J. OPERATOR THEORY
47(2002), 245–286

SYSTEM THEORY, OPERATOR MODELS AND SCATTERING: THE TIME-VARYING CASE

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Communicated by Nikolai K. Nikolski

ABSTRACT. It is well known that linear system theory, Lax-Phillips scattering theory, and operator model theory for a contraction operator are all intimately related. A common thread in all three theories is a contractive, analytic, operator-valued function on the unit disk $W(z)$ having a representation of the form $W(z) = D + zC(I - zA)^{-1}B$, known, depending on the context, as the transfer function, the scattering function, or the characteristic function. We present the time-varing analogue of this framework. Also included is a time-varying analogue of the Abstract Interpolation Problem of Katsnelson-Kheifets-Yuditskii.

KEYWORDS: *Time-varying system, realization, frequency response function, transfer function, isometric/coisometric/unitary system, Lax-Phillips scattering.*

MSC (2000): Primary 47A20, 47A40, 47A45, 47A57; Secondary 47A48, 93C05.

1. INTRODUCTION

A time-invariant, causal, bounded linear system (with state initialized to be 0 at time 0)

$$\Sigma : \begin{cases} x(n+1) = Ax(n) + Bu(n), \quad x(0) = 0, \\ y(n) = Cx(n) + Du(n), \end{cases} \tag{1.1}$$

can be viewed in two possible ways: in the time domain, as a lower triangular bounded Toeplitz operator $S = [s_{i-j}]_{i,j \geqslant 0}$ (with $s_n = 0$ for $n < 0$) acting on an ℓ^2 space (the input-output operator of the system) and, in the frequency domain as a multiplication operator M_S acting on the Hardy space H_2, the function S being the *transfer function* of the system given by

$$S(z) = D + zC(I - zA)^{-1}B = \sum_{n=0}^{\infty} s_n z^n. \tag{1.2}$$

Thus the Taylor coefficients $\{s_n\}_{n\in\mathbb{Z}}$ (also called Markov moments) of $S(z)$ can be read off from the input-output matrix $S = [s_{i-j}]$, and, as is seen from the equality in (1.2), are determined from the system matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ by

$$s_0 = D, \quad s_n = CA^{n-1}B \quad \text{for } n > 0. \tag{1.3}$$

A similar formula holds in the time-domain for the input-output operator S:

$$S = \mathcal{D} + \mathcal{C}(I - Z\mathcal{A})^{-1}Z\mathcal{B} \tag{1.4}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are diagonal operators acting on block ℓ^2 with the appropriate block sizes with constant diagonal entries equal to A, B, C, D respectively, and where Z is the forward shift operator on ℓ^2. (This formula requires some careful interpretation if the lower weighted shift operator $Z\mathcal{A}$ has spectral radius equal to 1 rather than strictly less than 1; this can be made precise via an appropriate limiting process.)

Of particular interest is the case where the system is dissipative (i.e., the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is contractive); then the transfer function S is analytic and contractive on the open unit disk $\mathbb{D}$. Such a function S also arises as the *scattering function* of a discrete-time Lax-Phillips scattering system (see [28]), as well as the *characteristic function* of a completely nonunitary contraction operator T on a Hilbert space $\mathcal{H}$ (see [29]). Indeed, there is a natural correspondence between a unitary system, a Lax-Phillips scattering system and a completely nonunitary contraction operator so that the same function S arises as the transfer function of the system, the scattering function of the scattering system and the characteristic operator function of the operator. Conversely, given a contractive, analytic function $S(z)$ of the unit disk, it is well understood (from various points of view) how to construct a dissipative, or more restrictedly, a conservative (also called unitary) linear system (Σ as in (1.1) with system matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ unitary), a model scattering system and a model completely nonunitary contraction operator, all corresponding to one another in the sense alluded to above, so that S is realized as the common transfer function, scattering function and characteristic operator function. One way to construct such models for a given contractive analytic function $S(z)$ is through the use of reproducing kernel Hilbert spaces. For example, if S is a contractive, analytic function on the unit disk $\mathbb{D}$, then the kernel $K_S(z,\omega) = \frac{I - S(z)S(\omega)^*}{1 - z\overline{\omega}}$ is positive (in the sense of reproducing kernels) on $\mathbb{D}$ and the associated reproducing kernel Hilbert space $\mathcal{H}(S)$ provides a coisometric realization of S. Indeed, one has

$$S(z) = D + zC(I_{\mathcal{H}(S)} - zA)^{-1}B$$

where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(S) \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(S) \\ \mathcal{E}_* \end{bmatrix} \tag{1.5}$$

is the backward shift realization defined by

$$(Af)(z) = \frac{f(z) - f(0)}{z}, \quad (Be)(z) = \frac{S(z) - S(0)}{z}e, \quad Cf = f(0), \quad D = S(0).$$

The pair (C, A) satisfies $\bigcap_{n=0}^{\infty} \ker CA^n = \{0\}$ (i.e. is *closely outer-connected*) and the operator matrix (1.5) is coisometric; these two conditions determine the realization uniquely up to a similarity operator, which moreover is unitary. See [12], [16], [10], [3], [1] for more on these coisometric realizations (and also the related isometric and unitary realizations) which were first introduced and studied by L. de Branges and J. Rovnyak.

In the setting of time-varying systems, the system (1.1) is replaced by a time-varying system

$$\Sigma_{tv} : \begin{cases} x(n+1) = A_n x(n) + B_n u(n), \\ y(n) = C_n x(n) + D_n u(n), \end{cases}$$

the Toeplitz operator $S = [s_{i-j}]$ is replaced by an upper triangular bounded operator $S = [s_{ij}]$ which is moreover contractive when the system is dissipative (i.e. the system matrix $\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ is contractive for each $n \in \mathbb{Z}$). One has

$$s_{ii} = D_i, \quad s_{ij} = C_i A_{i-1} \cdots A_{j+1} B_j \quad \text{if } i > j$$

as the analogue of (1.3) and $S = \mathcal{D} + \mathcal{C}(I - Z\mathcal{A})^{-1} Z\mathcal{B}$ where now

$$\mathcal{A} = \operatorname{diag}\{A_n\}, \quad \mathcal{B} = \operatorname{diag}\{B_n\}, \quad \mathcal{C} = \operatorname{diag}\{C_n\}, \quad \mathcal{D} = \operatorname{diag}\{D_n\}$$

are nonconstant diagonal block matrices and Z is the bilateral forward shift operator on the appropriate block ℓ^2 space.

The time-varying analogue of the transfer function in the frequency domain (1.2) is more problematical, but some progress has been made recently. An older approach is via the Zadeh transform (see [23] and [34]); we review this idea in Section 2.2. There is now a whole theory of nonstationary point evaluation and nonstationary matrix Nevanlinna-Pick interpolation (see e.g. [20], [19], [15]); moreover, this nonstationary interpolation theory has applications to time-varying systems which parallel the recently discovered applications of the time-invariant theory to robust control (see e.g. [14]), and to computational modeling (see [15] and [33]). In a somewhat different direction, the first and third authors ([4]) used the observation that multiplication by an upper triangular matrix on the left is a contraction operator from the Hilbert space $\mathcal{U}_{\mathrm{HS}}$ of upper triangular Hilbert-Schmidt operators into itself to define a nonstationary analogue of the de Branges-Rovnyak space $\mathcal{H}(S)$ and of the backward shift realization: see [2], [4]. In this way they recovered much of the reproducing kernel Hilbert space structure of the time-invariant setup in this formalism.

The main contribution of this paper is the systematic development of the full unified formalism of unitary systems, Lax-Phillips scattering systems and operator model theory for the time-varying case. A model of this synthesis of the three theories for the time-invariant case can be found in the work of Nikolskii and Vasyunin (see [30] and [31] for a recent update). We expect that this formalism eventually will become a powerful tool for further applications. Here we present two modest applications. First, we show how the scattering operator serves as a complete unitary invariant for a minimal, time-varying, scattering system, just as in the time-invariant case. A similar result holds for a contraction-operator family and a time-varying unitary system (see Section 6.4). As a second application, we present a time-varying version of the Abstract Interpolation Problem recently introduced by Katsnelson, Kheifets and Yuditskii (see [27], [25]), in both the original de Branges-Rovnyak model formulation, and the coordinate-free, scattering-theoretic form pointed out in [11] for the time-invariant case, and thereby further complete ideas already presented in [17] (see Section 7). The expert may find some new insight in the present paper even for the time-invariant case; when specialized to the time-invariant case, this paper can be considered an update of [21] and [10] giving the connections between systems, scattering and operator model theory which incorporates the ideas of Nikolskii and Vasyunin ([30]) on scattering and model transcriptions.

As we have explained above, the main theme of this paper is the identification of a time-varying analogue of the triptych of unitary systems, Lax-Phillips scattering and operator model theory. It is also possible to go the other way: one can view the time-varying setup as embedded in a time-invariant unitary system/scattering system/contraction operator having some extra structure. For the case of unitary systems and applications to interpolation, the "sparse embedding" of a time-varying, linear system into a time-invariant linear system is one of the main themes of the monograph of Foiaş, Frazho, Gohberg and Kaashoek ([19]). We allude to this connection between time-varying and time-invariant systems briefly in Section 6.4 in our discussion of transfer functions of unitary systems, but leave the details of this theme for the scattering and operator-model settings to be developed elsewhere.

The paper consists of six sections besides this Introduction. In Section 2 we introduce the main ideas concerning time-varying, unitary, linear systems, and associated objects (input-output operators, analogues of the transfer function and the frequency domain); the same basic formalism for the input-output operator and its elementary properties can also found in Section 4 of [6]. Section 3 introduces the notion of a Lax-Phillips scattering system for the discrete-time, time-varying case; the main invariant here is the scattering operator, a lower triangular operator acting on an appropriate ℓ^2 space. Section 4 presents the ideas from [30] concerning a coordinate-free models adapted to the time-varying setting; now we are modeling a family of contraction operators $T_n : \mathcal{H}_{n+1} \to \mathcal{H}_n$ rather than a single operator T on a Hilbert space $\mathcal{H}$. Section 5 lays out the equivalence between the scattering and model theory formalisms on the one side and unitary systems on the other, and establishes the identification between the scattering operator for the system, the characteristic operator for the contraction family $\mathcal{T} = \{T_n\}$, and the transfer function for the embedded unitary system. Section 6 discusses the time-varying versions of the Pavlov, Sz.-Nagy–Foiaş and de Branges-Rovnyak models from the unified perspective found in [30] for the time-invariant case. In particular, we show that any scattering system (respectively, family of contractions or unitary system) is unitarily equivalent to the model scattering system (respectively, family of contractions or unitary system) constructed from its scattering operator (respectively, characteristic operator or input-output operator), under appropriate nondegeneracy conditions. Finally Section 7 deals with the time-varying version of the Abstract Interpolation Problem from [27] in both the de Branges-Rovnyak model and coordinate-free forms and its application to the time-varying version of the right tangential Nevanlinna-Pick interpolation problem ([20], [19], [15]).

2. TIME-VARYING LINEAR SYSTEMS

2.1. Time-domain analysis. By a time-varying linear system we mean a system of equations of the form

$$\Sigma : \begin{cases} x(n+1) = A_n x(n) + B_n u(n), \\ y(n) = C_n x(n) + D_n u(n), \end{cases} \tag{2.1}$$

where, for each integer $n \in \mathbb{Z}$, $x(n)$ is the *state vector* at time n taking values in the time-n state space $\mathcal{H}_n$, $u(n)$ is the *input vector* at time n taking values in the time-n input space $\mathcal{E}_n$ and $y(n)$ is the *output vector* at time n taking values in the time-n output space $\mathcal{E}_{*n}$. Here $\mathcal{H}_n$, $\mathcal{E}_n$ and $\mathcal{E}_{*n}$ are all considered to be Hilbert spaces, and A_n, B_n, C_n and D_n are bounded, linear operators. The family of operators $\{U_n\}_{n\in\mathbb{Z}}$ with U_n given by

$$U_n := \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} : \begin{bmatrix} \mathcal{H}_n \\ \mathcal{E}_n \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_{n+1} \\ \mathcal{E}_{*n} \end{bmatrix}, \tag{2.2}$$

is called the time-varying *colligation* associated with the time-varying system Σ (2.1). We will consider only the case where the colligation $\{U_n\}$ is *contractive*, i.e.

$$\|x(n+1)\|^2 + \|y(n)\|^2 \leqslant \|x(n)\|^2 + \|u(n)\|^2 \tag{2.3}$$

for all $n \in \mathbb{Z}$. In this case, (2.3) can be rewritten as

$$\|x(n+1)\|^2 - \|x(n)\|^2 \leqslant \|u(n)\|^2 - \|y(n)\|^2$$

which, upon iteration, leads to

$$\|x(N_2+1)\|^2 - \|x(N_1)\|^2 \leqslant \sum_{n=N_1}^{n=N_2} (\|u(n)\|^2 - \|y(n)\|^2) \tag{2.4}$$

for all system trajectories $\{x(\cdot), u(\cdot), y(\cdot)\}$.

To discuss the input-output map for such a system, it will be convenient to introduce various Hilbert spaces associated with this setup as is done in [33], pp. 30–32 and [15], p. 23. We denote by $\mathcal{E}$ the whole aggregate $\{\mathcal{E}_n\}_{n\in\mathbb{Z}}$ of input Hilbert spaces, and similarly, $\mathcal{H} = \{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ and $\mathcal{E}_* = \{\mathcal{E}_{*n}\}_{n\in\mathbb{Z}}$. For $\mathcal{X}$ equal to $\mathcal{E}$, $\mathcal{H}$ or $\mathcal{E}_*$, we define Hilbert spaces $\ell^2(\mathbb{Z}, \mathcal{X})$, $\ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{X})$ and $\ell^2(\mathbb{Z}_{<n}, \mathcal{X})$ by

$$\begin{aligned} \ell^2(\mathbb{Z}, \mathcal{X}) &= \Big\{ \{x(k)\}_{k=-\infty}^{\infty} : x(k) \in \mathcal{X}_k \text{ and } \sum_{k\in\mathbb{Z}} \|x(k)\|^2 < \infty \Big\} \\ \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{X}) &= \Big\{ \{x(k)\}_{k=n}^{\infty} : x(k) \in \mathcal{X}_k \text{ and } \sum_{k=n}^{\infty} \|x(k)\|^2 < \infty \Big\} \\ \ell^2(\mathbb{Z}_{<n}, \mathcal{X}) &= \Big\{ \{x(k)\}_{k=-\infty}^{n-1} : x(k) \in \mathcal{X}_k \text{ and } \sum_{k=-\infty}^{n-1} \|x(k)\|^2 < \infty \Big\}. \end{aligned} \tag{2.5}$$

On occasion we shall also have need for the ℓ^∞ version of these spaces:

$$\ell^\infty(\mathbb{Z},\mathcal{X}) = \Big\{\{x(k)\}_{k=-\infty}^{\infty} : x(k) \in \mathcal{X}_k \text{ and } \sup_{k\in\mathbb{Z}} \|x(k)\| < \infty\Big\}$$
$$\ell^\infty(\mathbb{Z}_{\geqslant n},\mathcal{X}) = \Big\{\{x(k)\}_{k=n}^{\infty} : x(k) \in \mathcal{X}_k \text{ and } \sup_{k\geqslant n} \|x(k)\| < \infty\Big\}$$
$$\ell^\infty(\mathbb{Z}_{<n},\mathcal{X}) = \Big\{\{x(k)\}_{k=-\infty}^{n-1} : x(k) \in \mathcal{X}_k \text{ and } \sup_{k<n} \|x(k)\| < \infty\Big\};$$

and the shifted versions

$$\ell^2(\mathbb{Z},\mathcal{X}^{(1)}) = \Big\{\{x(k)\}_{k=-\infty}^{\infty} : x(k) \in \mathcal{X}_{k-1} \text{ and } \sum_{k=-\infty}^{\infty} \|x(k)\|^2 < \infty\Big\},$$
$$\ell^\infty(\mathbb{Z},\mathcal{X}^{(1)}) = \Big\{\{x(k)\}_{k=-\infty}^{\infty} : x(k) \in \mathcal{X}_{k-1} \text{ and } \sup_{k\in\mathbb{Z}} \|x(k)\| < \infty\Big\}.$$

We shall also have occasion to need the following notation for any families of Hilbert spaces $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{Z}}$ and $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{Z}}$:

$$\begin{aligned}
\mathcal{X}(\mathcal{F},\mathcal{G}) &= \text{the space of bounded operators from } \ell^2(\mathbb{Z},\mathcal{F}) \text{ into } \ell^2(\mathbb{Z},\mathcal{G})\\
\mathcal{L}(\mathcal{F},\mathcal{G}) &= \text{lower triangular elements of } \mathcal{X}(\mathcal{F},\mathcal{G})\\
\mathcal{U}(\mathcal{F},\mathcal{G}) &= \text{upper triangular elements of } \mathcal{X}(\mathcal{F},\mathcal{G})\\
\mathcal{U}_-(\mathcal{F},\mathcal{G}) &= \text{strictly upper triangular elements of } \mathcal{X}(\mathcal{F},\mathcal{G})\\
\mathcal{L}_-(\mathcal{F},\mathcal{G}) &= \text{strictly lower triangular elements of } \mathcal{X}(\mathcal{F},\mathcal{G})\\
\mathcal{D}(\mathcal{F},\mathcal{G}) &= \text{diagonal elements of } \mathcal{X}(\mathcal{F},\mathcal{G})\\
\mathcal{D}^{(n)}(\mathcal{F},\mathcal{G}) &= \{[F_{ij}] \in \mathcal{X}(\mathcal{F},\mathcal{G}) : F_{ij} = 0 \text{ for } j \neq i+n\};
\end{aligned}$$

and the Hilbert-Schmidt version of all these spaces:

$$\begin{aligned}
\mathcal{X}_{\mathrm{HS}}(\mathcal{F},\mathcal{G}) &= \text{Hilbert-Schmidt elements of } \mathcal{X}(\mathcal{F},\mathcal{G})\\
\mathcal{L}_{\mathrm{HS}}(\mathcal{F},\mathcal{G}) &= \text{Hilbert-Schmidt elements of } \mathcal{L}(\mathcal{F},\mathcal{G}), \text{ etc.}
\end{aligned}$$

Finally, it will be convenient to use $\mathbb{C}$ to denote the family of Hilbert spaces $\{\mathcal{C}_n : n \in \mathbb{Z}\}$ with $\mathcal{C}_n$ equal to the complex numbers $\mathbb{C}$ for all n. (No confusion should result as the meaning will be clear from the context.)

