpha y \in D_\alpha \subseteq D(U_s)$ and $x \in D_{\mathrm{c}} \subseteq D(U_s) = D(U_s^*)$, and the second holds since U is a $*$-representation. Moreover, $\langle Q_\alpha y, U_{(s^-)} x \rangle = \langle y, Q_\alpha U_{(s^-)} x \rangle = \langle y, U_{(s^-)} Q_\alpha x \rangle = \langle y, U_s^* Q_\alpha x \rangle$; the first equality holds as $Q_\alpha^* = Q_\alpha$, the second holds by (3.18), and the third as $\mathcal{U}$ is a $*$-representation. But, $\langle y, U_s^* Q_\alpha x \rangle = \langle U_s y, Q_\alpha x \rangle = \langle Q_\alpha U_s y, x \rangle$; the first equality holds by definition of adjoint operator (as $y \in D(U_s)$ and $Q_\alpha x \in D_\alpha \subseteq D(U_s^*)$), and the second holds since $Q_\alpha^* = Q_\alpha$. So, we have established $\langle U_s Q_\alpha y, x \rangle = \langle Q_\alpha U_s y, x \rangle$, for $x \in D_{\mathrm{c}}$. Since D_{c} is dense in H, the identity (3.17) follows. ∎

Let $\widehat{E} : S \to \mathcal{N}(H)$ be the canonical $*$-representation associated to E as in Section 2. Recall, in this case, that

$$D(\widehat{E}(s)) = \left\{ x \in H : \int_{S^*} |\widehat{s}|^2 \, \mathrm{d}E_{x,x} < \infty \right\}, \quad s \in S. \tag{3.19}$$

To complete the proof of Theorem 1.2 it suffices to show that

$$U_s = \widehat{E}(s), \quad s \in S. \tag{3.20}$$

Indeed, (3.15) then extends from D_{c} to $D(U_s)$, for $s \in S$ (see (2.2)), and the claim in Theorem 1.2 that $\operatorname{supp}(Ex)$ is compact iff $x \in D_{\mathrm{c}}$ follows from Proposition 2.1 (ix). Since both $\widehat{E}(s)$ and U_s are normal, to establish (3.20) it suffices to show $U_s \subseteq \widehat{E}(s)$, for $s \in S$ ([17], Theorem 13.32). So, let $y \in D(U_s)$. Since $\operatorname{supp}(E_{x,x})$ is compact whenever $x \in D_{\mathrm{c}}$, it is clear from (3.19) that $\{Q_\alpha y\}_{\alpha \in \mathcal{A}(S)} \subseteq D_{\mathrm{c}} \subseteq D(\widehat{E}(s))$. Accordingly, the definition of $\widehat{E}$ (see Section 2) implies $\widehat{E}(s) Q_\alpha y = \int_{S^*} \widehat{s}\, \mathrm{d}(E[Q_\alpha y]) = U_s Q_\alpha y$, where the last equality follows from (3.15) as $Q_\alpha y \in D_{\mathrm{c}}$

for $\alpha \in \mathcal{A}(S)$. Moreover, Lemma 3.8 (iii) shows $U_s Q_\alpha y = Q_\alpha U_s y$ and hence, $\widehat{E}(s) Q_\alpha y = Q_\alpha U_s y$ for $\alpha \in \mathcal{A}(S)$. Since $\lim\limits_{\alpha \in \mathcal{A}(S)} Q_\alpha y = y$ and $\lim\limits_{\alpha \in \mathcal{A}(S)} Q_\alpha U_s y = U_s y$ (see Lemma 3.8 (i)) we see that $\lim\limits_{\alpha \in \mathcal{A}(S)} (Q_\alpha y, \widehat{E}(s) Q_\alpha y) = (y, U_s y)$ in the graph of $\widehat{E}(s)$. Then the closedness of $\widehat{E}(s)$ implies $y \in D(\widehat{E}(s))$ and $\widehat{E}(s) y = U_s y$. This establishes (3.20) and thereby (finally!) completes the proof of Theorem 1.2. ∎

We conclude with a criterion useful for verifying (v) of Definition 1.1.

Definition 3.9. Let S be a commutative, unital semigroup with an involution and $\mathcal{U}$ be a map from S into $\mathcal{N}(H)$. Then $\mathcal{U}$ is called *orthogonally decomposable* if there exists an orthogonal family of selfadjoint projections $\{P_\beta\}_{\beta \in J} \subseteq \mathcal{L}(H)$ such that:

(i) $\sum\limits_{\beta \in J} P_\beta = I$, where the series is strong operator convergent.

(ii) $\bigcup\limits_{\beta \in J} P_\beta H \subseteq \bigcap\limits_{s \in S} D(U_s)$.

(iii) $P_\beta U_s \subseteq U_s P_\beta$, $s \in S$, $\beta \in J$.

Remark 3.10. Conditions (i) and (ii) imply that $\bigcap\limits_{s \in S} D(U_s)$ is dense in H. If H is separable, then (i) implies that J is a countable set.

Proposition 3.11. *Let S be a commutative, unital semigroup with involution and $\mathcal{U} : S \to \mathcal{N}(H)$ be an orthogonally decomposable map satisfying conditions* (i)–(iv) *of Definition* 1.1. *Then D_{c} is dense in H. In particular, $\mathcal{U}$ is a $*$-representation.*

Proof. Let $H_\beta = P_\beta H$, for $\beta \in J$. Fix $s \in S$. By (ii) of Definition 3.9 we have $H_\beta \subseteq D(U_s)$. Let $z \in H_\beta$. Then $P_\beta z = z$ and so (iii) of Definition 3.9 implies $U_s z = U_s P_\beta z = P_\beta U_s z \in H_\beta$. This shows $U_s H_\beta \subseteq H_\beta$, for $s \in S$ and $\beta \in J$, and so we may consider the restrictions $U_s^\beta = U_s | H_\beta$ from H_β into itself. Then U_s^β is a linear, everywhere defined operator on the closed subspace H_β. Since $U_s \in \mathcal{N}(H)$ is closed, it follows U_s^β is also closed and hence, by the closed graph theorem, we deduce $U_s^\beta \in \mathcal{L}(H_\beta)$. Using $U_{s^-} = U_s^*$ it follows that $U_{s^-}^\beta = (U_s^\beta)^*$, for $s \in S$ and $\beta \in J$. If $z \in H_\beta$, then (ii) of Definition 3.9 shows $z \in \bigcap\limits_{t \in S} D(U_t)$ and hence, by (iii) of Definition 1.1, we have $z \in D(U_s U_t) = D(U_{st}) \cap D(U_t)$ and $U_{st} z = U_s U_t z$, for $s, t \in S$. Then $U_{st}^\beta z = U_{st} z = U_s U_t z = U_s^\beta U_t^\beta z$. Since z is arbitrary we conclude $U_{st}^\beta = U_s^\beta U_t^\beta$, for $s, t \in S$ and $\beta \in J$.

For $\beta \in J$, define $\alpha_\beta : S \to [0, \infty)$ by $\alpha_\beta(s) := \|U_s^\beta\|_{\mathcal{L}(H_\beta)}$. Using the properties of $s \mapsto U_s^\beta$ established above it follows $\alpha_\beta \in \mathcal{A}(S)$. Fix $\beta \in J$. If $x \in H_\beta \subseteq \bigcap\limits_{s \in S} D(U_s)$, then $|\langle U_s x, x\rangle| = |\langle U_s^\beta x, x\rangle| \leqslant \|U_s^\beta\|_{\mathcal{L}(H_\beta)} \cdot \|x\|^2$, for $s \in S$, which shows $x \in D_{\alpha_\beta}$. Hence, $H_\beta \subseteq D_{\alpha_\beta}$ and so $\bigcup\limits_{\beta \in J} H_\beta \subseteq D_{\mathrm{c}}$. Since $\bigcup\limits_{\beta \in J} H_\beta$ is dense (cf. (i) and (ii) of Definition 3.9) it follows D_{c} is also dense. ∎

Remark 3.12. (a) The $*$-representation $\mathcal{U}$ of Example 2.7 is orthogonally decomposable. It suffices to take $J = \mathbb{N}$ and, for each $n \in \mathbb{N}$, define $P_n \in \mathcal{L}(H)$ to be the projection $P_n : h \mapsto \chi_{[n-1,n]} h$, for $h \in H$.