If we initialize the system (2.1) at time n by $x(n) = 0$ and feed in an input string $\vec{u} = \{u(k)\}_{k=n}^{\infty} \in \ell^2(\mathbb{Z}_{\geqslant n},\mathcal{E})$, the system equations (2.1) uniquely determine an output string $\vec{y} = \{y(k)\}_{k=n}^{\infty}$; from the dissipation inequality (2.4), we see that $\vec{y} \in \ell^2(\mathbb{Z}_{\geqslant n},\mathcal{E}_*)$ and $\|\vec{y}\|^2 \leqslant \|\vec{u}\|^2$. Thus we have a well-defined contractive linear input-output map T_Σ^n acting from $\ell^2(\mathbb{Z}_{\geqslant n},\mathcal{E})$ into $\ell^2(\mathbb{Z}_{\geqslant n},\mathcal{E}_*)$ such that $T_\Sigma^m = T_\Sigma^n|_{\ell^2(\mathbb{Z}_{\geqslant m},\mathcal{E})}$ for $m > n$. Thus we actually have a well-defined, linear contraction operator T_Σ acting from $\bigcup_{n\in\mathbb{Z}} \ell^2(\mathbb{Z}_{\geqslant n},\mathcal{E})$ into $\bigcup_{n\in\mathbb{Z}} \ell^2(\mathbb{Z}_{\geqslant n},\mathcal{E}_*)$. As $\bigcup_{n\in\mathbb{Z}} \ell^2(\mathbb{Z}_{\geqslant n},\mathcal{E})$ is dense in $\ell^2(\mathbb{Z},\mathcal{E})$, this map extends uniquely by continuity to a contraction operator

$$T_\Sigma : \ell^2(\mathbb{Z},\mathcal{E}) \to \ell^2(\mathbb{Z},\mathcal{E}_*).$$

We call this operator T_Σ the *input-output operator* for the system Σ. If we represent elements of $\ell^2(\mathbb{Z},\mathcal{E})$ and $\ell^2(\mathbb{Z},\mathcal{E}_*)$ as biinfinite block column vectors

$$\vec{u} = \begin{bmatrix} \vdots \\ u_{-1} \\ \boxed{u_0} \\ u_1 \\ \vdots \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} \vdots \\ y_{-1} \\ \boxed{y_0} \\ y_1 \\ \vdots \end{bmatrix},$$

then any bounded, linear operator from $\ell^2(\mathbb{Z},\mathcal{E})$ to $\ell^2(\mathbb{Z},\mathcal{E}_*)$ can be expressed as a biinfinite matrix $T = [T_{ij}]_{i,j\in\mathbb{Z}}$. When this is done for $T = T_\Sigma$, we see that T_Σ is lower triangular, i.e. $[T_\Sigma]_{ij} = 0$ for $i < j$.

It follows from the dissipation inequality (2.4) that the input-to-state-at-time-k map $\vec{u} \to R_{\Sigma,k}\vec{u} = x(k)$ (defined as the value $x(k)$ at time k if an input string $\vec{u} \in \ell^2(\mathbb{Z}_{\geqslant n},\mathcal{E})$ is fed into the system Σ (2.1) with initialization $x(n) = 0$) satisfies the estimate

$$\|R_{\Sigma,k}\vec{u}\|^2 = \|x(k)\|^2 \leqslant \sum_{j=-\infty}^{k-1} \|u(j)\|^2 \tag{2.6}$$

for all $\vec{u} \in \bigcup_{n\in\mathbb{Z}} \ell^2(\mathbb{Z}_{\geqslant n},\mathcal{E})$. By continuity, $R_{\Sigma,k}$ has a uniquely determined extension to all of $\ell^2(\mathbb{Z},\mathcal{E})$ such that the estimate (2.4) continues to hold for all $u \in \ell^2(\mathbb{Z},\mathcal{E})$. A consequence of (2.4) then is that $\lim_{k\to-\infty} \|R_{\Sigma,k}\vec{u}\| = 0$ for all $u \in \ell^2(\mathbb{Z},\mathcal{E})$. For this reason, we refer to the state trajectory $\vec{x} = R_\Sigma\vec{u} := \{R_{\Sigma,k}\vec{u}\}_{k\in\mathbb{Z}}$ arising in this way as the *state trajectory generated by the system* Σ *with input signal* $\vec{u} \in \ell^2(\mathbb{Z},\mathcal{E})$ *and with state initialization* $x(-\infty) = 0$. Note as another consequence of (2.4) that $\|R_\Sigma\vec{u}\|_\infty^2 \leqslant \|\vec{u}\|_2^2$, and hence the system trajectory $(\vec{u},\vec{x},\vec{y})$ is in the space of signals

$$\mathcal{S} = \ell^2(\mathbb{Z},\mathcal{E}) \times \ell^\infty(\mathbb{Z},\mathcal{H}) \times \ell^2(\mathbb{Z},\mathcal{E}_*). \tag{2.7}$$

Below we shall derive a sufficient condition for $\vec{x} = R_\Sigma\vec{u}$ to be in $\ell^2(\mathbb{Z},\mathcal{H})$ for each $\vec{u} \in \ell^2(\mathbb{Z},\mathcal{E})$.

Explicitly, the operator $R_{\Sigma,k} : \ell^2(\mathbb{Z}_{<k},\mathcal{E}) \to \mathcal{H}_k$ is given as the infinite row matrix $[\cdots \;\; [R_{\Sigma,k}]_{k-2} \;\; [R_{\Sigma,k}]_{k-1}]$ with $[R_{\Sigma,k}]_j = A_{k-1}A_{k-2}\cdots A_{j+1}B_j$ for $j < k$. (For $j = k-1$, we interpret the formula as $[R_{\Sigma,k}]_{k-1} = B_{k-1}$.) When the time-$k$ *reachability* (also called *controllability*) space

$$\mathcal{R}_{\Sigma,k} = \operatorname{im} R_{\Sigma,k} \tag{2.8}$$

is dense in $\mathcal{H}_k$ for each k, we say that the system Σ is *controllable* (the term *closely inner-connected* is also used, see [3]). The dual notion is the same idea for the adjoint system: we say that Σ is *observable* (the term *closely outer-connected* is also used) if the time-k observability space

$$\mathcal{O}_{\Sigma,k} = \operatorname{span}\{\operatorname{im} A_k^* A_{k+1}^* \cdots A_{\ell-1}^* C_\ell^* : \ell \geqslant k\} \tag{2.9}$$

is dense in $\mathcal{H}_k$ for each k. In case the span $\mathcal{R}_{\Sigma,k} + \mathcal{O}_{\Sigma,k}$ is dense in $\mathcal{H}_k$ for each k, we say that the system Σ is *closely connected.*

The input-output operator T_Σ can also be represented in a more explicit operator-theoretic form as follows. For $\chi = \{\chi_n\}_{n\in\mathbb{Z}}$ any system of Hilbert spaces, define the forward bilateral shift operator $Z : \ell^\infty(\mathbb{Z},\mathcal{H}) \to \ell^\infty(\mathbb{Z},\mathcal{H})^{(-1)}$ by

$$Z : \begin{bmatrix} \vdots \\ x_{-1} \\ \boxed{x_0} \\ x_1 \\ \vdots \end{bmatrix} \to \begin{bmatrix} \vdots \\ x_{-2} \\ \boxed{x_{-1}} \\ x_0 \\ \vdots \end{bmatrix}.$$

Define operators

$$\begin{array}{ll} \mathcal{A} : \ell^2(\mathbb{Z},\mathcal{H}) \to \ell^2(\mathbb{Z},\mathcal{H})^{(-1)}, & \mathcal{B} : \ell^2(\mathbb{Z},\mathcal{E}) \to \ell^2(\mathbb{Z},\mathcal{H})^{(-1)}, \\ \mathcal{C} : \ell^2(\mathbb{Z},\mathcal{H}) \to \ell^2(\mathbb{Z},\mathcal{E}_*), & \mathcal{D} : \ell^2(\mathbb{Z},\mathcal{E}) \to \ell^2(\mathbb{Z},\mathcal{E}_*), \end{array}$$

by

$$(2.10) \qquad \begin{array}{ll} \mathcal{A} = \mathrm{diag}(A_k)_{k\in\mathbb{Z}}, & \mathcal{B} = \mathrm{diag}(B_k)_{k\in\mathbb{Z}}, \\ \mathcal{C} = \mathrm{diag}(C_k)_{k\in\mathbb{Z}}, & \mathcal{D} = \mathrm{diag}(D_k)_{k\in\mathbb{Z}}. \end{array}$$

By assumption each system matrix U_n (2.2) is contractive, and hence each of $\mathcal{A},\mathcal{B},\mathcal{C}$ and $\mathcal{D}$ are contractive as operators between the relevant ℓ^2-spaces. If it happens that the state trajectory $\vec{x} = \{x(k)\}_{k\in\mathbb{Z}}$ is in $\ell^2(\mathbb{Z},\mathcal{H})$, then the aggregate of the system equations (2.1) for all $n = \dots,-1,0,1,\dots$ can be viewed as the system of equations

$$(2.11) \qquad \Sigma_{\mathrm{agg}} : \begin{cases} Z^{-1}\vec{x} = \mathcal{A}\vec{x} + \mathcal{B}\vec{u}, \\ \vec{y} = \mathcal{C}\vec{x} + \mathcal{D}\vec{u}. \end{cases}$$

As noted above, in general we are only guaranteed that $\vec{x} = \mathcal{R}\vec{u} \in \ell^\infty(\mathbb{Z},\mathcal{H})$ for $u \in \ell^2(\mathbb{Z},\mathcal{E})$, so the first of the aggregate system equations must be interpreted on $\ell^\infty(\mathbb{Z},\mathcal{H})$ rather than on $\ell^2(\mathbb{Z},\mathcal{H})$. A sufficient condition for $\vec{x} = \mathcal{R}\vec{u}$ to be in $\ell^2(\mathbb{Z},\mathcal{H})$ is given by the following lemma (see also [22], Theorem 2.1 and [32], Lemma 4.2).

LEMMA 2.1. *Let Σ be a contractive system as in* (2.2). *Then $\vec{x} = R_\Sigma u \in \ell^2(\mathbb{Z},\mathcal{H})$ for all $u \in \ell^2(\mathbb{Z},\mathcal{E})$ if the block diagonal operator $Z\mathcal{A}$ on $\ell^2(\mathbb{Z},\mathcal{H})$ has spectral radius less that* 1. *Moreover, if this is the case and Σ is isometric, then the input-output operator $T_\Sigma : \ell^2(\mathbb{Z},\mathcal{E}) \to \ell^2(\mathcal{Z},\mathcal{E}_*)$ is isometric.*

Proof. We first collect some needed preliminaries. Note that the operator $Z\mathcal{A}$ is a weighted shift operator with all nonzero block matrix entries on the first subdiagonal below the main diagonal. The n^{th} power $(ZA)^n$ of ZA is a weighted shift operator with all nonzero block matrix entries on the n^{th} diagonal below the main diagonal. The norm of $(ZA)^n$ is equal to the supremum $\sup_{k\in\mathbb{Z}} \|[(ZA)^n]_{k+n,k}\|$ of the norms of these nonzero n^{th}-subdiagonal block entries. By explicit computation one finds that $[(Z\mathcal{A})^n]_{k+n,k} = A_{k+n-1}\cdots A_{k+1}A_k$, and hence $\|(Z\mathcal{A})^n\| = \sup_{k\in\mathbb{Z}} \|A_{k+n-1}\cdots A_{k+1}A_k\|$. It follows that the spectral radius of $Z\mathcal{A}$ is given by

$$r_\sigma(Z\mathcal{A}) = \lim_{n\to\infty}\left[\sup_{k\in\mathbb{Z}}\|A_{k+n-1}\cdots A_{k+1}\|\right]^{1/n}.$$

Now let us suppose that $r_\sigma(Z\mathcal{A}) < 1$ and an input signal $\vec{u} \in \ell^2(\mathbb{Z},\mathcal{E})$ is fed into the system Σ (2.1) such that $u(k) = 0$ for all k with $N_1 \leqslant k \leqslant N_2$ for some finite $N_1 < N_2$. If $\vec{x} = \{x(k)\}_{k=-\infty}^{\infty}$ is the resulting system trajectory, we have that $x(k) = 0$ for $k \leqslant N_1$ and that $x(k) = A_{k-1}A_{k-2}\cdots A_{N_2}x(N_2)$ for $k > N_2$. From the condition that $r_\sigma(Z\mathcal{A}) < 1$, from the root test we see that $\sum_{k=N_2}^{\infty}\|x(k)\|^2 < \infty$, and hence that $\vec{x} \in \ell^2(\mathbb{Z},\mathcal{H})$. However since $r_\sigma(ZA) < 1$, we see that the first aggregate system equation in (2.11) has a unique solution in $\ell^2(\mathbb{Z},\mathcal{H})$, namely

$$\vec{x} = (I - Z\mathcal{A})^{-1}Z\mathcal{B}\vec{u}, \tag{2.12}$$

and hence must be equal to the $\vec{x}$ in the system trajectory generated by $\vec{u}$. By entrywise continuity and uniqueness, this same formula (2.12) must continue to hold for the system trajectory $\vec{x}$ generated by any input signal $\vec{u}$ in $\ell^2(\mathbb{Z},\mathcal{E})$. We conclude that $\vec{x}$ is in $\ell^2(\mathbb{Z},\mathcal{H})$ for any input signal $\vec{u} \in \ell^2(\mathbb{Z},\mathcal{E})$ whenever $r(Z\mathcal{A}) < 1$.

If we now assume in addition that Σ is isometric, then the dissipation inequality (2.4) holds with equality. From the fact that $\|x(k)\| \to 0$ for $\vec{u} \in \ell^2(\mathbb{Z},\mathcal{E})$ with finite support and the fact that the dissipation inequality (2.4) holds with equality, we see that T_Σ is isometric on a dense subspace of $\ell^2(\mathbb{Z},\mathcal{E})$, and hence on all of $\ell^2(\mathbb{Z},\mathcal{E})$ by an easy approximation argument. ∎

If $Z\mathcal{A}$ has spectral radius less than 1, then we have seen that the input-to-state map $\vec{u} \to \vec{x}$ is given by (2.12). We can then substitute this into the second of equations (2.11) to arrive at $\vec{y} = T_\Sigma\vec{u} = [\mathcal{D} + \mathcal{C}(I - Z\mathcal{A})^{-1}Z\mathcal{B}]\vec{u}$. If $r_\sigma(Z\mathcal{A})$ is not less than 1, this formula must be interpreted via a limiting process

$$T_\Sigma = \lim_{r\uparrow 1}(\mathcal{D} + \mathcal{C}(I - rZ\mathcal{A})^{-1}Z\mathcal{B}). \tag{2.13}$$

We view the operator-theoretic formula (2.13) for the input-output operator T_Σ of the system Σ as the *time-domain version of the transfer function* of the time-varying system Σ.

Rather than considering inputs $\vec{u}$ supported on the whole time line $\mathbb{Z}$, it is often useful to consider the system with initialization of the state vector at some time n_0 and then driven by input signals $u(n)$ with $n \geqslant n_0$ to determine an output signal $y(n)$ for $n \geqslant n_0$. In the time-invariant case, one usually takes the initialization time n_0 to be $n_0 = 0$ since any other choice will lead to the same results after a translation due to the time-invariance of the system. In the nonstationary case, different choices of initialization point n_0 lead to different results in general. This makes it natural to consider the time-varying system really as a collection of different systems, one placed at each point in time n_0, depending on the point in time which one considers as the present (i.e., the point n_0 at which one imposes the initialization of the state). Alternatively, one can think of this point n_0 as the point relative to which one measures past, present and future.

The n_0^{th} system $\Sigma^{(n_0)}$ is that system obtained from Σ by measuring past, present and future relative to time n_0; in the time-invariant case $\Sigma^{(n_0)} = \Sigma^{(0)} = \Sigma$, but this will fail for the general time-varying case. With this motivation we are led naturally to consideration of the system Σ (2.1) where the input vector $\mathbf{u}(n)$ is assumed to be a biinfinite matrix supported on the n^{th} diagonal below the main diagonal, i.e. $\mathbf{u}(n)$ has the form

$$[\mathbf{u}(n)]_{ij} = 0 \quad \text{unless } i = n + j.$$

Here we think of the j^{th} column of $\mathbf{u}(n)$ as registering the input, say $u^j(n)$, to the j^{th} system at n time units past the present time as measured in the j^{th} system, i.e., at absolute time $j + n$; hence the j^{th} column of $\mathbf{u}(n)$ has the entry $u^j(n)$ in row $j+n$ with all other entries equal to 0. In this scheme, with the block diagonal operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ defined as in (2.10), the system update equations take the form

$$\Sigma_{\text{aug}} : \begin{cases} \mathbf{x}(n+1) = Z\mathcal{A}\mathbf{x}(n) + Z\mathcal{B}\mathbf{u}(n), \\ \mathbf{y}(n+1) = \mathcal{C}\mathbf{x}(n) + \mathcal{D}\mathbf{u}(n). \end{cases} \tag{2.14}$$

Here we may consider this evolution of diagonal operators to be initialized at some particular finite time n_0 with $x_{n_0} = 0$. Note that this is a time-invariant system with system operator

$$\mathbf{U}^{\text{aug}} : \begin{bmatrix} \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{H}) \\ \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{E}) \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{H}) \\ \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{E}_*) \end{bmatrix}$$

equal to left multiplication by the matrix U^{aug} ($\mathbf{U}^{\text{aug}} = L_{U^{\text{aug}}}$) where

$$U^{\text{aug}} = \begin{bmatrix} Z\mathcal{A} & Z\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} : \begin{bmatrix} \ell^2(\mathbb{Z}, \mathcal{H}) \\ \ell^2(\mathbb{Z}, \mathcal{E}) \end{bmatrix} \to \begin{bmatrix} \ell^2(\mathbb{Z}, \mathcal{H}) \\ \ell^2(\mathbb{Z}, \mathcal{E}_*) \end{bmatrix},$$

but with location of the input signal at time n severely restricted. We note that the full time-invariant linear system $\mathbf{\Sigma}^{\text{aug}}$ associated with the system operator $\mathbf{U}^{\text{aug}}$ (with input space equal to $\mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{E})$ and output space equal to $\mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{E}_*)$) can be viewed simply as the direct sum of infinitely many copies of the system $\Sigma^{\text{aug}\prime}$ associated with the system matrix U^{aug} (with state space equal to $\ell^2(\mathbb{C}, \mathcal{H})$, input space equal to $\ell^2(\mathbb{C}, \mathcal{E})$ and output space equal to $\ell^2(\mathbb{C}, \mathcal{E}_*)$). This latter system is the time-invariant system associated with a time-varying system Σ by the method of "sparse embedding" described in [19] and mentioned in the Introduction, and is used in [19] as a tool to reduce the study of problems concerning time-varying systems to known results in the theory of time-invariant systems.

Let us assume that each $\mathbf{u}(n)$ is a Hilbert-Schmidt diagonal operator such that

$$\sum_{n=-\infty}^{\infty} \|\mathbf{u}(n)\|^2 < \infty.$$

Then we may define the aggregate input signal $\mathbf{u} \in \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{E})$ as the sum

$$\mathbf{u} = \sum_{n=-\infty}^{\infty} \mathbf{u}(n)$$

with convergence in Hilbert-Schmidt norm. By an analysis similar to what we did for the conventional case, we can define $\mathbf{x} \in \mathcal{X}(\mathbb{C}, \mathcal{H})$ as the entry-wise limit

$x = \sum_{n=-\infty}^{\infty} \mathbf{x}(n)$ and $\mathbf{y} \in \mathcal{X}_{\rm HS}(\mathbb{C}, \mathcal{E}_*)$ as the sum $\sum_{n=-\infty}^{\infty} \mathbf{y}(n)$ with series convergence in Hilbert-Schmidt norm. Then we have that

$$\|\mathbf{y}\|_{\rm HS} \leqslant \|\mathbf{u}\|_{\rm HS}$$

with equality in the stable case described above for the conventional setting. Also, by resorting to a limiting argument as we did above for the conventional case, we may extend the ideas to consider input strings $\mathbf{u} = \{\mathbf{u}(n)\}_{n\in\mathbb{Z}}$ with $\mathbf{u}(n) \neq 0$ for infinitely many negative values of n. We are thus able to arrive at an arbitrary element of $\mathcal{X}_{\rm HS}(\mathbb{C}, \mathcal{E})$ as being equal to an admissible input string $[\mathbf{u}]_{ij} = \sum_{n=-\infty}^{\infty} [\mathbf{u}(n)]_{ij}$ for our augmented system.

It is interesting to consider the aggregate of the system equations as written down in (2.11) for the setting of the augmented system (2.14). However, rather than introducing a large amount of sparsity by considering the aggregate input signal $\mathbf{u}$ as an element of $\ell^2(\mathbb{Z}, \mathcal{D}^n(\mathbb{C}, \mathcal{E}))$, we define the aggregate input signal $\mathbf{u}$ as the infinite sum $\mathbf{u} = \sum_{n=-\infty}^{\infty} \mathbf{u}(n)$ inside $\mathcal{X}_{\rm HS}(\mathbb{C}, \mathcal{E})$, and similarly for the state and output trajectories. When this is done, we get simply the same linear system as in (2.11)

$$\Sigma^{\rm aug}_{\rm agg} : \begin{cases} Z^{-1}\mathbf{x} = \mathcal{A}\mathbf{x} + \mathcal{B}\mathbf{u}, \\ \mathbf{y} = \mathcal{C}\mathbf{x} + \mathcal{D}\mathbf{u}, \end{cases} \tag{2.15}$$

where now $\mathbf{x} \in \mathcal{X}(\mathbb{C}, \mathcal{H})$, $\mathbf{u} \in \mathcal{X}_{\rm HS}(\mathbb{C}, \mathcal{E})$ and $\mathbf{y} \in \mathcal{X}_{\rm HS}(\mathbb{C}, \mathcal{E}_*)$. The resulting input-output operator $T_{\Sigma_{\rm aug}}$ then maps $\mathcal{X}_{\rm HS}(\mathcal{C}, \mathcal{E})$ contractively into $\mathcal{X}_{\rm HS}(\mathcal{C}, \mathcal{E}_*)$ and is given simply as multiplication on the left by T_Σ, where T_Σ is the input-output operator for the original (conventional) system Σ, as given by (2.13), i.e.

$$T_{\Sigma_{\rm aug}} = L_{T_\Sigma} \quad \text{where } T_\Sigma \text{ is given by (2.13).} \tag{2.16}$$

Note in particular that, since T_Σ is lower triangular, it follows that $T_{\Sigma_{\rm aug}}$ takes $\mathcal{L}_{\rm HS}(\mathbb{C}, \mathcal{E})$ into $\mathcal{L}_{\rm HS}(\mathbb{C}, \mathcal{E}_*)$.

2.2. Frequency-domain analysis. A standard and productive technique for the analysis of time-invariant (discrete-time) systems is to use the so-called Z-transform (or discrete-time Fourier transform in mathematical terminology) $\{x(n)\} \to \widehat{x}(\lambda) := \sum_{n=-\infty}^{\infty} x(n)\lambda^n$. It is well accepted that this technique is not so effective for the time-varying case. Nevertheless, a partial substitute which has been studied in the literature is the Zadeh transform (see [23], [34]), which amounts to applying the Z-transform to what we have called the augmented system $\Sigma_{\rm aug}$ (2.14) rather than to the original system Σ (2.1) itself. Thus, in general, if $\mathbf{f} = \{\mathbf{f}(n)\}_{n\in\mathbb{Z}}$ is a string of Hilbert-Schmidt operators with $\mathbf{f}(n) \in \mathcal{D}^n_{\rm HS}(\mathcal{F}, \mathcal{G})$, we define the *Zadeh transform* of $\mathbf{f}$ to be the operator-valued function of the complex variable λ given by

$$\mathbf{f}^{\wedge Z}(\lambda) = \sum_{n=-\infty}^{\infty} \mathbf{f}(n)\lambda^n \in \mathcal{X}(\mathcal{F}, \mathcal{G})$$

whenever the sum converges. The associated Zadeh transfer function of the system Σ is given by

$$S_{\Sigma}^{Z}(\lambda) := \mathcal{D} + \lambda\mathcal{C}(I - \lambda Z\mathcal{A})^{-1}Z\mathcal{B}. \tag{2.17}$$

Note that (as a consequence of (2.13)) $\lim_{r\uparrow 1} S_{\Sigma}^{Z}(r) = T_{\Sigma}$, as pointed out in [17]. The main result concerning the Zadeh transfer function (and the justification for the terminology) is the following. This result holds without any assumptions on Σ being unitary or contractive, but for simplicity we do not go into these side issues. We omit the proof as we will prove a different but formally similar result shortly.

THEOREM 2.2. *Let Σ be a time-varying linear system as in* (2.1) *with associated input-output operator T_{Σ} and input-output operator $T_{\Sigma_{\mathrm{aug}}} = L_{T_{\Sigma}}$ for the augmented system* (2.14). *Let $\mathbf{u} = \{\mathbf{u}(n)\}_{n\in\mathbb{Z}}$ be any admissible input string for Σ_{aug} with corresponding output string $\mathbf{y} = T_{\Sigma_{\mathrm{aug}}}\mathbf{u}$. Then*

$$\mathbf{y}^{\wedge Z}(\lambda) = S_{\Sigma}^{Z}(\lambda)\mathbf{u}^{\wedge Z}(\lambda)$$

for all λ in the unit disk $\mathbb{D}$.

More recently a somewhat different version of the Zadeh transfer function has been introduced by Alpay-Dewilde-Dym (see e.g. [20] and [19]). Specifically, suppose that $\mathbf{f} \in \mathcal{L}_{\mathrm{HS}}(\mathcal{F},\mathcal{G})$ and that $W = \operatorname{diag}\{W_n\}_{n\in\mathbb{Z}} \in \mathcal{D}(\mathcal{F},\mathcal{F}^{(1)})$ (and thus $W_n : \mathcal{F}_n \to \mathcal{F}_{n-1}$) and $Z^{-1}W \in \mathcal{D}^{-1}(\mathcal{F},\mathcal{F})$. We assume also that the spectral radius $r_{\sigma}(Z^{-1}W)$ of $Z^{-1}W$ is less than 1. In this case, we think of W as a "time-varying" analogue of a point in the unit disk and define the right transform of $\mathbf{f}$ evaluated at the point W by

$$\mathbf{f}^{\wedge R}(W) = \sum_{n=0}^{\infty} \mathbf{f}(n)(Z^{-1}W)^{n} \quad \text{if } \mathbf{f} = \sum_{n=0}^{\infty}\mathbf{f}(n) \text{ with } \mathbf{f}(n) \in \mathcal{D}^{n}(\mathcal{G},\mathcal{F}). \tag{2.18}$$

As is explained in [19], this can be viewed as point evaluation for the Zadeh transform $\mathbf{f}^{\wedge Z}(\lambda) = \sum_{n=0}^{\infty} \mathbf{f}(n)\lambda^{n}$ (a function of a scalar complex variable), but with operator argument $Z^{-1}W$.

Now let us suppose that we are given a time-varying linear system as in (2.1), and suppose that $W \in \mathcal{D}(\mathbb{C},\mathbb{C})$ is such that $r_{\sigma}(Z^{-1}W) < 1$. The *right transfer function* introduced by the first and third authors in [4] is given by

$$S_{\Sigma}^{\wedge R}(W) = L_{\mathcal{D}} + L_{\mathcal{C}}(I - R_{Z^{-1}W}L_{Z\mathcal{A}})^{-1}R_{Z^{-1}W}L_{Z\mathcal{B}} : \mathcal{D}_{\mathrm{HS}}(\mathbb{C},\mathcal{E}) \to \mathcal{D}_{\mathrm{HS}}(\mathbb{C},\mathcal{E}_{*}) \tag{2.19}$$

where L_X and R_Y are the operators defined on a space of block operator matrices of left multiplication and right multiplication:

$$L_X(M) = X\cdot M, \quad R_Y(M) = M\cdot Y$$

(under the assumption that X, M, Y are block operator matrices of compatible sizes). Note that the Alpay-Peretz right transfer function collapses to the Zadeh transfer function if we formally set $Z^{-1}W$ to be of the form $Z^{-1}W = \lambda I$. (Note that the Zadeh transform does not correspond to a special case of the right transform since $Z^{-1}W$ is necessarily strictly upper triangular while λI is diagonal.) The main result concerning the right transfer function is the following generalization of Theorem 2.2 obtained in [4]. For completeness we include a derivation here.

THEOREM 2.3. *Let Σ be a time-varying linear system as in* (2.1) *with associated input-output operator T_Σ and input-output operator $T_{\Sigma_{\text{aug}}} = L_{T_\Sigma}$ for the augmented system* (2.14). *Let $\mathbf{u} = \{\mathbf{u}(n)\}_{n\in\mathbb{Z}}$ be any admissible input string for Σ_{aug} with corresponding output string $\mathbf{y} = T_{\Sigma_{\text{aug}}}\mathbf{u}$. Then*

$$\mathbf{y}^{\wedge R}(W) = S_\Sigma^{\wedge R}(W)\mathbf{u}^{\wedge R}(W)$$

for all W in $\mathcal{D}(\mathbb{C},\mathbb{C})$ with $r_\sigma(Z^{-1}W) < 1$.

Proof. Let $\mathbf{u} = \{\mathbf{u}(n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ be an admissible input string for Σ_{aug} with associated output string $\mathbf{y} = \{\mathbf{y}(n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ and state trajectory $\mathbf{x} = \{\mathbf{x}(n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$. Then we have the system equations

$$\begin{aligned} \mathbf{x}(n+1) &= Z\mathcal{A}\mathbf{x}(n) + Z\mathcal{B}\mathbf{u}(n) \\ \mathbf{y}(n) &= \mathcal{C}\mathbf{x}(n) + \mathcal{D}\mathbf{u}(n). \end{aligned} \tag{2.20}$$

If we apply the right transform at value W to both sides of the first of equations (2.20), we get

$$\sum_{n=0}^{\infty} \mathbf{x}(n+1)(Z^{-1}W)^n = Z\mathcal{A}\mathbf{x}^{\wedge R}(W) + Z\mathcal{B}\mathbf{u}^{\wedge R}(W).$$

Let $\mathbf{x}' \in \mathcal{D}^1(\mathbb{C},\mathcal{H})$ denote the infinite sum on the left hand side of this equation. From the definition of $\mathbf{x}^{\wedge R}(W)$ and the fact that $\mathbf{x}(0) = 0$, it is easy to see that

$$\mathbf{x}' \cdot (Z^{-1}W) = x^{\wedge R}(W). \tag{2.21}$$

Hence we may rewrite the first of equations (2.20) as $\mathbf{x}' = R_{Z^{-1}W}L_{Z\mathcal{A}}\mathbf{x}' + L_{Z\mathcal{B}}\mathbf{u}^{\wedge R}(W)$. We may then solve for $\mathbf{x}'$ to get $\mathbf{x}' = (I - R_{Z^{-1}W}L_{Z\mathcal{A}})^{-1}L_{Z\mathcal{B}}\mathbf{u}^{\wedge R}(W)$. Substituting this expression back into the second of equations (2.20) and remembering (2.21) gives

$$\mathbf{y}^{\wedge R}(W) = [\mathcal{C}(I - R_{Z^{-1}W}L_{Z\mathcal{A}})^{-1}R_{Z^{-1}W}L_{Z\mathcal{B}} + \mathcal{D}]u^{\wedge R}(W)$$

and the theorem follows. ∎

REMARK 2.4. The first and third author in [4] obtained a converse result on the realization of the type (2.19) starting with any contractive element S of $\mathcal{L}(\mathcal{E},\mathcal{E}_*)$. Here we shall arrive at a time-domain realization of the type (2.13) for a given contractive $S \in \mathcal{L}(\mathcal{E},\mathcal{E}_*)$, from which will follow a realization of the type (2.19) by the analysis above.

REMARK 2.5. Theorem 2.3 provides a time-varying analogue of the fact for the time-invariant case that the Laplace transform of the output signal is equal to the transfer function times the Laplace transform of the input signal (assuming zero initial condition). In the time-invariant case, there is a second interpretation of the transfer function, namely its role as the "frequency response function", whereby the steady-state output of the system resulting from a periodic input signal is a periodic signal of the same frequency but with amplitude equal to the modulus of the transfer function at the given frequency times the amplitude of the input signal and with phase shift equal to the phase of the value of the transfer function at the given frequency. A time-varying analogue of this property has been derived in [9].

2.3. PRELIMINARIES FOR CONNECTIONS WITH SCATTERING. To conclude this section, we derive a proposition concerning a parametrizing system trajectories which will be needed in Section 3. In the derivation of the input-output operator given above (for the (unaugmented) system Σ (2.1)), we indicated how an arbitrary element $u \in \ell^2(\mathbb{Z}, \mathcal{E})$ generates a whole system trajectory

$$(\vec{u}, \vec{x}, \vec{y}) = (\vec{u}, \mathcal{C}\vec{u}, T_\Sigma \vec{u})$$

for the system (2.1), essentially by running the system forward with the state initialized to be 0 at time $n = -\infty$. We now present another method for generating trajectories also lying in the signal space $\mathcal{S}$. For this discussion we assume that each system matrix $U_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$ is *unitary* rather than merely contractive, i.e., that

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \text{ is invertible with } \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}^{-1} = \begin{bmatrix} A_k^* & C_k^* \\ B_k^* & D_k^* \end{bmatrix} : \begin{bmatrix} \mathcal{H}_{k+1} \\ \mathcal{E}_{*k} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_k \\ \mathcal{E}_k \end{bmatrix}.$$

Fix a time $n \in \mathbb{Z}$; as free parameters for our trajectory, we consider an arbitrary element $(\vec{y}, x(n), \vec{u})$ in the *scattering data space at time* n, $\mathcal{K}_n$, given by

$$\mathcal{K}_n = \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*) \oplus \mathcal{H}_n \oplus \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}). \tag{2.22}$$

From $x(n) \in \mathcal{H}_n$ and $\vec{u} = \{u(j)\}_{j \geqslant n}$, we determine $x(j)$ for $j > n$ and $y(j)$ for $j \geqslant n$ from the recursion

$$x(j+1) = A_j x(j) + B_j u(j), \quad y(j) = C_j x(j) + D_j u(j). \tag{2.23}$$

Similarly, from $x(n)$ and $\vec{y} = \{y(j)\}_{j<n}$ we determine $x(j)$ and $u(j)$ for $j < n$ from the recursion

$$x(j) = A_j^* x(j+1) + C_j^* y(j), \quad u(j) = B_j^* x(j+1) + D_j^* y(j). \tag{2.24}$$

In this way we generate biinfinite extended sequences

$$(\vec{u}_{\rm e}, \vec{x}_{\rm e}, \vec{y}_{\rm e}) = (\{u(j)\}_{j \in \mathbb{Z}}, \{x(j)\}_{j \in \mathbb{Z}}, \{y(j)\}_{j \in \mathbb{Z}}).$$

From the assumed unitary property of each

$$U_j = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} : \begin{bmatrix} \mathcal{H}_j \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_{j+1} \\ \mathcal{E}_{*j} \end{bmatrix}$$

it is easy to see that the resulting triple $(\vec{u}, \vec{x}, \vec{y})$ is in the signal space $\mathcal{S}$ given by (2.7). Conversely, for each choice of time $n \in \mathbb{Z}$, it is clear that any system trajectory of the system Σ (given by (2.1)) lying in $\mathcal{S}$ arises in this way from an element of the associated scattering space $\mathcal{K}_n$ at time n. We summarize this discussion in the following.

PROPOSITION 2.6. *Let Σ be the time-varying system given by* (2.1) *with associated signal space $\mathcal{S}$ and scattering space at time n equal to $\mathcal{K}_n$ as given by* (2.1) *and* (2.22). *Define the map Π_n (the window map at time n) from $\mathcal{S}$ to $\mathcal{K}_n$ by*

$$\Pi_n : (\vec{u}, \vec{x}, \vec{y}) \to (\vec{y}|_{\{j : j<n\}}, x(n), \vec{u}|_{\{j : j \geqslant n\}}).$$

Then the restriction of Π_n to system trajectories in $\mathcal{S}$ is bijective from system trajectories in $\mathcal{S}$ onto $\mathcal{K}_n$ with inverse given by the two recursions (2.23) *and* (2.24).

3. TIME-VARYING SCATTERING SYSTEMS

We introduce here a time-varying version of a discrete-time Lax-Phillips scattering system. By a *time-varying scattering system* (TVSS) we shall mean a collection of objects

$$\mathfrak{S} = \{\mathcal{K} = \{\mathcal{K}_n\}, \mathcal{G} = \{\mathcal{G}_n\}, \mathcal{G}_* = \{\mathcal{G}_{*n}\}, \mathcal{U} = \{\mathcal{U}_n\}\} \tag{3.1}$$

such that, for each $n \in \mathbb{Z}$:

(1) Each $\mathcal{K}_n$ (called the *ambient space for* $\mathfrak{S}$ *at time* n) is a Hilbert space and $\mathcal{U}_n$ is a unitary operator from $\mathcal{K}_{n+1}$ onto $\mathcal{K}_n$.

(2) Each *outgoing subspace* $\mathcal{G}_n$ is a closed subspace for $\mathcal{K}_n$ such that $\mathcal{U}_n : \mathcal{G}_{n+1} \to \mathcal{G}_n$ and

$$\bigcap_{k=0}^{\infty} \mathcal{U}_n \mathcal{U}_{n+1} \cdots \mathcal{U}_{n+k} \mathcal{G}_{n+k+1} = \{0\} \quad \text{in } \mathcal{K}_n.$$

(3) Each *incoming subspace* $\mathcal{G}_{*n}$ is a closed subspace for $\mathcal{K}_n$ such that $\mathcal{U}_n^* : \mathcal{G}_{*n} \to \mathcal{G}_{*n+1}$ and

$$\bigcap_{k=0}^{\infty} \mathcal{U}_n^* \mathcal{U}_{n-1}^* \cdots \mathcal{U}_{n-k}^* \mathcal{G}_{*n-k} = \{0\} \quad \text{in } \mathcal{K}_{n+1}.$$

(4) $\mathcal{G}_n$ is orthogonal to $\mathcal{G}_{*n}$ in $\mathcal{K}_n$ for each n.

We shall occasionally have need of the subspace

$$\mathcal{H}_n := \mathcal{K}_n \ominus [\mathcal{G}_{*n} \oplus \mathcal{G}_n],$$

called the *scattering space* and sometimes also the *model space* for the TVSS $\mathfrak{S}$.