(b) The $*$-representation $\mathcal{U}$ of Example 2.8 is also orthogonally decomposable. It suffices to take $J = \mathbb{N}_0$ and, for each $n \in \mathbb{N}_0$, define $P_n \in \mathcal{L}(H)$ to be the projection $P_n : \xi \to \langle \xi, e_n \rangle e_n$, for $\xi \in H$, with $\{e_n\}_{n=0}^{\infty}$ the standard basis of $H = \ell^2(\mathbb{N}_0)$.

Example 3.13. Let S be the family of all locally bounded Borel functions $\varphi : [0, \infty) \to \mathbb{C}$. With respect to pointwise multiplication S is a commutative semigroup with unit $e = 1\!\!1$. As involution take complex conjugation $\varphi \mapsto \overline{\varphi}$ (defined pointwise). Let $H = L^2([0, \infty))$. For $\varphi \in S$, let U_φ be the operator with domain $D(U_\varphi) = \{h \in H : \varphi h \in H\}$ given by $U_\varphi : h \mapsto \varphi h$, for $h \in H$. Then $\mathcal{U} : S \to \mathcal{N}(H)$ so defined has properties (i)–(iv) of Definition 1.1. Note that $C_c([0, \infty)) \subseteq D_{\mathcal{U}}$ and so $D_{\mathcal{U}}$ is dense. It is routine to check the projections $\{P_n\}_{n \in \mathbb{N}} \subseteq \mathcal{L}(H)$ given in Remark 3.12 (a) form an orthogonal decomposition for $\mathcal{U}$. By Proposition 3.11, $\mathcal{U} : S \to \mathcal{N}(H)$ is a $*$-representation. ∎

4. THE SPECTRAL THEOREM

An examination of the proof of Theorem 1.2 shows it did *not* use the fact that a normal or selfadjoint operator has a spectral decomposition via its resolution of the identity. In fact, we now *deduce* the spectral theorem as a consequence of Theorem 1.2.

Let $T \in \mathcal{N}(H)$ be selfadjoint. If $S = \mathbb{N}_0$ is the semigroup of Lemma 2.6, then it is routine to check that $U_n = T^n$, for $n \in \mathbb{N}_0$, satisfies (i)–(iv) of Definition 1.1. Moreover, if $\alpha_\lambda \in \mathcal{A}(S)$ is defined by $\alpha_\lambda(n) = \lambda^n$, $n \in \mathbb{N}_0$, for each $\lambda > 0$, then (1.2) implies

$$D_{\alpha_\lambda} = \Big\{ x \in \bigcap_{n=0}^{\infty} D(T^n) : \|T^n x\| \leqslant \lambda^n \|x\| \text{ for all } n \in \mathbb{N}_0 \Big\}.$$

Lemma 2.6 implies that $D_c = \bigcup_{\lambda > 0} D_{\alpha_\lambda}$. Since $D_{\alpha_\lambda} = F(T, \lambda)$, where $F(T, \lambda)$ is the space introduced in [13], p. 86, Lemma 4 of [13] shows that D_c is dense. Hence, $\mathcal{U} : \mathbb{N}_0 \to \mathcal{N}(H)$ given by $n \mapsto U_n$, $n \in \mathbb{N}_0$, is indeed a $*$-representation in our sense. By Theorem 1.2 there is a unique selfadjoint Radon spectral measure $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$ such that $U_n x = T^n x = \int_{S^*} \widehat{n} \, \mathrm{d}Ex$, for $n \in \mathbb{N}_0$ and $x \in D(T^n)$. But, S^* can be identified with $\mathbb{R}$ via the isomorphism $\xi \mapsto \widetilde{\xi}$, where $\widetilde{\xi}(n) = \xi^n$, $n \in \mathbb{N}_0$, for $\xi \in \mathbb{R}$ ([2], p. 115), and so $T^n x = \int_{\mathbb{R}} \xi^n \, \mathrm{d}E(\xi)x$, for $n \in \mathbb{N}_0$ and $x \in D(T^n)$. In particular, the choice $n = 1$ yields $Tx = \int_{\mathbb{R}} \xi \, \mathrm{d}E(\xi)x$, for $x \in D(T)$. Remark 2.4 and (3.20), applied to $\mathcal{U} : \mathbb{N}_0 \to \mathcal{N}(H)$, show that $\sigma(T) = \operatorname{supp}(E)$ and so $Tx = \int_{\sigma(T)} \xi \, \mathrm{d}E(\xi)x$, for $x \in D(T)$; this is the spectral theorem for T.