A convenient compact notation will be to define, for i and j any integers, the operator $\mathcal{U}_{[i,j]} : \mathcal{K}_j \to \mathcal{K}_i$ by

$$\mathcal{U}_{[i,j]} = \begin{cases} I: & i = j, \\ \mathcal{U}_i \mathcal{U}_{i+1} \cdots \mathcal{U}_{j-1}: & i < j, \\ \mathcal{U}_{i-1}^* \mathcal{U}_{i-2}^* \cdots \mathcal{U}_j^*: & i > j. \end{cases}$$

Note that $\mathcal{U}$ has the two-parameter semigroup property $\mathcal{U}_{[i,j]}\mathcal{U}_{[j,k]} = \mathcal{U}_{[i,k]}$ and the unitary representation property $(\mathcal{U}_{[i,j]})^* = \mathcal{U}_{[j,i]}$. Axioms (2) and (3) may be expressed more succinctly as

$$\bigcap_{j: j \geqslant n} \mathcal{U}_{[n,j]} \mathcal{G}_j = \{0\}, \qquad \bigcap_{j: j \leqslant n} \mathcal{U}_{[n,j]} \mathcal{G}_{*j} = \{0\},$$

for each $n \in \mathbb{Z}$. We shall say that the TVSS $\mathfrak{S}$ is *minimal* if it happens that $\widetilde{\mathcal{G}}_{*n} + \widetilde{\mathcal{G}}_n$ is dense in $\mathcal{K}_n$ for each n, where we have set

$$\widetilde{\mathcal{G}}_{*n} = \text{ closure} \bigcup_{j \geqslant n} \mathcal{U}_{[n,j]} \mathcal{G}_{*j},$$

$$\widetilde{\mathcal{G}}_n = \text{ closure} \bigcup_{j \leqslant n} \mathcal{U}_{[n,j]} \mathcal{G}_j.$$

As an example we now introduce the *free* TVSS $\mathfrak{S}^f$. Let $\{\mathcal{E}_k\}_{k \in \mathbb{Z}}$ be any family of Hilbert spaces indexed by the integers $k \in \mathbb{Z}$. Denote by $\mathcal{E}$ the whole aggregate of spaces $\mathcal{E} = \{\mathcal{E}_k\}_{k \in \mathbb{Z}}$ with associated ℓ^2 spaces $\ell^2(\mathbb{Z}, \mathcal{E})$, $\ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E})$

and $\ell^2(\mathbb{Z}_{<n}, \mathcal{E})$ as in Section 2. By the free TVSS $\mathfrak{S}^f$ (associated with the family $\{\mathcal{E}_k\}$), we mean the TVSS $\{\{\mathcal{K}_n^f\}, \{\mathcal{G}_n^f\}, \{\mathcal{G}_*^f\}, \{\mathcal{U}_n^f\}\}$ where

$$(3.2)\qquad \begin{aligned} &\mathcal{K}_n^f = \ell^2(\mathbb{Z}, \mathcal{E}) \text{ (independent of } n), \\ &\mathcal{G}_n^f = \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}), \\ &\mathcal{G}_{*n}^f = \ell^2(\mathbb{Z}_{<n}, \mathcal{E}), \text{ and} \\ &\mathcal{U}_n^f : \mathcal{K}_{n+1} \to \mathcal{K}_n \text{ is equal to the identity operator.} \end{aligned}$$

Here of course we are identifying $\ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E})$ and $\ell^2(\mathbb{Z}_{<n}, \mathcal{E})$ as subspaces of $\ell^2(\mathbb{Z}, \mathcal{E})$ in the canonical way. As $\mathcal{K}_n^f$ is in fact independent of n, it makes sense to use the simpler notation $\mathcal{K}^f$ for this space. Thus we see that indeed $\mathcal{U}_n^f = I : \mathcal{G}_{n+1}^f \to \mathcal{G}_n^f$ since $\mathcal{G}_{n+1}^f \subset \mathcal{G}_n^f$ and that $\mathcal{U}_n^{f*} = I : \mathcal{G}_{*n}^f \to \mathcal{G}_{*n+1}^f$ since $\mathcal{G}_{*n}^f \subset \mathcal{G}_{*n+1}^f$ under our canonical identifications. The rest of the axioms (1)–(4) for an TVSS are easily checked and it is also clear that $\mathfrak{S}^f$ is minimal.

The next goal is to understand how to view a general TVSS as a "scattering" between two free TVSSs. Let therefore $\{\mathcal{K}_n\}, \{\mathcal{G}_n\}, \{\mathcal{G}_{*n}\}, \{\mathcal{U}_n\}$ be a general TVSS. For $n \in \mathbb{Z}$, define subspaces $\mathcal{E}_n$ and $\mathcal{E}_{*n}$ by

$$\mathcal{E}_n = \mathcal{G}_n \ominus \mathcal{U}_n \mathcal{G}_{n+1} \subset \mathcal{K}_n, \quad \mathcal{E}_{*n} = \mathcal{G}_{*n+1} \ominus \mathcal{U}_n^* \mathcal{G}_{*n} \subset \mathcal{K}_{n+1}.$$

By using axioms (1)–(4) one can deduce that $\mathcal{G}_n$ and $\mathcal{G}_{*n}$ have the internal orthogonal direct sum decompositions

$$(3.3)\qquad \mathcal{G}_n = \bigoplus_{j \geqslant n} \mathcal{U}_{[n,j]} \mathcal{E}_j, \quad \mathcal{G}_{*n} = \bigoplus_{j \leqslant n} \mathcal{U}_{[n,j+1]} \mathcal{E}_{*j}.$$

Further application of the axioms (1)–(4) leads to the biinfinite internal orthogonal direct sum decompositions for $\widetilde{\mathcal{G}}_n$ and $\widetilde{\mathcal{G}}_{*n}$:

$$\widetilde{\mathcal{G}}_n = \bigoplus_{j \in \mathbb{Z}} \mathcal{U}_{[n,j]} \mathcal{E}_j, \quad \widetilde{\mathcal{G}}_{*n} = \bigoplus_{j \in \mathbb{Z}} \mathcal{U}_{[n,j+1]} \mathcal{E}_{*j}.$$

Let us define Fourier representations

$$\Phi_n : \mathcal{K}_n \to \mathcal{K}^f := \ell^2(\mathbb{Z}, \mathcal{E}), \quad \Phi_{*n} : \mathcal{K}_n \to \mathcal{K}^{*f} := \ell^2(\mathbb{Z}, \mathcal{E}_*)$$

by $\Phi_n(k_n) = \{[\Phi_n k_n]_j\}_{j=-\infty}^{\infty}$ and $\Phi_{*n}(k_n) = \{[\Phi_{*n} k_n]_j\}_{j=-\infty}^{\infty}$ (for $k_n \in \mathcal{K}_n$) by

$$(3.4)\qquad [\Phi_n k_n]_j = \pi_j \mathcal{U}_{[j,n]} k_n, \quad [\Phi_{*n} k_n]_j = \pi_{*j} \mathcal{U}_{[j+1,n]} k_n$$

where we have set

$$\pi_n : \mathcal{K}_n \to \mathcal{E}_n, \quad \pi_{*n} : \mathcal{K}_{n+1} \to \mathcal{E}_{*n}$$

equal to the orthogonal projection operators. Then Φ_n is a partial isometry from $\mathcal{K}_n$ onto $\mathcal{K}^f = \ell^2(\mathbb{Z}, \mathcal{E})$ with initial space equal to $\widetilde{\mathcal{G}}_n$ such that

$$\Phi_n(\mathcal{G}_n) = \mathcal{G}_n^f := \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E})$$

while Φ_{*n} is a partial isometry from $\mathcal{K}_n$ onto $\mathcal{K}^{*f} = \ell^2(\mathbb{Z}, \mathcal{E}_*)$ with initial space equal to $\widetilde{\mathcal{G}}_{*n}$ such that

$$\Phi_{*n}(\mathcal{G}_{*n}) = \mathcal{G}_{*n}^{*f} := \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*).$$

We next define the *scattering operator at time* n $S_{\mathfrak{S},n}$ for the TVSS $\mathfrak{S}$ as the operator

$$S_{\mathfrak{S},n} = \Phi_{*n}\Phi_n^* : \mathcal{K}^f \to \mathcal{K}^{*f}. \tag{3.5}$$

As $S_{\mathfrak{S},n}$ is an operator between the ℓ^2 spaces $\ell^2(\mathbb{Z},\mathcal{E})$ and $\ell^2(\mathbb{Z},\mathcal{E}_*)$ one can view $S_{\mathfrak{S},n}$ as multiplication by a biinfinite block matrix $[S^n_{i,j}]_{i,j\in\mathbb{Z}}$. Our next task is to compute these matrix entries.

Lemma 3.1. *Let the scattering operator at time* n*,* $S_{\mathfrak{S},n}$*, be defined as in* (3.5)*, and let*

$$S^n_{i,j} : \mathcal{E}_j \to \mathcal{E}_{*i}, \quad i,j \in \mathbb{Z}$$

be the matrix entries for $S_{\mathfrak{S},n}$ *when considered as an operator from* $\ell^2(\mathbb{Z},\mathcal{E}) \to \ell^2(\mathbb{Z},\mathcal{E}_*)$*. Then*

$$S^n_{i,j} = \begin{cases} 0 : & i < j \\ \pi_{*i}\mathcal{U}_{[i+1,j]}\pi_j^* : & i \geqslant j. \end{cases}$$

In particular, $S_{\mathfrak{S}} := S_{\mathfrak{S},n}$ *is independent of* n *and has a block lower triangular matrix representation.*

Proof. From the explicit form of the matrix entries Φ_n and Φ_{*n} for Φ and Φ_{*n} in (3.4), we can compute explicitly that

$$S^n_{i,j} = [\Phi_{*n}]_i[\Phi_n]_j^* = \pi_{*i}\mathcal{U}_{[i+1,n]}(\mathcal{U}_{[j,n]})^*\pi_j^* = \pi_{*i}\mathcal{U}_{[i+1,n]}\mathcal{U}_{[n,j]}\pi_j^* = \pi_{*i}\mathcal{U}_{[i+1,j]}\pi_j^*$$

and hence in particular $S_{i,j} := S^n_{i,j}$ is independent of n and we have the formula for $S_{i,j}$ for $i \geqslant j$. It remains only to verify that this formula produces $S_{i,j} = 0$ for $i < j$.

For $i = j-1$ we have $S_{j-1,j} = \pi_{*j-1}\pi_j^*$. This quantity being equal to 0 is the same as the subspaces $\mathcal{E}_{*j-1}$ and $\mathcal{E}_j$ being orthogonal in $\mathcal{K}_j$. But, by definition, $\mathcal{E}_{*j-1} \subset \mathcal{G}_{*j}$ and $\mathcal{E}_j \subset \mathcal{G}$ where $\mathcal{G}_{*j}$ and $\mathcal{G}_j$ are orthogonal in $\mathcal{K}_j$ by axiom (4) in the definition of TVSS. Hence $S_{j-1,j} = 0$ as asserted. For $i < j-1$, we have

$$S_{i,j} = \pi_{*i}\mathcal{U}_{[i+1,j]}\pi_j^* = \pi_{*i}\mathcal{U}_{i+1}\mathcal{U}_{i+2}\cdots\mathcal{U}_{j-1}\pi_j^*$$

where

$$\mathcal{U}_{i+1}\mathcal{U}_{i+2}\cdots\mathcal{U}_{j-1}\mathcal{E}_j \subset \mathcal{U}_{i+1}\mathcal{U}_{i+2}\cdots\mathcal{U}_{j-1}\mathcal{G}_j \subset \mathcal{G}_{i+1}$$

by an iteration of the first part of Axiom (2). Since $\mathcal{E}_{*i} \subset \mathcal{G}_{*i+1}$ and $\mathcal{G}_{*i+1}$ is orthogonal to $\mathcal{G}_{i+1}$, it again follows that $S_{i,j}$ must be 0 as before. The lemma follows. ∎

4. OPERATOR MODEL THEORY: COORDINATE-FREE VERSION

We now discuss a time-varying version of the coordinate-free model theory as developed in the work of Nikolskii and Vasyunin (see e.g. [30] and [31]). In place of a single contraction operator T on a Hilbert space $\mathcal{H}$, we are given a collection of Hilbert spaces $\{\mathcal{H}_n : n \in \mathbb{Z}\}$ and a collection of contraction operators (or *contractive family* $T_n : \mathcal{H}_{n+1} \to \mathcal{H}_n$.) In place of a unitary dilation $\mathcal{U}$ of the single contraction operator T on a Hilbert space $\mathcal{K} \supset \mathcal{H}$, we consider a collection of Hilbert spaces $\mathcal{K}_n \supset \mathcal{H}_n$ ($n \in \mathbb{Z}$) and a collection of unitary operators (or *unitary family*) $\mathcal{U}_n : \mathcal{K}_{n+1} \to \mathcal{K}_n$ such that

$$T_n P^{\mathcal{K}_{n+1}}_{\mathcal{H}_{n+1}} = P^{\mathcal{K}_n}_{\mathcal{H}_n} \mathcal{U}_n \quad \text{for all } n \in \mathbb{Z}$$

where in general $P^{\mathcal{K}}_{\mathcal{H}}$ is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$ whenever $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces with $\mathcal{H} \subset \mathcal{K}$. One way to construct such a unitary-family dilation $\{\mathcal{U}_n : n \in \mathbb{Z}\}$ of the contractive family $\{T_n : n \in \mathbb{Z}\}$ is to construct a Halmos unitary dilation of T_n for each n, namely, a unitary operator V_n of the form

$$V_n = \begin{bmatrix} T_n & \beta_n \\ \gamma_n & \delta_n \end{bmatrix} : \begin{bmatrix} \mathcal{H}_{n+1} \\ \mathcal{E}_{*n} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_n \\ \mathcal{E}_n \end{bmatrix}$$

for appropriate defect spaces $\mathcal{E}_{*n}$ and $\mathcal{E}_n$, where β_n is injective and γ_n is surjective. The Halmos dilation is unique up to unitary equivalence; a particular representation is obtained by taking $\mathcal{E}_n = \mathcal{D}_{T_n}$ and $\mathcal{E}_{*n} = \mathcal{D}_{T_n^*}$ to be the defect spaces given for a general contraction operator T by

$$\mathcal{D}_T = \text{clos im}\, D_T \quad \text{where } D_T = (I - T^*T)^{\frac{1}{2}}$$

and then set

$$\begin{aligned} \beta_n &= D_{T_n^*} : \mathcal{D}_{T_n^*} \to \mathcal{H}_n, \\ \gamma_n &= D_{T_n} : \mathcal{H}_{n+1} \to \mathcal{D}_{T_n}, \\ \delta_n &= -T_n^* : \mathcal{D}_{T_n^*} \to \mathcal{D}_{T_n}. \end{aligned} \tag{4.1}$$

One then defines $\mathcal{K}_n$ by

$$\mathcal{K}_n = \begin{bmatrix} \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*) \\ \mathcal{H}_n \\ \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}) \end{bmatrix}$$

where we have set $\mathcal{E}$ equal to the aggregate $\{\mathcal{E}_n : n \in \mathbb{Z}\}$ and $\mathcal{E}_*$ equal to $\{\mathcal{E}_{*n} : n \in \mathbb{Z}\}$ and are using the notation (2.5) introduced in Section 2. Define the projection operators (as in Section 3)

$$\pi_{*n} : \ell^2(\mathbb{Z}_{<n+1}, \mathcal{E}_*) \to \mathcal{E}_{*n}, \quad \pi_n : \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}) \to \mathcal{E}_n$$

by

$$\pi_{*n} : \begin{bmatrix} \vdots \\ e_{*n-1} \\ e_{*n} \end{bmatrix} \to e_{*n}, \quad \pi_n : \begin{bmatrix} e_n \\ e_{n+1} \\ \vdots \end{bmatrix} \to e_n$$

with adjoints equal to the inclusion maps

$$\pi_{*n+1}^* : e_{*n} \to \begin{bmatrix} \vdots \\ 0 \\ e_{*n} \end{bmatrix}, \quad \pi_n^* : e_n \to \begin{bmatrix} e_n \\ 0 \\ \vdots \end{bmatrix}.$$

We then consider $\mathcal{H}_n$ as a subspace of $\mathcal{K}_n$ in the canonical way and define $\mathcal{U}_n : \mathcal{K}_{n+1} \to \mathcal{K}_n$ by

$$\mathcal{U}_n = \begin{bmatrix} S_{-,n} & 0 & 0 \\ \beta\pi_{*n+1} & T_n & 0 \\ \pi_n^*\delta & \pi_n^*\gamma & S_n \end{bmatrix},$$

where we have set

$$S_{-,n} : (\dots, y_{n-2}, y_{n-1}, y_n) \to (\dots, y_{n-2}, y_{n-1})$$
$$S_n : (u_{n+1}, u_{n+2}, u_{n+3}, \dots) \to (0, u_{n+1}, u_{n+2}, u_{n+3}, \dots).$$

In more coordinate-free form, we simply assume that we have such a unitary-family dilation $\{\mathcal{U}_n\}$. For each $n \in \mathbb{Z}$ set

$$\mathcal{K}_{+n} = \operatorname{clos} \cup \{\mathcal{U}_{[n,j]}\mathcal{H}_j : j \geqslant n\}, \quad \mathcal{K}_{-n} = \operatorname{clos} \cup \{\mathcal{U}_{[n,j]}\mathcal{H}_j : j \leqslant n\}$$

and then define subspaces $\mathcal{G}_n$ and $\mathcal{G}_{*n}$ of $\mathcal{K}_n$ by

$$\mathcal{G}_n = \mathcal{K}_{+n} \ominus \mathcal{H}_n, \quad \mathcal{G}_{*n} = \mathcal{K}_{-n} \ominus \mathcal{H}_n.$$

We may then define wandering subspaces

$$\mathcal{E}_n = \mathcal{G}_n \ominus \mathcal{U}_n\mathcal{G}_{n+1}, \quad \mathcal{E}_{*n} = \mathcal{G}_{*n+1} \ominus \mathcal{U}_n^*\mathcal{G}_{*n}$$

and observe that we have the internal, orthogonal direct sum decompositions

$$\mathcal{G}_n = \bigoplus_{j=n}^{\infty} \mathcal{U}_{[n,j]}\mathcal{E}_j, \quad \mathcal{G}_{*n} = \bigoplus_{j=-\infty}^{n-1} \mathcal{U}_{[n,j+1]}\mathcal{E}_{*j}.$$

In short, it follows that $\{\mathcal{U}_n, \mathcal{G}_n, \mathcal{G}_{*n}\}$ is a TVSS as defined in Section 3. If the contractive family $\{T_n\}$ is *completely nonunitary* in the sense that

$$\mathcal{H}_n^{(2)} := \{h_n \in \mathcal{H}_n : \dots = \|T_{n-1}h_n\| = \|h_n\| = \|T_n^*h_n\| = \|T_{n+1}^*T_n^*h_n\| = \dots\} = \{0\}$$

and if the unitary-family dilation $\{\mathcal{U}_n\}$ is minimal (as is the case if one uses the above construction with a Halmos dilation V_n of T_n for each n), then it can be shown that the associated TVSS is minimal, i.e.

$$\mathcal{K}_n = \operatorname{clos}[\widetilde{\mathcal{G}}_{*n} + \widetilde{\mathcal{G}}_n]$$

for each $n \in \mathbb{Z}$.

We then have Fourier representations $\Phi_n : \mathcal{K}_n \to \ell^2(\mathbb{Z}, \mathcal{E})$ and $\Phi_{*n} : \mathcal{K}_n \to \ell^2(\mathbb{Z}, \mathcal{E}_*)$ and a scattering operator

$$S_{\mathfrak{S}} = \Phi_{*n}\Phi_n^* : \ell^2(\mathbb{Z}, \mathcal{E}) \to \ell^2(\mathbb{Z}, \mathcal{E}_*)$$

as in Section 3. In this context, the scattering operator is called the *characteristic operator* for the contractive family $\mathcal{T} := \{T_n : n \in \mathbb{Z}\}$ and denoted by $\Theta_{\mathcal{T}}$:

$$\Theta_{\mathcal{T}} = \Phi_{*n}\Phi_n^* = S_{\mathfrak{S}}.$$

A *functional model* associated with a given lower triangular, contractive element Θ of $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ is a method for constructing a particular contractive family $\mathcal{T}(\Theta)$ in such a way that we recover a given contractive family $\mathcal{T}$ from the model built from its characteristic operator $\Theta_{\mathcal{T}}$ up to unitary equivalence:

$$\mathcal{T} \cong \mathcal{T}(\Theta_{\mathcal{T}}).$$

Generally, there will also be a model for the associated TVSS incorporated in any such construction. We shall return to this topic of models in Section 6.

We have seen that operator model theory and scattering are closely connected, and that by definition the characteristic operator $\Theta_{\mathcal{T}}$ for a contractive family $\mathcal{T}$ is equal to the scattering operator $S_{\mathfrak{S}}$ for the associated scattering system $\mathfrak{S}$. In the next section we make explicit the connections between scattering/model theory and the time-varying unitary linear systems discussed in Section 2.

5. EQUIVALENCE OF SCATTERING AND MODEL THEORY WITH UNITARY LINEAR SYSTEMS

Given a unitary time-varying system (2.1) (where now we assume that each $U_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$ is *unitary*), we associate a time-varying scattering system (TVSS)

$$\mathfrak{S} = \mathfrak{S}(\Sigma) = \{\mathcal{K} = \{\mathcal{K}_n\}_{n\in\mathbb{Z}},\, \mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{Z}},\, \mathcal{G}_* = \{\mathcal{G}_{*n}\}_{n\in\mathbb{Z}},\, \mathcal{U} = \{\mathcal{U}_n\}_{n\in\mathbb{Z}}\}$$

as follows. Define

$$(5.1)\qquad \begin{aligned} \mathcal{K}_n &= \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*) \oplus \mathcal{H}_n \oplus \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}), \\ \mathcal{G}_{*n} &= \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*), \quad \mathcal{G}_n = \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}) \end{aligned}$$

with $\mathcal{U}_n : \mathcal{K}_{n+1} \to \mathcal{K}_n$ defined by

$$(5.2)\qquad \mathcal{U}_n : \Pi_{n+1}(\vec{u}, \vec{x}, \vec{y}) \to \Pi_n(\vec{u}, \vec{x}, \vec{y})$$

where $\Pi_n : \mathcal{S} \to \mathcal{K}_n$ is the "window operator" at time n for the system Σ defined in Proposition 2.4, and where $(\vec{u}, \vec{x}, \vec{y})$ is an arbitrary system trajectory for the system Σ in the signal space $\mathcal{S}$ (see (2.7)). To see that the expression (5.2) gives rise to a well-defined unitary operator, it suffices to offer a more explicit alternative representation for $\mathcal{U}_n$. Indeed, note that

$$\begin{aligned} \Pi_{n+1}(\vec{u}, \vec{x}, \vec{y}) &= \vec{y}|_{\{j: j\leqslant n\}} \oplus x(n+1) \oplus \vec{u}|_{\{j:j>n\}}), \\ \Pi_n(\vec{u}, \vec{x}, \vec{y}) &= \vec{y}_{\{j:j<n\}} \oplus x(n) \oplus \vec{u}|_{\{j:j\geqslant n\}} \end{aligned}$$

where, in addition,

$$x(n) = A_n^* x(n+1) + C_n^* y(n), \quad u(n) = B_n^* x(n+1) + D_n^* y(n).$$

Hence, a more explicit representation for $\mathcal{U}_n$, viewed as a 3×3 block operator matrix

$$\mathcal{U}_n : \begin{bmatrix} \ell^2(\mathbb{Z}_{\leqslant n}, \mathcal{E}_*) \\ \mathcal{H}_{n+1} \\ \ell^2(\mathbb{Z}_{>n}, \mathcal{E}) \end{bmatrix} \to \begin{bmatrix} \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*) \\ \mathcal{H}_n \\ \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}) \end{bmatrix}$$

is

$$\mathcal{U}_n = \begin{bmatrix} S_{-,n} & 0 & 0 \\ C_n^* \pi_{*n} & A_n^* & 0 \\ \pi_n^* D_n^* \pi_{*n} & B_n^* & S_n \end{bmatrix} \tag{5.3}$$

where the operators

$$\begin{aligned} S_{-,n} &: \ell^2(\mathbb{Z}_{\leqslant n}, \mathcal{E}_*) \to \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*) \\ \pi_{*n} &: \ell^2(\mathbb{Z}_{\leqslant n}, \mathcal{E}_*) \to \mathcal{E}_{*n} \\ \pi_n^* &: \mathcal{E}_n \to \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}) \\ S_n &: \ell^2(\mathbb{Z}_{>n}, \mathcal{E}) \to \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}) \end{aligned}$$

are given explicitly by

$$\begin{aligned} S_{-,n} &: (\dots, y_{n-2}, y_{n-1}, y_n) \to (\dots, y_{n-2}, y_{n-1}) \\ \pi_{*n} &: (\dots, y_{n-2}, y_{n-1}, y_n) \to y_n \\ \pi_n^* &: u_n \to (u_n, 0, 0, 0, \dots) \\ S_n &: (u_{n+1}, u_{n+2}, u_{n+3}, \dots) \to (0, u_{n+1}, u_{n+2}, u_{n+3}, \dots). \end{aligned}$$

From this representation and the unitary property of $\begin{bmatrix} A_n^* & C_n^* \\ B_n^* & D_n^* \end{bmatrix}$, it is easily checked directly that $\mathcal{U}_n$ is well-defined and unitary from $\mathcal{K}_{n+1}$ onto $\mathcal{K}_n$. One can also check that the TVSS axioms (1)–(4) hold for the system $\mathfrak{S} = \mathfrak{S}(\Sigma)$ as defined by (5.1) for a given unitary system Σ (2.1).

Conversely, given a scattering system $\mathfrak{S}$ as in (3.1) (for which axioms (1)–(4) are satisfied), we associate a unitary system

$$\Sigma = \Sigma(\mathfrak{S}) = \left\{ U_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} : \begin{bmatrix} \mathcal{H}_n \\ \mathcal{E}_n \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_{n+1} \\ \mathcal{E}_{*n} \end{bmatrix} \right\}$$

by the following procedure. Define Hilbert spaces $\mathcal{E}_n$, $\mathcal{E}_{*n}$ and $\mathcal{H}_n$ (equal to subspaces of $\mathcal{K}_n$) by

$$\mathcal{E}_n = \mathcal{G}_n \ominus \mathcal{U}_n \mathcal{G}_{n+1}, \quad \mathcal{E}_{*n} = \mathcal{G}_{*n+1} \ominus \mathcal{U}_n^* \mathcal{G}_{*n}, \quad \mathcal{H}_n = \mathcal{K}_n \ominus [\mathcal{G}_n \oplus \mathcal{G}_{*n}],$$

and the operator $U_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ by

$$U_n = P_{\mathcal{H}_{n+1} \oplus \mathcal{E}_{*n}} \mathcal{U}_n^* |_{\mathcal{H}_n \oplus \mathcal{E}_n}. \tag{5.4}$$

where $P_{\mathcal{H}_{n+1} \oplus \mathcal{E}_{*n}}$ is the orthogonal projection of $\mathcal{K}_{n+1}$ onto $\mathcal{H}_{n+1} \oplus \mathcal{E}_{*n}$. A consequence of the scattering axioms is that in fact $\mathcal{U}_n^*(\mathcal{H}_n \oplus \mathcal{E}_n) = \mathcal{E}_{*n} \oplus \mathcal{H}_{n+1}$ from which it follows that U_n is unitary since $\mathcal{U}_n^*$ is unitary. We may view the time-varying unitary linear system associated with $\{U_n\}$

$$\Sigma = \Sigma(\mathfrak{S}) : \begin{cases} x(n+1) = A_n x(n) + B_n u(n), \\ y(n) = C_n x(n) + D_n u(n), \end{cases}$$

as being associated with our original TVSS $\mathfrak{S}$. Moreover, it is clear that

$$\Sigma(\mathfrak{S}(\Sigma)) = \Sigma, \quad \mathfrak{S}(\Sigma(\mathfrak{S})) = \mathfrak{S}$$

for any unitary time-varying system Σ and TVSS $\mathfrak{S}$. We have thus established a one-to-one correspondence $\Sigma \to \mathfrak{S}(\Sigma)$ and $\mathfrak{S} \to \Sigma(\mathfrak{S})$ between unitary time-varying systems Σ and TVSSs $\mathfrak{S}$.

Recall that we have associated an input-output operator T_Σ with any contractive (in particular, unitary) time-varying system Σ and a scattering operator $S_\mathfrak{S}$ with any TVSS $\mathfrak{S}$. The next result establishes that these objects are identical if $\Sigma = \Sigma(\mathfrak{S})$, or equivalently, if $\mathfrak{S} = \mathfrak{S}(\Sigma)$.

THEOREM 5.1. *Let Σ be a time-varying linear system as in* (2.1) *for which* $U_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ *is unitary for each n, let $\mathfrak{S}$ be an* TVSS *as in* (3.1)*, and suppose that Σ and $\mathfrak{S}$ correspond to each other in the correspondence described above:*

$$\Sigma = \Sigma(\mathfrak{S}), \quad \mathfrak{S} = \mathfrak{S}(\Sigma).$$

Let T_Σ be the input-output operator associated with the system Σ and let $S_\mathfrak{S}$ be the scattering operator associated with the TVSS $\mathfrak{S}$*. Then*

$$T_\Sigma = S_\mathfrak{S} : \ell^2(\mathbb{Z}, \mathcal{E}) \to \ell^2(\mathbb{Z}, \mathcal{E}_*).$$

Moreover, the scattering system $\mathfrak{S}$ is minimal if and only if the unitary input-state-output system Σ is closely connected, i.e., if and only if the span of the time-n reachability space $\mathcal{R}_{\Sigma,n}$ given by (2.8) *and the time-n observability space $\mathcal{O}_{\Sigma,n}$ given by* (2.9) *is dense in $\mathcal{H}_n$ for each $n = \dots, -1, 0, 1, \dots$.*

Proof. It suffices to show that $T_\Sigma = S_\mathfrak{S}$ on $\ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E})$ for each $n \in \mathbb{Z}$. Therefore we fix an $n \in \mathbb{Z}$, choose a $\vec{u} = \{u(j)\}_{j \geqslant n} \in \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E})$ and set $\vec{y}_{\rm scat} = \{y_{\rm scat}(i)\}_{i \geqslant n} = S_\mathfrak{S}\vec{u}$ and $\vec{y} = \{y(j)\}_{j \geqslant n} = T_\Sigma \vec{u}$. The goal is to show that $y_{\rm scat}(i) = y(i)$ for each $i = n, n+1, \dots$. First we note that

$$\begin{aligned} y_{\rm scat}(i) &= \sum_{k=n}^{i} S_{i,k} u(k) = \sum_{k=n}^{i} \pi_{*i} \mathcal{U}_{[i+1,k]} \pi_k^* u(k) \\ &= \sum_{k=n}^{i} \pi_{*i} \mathcal{U}_i^* \mathcal{U}_{i-1}^* \cdots \mathcal{U}_k^* \pi_k^* u(k). \end{aligned} \tag{5.5}$$

On the other hand, $y(i)$ for $i \geqslant n$ is determined by the recursion

$$\begin{aligned} x(i+1) &= A_i x(i) + B_i u(i), \quad x(n) = 0 \\ y(i) &= C_i x(i) + D_i u(i). \end{aligned} \tag{5.6}$$

In particular, by using (5.3) or (5.4), we see that

$$y(n) = D_n u(n) = \pi_{*n} \mathcal{U}_n^* u(n) = [S_{n,n}] u(n) = y_{\rm scat}(n)$$

so the assertion holds for $i = n$. More generally a simple induction argument using (5.3) and (5.4) gives that the solution of the recursion (5.6) satisfies

$$\begin{bmatrix} x(i+1) \\ y(i) \end{bmatrix} = P_{\mathcal{H}_{i+1} \oplus \mathcal{E}_{*i}} \mathcal{U}_i^* \mathcal{U}_{i-1}^* \cdots \mathcal{U}_n^* \vec{u}$$

for $i = n, n+1, \dots$. In particular it follows that $y(i) = P_{\mathcal{E}_{*i}} \mathcal{U}_i^* \mathcal{U}_{i-1}^* \cdots \mathcal{U}_n^* \vec{u}$ (where $P_{\mathcal{E}_{*i}}$ is just another notation for π_{*i}). Comparison now with (5.5) shows that $y(i) = y_{\rm scat}(i)$ as claimed.

To prove the last statement, one notes that in general $(\widetilde{\mathcal{G}}_n + \widetilde{\mathcal{G}}_{*n})^\perp$ is contained in the time-n scattering space $\mathcal{H}_n$, and is identical to the orthogonal complement of $\mathcal{R}_{\Sigma,n} + \mathcal{O}_{\Sigma,n}$ inside $\mathcal{H}_n$. ∎

As a corollary we obtain the following.

COROLLARY 5.2. *Let $\mathcal{T} = \{T_n : \mathcal{H}_{n+1} \to \mathcal{H}_n : n \in \mathbb{Z}\}$ be a contractive family. Let*

$$U_n = \begin{bmatrix} T_n^* & B_n \\ C_n & D_n \end{bmatrix} : \begin{bmatrix} \mathcal{H}_n \\ \mathcal{E}_n \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_{n+1} \\ \mathcal{E}_{*n} \end{bmatrix}$$

be a Halmos unitary dilation of T_n^* and let $\mathcal{U}_n : \mathcal{K}_{n+1} \to \mathcal{K}_n$ be the unitary-family dilation of $\mathcal{T}$ associated with U_n via the construction above. Then the characteristic operator $\Theta_{\mathcal{T}}$ for $\mathcal{T}$ coincides with the input-output operator T_Σ for the time-varying, unitary, linear system

$$x(n+1) = T_n^* x(n) + B_n u(n), \quad y(n) = C_n x(n) + D_n u(n).$$

6. MODEL TRANSCRIPTIONS

Let us consider an TVSS $\mathfrak{S}$ as in (3.1). We have seen that there are then two Fourier representations Φ_n and Φ_{*n} which are partial isometries mapping $\mathcal{K}_n$ onto $\ell^2(\mathbb{Z}, \mathcal{E})$ and $\ell^2(\mathbb{Z}, \mathcal{E}_*)$ respectively. Furthermore, the initial space for Φ_n is $\widetilde{\mathcal{G}}_n$ while the initial space for Φ_{*n} is $\widetilde{\mathcal{G}}_{*n}$, and, if we assume that our TVSS is minimal, we have that $\widetilde{\mathcal{G}}_n + \widetilde{\mathcal{G}}_{*n}$ is dense in $\mathcal{K}_n$. Hence under this minimality assumption, if we define a map $\widehat{\Phi}_n : \widetilde{\mathcal{G}}_{*n} + \widetilde{\mathcal{G}}_n \to \ell^2(\mathbb{Z}, \mathcal{E}_*) \oplus \ell^2(\mathbb{Z}, \mathcal{E})$ by

$$\widehat{\Phi}_n : g_{*n} + g_n \to \begin{bmatrix} \Phi_{*n} g_{*n} \\ \Phi_n g_n \end{bmatrix} \quad \text{for } g_{*n} \in \widetilde{\mathcal{G}}_{*n} \text{ and } g_n \in \widetilde{\mathcal{G}}_n,$$

then $\widehat{\Phi}_n$ is a (not necessarily well-defined) linear mapping from a dense subset of $\mathcal{K}_n$ onto $\ell^2(\mathbb{Z}, \mathcal{E}_*) \oplus \ell^2(\mathbb{Z}, \mathcal{E})$. In the terminology of [30] (adapted to our time-varying setting), various model transcriptions then amount to a particular choice of linear mapping (involving only the scattering operator S)

$$\widehat{\Pi}_S^M = \begin{bmatrix} \Pi_{S*}^M & \Pi_S^M \end{bmatrix} : \begin{bmatrix} \ell^2(\mathbb{Z}, \mathcal{E}_*) \\ \ell^2(\mathbb{Z}, \mathcal{E}) \end{bmatrix} \to \widetilde{\mathcal{K}}^M(S) \tag{6.1}$$

where $\widetilde{\mathcal{K}}^M(S)$ is a pre-model space for the model M. One then defines an inner product on $\widetilde{\mathcal{K}}_S^M$ so as to make the composite identification map

$$\mathcal{I}_{S,n}^M := \widehat{\Pi}_S^M \circ \widehat{\Phi}_n : \widetilde{\mathcal{G}}_{*n} + \widetilde{\mathcal{G}}_n \to \mathcal{K}^M(S)$$

an isometry (where $\widetilde{\mathcal{G}}_{*n} + \widetilde{\mathcal{G}}_n$ is considered with the inner product inherited from $\mathcal{K}_n$); furthermore, in all the examples this inner product turns out to be independent of n. Then the scattering model space $\mathcal{K}_S^M$ is obtained as the completion (if necessary) of $\widetilde{\mathcal{K}}_S^M$ in this inner product (after identification of elements of zero norm with the zero element, if necessary). Then $\mathcal{I}_{S,n}^M$ extends uniquely to a unitary operator (also denoted by $\mathcal{I}_{S,n}^M$) from $\mathcal{K}_n$ onto $\mathcal{K}^M(S)$. One then defines the spaces

$$\mathcal{H}^M(S) = \mathcal{I}_{S,n}^M \mathcal{H}_n, \quad \mathcal{G}_{*n}^M(S) = \mathcal{I}_{S,n}^M \mathcal{G}_{*n}, \quad \mathcal{G}_n^M(S) = \mathcal{I}_{S,n}^M \mathcal{G}_n$$

and the operators

$$\mathcal{U}^M_{S,n} = I \text{ on } \mathcal{K}^M(S), \quad T^M_{S,n} = P_{\mathcal{H}^M_n(S)}|_{\mathcal{H}^M_{n+1}(S)}$$

to get a TVSS $\mathfrak{S}^M_S = \{\mathcal{U}^M_{S,n}\}, \{\mathcal{G}^M_n(S)\}, \{\mathcal{G}^M_{*n}(S))\}\}$ with associated model contractive family $\mathcal{T}^M_S = \{T^M_{S,n} : n \in \mathbb{Z}\}$.

We now illustrate instances of this construction with three popular examples (see [30]), namely: (1) the *Pavlov model*, (2) the *Sz.-Nagy–Foiaş model* and (3) the *de Branges-Rovnyak model.* We first note the following general inner product identity concerning TVSSs.