Suppose $T \in \mathcal{N}(H)$ is normal. Let $S = \mathbb{N}_0 \times \mathbb{N}_0$ be the commutative, unital semigroup with involution $(m, n) \mapsto (n, m)$, as defined in Lemma 2.6. Define

$\mathcal{U} : S \to \mathcal{N}(H)$ by $U_{(m,n)} = T^m(T^*)^n$, where $T^0 := I$. It is routine to verify $\mathcal{U}$ satisfies (i)–(iv) of Definition 1.1. Let $D_{\mathcal{U}} = \bigcap_{(m,n)} D(U_{(m,n)})$; using the definition of the domain of products (and powers) of unbounded operators it follows $D_{\mathcal{U}} = \bigcap_{n=1}^{\infty} D((T^*T)^n)$. Define

$$D_a = \{x \in D_{\mathcal{U}} : |\langle T^m(T^*)^n x, x\rangle| \leqslant a^{m+n}\|x\|^2 \text{ for all } (m,n) \in S\},$$

for each $a > 0$ and, for each $b > 0$, define

$$C_b = \{x \in D_{\mathcal{U}} : |\langle (T^*T)^k x, x\rangle| \leqslant b^k\|x\|^2 \text{ for all } k \in \mathbb{N}_0\}.$$

LEMMA 4.1. *$C_b = D_{b^{1/2}}$, for each $b > 0$.*

Proof. Fix $x \in C_b$. Suppose $(m,n) \in S$ and $m+n$ is even. Using the property $\|Rx\| = \|R^*x\|$ for $x \in D(R) = D(R^*)$, whenever $R \in \mathcal{N}(H)$, it follows that

$$\|T^m(T^*)^n x\| = \|(T^*T)^{(m+n)/2}x\| \leqslant b^{(m+n)/2}\|x\| = (b^{1/2})^{m+n}\|x\|. \tag{4.1}$$

Define the positive definite function $\varphi_x : S \to \mathbb{C}$ by $\varphi_x(m,n) = \langle T^m(T^*)^n x, x\rangle$, for $(m,n) \in S$. By the Cauchy-Schwarz inequality and (4.1)

$$|\varphi_x(m,n)| \leqslant (b^{1/2})^{m+n}\|x\|^2, \quad m+n \text{ even}. \tag{4.2}$$

Since φ_x is positive definite, $|\varphi_x(s+t^-)|^2 \leqslant \varphi_x(s+s^-)\varphi_x(t+t^-)$, for $s,t \in S$ ([2], Chapter 4). Hence, with $(m,n) = s$ and $(0,0) = t$, we have

$$\begin{aligned}|\varphi_x(m,n)|^2 &= |\varphi_x((m,n)+(0,0))|^2 \leqslant \varphi_x((m,n)+(n,m))\varphi_x(0,0)\\ &= \varphi_x(m+n,m+n)\|x\|^2 \leqslant (b^{1/2})^{(2m+2n)}\|x\|^4;\end{aligned}$$

the last inequality follows from (4.2). Since $s = (m,n) \in S$ is arbitrary we see (via Proposition 3.2 (iv)) that $x \in D_{b^{1/2}}$. The reverse containment $D_{b^{1/2}} \subseteq C_b$ is obvious. ∎

Lemma 4.1 implies $\bigcup_{b>0} C_b = \bigcup_{a>0} D_a = D_{\mathrm{c}}$, where $D_{\mathrm{c}} = \bigcup_{a>0} D_a$ follows from Lemma 2.6 (ii). But, the above consideration for selfadjoint operators implies $\bigcup_{b>0} C_b$ is dense (as T^*T is selfadjoint) and hence, D_{c} is dense. Accordingly, $\mathcal{U}$ is a $*$-representation. By Theorem 1.2 there is a unique selfadjoint Radon spectral measure $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$ with $T^m(T^*)^n x = \int_{S^*} (m,n)^\wedge \,\mathrm{d}Ex$, for $(m,n) \in S$ and $x \in D(T^m(T^*)^n)$. But, S^* can be identified with $\mathbb{C}$ via the isomorphism $z \mapsto \widetilde{z}$, for $z \in \mathbb{C}$, where $\widetilde{z}(m,n) = z^m\overline{z}^n$, $(m,n) \in S$, and so $T^m(T^*)^n x = \int_{\mathbb{C}} z^m\overline{z}^n \,\mathrm{d}E(z)x$ for $(m,n) \in S$ and $x \in D(T^m(T^*)^n)$. In particular, the choice $(m,n) = (1,0)$ yields $Tx = \int_{\mathbb{C}} z \,\mathrm{d}E(z)x$, for $x \in D(T)$. Again Remark 2.4 and (3.20), applied this time to $\mathcal{U} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathcal{N}(H)$, show $\sigma(T) = \operatorname{supp}(E)$ and so $Tx = \int_{\sigma(T)} z \,\mathrm{d}E(z)x$, for $x \in D(T)$; this is the spectral theorem for T.

5. PERFECT SEMIGROUPS

A commutative semigroup S (still unital and with an involution) is called *perfect* if *every* positive definite function on it is the generalized Laplace transform of a unique positive Radon measure on S^*, ([2], p. 203). Examples are finite semigroups, idempotent semigroups, abelian groups with the involution $s^- = s^{-1}$, but also (and somewhat surprising) the semigroups $(\mathbb{Q}_+, +)$ and $(\mathbb{Q}, +)$, with the identity function as involution! Finite products, countable direct sums, and homomorphic images of perfect semigroups are again perfect ([2], Chapter 6, Section 5). We shall see that for perfect semigroups S an analogue of Theorem 1.2 holds under considerably weaker assumptions.

Let $D \subseteq H$ be any subspace of a Hilbert space H, and denote by $L(D)$ the set of all linear operators $T : D \to D$, without any continuity assumptions. We can and do assume that $\overline{D} = H$ in the sequel. Let $T \in L(D)$. If, for every $y \in D$ there exists $Ry \in D$ such that $\langle Tx, y\rangle = \langle x, Ry\rangle$, for $x \in D$, then Ry is unique with this property and $R \in L(D)$. In this case we write $R = T^{\#}$. If $T^{\#} = T$, the operator T is called *symmetric*.

A map $\mathcal{U} : S \to L(D)$ satisfying $U_e = \mathrm{I}$ will be called a $*$-*homomorphism* if $U_s^{\#}$ exists for each $s \in S$ and $U_s^{\#} = U_{s^-}$, and if $U_{st} = U_s U_t$ for all $s, t \in S$ (in the usual sense of composing maps).

Theorem 5.1. *Let D be a dense linear subspace of a Hilbert space H, let S be a perfect, commutative, unital semigroup with involution and let $\mathcal{U} : S \to L(D)$ be a $*$-homomorphism. Then there exists a unique selfadjoint Radon spectral measure $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$ such that*

$$U_s x = \int_{S^*} \widehat{s} \,\mathrm{d}Ex, \quad x \in D,\ s \in S.$$

Proof. Again we use the functions $\varphi_x(s) := \langle U_s x, x\rangle$, for $s \in S$ and $x \in D$, which are positive definite since $\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi_x(s_j s_k^-) = \left\| \sum_{j=1}^{n} c_j U_{s_j} x \right\|^2$. By perfectness of S there is a unique Radon measure $\mu_x \geqslant 0$ on S^* such that $\widehat{s} \in L^1(\mu_x)$, for $s \in S$, and $\varphi_x = \widehat{\mu}_x$ ([2], Definition 6.5.1). Repeating partly the proof of Theorem 2 in [16] we define complex Radon measures $\mu_{x,y} := \frac{1}{4}(\mu_{x+y} - \mu_{x-y} + \mathrm{i}\mu_{x+\mathrm{i}y} - \mathrm{i}\mu_{x-\mathrm{i}y})$, leading to $\widehat{\mu}_{x,y}(s) = \langle U_s x, y\rangle$ for $x, y \in D$ and $s \in S$ (note $\mu_{x,x} = \mu_x$). The kernel from $D \times D$ to $\mathbb{C}$ given by $(x, y) \mapsto \mu_{x,y}(B)$, for each fixed $B \in \mathcal{B}(S^*)$, turns out again to be positive (semi-)definite, and is a bounded sesquilinear form since