LEMMA 6.1. *Let $\mathfrak{S} = \{\mathcal{U}_n, \mathcal{G}_n, \mathcal{G}_{*n}\}$ be a* TVSS *with associated Fourier operators Φ_{*n}, Φ_n and with scattering operator S. Then, for any pair of elements g_n, g'_n in $\widetilde{\mathcal{G}}_n$ and pair of elements g_{*n}, g'_{*n} in $\widetilde{\mathcal{G}}_{*n}$, we have*

$$\langle g_{*n} + g_n, g'_{*n} + g'_n \rangle_{\mathcal{K}_n} = \left\langle \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} \Phi_{*n} g_{*n} \\ \Phi_n g_n \end{bmatrix}, \begin{bmatrix} \Phi_{*n} g'_{*n} \\ \Phi_n g'_n \end{bmatrix} \right\rangle_{\ell^2(\mathbb{Z},\mathcal{E}_*)\oplus\ell^2(\mathbb{Z},\mathcal{E})}.$$

Proof. Recall that g_{*n} and g'_{*n} are in the initial space of Φ_{*n}, that g_n and g'_n are in the initial space of Φ_n, and that $S = \Phi_{*n}\Phi^*_n$ (independently of n). Hence,

$$\begin{aligned} \langle g_{*n} + g_n, g'_{*n} + g'_n \rangle_{\mathcal{K}_n} &= \langle g_{*n}, g'_{*n}\rangle + \langle g_{*n}, g'_n\rangle + \langle g_n, g'_{*n}\rangle + \langle g_n, g'_n\rangle \\ &= \langle \Phi_{*n} g_{*n}, \Phi_{*n} g'_{*n}\rangle + \langle \Phi^*_{*n}\Phi_{*n} g_{*n}, \Phi^*_n \Phi_n g'_n\rangle \\ &\quad + \langle \Phi^*_n \Phi_n g_n, \Phi^*_{*n}\Phi_{*n} g'_{*n}\rangle + \langle \Phi_n g_n, \Phi_n g'_n\rangle \\ &= \langle \Phi_{*n} g_{*n}, \Phi_{*n} g'_{*n}\rangle + \langle S^*\Phi_{*n} g_{*n}, \Phi_n g'_n\rangle \\ &\quad + \langle S\Phi_n g_n, \Phi_{*n} g'_{*n}\rangle + \langle \Phi_n g_n, \Phi_n g'_n\rangle \\ &= \left\langle \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} \Phi_{*n} g_{*n} \\ \Phi_n g_n \end{bmatrix}, \begin{bmatrix} \Phi_{*n} g'_{*n} \\ \Phi_n g'_n \end{bmatrix} \right\rangle_{\ell^2(\mathbb{Z},\mathcal{E}_*)\oplus\ell^2(\mathbb{Z},\mathcal{E})}. \quad \blacksquare \end{aligned}$$

6.1. THE PAVLOV MODEL. The Pavlov model (see [30]) is the most convenient model for studying scattering systems (as opposed to the study of models for a contractive family $\mathcal{T}$). For this case we simply take $\widetilde{\mathcal{K}}^{\mathrm{P}}(S)$ to be equal to $\ell^2(\mathbb{Z}, \mathcal{E}_*) \oplus \ell^2(\mathbb{Z}, \mathcal{E})$ with $\widehat{\Pi}^{\mathrm{P}}_S$ equal to the identity operator. As a consequence of Lemma 6.1, we take the inner product on $\widetilde{\mathcal{K}}^{\mathrm{P}}(S)$ to be given by

$$\left\langle \begin{bmatrix} h_* \\ h \end{bmatrix}, \begin{bmatrix} h'_* \\ h' \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} h_* \\ h \end{bmatrix}, \begin{bmatrix} h'_* \\ h' \end{bmatrix} \right\rangle_{\ell^2(\mathbb{Z},\mathcal{E}_*)\oplus\ell^2(\mathbb{Z},\mathcal{E})} \tag{6.2}$$

with $\mathcal{K}^{\mathrm{P}}(S)$ equal to the completion of $\widetilde{\mathcal{K}}^{\mathrm{P}}(S)$ in this inner product, where elements of zero norm are identified to 0. Then the associated incoming and outgoing spaces are given by

$$\mathcal{G}^{\mathrm{P}}_n(S) = \begin{bmatrix} 0 \\ \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}) \end{bmatrix}, \quad \mathcal{G}^{\mathrm{P}}_{*n}(S) = \begin{bmatrix} \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*) \\ 0 \end{bmatrix} \tag{6.3}$$

with model space family $\mathcal{H}^{\mathrm{P}}_n(S)$ given formally by

$$\mathcal{H}^{\mathrm{P}}_n(S) = \begin{bmatrix} I & S \\ S^* & I \end{bmatrix}^{-1} \begin{bmatrix} \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}_*) \\ \ell^2(\mathbb{Z}_{<n}, \mathcal{E}) \end{bmatrix}$$

with model contraction family equal to

$$T^{\mathrm{P}}_{S,n} = P_{\mathcal{H}^{\mathrm{P}}_{S,n}}|_{\mathcal{H}^{\mathrm{P}}_{S,n+1}}.$$

We will not compute $T^{\mathrm{P}}_{S,n}$ more explicitly here.

By using the Pavlov model it is straightforward to see that there always exists a TVSS with given scattering function $S \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$. The result is as follows.

THEOREM 6.2. *Let S be a given contractive element of $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ (for given families $\mathcal{E} = \{\mathcal{E}_n\}_{n\in\mathbb{Z}}$ and $\mathcal{E}_* = \{\mathcal{E}_{*n}\}_{n\in\mathbb{Z}}$). Let $\mathcal{K}^{\mathrm{P}}_n(S)$ be the completion of equivalence classes of $\ell^2(\mathbb{Z}, \mathcal{E}_*) \oplus \ell^2(\mathbb{Z}, \mathcal{E})$ in the inner product* (6.2) *(independent of n) with $\mathcal{G}^{\mathrm{P}}_n(S)$ and $\mathcal{G}^{\mathrm{P}}_{*n}(S)$ given by* (6.3), *and with $\mathcal{U}^{\mathrm{P}}_{S,n}$ equal to the identity operator on $\mathcal{K}^{\mathrm{P}}(S)$ for all n.*

(i) *Then*

$$\mathfrak{S}^{\mathrm{P}}_S := \{\{\mathcal{U}^{\mathrm{P}}_{S,n}\}, \{\mathcal{G}^{\mathrm{P}}_n(S)\}, \{\mathcal{G}^{\mathrm{P}}_{*n}(S)\}\}$$

is a TVSS *with scattering operator coinciding with S. More precisely, there are unitary identification maps*

$$i_n : \mathcal{E}_n \to \mathcal{E}^{\mathrm{P}}_n := \mathcal{G}^{\mathrm{P}}_n(S) \ominus \mathcal{G}^{\mathrm{P}}_{n+1}(S)$$
$$i_{*n} : \mathcal{E}_{*n} \to \mathcal{E}^{\mathrm{P}}_{*n} := \mathcal{G}^{\mathrm{P}}_{*n+1}(S) \ominus \mathcal{G}^{\mathrm{P}}_{*n}(S)$$

so that

$$i_*^* S_{\mathfrak{S}^{\mathrm{P}}_S} i = S,$$

where $i_* : \ell^2(\mathbb{Z}, \mathcal{E}_*) \to \ell^2(\mathbb{Z}, \mathcal{E}^{\mathrm{P}}_*)$ and $i : \ell^2(\mathbb{Z}, \mathcal{E}) \to \ell^2(\mathbb{Z}, \mathcal{E}^{\mathrm{P}})$ are obtained as

$$i_* = \operatorname{diag}\{i_n\}_{n\in\mathbb{Z}}, \quad i = \operatorname{diag}\{i_n\}_{n\in\mathbb{Z}}.$$

(ii) *Then S can be realized as the input-output operator (or transfer function in the time domain)*

$$S = \lim_{r\uparrow 1}(\mathcal{D} + \mathcal{C}(I - rZ\mathcal{A})^{-1}Z\mathcal{B})$$

(with $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ diagonal operators as in (2.10)*) of the time-varying unitary system*

$$\Sigma^{\mathrm{P}}_S : \begin{cases} x(n+1) = A(n)x(n) + B(n)u(n), \\ y(n) = C(n)x(n) + D(n)u(n), \end{cases}$$

where $U^{\mathrm{P}}_{S,n} = \begin{bmatrix} A(n) & B(n) \\ C(n) & D(n) \end{bmatrix}$ *is given by*

$$\begin{bmatrix} A(n) & B(n) \\ C(n) & D(n) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & i_{*n} \end{bmatrix} \left(P^{\mathcal{K}^{\mathrm{P}}(S)}_{\mathcal{H}^{\mathrm{P}}_{n+1}(S)\oplus\mathcal{E}^{\mathrm{P}}_{*n}} |_{\mathcal{H}^{\mathrm{P}}_n(S)\oplus\mathcal{E}^{\mathrm{P}}_n} \right) \begin{bmatrix} I & 0 \\ 0 & i_n \end{bmatrix}.$$

Proof. We first verify that Σ^{P}_S is a TVSS, i.e. that Σ^{P}_S satisfies Axioms (1)–(4) for a TVSS as set forth in Section 3. The first axiom is trivial. Axiom (2) follows easily from the fact the $\mathcal{G}^{\mathrm{P}}_{n+1}(S) \subset \mathcal{G}^{\mathrm{P}}_{n}(S)$ for all $n \in \mathbb{Z}$ and the fact that ℓ^2 spaces have the property

$$\bigcap_{n\in\mathbb{Z}} \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}) = \{0\}.$$

The second axiom follows similarly from the fact that $\mathcal{G}^{\mathrm{P}}_{*n}(S) \subset \mathcal{G}^{\mathrm{P}}_{*n+1}(S)$ for all n and the reverse property for ℓ^2 spaces that

$$\bigcap_{n\in\mathbb{Z}} \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*) = \{0\}.$$

Finally Axiom (4) follows easily from the definition (6.2) of the inner product and the assumption that S is a lower triangular element ($S \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$) of $\mathcal{X}(\mathcal{E}, \mathcal{E}_*)$.

It remains to compute the scattering operator for the TVSS Σ^{P}_{S}. It is easy to see that the spaces $\widetilde{\mathcal{G}}^{\mathrm{P}}_{n}(S)$ and $\widetilde{\mathcal{G}}^{\mathrm{P}}_{*n}(S)$ can be identified explicitly as

$$\widetilde{\mathcal{G}}^{\mathrm{P}}_{n}(S) = \begin{bmatrix} \ell^2(\mathbb{Z}, \mathcal{E}_*) \\ 0 \end{bmatrix}, \quad \widetilde{\mathcal{G}}^{\mathrm{P}}_{*n}(S) = \begin{bmatrix} 0 \\ \ell^2(\mathbb{Z}, \mathcal{E}) \end{bmatrix} \text{ independent of } n.$$

Moreover it is easily seen that

$$\mathcal{G}^{\mathrm{P}}_{n}(S) \ominus \mathcal{U}_n \mathcal{G}^{\mathrm{P}}_{n+1}(S) = \begin{bmatrix} 0 \\ \mathcal{E}_n \end{bmatrix}, \quad \mathcal{G}^{\mathrm{P}}_{*n+1}(S) \ominus \mathcal{U}_n^* \mathcal{G}^{\mathrm{P}}_{*n}(S) = \begin{bmatrix} \mathcal{E}_{*n} \\ 0 \end{bmatrix}.$$

It is therefore natural to define identification maps i_{*n} and i_n by

$$i_{*n} : e_{*n} \to \begin{bmatrix} e_{*n} \\ 0 \end{bmatrix}, \quad i_n : e_n \to \begin{bmatrix} 0 \\ e_n \end{bmatrix}.$$

Next one can check that the full incoming and outgoing spaces for the Pavlov model $\widetilde{\mathcal{G}}^{\mathrm{P}}_{*n}(S)$ and $\widetilde{\mathcal{G}}^{\mathrm{P}}_{n}(S)$ work out to be

$$\widetilde{\mathcal{G}}^{\mathrm{P}}_{*n}(S) = \begin{bmatrix} \ell^2(\mathbb{Z}, \mathcal{E}_*) \\ 0 \end{bmatrix}, \quad \widetilde{\mathcal{G}}^{\mathrm{P}}_{n}(S) = \begin{bmatrix} 0 \\ \ell^2(\mathbb{Z}, \mathcal{E}) \end{bmatrix} \text{ (independent of } n).$$

It is now easy to check that the Fourier operators Φ^{P}_{*n} and Φ^{P}_{n} for the Pavlov model satisfy

$$\begin{aligned} i_*^* \Phi^{\mathrm{P}}_{*n} &: \begin{bmatrix} g_* \\ 0 \end{bmatrix} \to g_* \qquad \text{for } g_* \in \ell^2(\mathbb{Z}, \mathcal{E}_*) \\ i^* \Phi^{\mathrm{P}}_{n} &: \begin{bmatrix} 0 \\ g \end{bmatrix} \to g \qquad \text{for } g \in \ell^2(\mathbb{Z}, \mathcal{E}). \end{aligned}$$

We are now ready to compute, for g and g_* as above,

$$\begin{aligned} \langle i_*^* S_{\mathfrak{S}^{\mathrm{P}}_S} i g, g_* \rangle &= \langle i_*^* \Phi^{\mathrm{P}}_{*n} \Phi^{P*}_{n} i g, g_* \rangle = \langle \Phi^{P*}_{n} i g, \Phi^{P*}_{*n} i_* g_* \rangle_{\mathcal{K}^{\mathrm{P}}(S)} = \left\langle \begin{bmatrix} 0 \\ g \end{bmatrix}, \begin{bmatrix} g_* \\ 0 \end{bmatrix} \right\rangle_{\mathcal{K}^{\mathrm{P}}(S)} \\ &= \left\langle \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix}, \begin{bmatrix} g_* \\ 0 \end{bmatrix} \right\rangle_{\ell^2(\mathbb{Z}, \mathcal{E}_*) \oplus \ell^2(\mathbb{Z}, \mathcal{E})} = \langle S g, g_* \rangle_{\ell^2(\mathbb{Z}, \mathcal{E})} \end{aligned}$$

and the first assertion follows. The second assertion is then a direct consequence of Theorem 5.1. ∎

6.2. The Sz.-Nagy–Foiaş model. For a full treatment of the time-invariant version of the Sz.-Nagy–Foiaş model, we refer to [29]. To obtain our time-varying adaptation of the Sz.-Nagy–Foiaş model, following [30] we set

$$\widetilde{\mathcal{K}}^{\mathrm{NF}}(S) = \begin{bmatrix} \ell^2(\mathbb{Z}, \mathcal{E}_*) \\ D_S \ell^2(\mathbb{Z}, \mathcal{E}) \end{bmatrix}$$

(where D_S is the defect operator $(I - S^*S)^{\frac{1}{2}}$ on $\ell^2(\mathbb{Z}, \mathcal{E})$) with

$$\widehat{\Psi}_S^{\mathrm{NF}} = \begin{bmatrix} I & S \\ 0 & D_S \end{bmatrix}.$$

By again applying Lemma 6.1, we see that, for $g_{*n}, g'_{*n} \in \widetilde{\mathcal{G}}_{*n}$ and $g_n, g'_n \in \widetilde{\mathcal{G}}_n$,

$$\begin{aligned}
\langle g_{*n} + g_n, g'_{*n} + g'_n \rangle &= \left\langle \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} \Phi_{*n} g_{*n} \\ \Phi_n g_n \end{bmatrix}, \begin{bmatrix} \Phi_{*n} g'_{*n} \\ \Phi_n g'_n \end{bmatrix} \right\rangle_{\ell^2(\mathbb{Z}, \mathcal{E}_*) \oplus \ell^2(\mathbb{Z}, \mathcal{E})} \\
&= \left\langle \begin{bmatrix} I & 0 \\ S^* & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I - S^*S \end{bmatrix} \begin{bmatrix} I & S \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi_{*n} g_{*n} \\ \Phi_n g_n \end{bmatrix}, \begin{bmatrix} \Phi_{*n} g'_{*n} \\ \Phi_n g_n \end{bmatrix} \right\rangle \\
&= \langle \mathcal{I}_{S,n}^{\mathrm{NF}}(g_{*n} + g_n), \mathcal{I}_{S,n}^{\mathrm{NF}}(g'_{*n} + g'_n) \rangle_{\ell^2(\mathbb{Z}, \mathcal{E}_*) \oplus \ell^2(\mathbb{Z}, \mathcal{E})}
\end{aligned}$$

and hence we define the scattering Sz.-Nagy–Foiaş model space to be

$$\mathcal{K}^{\mathrm{NF}}(S) = \begin{bmatrix} \ell^2(\mathbb{Z}, \mathcal{E}_*) \\ \mathcal{D}_S \end{bmatrix}$$

(where $\mathcal{D}_S$ is the closure of the image of D_S) with inner product equal to that inherited from $\ell^2(\mathbb{Z}, \mathcal{E}_*) \oplus \ell^2(\mathbb{Z}, \mathcal{E})$. The Sz.-Nagy–Foiaş model outgoing, incoming, and model spaces are then given by

$$\begin{aligned}
\mathcal{G}_n^{\mathrm{NF}}(S) &= \begin{bmatrix} S \\ D_S \end{bmatrix} \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}) \\
\mathcal{G}_{*n}^{\mathrm{NF}}(S) &= \begin{bmatrix} \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*) \\ 0 \end{bmatrix}, \\
\mathcal{H}_n^{\mathrm{NF}}(S) &= \begin{bmatrix} \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}_*) \\ \mathcal{D}_S \end{bmatrix} \ominus \begin{bmatrix} S \\ D_S \end{bmatrix} \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E})
\end{aligned}$$

with model contractive family given by

$$T_{S,n}^{\mathrm{NF}} = P_{\mathcal{H}_n^{\mathrm{NF}}(S)}|_{\mathcal{H}_{n+1}^{\mathrm{NF}}(S)}.$$

Conversely, one can derive an analogue of Theorem 6.2 for the Sz.-Nagy–Foiaş model. Specifically, given a contractive element S of $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$, it is straightforward to check directly that

$$\mathfrak{S}_S^{\mathrm{NF}} = \{\{\mathcal{U}_{S,n}^{\mathrm{NF}}\}, \{\mathcal{G}_n^{\mathrm{NF}}(S)\}, \{\mathcal{G}_{*n}^{\mathrm{NF}}(S)\}\}$$

is a scattering system with scattering operator coinciding with S:

$$i_*^* S_{\mathfrak{S}_S^{\mathrm{NF}}} i = S \tag{6.4}$$

where this time

$$(6.5)\qquad \begin{aligned} i &= \operatorname{diag}\{i_n\} \quad \text{with } i_n : e_n \to \begin{bmatrix} S \\ I \end{bmatrix} e_n \\ i_* &= \operatorname{diag}\{i_{*n}\} \quad \text{with } i_{*n} : e_{*n} \to \begin{bmatrix} e_{*n} \\ 0 \end{bmatrix}. \end{aligned}$$

A consequence of Theorem 5.1 then is that S can be realized as the input-output operator

$$(6.6)\qquad S = T_{\Sigma_S^{\rm NF}}$$

of the time-varying unitary system

$$\Sigma_S^{\rm NF} : \begin{cases} x(n+1) = A_n x(n) + B_n u(n), \\ y(n) = C_n x(n) + D_n u(n), \end{cases}$$

where now $U_{S,n}^{\rm NF} = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ is given by

$$U_{S,n}^{\rm NF} = \begin{bmatrix} I & 0 \\ 0 & i_{*n}^* \end{bmatrix} P^{\mathcal{K}^{\rm NF}}_{\mathcal{H}_n^{\rm NF} \oplus \mathcal{E}_{*n}^{\rm NF}} \Big|_{\mathcal{H}_{n+1}^{\rm NF}(S) \oplus \mathcal{E}_n^{\rm NF}} \begin{bmatrix} I & 0 \\ 0 & i_n \end{bmatrix}.$$

REMARK 6.3. This model for time-varying scattering (also known as "unitary coupling") and contractive families is discussed in some detail in [13], but from the point of view of Kolmogorov decompositions of positive-semidefinite block matrices and Schur parameters rather than directly from the point of view of dilation theory as is done here. The time-variant version of the Sz.-Nagy–Foiaş model from the viewpoint of dilation theory also appears as Theorem 1.8.1 in [32], and as Theorem 4.3 in [6].

6.3. DE BRANGES-ROVNYAK MODEL. The original treatment of the de Branges-Rovnyak model can be found in [12]. For our time-varying version, as suggested by the transcription methodology from [30], we take

$$\widetilde{\mathcal{K}}^{\rm dBR}(S) = \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} \ell^2(\mathbb{Z}, \mathcal{E}_*) \\ \ell^2(\mathbb{Z}, \mathcal{E}) \end{bmatrix}$$

with transcription operator

$$\widehat{\Pi}_S^{\rm dBR} = \begin{bmatrix} I & S \\ S^* & I \end{bmatrix}.$$

Thus for $g_{*n}, g'_{*n} \in \widetilde{\mathcal{G}}_{*n}$ and $g_n, g'_n \in \widetilde{\mathcal{G}}_n$, we define the inner product on $\widetilde{\mathcal{K}}^{\rm dBR}(S)$ so that

$$\left\langle \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} \Phi_{*n} g_{*n} \\ \Phi_n g_n \end{bmatrix}, \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} \Phi_{*n} g'_{*n} \\ \Phi_n g'_n \end{bmatrix} \right\rangle_{\mathcal{K}^{\rm dBR}(S)} = \langle g_{*n} + g_n, g'_{*n} + g'_n \rangle_{\mathcal{K}_n}$$
$$= \left\langle \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} \Phi_{*n} g_{*n} \\ \Phi_n g_n \end{bmatrix}, \begin{bmatrix} \Phi_{*n} g'_{*n} \\ \Phi_n g'_n \end{bmatrix} \right\rangle_{\ell^2(\mathbb{Z}, \mathcal{E}_*) \oplus \ell^2(\mathbb{Z}, \mathcal{E})}$$

where again we used Lemma 6.1. Upon analyzing the completion process, one sees that one should take the de Branges-Rovnyak model scattering space to be

$$\mathcal{K}^{\rm dBR}(S) = \operatorname{Ran} \begin{bmatrix} I & S \\ S^* & I \end{bmatrix}^{\frac{1}{2}} \tag{6.7}$$

with so-called lifted norm (or inner product) given by

$$\left\langle \begin{bmatrix} I & S \\ S^* & I \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} h_{*n} \\ h_n \end{bmatrix}, \begin{bmatrix} I & S \\ S^* & I \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} h'_{*n} \\ h'_n \end{bmatrix} \right\rangle_{\mathcal{K}^{\rm dBR}(S)} = \left\langle Q \begin{bmatrix} h_{*n} \\ h_n \end{bmatrix}, \begin{bmatrix} h'_{*n} \\ h'_n \end{bmatrix} \right\rangle \tag{6.8}$$

where Q is the orthogonal projection from $\ell^2(\mathbb{Z}, \mathcal{E}_*) \oplus \ell^2(\mathbb{Z}, \mathcal{E})$ onto the orthogonal complement of the kernel of $\begin{bmatrix} I & S \\ S^* & I \end{bmatrix}$. The outgoing, incoming and model spaces for the de Branges-Rovnyak model then work out to be

$$\begin{aligned} \mathcal{G}_n^{\rm dBR}(S) &= \begin{bmatrix} S \\ I \end{bmatrix} \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}) \\ \mathcal{G}_{*n}^{\rm dBR}(S) &= \begin{bmatrix} I \\ S^* \end{bmatrix} \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*) \\ \mathcal{H}_n^{\rm dBR}(S) &= \begin{bmatrix} I & S \\ S^* & I \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} \ell^2(\mathbb{Z}, \mathcal{E}_*) \\ \ell^2(\mathbb{Z}, \mathcal{E}) \end{bmatrix} \bigcap \begin{bmatrix} \ell^2(\mathbb{Z}_{\geqslant n}, \mathcal{E}_*) \\ \ell^2(\mathbb{Z}_{<n}, \mathcal{E}) \end{bmatrix}. \end{aligned} \tag{6.9}$$

An analogue of Theorem 6.2 for the de Branges-Rovnyak model with similar proof holds; namely, given a contractive element S of $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$, one can define

$$\mathfrak{S}_S^{\rm dBR} = \{\{\mathcal{U}_{S,n}^{\rm dBR}\}, \{\mathcal{G}_n^{\rm dBR}(S)\}, \{\mathcal{G}_{*n}^{\rm dBR}(S)\}\}$$

and check directly that $\Sigma_{S,{\rm scat}}^{\rm dBR}$ satisfies the axioms of a scattering system with scattering operator coinciding with S:

$$i_*^* S_{\mathfrak{S}_S^{\rm dBR}} i = S \tag{6.10}$$

where this time

$$\begin{aligned} i &= \operatorname{diag}\{i_n\} \quad \text{with } i_n : e_n \to \begin{bmatrix} S \\ I \end{bmatrix} e_n \\ i_* &= \operatorname{diag}\{i_{*n}\} \quad \text{with } i_{*n} : e_{*n} \to \begin{bmatrix} I \\ S^* \end{bmatrix} e_{*n}. \end{aligned} \tag{6.11}$$

A consequence of Theorem 5.1 again then is that S can be realized as the input-output operator

$$S = T_{\Sigma_S^{\rm dBR}} \tag{6.12}$$

of the time-varying unitary system

$$\Sigma_S^{\rm dBR} : \begin{cases} x(n+1) = A_n x(n) + B_n u(n), \\ y(n) = C_n x(n) + D_n u(n), \end{cases}$$

where now $U_{S,n}^{\rm dBR} = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ is given by

$$U_{S,n}^{\rm dBR} = \begin{bmatrix} I & 0 \\ 0 & i_{*n}^* \end{bmatrix} P^{\mathcal{K}^{\rm dBR}}_{\mathcal{H}_n^{\rm dBR} \oplus \mathcal{E}_{*n}^{\rm dBR}} \begin{bmatrix} I & 0 \\ 0 & i_n \end{bmatrix} : \begin{bmatrix} \mathcal{H}_N^{\rm dBR}(S) \\ \mathcal{E}_n \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_{n+1}^{\rm dBR}(S) \\ \mathcal{E}_{*n} \end{bmatrix}. \tag{6.13}$$

Our next goal is the derivation of explicit formulas for the de Branges-Rovnyak model unitary colligation $U_{S,n}^{\rm dBR}$ (6.13). A first step in this direction is the following more convenient form of the unitary identification map $\mathcal{I}_{S,n}^{\rm dBR} := \widehat{\Pi}_S^{\rm dBR} \circ \widehat{\Phi}_n$ between the scattering space $\mathcal{K}_n$ and the de Branges-Rovnyak model scattering space $\mathcal{K}^{\rm dBR}(S)$.

PROPOSITION 6.4. *Let $\mathfrak{S}$ be a* TVSS *with scattering operator $S = \Phi_{*n}\Phi_n^* \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ and let $\mathcal{I}_{S,n}^{\rm dBR}$ be the unitary identification map between $\mathcal{K}_n$ and $\mathcal{K}^{\rm dBR}(S)$ as above. Then*

$$\mathcal{I}_{S,n}^{\rm dBR} = \begin{bmatrix} \Phi_{*n} \\ \Phi_n \end{bmatrix} : \mathcal{K}_n \to \mathcal{K}^{\rm dBR}(S)$$

for all n.

Proof. We check , for $g_{*n} \in \mathcal{G}_{*n}$,

$$\mathcal{I}_{S,n}^{\rm dBR} g_{*n} = \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} \Phi_{*n} g_{*n} \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ \Phi_n \Phi_{*n}^* \end{bmatrix} \Phi_{*n} g_{*n} = \begin{bmatrix} \Phi_{*n} \\ \Phi_n \end{bmatrix} g_{*n}.$$

Similarly, for $g_n \in \mathcal{G}_n$, we have

$$\mathcal{I}_{S,n}^{\rm dBR} g_n = \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} 0 \\ \Phi_n g_n \end{bmatrix} = \begin{bmatrix} \Phi_{*n}\Phi_n^* \\ I \end{bmatrix} \Phi_n g_n = \begin{bmatrix} \Phi_{*n} \\ \Phi_n \end{bmatrix} g_n.$$

The assertion now follows by linearity and continuity. ∎

From Proposition 6.3 we are next able to get the following explicit formulas for the restriction $\Gamma_n = \mathcal{I}_{S,n}^{\rm dBR} | \mathcal{H}_n$ of the de Branges-Rovnyak identification map to the state space $\mathcal{H}_n$ at time n.

PROPOSITION 6.5. *Let us suppose that*

$$\mathfrak{S} = \{\{\mathcal{U}_n\}, \{\mathcal{G}_n\}, \{\mathcal{G}_{*n}\}\}$$

is a TVSS *with scattering operator equal to $S \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ and with associated time-varying unitary colligation*

$$U_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} = P^{\mathcal{K}_{n+1}}_{\mathcal{H}_{n+1} \oplus \mathcal{E}_{*n}} \mathcal{U}_n^* |_{\mathcal{H}_n \oplus \mathcal{E}_n} : \begin{bmatrix} \mathcal{H}_n \\ \mathcal{E}_n \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_{n+1} \\ \mathcal{E}_{*n} \end{bmatrix}$$

and with Fourier transforms $\Phi_n : \mathcal{K}_n \to \ell^2(\mathbb{Z}, \mathcal{E})$ and $\Phi_{*n} : \mathcal{K}_n \to \ell^2(\mathbb{Z}, \mathcal{E}_*)$. Then, for $h_n \in \mathcal{H}_n$, $\Phi_n h_n = \{\{\Phi_n h_n\}_j\}_{j \in \mathbb{Z}}$ and $\Phi_{*n} h_n = \{\{\Phi_{*n} h_n\}_j\}_{j \in \mathbb{Z}}$ are given by

$$\{\Phi_{*n} h_n\}_j = \begin{cases} 0 & \text{for } j < n, \\ C_j A_{j-1} \cdots A_n h_n & \text{for } j \geqslant n; \end{cases}$$

$$\{\Phi_n h_n\}_j = \begin{cases} B_j^* A_{j+1}^* \cdots A_{n-1}^* h_n & \text{for } j < n, \\ 0 & \text{for } j \geqslant n. \end{cases}$$

Proof. Since $\mathcal{H}_n \perp \mathcal{G}_{*n}$, it follows that $P^{\mathcal{K}_n}_{\widetilde{\mathcal{G}}_n}\mathcal{H}_n \perp \mathcal{G}_{*n}$. Since $\Phi_{*n} : \mathcal{G}_{*n} \to \ell^2(\mathbb{Z}_{<n}, \mathcal{E}_*)$, we see that $\Phi_{*n} : \mathcal{H}_n \subset \ell^2(\mathbb{Z}, \mathcal{E}_*)$, or $\{\Phi_{*n}h_n\}_j = 0$ for $j < n$. Similarly we see from $\mathcal{H}_n \perp \mathcal{G}_n$ and $\Phi_n : \mathcal{G}_n \to \ell^2(\mathbb{Z}_{\text{gen}}, \mathcal{E})$ that $\Phi_n\mathcal{H}_n \subset \ell^2(\mathbb{Z}_{<n}, \mathcal{E})$, or $\{\Phi_n h_n\}_j = 0$ for $j \geqslant n$. To compute $\{\Phi_{*n}h_n\}_j$ for $j \geqslant n$, we use the formula

$$\{\Phi_{*n}h_n\}_j = P^{\mathcal{K}_{j+1}}_{\mathcal{E}_{*j}}\mathcal{U}_j^* \cdots \mathcal{U}_n^* h_n$$

together with the relation

$$P^{\mathcal{K}_{n+1}}_{\mathcal{H}_{n+1}\oplus\mathcal{E}_{*n}}\mathcal{U}_n^*|_{\mathcal{H}_n\oplus\mathcal{E}_n} = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}.$$

In particular we see that $P^{\mathcal{K}_{n+1}}_{\mathcal{E}_{*n}}\mathcal{U}_n^*|_{\mathcal{H}_n} = C_n$ and hence

$$\{\Phi_{*n}h_n\}_n = P^{\mathcal{K}_{n+1}}_{\mathcal{E}_{*n}}\mathcal{U}_n^* h_n = C_n h_n.$$

For the general case, note the semiinvariance properties

$$P^{\mathcal{K}_{k+2}}_{\mathcal{E}_{*k+1}}\mathcal{U}_{k+1}^*\mathcal{U}_k^*|_{\mathcal{H}_k} = P^{\mathcal{K}_{k+2}}_{\mathcal{E}_{*k+1}}\mathcal{U}_{k+1}^* P^{\mathcal{K}_{k+1}}_{\mathcal{H}_{k+1}}\mathcal{U}_k^*|_{\mathcal{H}_k}$$
$$P^{\mathcal{K}_{k+2}}_{\mathcal{H}_{*k+2}}\mathcal{U}_{k+1}^*\mathcal{U}_k^*|_{\mathcal{H}_k} = P^{\mathcal{K}_{k+2}}_{\mathcal{H}_{*k+2}}\mathcal{U}_{k+1}^* P^{\mathcal{K}_{k+1}}_{\mathcal{H}_{k+1}}\mathcal{U}_k^*|_{\mathcal{H}_k}.$$

Thus for $k > 0$ we have

$$\begin{aligned}\{\Phi_{*n}h_n\}_{n+k} &= P^{\mathcal{K}_{n+k+1}}_{\mathcal{E}_{*n+k}}\mathcal{U}_{n+k}^* P^{\mathcal{K}_{n+k}}_{\mathcal{H}_{n+k}}\mathcal{U}_{n+k-1}^* \cdots P^{\mathcal{K}_{n+1}}_{\mathcal{H}_{n+1}}\mathcal{U}_n^* h_n \\ &= C_{n+k}A_{n+k-1}^* \cdots A_n^* h_n\end{aligned}$$

as asserted. A similar argument arrives at the formula for $\{\Phi_n h_n\}_j$ for $j \geqslant n$, by using the dual relations

$$\begin{bmatrix} A_k^* & C_k^* \\ B_k^* & D_k^* \end{bmatrix} = P^{\mathcal{K}_k}_{\mathcal{H}_k\oplus\mathcal{E}_k}\mathcal{U}_k|_{\mathcal{H}_{k+1}\oplus\mathcal{E}_{*k}}$$
$$P^{\mathcal{K}_k}_{\mathcal{E}_k}\mathcal{U}_k\mathcal{U}_{k+1}|_{\mathcal{H}_{k+2}} = P^{\mathcal{K}_k}_{\mathcal{E}_k}\mathcal{U}_k P^{\mathcal{K}_{k+1}}_{\mathcal{H}_{k+1}}\mathcal{U}_{k+1}|_{\mathcal{H}_{k+1}}$$
$$P^{\mathcal{K}_k}_{\mathcal{H}_k}\mathcal{U}_k\mathcal{U}_{k+1}|_{\mathcal{H}_{k+2}} = P^{\mathcal{K}_k}_{\mathcal{H}_k}\mathcal{U}_k P^{\mathcal{K}_{k+1}}_{\mathcal{H}_{k+1}}\mathcal{U}_{k+1}|_{\mathcal{H}_{k+2}}. \quad \blacksquare$$

We are now ready to compute the de Branges-Rovnyak unitary colligation $U_{S,n}^{\text{dBR}}$ explicitly.

Theorem 6.6. *The de Branges-Rovnyak colligation*

$$U_{S,n}^{\text{dBR}} = \begin{bmatrix} A_n^{\text{dBR}} & B_n^{\text{dBR}} \\ C_n^{\text{dBR}} & D_n^{\text{dBR}} \end{bmatrix} : \begin{bmatrix} \mathcal{H}_n^{\text{dBR}}(S) \\ \mathcal{E}_n \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_{S,n+1}^{\text{dBR}} \\ \mathcal{E}_{*n} \end{bmatrix}$$

(see (6.13)*) is given explicitly by*

$$\begin{aligned} A_n^{\text{dBR}} &: \begin{bmatrix} f_n \\ g_n \end{bmatrix} \to \begin{bmatrix} f_n - \{f_n\}_n\delta_n \\ g_n - S^*\pi_n^*\{f_n\}_n \end{bmatrix} \\ B_n^{\text{dBR}} &: e_n \to \begin{bmatrix} (S - S^R(0))\pi_n^* \\ (I - S^*S^{\wedge R}(0))\pi_n^* \end{bmatrix} e_n \\ C_n^{\text{dBR}} &: \begin{bmatrix} f_n \\ g_n \end{bmatrix} \to \{f_n\}_n \\ D_n^{\text{dBR}} &= S_{nn} = \pi_{*n}S^{\wedge R}(0)\pi_n^* = \pi_{*n}S\pi_n^* \end{aligned} \tag{6.14}$$

with adjoint given by

$$
\begin{aligned}
&(A_n^{\mathrm{dBR}})^* : \begin{bmatrix} f_{n+1} \\ g_{n+1} \end{bmatrix} \to \begin{bmatrix} f_{n+1} - S\pi_n^*\{g_n\}_n \\ g_{n+1} - \{g_n\}_n \end{bmatrix} \\
&(C_n^{\mathrm{dBR}})^* : e_{*n} \to \begin{bmatrix} (I - SS^{\wedge R}(0)^*)\pi_{*n}^* \\ (S^* - S^{\wedge R}(0)^*)\pi_{*n}^* \end{bmatrix} e_n \\
&(B_n^{\mathrm{dBR}})^* : \begin{bmatrix} f_{n+1} \\ g_{n+1} \end{bmatrix} \to \{g_n\}_n \\
&(D_n^{\mathrm{dBR}})^* = S_{nn}^* = \pi_n S^{\wedge R}(0)^* \pi_{*n}^* = \pi_n S^* \pi_{*n}^*
\end{aligned} \tag{6.15}
$$

where δ_n is the Kronecker delta ($\{\delta_n\}_j = 1$ for $j \neq n$ and 0 otherwise), and where π_n^* is the canonical injection of $\mathcal{E}_n$ into $\ell^2(\mathbb{Z}, \mathcal{E})$ and π_{*n}^* is the canonical injection of $\mathcal{E}_{*n}$ into $\ell^2(\mathbb{Z}, \mathcal{E}_*)$.