$$\begin{aligned} |\mu_{x,y}(B)| &\leqslant [\mu_{x,x}(B)\mu_{y,y}(B)]^{1/2} \leqslant [\mu_{x,x}(S^*)\mu_{y,y}(S^*)]^{1/2} \\ &= [\varphi_x(0)\varphi_y(0)]^{1/2} = \|x\| \cdot \|y\|. \end{aligned}$$

So, there is a unique operator $E(B) \in \mathcal{L}(H)$ such that $\langle E(B)x, y\rangle = \widetilde{\mu}_{x,y}(B)$, for $x, y \in H$, where each $\widetilde{\mu}_{x,y}$ is a Radon measure and $\widetilde{\mu}_{x,y} = \mu_{x,y}$ whenever $x, y \in D$. The same calculations and arguments used in the proof of Theorem 2 of [16] lead to the fact that $E : \mathcal{B}(S^*) \to \mathcal{L}(H)$ is actually a selfadjoint, Radon spectral measure; here we make crucial use of the injectivity of the generalized Laplace transformation $\mu \mapsto \widehat{\mu}$ on the complex linear space generated by

$\{\mu \in M_+(S^*) : \widehat{s} \in L^1(\mu) \text{ for all } s \in S\}$ ([2], Proposition 6.5.2). Now $E_{x,y} = \mu_{x,y}$ whenever $x, y \in D$ and so $\langle U_s x, y\rangle = \widehat{\mu}_{x,y}(s) = \int_{S^*} \widehat{s}\,\mathrm{d}\mu_{x,y} = \int_{S^*} \widehat{s}\,\mathrm{d}E_{x,y}$, for $x, y \in D$ and $s \in S$. On the other hand, the vector integral $\int_{S^*} \widehat{s}\,\mathrm{d}Ex$ exists in H since $|\widehat{s}|^2 = (ss^-)^\wedge \in L^1(\mu_x)$, i.e. $\widehat{s} \in L^2(\mu_x) = L^2(E_{x,x})$ for $x \in D$. Therefore $\int_{S^*} \widehat{s}\,\mathrm{d}E_{x,y} = \langle \int_{S^*} \widehat{s}\,\mathrm{d}Ex, y\rangle$ for $x, y \in D$ and so indeed $U_s x = \int_{S^*} \widehat{s}\,\mathrm{d}Ex$ for $x \in D$ and $s \in S$. ∎

Remark 5.2. The above proof shows $D \subseteq D(\widehat{E}(s))$ for $s \in S$, where $\widehat{E} : S \to \mathcal{N}(H)$ is the canonical $*$-representation associated to E, and $\widehat{E}(s)|D = U_s$. Hence, D is contained in $\bigcap_{s\in S} D(\widehat{E}(s))$ and invariant under $\widehat{E}(s)$, for each $s \in S$. As mentioned in the introduction, the conclusion of Theorem 5.1 does not hold, in general, for non-perfect semigroups. In [4], Proposition 2 a result with some similarity to Theorem 5.1 is shown for "operator semiperfect" semigroups, with $L(D)$ replaced by the set of sesquilinear forms on some vector space. The relation between semiperfect (a weakened version of "perfect") and operator semiperfect semigroups is only partly clarified; for finitely generated semigroups the two notions coincide ([4], Theorem 1). As further related work we mention [18] where the author studies "$*$-representations" $\mathcal{U} : \mathcal{A} \to L(D)$ defined on an *algebra* $\mathcal{A}$ with involution, rather than on the more general notion of a semigroup.

Let us now consider the particular semigroup $S = (\mathbb{Q}_+, +)$ of non-negative rational numbers which is perfect ([2], Proposition 6.5.6), but not finitely generated. Its dual S^* can be identified with $([-\infty, \infty), +)$ via $\rho_t(s) = \mathrm{e}^{st}$ and $\rho_{-\infty} = \chi_{\{0\}}$ for $t \in \mathbb{R}$, $s \in \mathbb{Q}_+$. On $\mathbb{Q}_+$ there is no involution other than the identity. Hence, a $*$-homomorphism $\mathcal{U} : \mathbb{Q}_+ \to L(D)$ takes its values in the symmetric operators on D.

Corollary 5.3. (i) *Let D be a dense linear subspace of a Hilbert space H, and let $\mathcal{U} : \mathbb{Q}_+ \to L(D)$ be a $*$-homomorphism. Then there is a unique selfadjoint Radon spectral measure $E : \mathcal{B}([-\infty, \infty)) \to \mathcal{L}(H)$ such that*

$$U_s x = \chi_{\{0\}}(s) \cdot E(\{-\infty\})x + \int_{\mathbb{R}} \mathrm{e}^{st}\,\mathrm{d}E(t)x, \quad s \in \mathbb{Q}_+,\ x \in D.$$

(ii) *Weak continuity of $\mathcal{U}$, i.e. continuity of $s \mapsto \langle U_s x, y\rangle$, $s \in \mathbb{Q}_+$, for each $x, y \in D$, is equivalent with $E(\mathbb{R}) = \mathrm{I}$ (i.e. $E(\{-\infty\}) = 0$).*

(iii) *Let $\mathcal{U} : \mathbb{R}_+ \to L(D)$ be a weakly continuous $*$-homomorphism. Then there is a unique selfadjoint Radon spectral measure $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(H)$ such that*

$$U_s x = \int_{\mathbb{R}} \mathrm{e}^{st}\,\mathrm{d}E(t)x, \quad s \in \mathbb{R}_+,\ x \in D.$$

Proof. Part (i) is an immediate consequence of Theorem 5.1. From

$$\langle U_s x, x\rangle = \chi_{\{0\}}(s) \cdot \|E(\{-\infty\})x\|^2 + \int_{\mathbb{R}} \mathrm{e}^{st}\,\mathrm{d}E_{x,x}(t), \quad s \in \mathbb{Q}_+,\ x \in D,$$

one easily deduces (ii). To see the last part we note that the restriction of $\mathcal{U}$ to $\mathbb{Q}_+$ has (by (i)) the representation $U_s x = \int_{\mathbb{R}} e^{st}\, dE(t)x$, for $s \in \mathbb{Q}_+$ and $x \in D$, implying $\langle U_s x, y\rangle = \int_{\mathbb{R}} e^{st}\, dE_{x,y}(t)$, for $s \in \mathbb{Q}_+$ and $x, y \in D$. Since both sides of this equation are continuous functions of s on $\mathbb{R}_+$ (for $x, y \in D$) and agree on $\mathbb{Q}_+$, they must agree on $\mathbb{R}_+$. ∎

The semigroup $(\mathbb{Q}, +)$ of *all* rational numbers (with the identity function as involution!) is also perfect ([2], Proposition 6.5.10), and $\mathbb{Q}^* \cong (\mathbb{R}, +)$ via $\rho_t(s) = e^{st}$. In particular, *all* characters on $(\mathbb{Q}, +)$ are continuous (a rare exception), and so every $*$-homomorphism $\mathcal{U} : (\mathbb{Q}, +) \to L(D)$ has a unique representation

$$U_s x = \int_{\mathbb{R}} e^{st}\, dE(t)x, \quad s \in \mathbb{Q},\, x \in D,$$

as does every weakly continuous $*$-homomorphism $\mathcal{U} : (\mathbb{R}, +) \to L(D)$. Corresponding results hold likewise for the semigroups $\mathbb{Q}_+^k, \mathbb{Q}^k, \mathbb{R}_+^k$ and $\mathbb{R}^k$ since, as already mentioned, finite products of perfect semigroups are again perfect ([2], Theorem 6.5.4).

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