Proof. It is convenient to assume that we are given a TVSS $\mathfrak{S} = \{\{\mathcal{U}_n\}, \{\mathcal{G}_n\}, \{\mathcal{G}_{*n}\}\}$ with scattering operator equal to S and with associated time-varying unitary colligation $U_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ as in the hypotheses of Proposition 6.4. Let us use the notation $\widehat{\Gamma}_n$ for the restriction of the de Branges-Rovnyak identification map $\mathcal{I}_{S,n}^{\mathrm{dBR}}$ to the scattering space $\mathcal{H}_n$. Thus, by Proposition 6.3 we may write

$$
\widehat{\Gamma}_n := \begin{bmatrix} \Gamma_{*n} \\ \Gamma_n \end{bmatrix} = \begin{bmatrix} \Phi_{*n} \\ \Phi_n \end{bmatrix} |_{\mathcal{H}_n}. \tag{6.16}
$$

By definition, the de Branges-Rovnyak colligation $U_{S,n}^{\mathrm{dBR}}$ is determined by the intertwining condition

$$
\begin{bmatrix} \widehat{\Gamma}_{n+1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} = \begin{bmatrix} A_n^{\mathrm{dBR}} & B_n^{\mathrm{dBR}} \\ C_n^{\mathrm{dBR}} & D_n^{\mathrm{dBR}} \end{bmatrix} \begin{bmatrix} \widehat{\Gamma}_n & 0 \\ 0 & I \end{bmatrix} \tag{6.17}
$$

where, by Proposition 6.4 we have, for $h_n \in \mathcal{H}_n$,

$$
\{\widehat{\Gamma}_n h_n\}_j = \begin{cases} \begin{bmatrix} C_j A_{j-1} \cdots A_n h_n \\ 0 \end{bmatrix} & \text{for } j \geqslant n, \\ \begin{bmatrix} 0 \\ B_j^* A_{j+1}^* \cdots A_{n-1}^* h_n \end{bmatrix} & \text{for } j < n. \end{cases} \tag{6.18}
$$

Let us write a generic element of $\mathcal{H}_n^{\mathrm{dBR}}(S)$ as $\begin{bmatrix} f_n \\ g_n \end{bmatrix} = \begin{bmatrix} \Gamma_{*n} \\ \Gamma_n \end{bmatrix} h_n$ where h_n is a generic element of $\mathcal{H}_n$. As elements of the spaces $\mathcal{H}_n^{\mathrm{dBR}}(S)$ and $\mathcal{H}_n^{\mathrm{dBR}}(S)$ are block columns with two components, it will be useful to have a notation for the finer decompositions of the de Branges-Rovnyak colligation operators:

$$
A_n^{\mathrm{dBR}} = \begin{bmatrix} A_{n,11}^{\mathrm{dBR}} & A_{n,12}^{\mathrm{dBR}} \\ A_{n,21}^{\mathrm{dBR}} & A_{n,22}^{\mathrm{dBR}} \end{bmatrix}, \quad B_n^{\mathrm{dBR}} = \begin{bmatrix} B_{n,1}^{\mathrm{dBR}} \\ B_{n,2}^{\mathrm{dBR}} \end{bmatrix}, \quad C_n^{\mathrm{dBR}} = \begin{bmatrix} C_{n,1}^{\mathrm{dBR}} & C_{n,2}^{\mathrm{dBR}} \end{bmatrix}.
$$

From (6.18) we see that

$$
\begin{aligned}
\{\Gamma_{*n+1} A_n h_n\}_j &= \begin{cases} C_j A_{j-1} \cdots A_{n+1} A_n h_n & \text{for } j \geqslant n+1, \\ 0 & \text{otherwise} \end{cases}; \\
&= \begin{cases} \{f_n\}_j & \text{for } j \geqslant n, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

This combined with (6.17) gives us $A^{\mathrm{dBR}}_{n,11} : f_n \to f_n - \{f_n\}_n \delta_n$, $A^{\mathrm{dBR}}_{n,12} = 0$.

Next we use (6.18) to see that

$$\Gamma_{*n+1} B_n e_n = \begin{cases} C_i A_{i-1} \cdots A_{n+1} B_n e_n & \text{for } i \geqslant n+1 , \\ 0 & \text{otherwise.} \end{cases}$$

From this combined with (6.17) we see that $B^{\mathrm{dBR}}_{n,1} e_n = (S - S^{\wedge R}(0)) \pi_n^* e_n$. Similarly, from (6.17) and the formula (6.18) for Γ_{n+1} we have

$$B^{\mathrm{dBR}}_{n,2} e_n = \Gamma_{n+1} B_n e_n = \begin{cases} B_j^* A_{j+1}^* \cdots A_n^* B_n e_n & \text{for } j \leqslant n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.19}$$

On the other hand by Theorem 5.1 we see that

$$[I - SS^{\wedge R}(0)]_{ij} = \delta_{ij} I - S_{ji}^* D_j = \begin{cases} 0 & \text{for } i > j, \\ I - D_i^* D_i & \text{for } i = j, \\ -B_i^* A_i^* \cdots A_{j+1}^* C_j^* D_j & \text{for } i < j. \end{cases} \tag{6.20}$$

From the fact that $U_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ is unitary we get the relations $I - D_i^* D_i = B_i^* B_i$, $C_j^* D_j = A_j^* B_j$. Hence (6.20) becomes

$$[I - SS^{\wedge R}(0)]_{ij} = \begin{cases} 0 & \text{for } i > j, \\ B_i^* B_i & \text{for } i = j, \\ B_i^* A_i^* \cdots A_{j+1}^* A_j^* B_j & \text{for } i < j. \end{cases}$$

Comparison of this with (6.19) now gives

$$B^{\mathrm{dBR}}_{n,2} e_n = (I - S^* S^{\wedge R}(0)) \pi_n^* e_n$$

as wanted.

From (6.18) and (6.17) we get $C^{\mathrm{dBR}}_n \Gamma_{*n} h_n = C_n h_n = C_n \{f_n\}_n$ and then also $C^{\mathrm{dBR}}_{n,1} f_n = \{f_n\}_n$, $C^{\mathrm{dBR}}_{n,2} = 0$. From (6.17) we read off $D^{\mathrm{dBR}}_n = D_n = S_{nn}$ as asserted.

Next we use (6.18) to see that

$$\{\Gamma_{n+1} A_n h_n\}_j = \begin{cases} B_j^* A_{j+1}^* \cdots A_n^* A_n h_n & \text{for } j \leqslant n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.21}$$

We now use the remaining relations coming from the fact that $\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ is isometric, namely: $A_n^* A_n = I - C_n^* C_n$, $B_n^* A_n = -D_n^* C_n$. Hence (6.21) becomes

$$\{\Gamma_{n+1} A_n h_n\}_j = \begin{cases} B_n^* A_n h_n = -D_n^* C_n h_n & \text{for } j = n, \\ B_j^* A_{j+1}^* \cdots A_{n-1}^* h_n - B_j^* A_{j+1}^* \cdots A_{n-1}^* C_n^* C_n h_n & \text{for } j < n, \\ 0 & \text{otherwise.} \end{cases}$$

Recalling now that $C_n h_n = \{f_n\}_n$, the definition of $g_n = \Gamma_n h_n$ and the intertwining condition (6.17), we arrive at

$$A^{\mathrm{dBR}}_{n,21} : f_n \to -S^* \pi_n^* \{f_n\}_n, \quad A^{\mathrm{dBR}}_{n,22} : g_n \to g_n$$

and all the formulas (6.14) hold as asserted.

The adjoint formulas can be proved by the computations done with the roles of the inputs and outputs interchanged and working with the adjoint system matrix, which by construction is also isometric. ∎

REMARK 6.7. This time-varying de Branges-Rovnyak model is also discussed briefly in [13] from the point of view of Kolmogorov decompositions and Schur parameters. It appears as well in [7] (see Theorem 8.1) and in [32] (see Theorem 3.2.1), where, in addition, the authors show that the resulting system is minimal and optimal in the sense of Arov.

REMARK 6.8. When the time-varying system is embedded in a time-invariant system by working with the aggregate form of the system equations as in (2.15), one arrives at the time-varying versions of the de Branges-Rovnyak model studied in [4].

6.4. APPLICATION: THE SCATTERING OPERATOR AS A COMPLETE UNITARY INVARIANT. As a corollary of any of the three model theories described above, it is easy to see that the scattering operator is a *complete unitary invariant* for a scattering system $\mathfrak{S}$. To make this precise requires a couple elementary definitions. Let us say that two operators $S \in \mathcal{L}(\{\mathcal{E}_n\}, \{\mathcal{E}_{*n}\})$ and $S \in \mathcal{L}(\{\mathcal{E}'_n, \{\mathcal{E}'_{*n}\})$ *coincide* if there are unitary operators $\iota_n : \mathcal{E}_n \to \mathcal{E}'_n$ and $\iota_{*n} : \mathcal{E}_{*n} \to \mathcal{E}'_{*n}$ so that $S_{ij} = \iota_i^* S'_{ij} \iota_j$ for all $i, j \in \mathbb{Z}$. Let us say that two scattering systems $\mathfrak{S} = \{\{\mathcal{K}_n\}, \{\mathcal{G}_n\}, \{\mathcal{G}_{*n}\}, \{\mathcal{U}_n\}\}$ and $\mathfrak{S}' = \{\{\mathcal{K}_n\}', \{\mathcal{G}'_n\}, \{\mathcal{G}'_{*n}\}, \{\mathcal{U}'_n\}\}$ are *unitarily equivalent* if there are unitary operators $V_n : \mathcal{K}_n \to \mathcal{K}'_n$ such that $V_n(\mathcal{G}_n) = \mathcal{G}'_n$, $V_n(\mathcal{G}_{*n}) = \mathcal{G}'_{*n}$ and $V_n \mathcal{U}_n = \mathcal{U}'_n V_{n+1}$. Then we have the following result.

THEOREM 6.9. *Two minimal* TVSSs $\mathfrak{S}$ *and* $\mathfrak{S}'$ *are unitarily equivalent if and only if their associated scattering operators* $S_{\mathfrak{S}}$ *and* $S_{\mathfrak{S}'}$ *coincide.*

Proof. It is easy to see directly that if two scattering systems are unitarily equivalent, then their scattering operators coincide. Conversely, given two minimal, scattering systems $\mathfrak{S}$ and $\mathfrak{S}'$ with scattering operators S and S' respectively, since the operator $\mathcal{I}^M_{S,n}$ given by (6.1) is unitary (in case $\mathfrak{S}$ is minimal), we see that the given scattering system $\mathfrak{S}$ is unitarily equivalent to the associated model scattering system $\mathfrak{S}^M_S$ (where M can stand for any of P, NF or dBR corresponding to the Pavlov, Sz.-Nagy–Foiaş or de Branges-Rovnyak models). Similarly, $\mathfrak{S}'$ is unitarily equivalent to $\mathfrak{S}^M_{S'}$. In case S and S' coincide, it is easy to check that $\mathfrak{S}^M_S$ and $\mathfrak{S}^M_{S'}$ are unitarily equivalent scattering systems. As unitary equivalence of scattering systems is an equivalence relation, it follows that $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{S}'$ as asserted. ∎

In a similar way, one can show: (i) *two completely nonunitary contractive-operator families* $\mathcal{T} = \{T_n : \mathcal{H}_{n+1} \to \mathcal{H}_n\}$ *and* $\mathcal{T}' = \{T'_n : \mathcal{H}'_{n+1} \to \mathcal{H}'_n\}$ *are unitarily equivalent if and only if their associated characteristic operators* $\Theta_{\mathcal{T}}$ *and* $\Theta_{\mathcal{T}'}$ *coincide*, and (ii) *two closely connected, time-varying, unitary systems* Σ *and* Σ' *are unitarily equivalent if and only if their input-output operators* T_Σ *and* $T_{\Sigma'}$ *coincide.* We leave the formulation of the relevant definitions of "unitary equivalence" in the context of "contractive-operator family" and "time-varying unitary system" to the reader.

7. THE ABSTRACT INTERPOLATION PROBLEM

A high-level interpolation problem with slick solution procedure incorporating much of classical and modern matrix-theoretic interpolation theory as special cases has been formulated by Katsnelson, Kheifets and Yuditskii (see [27] and [25] for a recent survey) and termed by them the Abstract Interpolation Problem (AIP). Here we formulate the time-varying analogue of the AIP and show how the time-varying version of matrix bitangential Nevanlinna-Pick interpolation problem recently studied in the literature (see [20] and [19]) can be captured as a special case of this *time-varying Abstract Interpolation Problem* (TVAIP). Preliminary results in this direction have already appeared in [17].

The data for the TVAIP is as follows. We assume that we are given a family of linear spaces $\{\mathcal{X}_n^0\}$ and families of Hilbert spaces $\mathcal{X} = \{\mathcal{X}_n\}$, $\mathcal{E} = \{\mathcal{E}_n\}$, $\mathcal{E}_* = \{\mathcal{E}_{*n}\}$ ($n \in \mathbb{Z}$) together with linear operators

$$T_{1,n} : \mathcal{X}_n^0 \to \mathcal{X}_n, \quad T_{2,n} : \mathcal{X}_n^0 \to \mathcal{X}_{n+1}, \quad M_n : \mathcal{X}_n^0 \to \mathcal{E}_n, \quad M_{*n} : \mathcal{X}_n^0 \to \mathcal{E}_{*n}.$$

These data are assumed to satisfy the time-varying version of the so-called Potapov identity given by:

$$\|T_{1,n}x_n\|^2_{\mathcal{X}_n} + \|M_n x_n\|^2_{\mathcal{E}_n} = \|T_{2,n}x_n\|^2_{\mathcal{X}_{n+1}} + \|M_{*n}x_n\|^2_{\mathcal{E}_{*n}} \tag{7.1}$$

for all $x_n \in \mathcal{X}_n^0$. We now state the TVAIP in coordinate-free form: *given a* TVAIP *data set*

$$\omega_{\text{TVAIP}} = \{\{\mathcal{X}_n^0\}, \mathcal{X} = \{\mathcal{X}_n\}, \mathcal{E} = \{\mathcal{E}_n\}, \mathcal{E}_* = \{\mathcal{E}_{*n}\}, \{T_{1,n}\}, \{T_{2,n}\}, \{M_n\}, \{M_{*n}\}\} \tag{7.2}$$

satisfying the hypothesis (7.1), *find a minimal* TVSS *as in* (3.1)

$$\mathfrak{S} = \{\{\mathcal{U}_n\}, \{\mathcal{G}_n\}, \{\mathcal{G}_{*n}\}\} \tag{7.3}$$

with $\mathcal{E}_n = \mathcal{G}_n \ominus \mathcal{U}_n \mathcal{G}_{n+1}$ *and* $\mathcal{E}_{*n} = \mathcal{G}_{*n+1} \ominus \mathcal{U}_n^* \mathcal{G}_{*n}$ *and contraction operators*

$$\mathbf{F}_n : \mathcal{X}_n \to \mathcal{H}_n := \mathcal{K}_n \ominus [\mathcal{G}_{*n} \oplus \mathcal{G}_n]$$

so that the identity

$$(\mathbf{F}_n T_{1,n} + M_n)x_n = \mathcal{U}_n(\mathbf{F}_{n+1}T_{2,n} + M_{*n})x_n \tag{7.4}$$

holds for all $x_n \in \mathcal{X}_n^0$ *for all* $n \in \mathbb{Z}$.

The de Branges-Rovnyak model version of the TVAIP is as follows: *given a* TVAIP *data set* ω_{TVAIP} *as above satisfying* (7.1), *find a contractive element* S *of* $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ *and a family of contraction operators* F_n *from the space* $\mathcal{X}_n$ *into the appropriate de Branges-Rovnyak model space*

$$F_n : \mathcal{X}_n \to \mathcal{H}_n^{\text{dBR}}(S) \quad \text{for } n \in \mathbb{Z}$$

so that the identity

$$F_n T_{1,n} x_n = F_{n+1} T_{2,n} x_n - \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} -\pi_{*n}^* M_{*n} \\ \pi_n^* M_n \end{bmatrix} x_n, \tag{7.5}$$

holds for all $x_n \in \mathcal{X}_n^0$. The mapping $\mathcal{I}_{S,n}^{\mathrm{dBR}}$ between a TVSS $\mathfrak{S}$ and the de Branges-Rovnyak model TVSS with scattering operator S implements the equivalence between these two formulations of the TVAIP, as was shown in [11] for the time-invariant case.

The system of equations arising in the de Branges-Rovnyak-model version of the TVAIP can be expressed more compactly as follows. Given a data set ω_{TVAIP} as in (7.2), let $\mathcal{D}_f(\mathbb{C}, \mathcal{X}^0)$ be the space of diagonal matrices with j^{th} diagonal entry x_j an element of $\mathcal{X}_j^0$ and all but finitely many of these x_j's equal to 0. Define operators T_1, T_2, M, and M_* by

$$
\begin{aligned}
T_1 &= \operatorname{diag}\{T_{1,n}\}_{n\in\mathbb{Z}} : \mathcal{D}_f(\mathbb{C}, \mathcal{X}^0) \to \mathcal{D}(\mathbb{C}, \mathcal{X}) \\
T_2 &= \operatorname{diag}\{T_{2,n-1}\}_{n\in\mathbb{Z}} : \mathcal{D}_f(\mathbb{C}, \mathcal{X}^0) \to \mathcal{D}(\mathbb{C}, \mathcal{X}^{(-1)}) \\
M &= \operatorname{diag}\{M_n\}_{n\in\mathbb{Z}} : \mathcal{D}_f(\mathbb{C}, \mathcal{X}^0) \to \mathcal{D}(\mathbb{C}, \mathcal{E}) \\
M_* &= \operatorname{diag}\{M_{*n}\}_{n\in\mathbb{Z}} : \mathcal{D}_f(\mathbb{C}, \mathcal{X}^0) \to \mathcal{D}(\mathbb{C}, \mathcal{E}).
\end{aligned} \tag{7.6}
$$

Then the TVAIP in *augmented de Branges-Rovnyak-model form* can be alternatively formulated as: *given a* TVAIP *data set* ω_{TVAIP} (7.2) *with associated operators* T_1, T_2, M *and* M_* *as in* (7.6), *find a contractive element* S *of* $\mathcal{L}_{\mathrm{HS}}(\mathcal{E}, \mathcal{E}_*)$ *and a contraction operator*

$$
F = \begin{bmatrix} \cdots & F_{-1} & F_0 & F_1 & \cdots \end{bmatrix} : \mathcal{D}_{\mathrm{HS}}(\mathbb{C}, \mathcal{X}) \to \mathcal{H}_S^{\mathrm{dBR,aug}}
$$

so that

$$
FT_1 x = F(R_Z T_2)x - \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} M_* \\ M \end{bmatrix} x \tag{7.7}
$$

for all $x \in \mathcal{D}_f(\mathbb{C}, \mathcal{X}^0)$. Here we have set $\mathcal{H}_S^{\mathrm{dBR,aug}}$ equal to the space of all biinfinite block row matrices $\begin{bmatrix} \cdots & h_{-1} & \boxed{h_0} & h_1 & \cdots \end{bmatrix}$ where $h_j \in \mathcal{H}_j^{\mathrm{dBR}}(S)$ and $\sum_{j=-\infty}^{\infty} \|h_j\|^2 < \infty$.

As is the case for the time-invariant case, the solution of the TVAIP is rather straightforward. It would appear that the more difficult part for applications is to determine the TVAIP data set ω_{TVAIP} which gives rise to a given concrete interpolation or extension problem in an applications setting. The main consequence of the hypothesis (7.1) is that the family of partial operators $V_n : \mathcal{D}_{V_n} \to \mathcal{R}_{V_n}$ with

$$
\mathcal{D}_{V_n} = \operatorname{im} \begin{bmatrix} T_{1n} \\ M_n \end{bmatrix} \subset \begin{bmatrix} \mathcal{X}_n \\ \mathcal{E} \end{bmatrix}, \quad \mathcal{R}_{V_n} = \operatorname{im} \begin{bmatrix} T_{2n} \\ M_{*n} \end{bmatrix} \subset \begin{bmatrix} \mathcal{X}_{n+1} \\ \mathcal{E}_* \end{bmatrix}
$$

given by

$$
V_n : \begin{bmatrix} T_{1n} \\ M_n \end{bmatrix} x_n \to \begin{bmatrix} T_{2n} \\ M_{*n} \end{bmatrix} x_n \quad \text{for } x_n \in \widetilde{\mathcal{X}}_n \tag{7.8}
$$

consists of isometries:

$$
\|V_n d_n\|_{\mathcal{X}_{n+1} \oplus \mathcal{E}_{*n}} = \|d_n\|_{\mathcal{X}_n \oplus \mathcal{E}_n} \quad \text{for } d_n \in \mathcal{D}_{V_n}.
$$

The main result concerning the solution of the TVAIP is that solutions of the TVAIP correspond in a simple way to unitary-family extensions of the partial-isometry family $\{V_n\}$.

THEOREM 7.1. *Let* ω_{TVAIP} *be a* TVAIP *data set as in* (7.2) *and let* $\{V_n\}$ *be the partial-isometry family of operators given as in* (7.8). *Then solutions of the* TVAIP *with data set* ω_{TVAIP} *are in one-to-one correspondence with unitary-family extensions of the partial-isometry family* $V_n : \mathcal{D}_V \to \mathcal{R}_V$. *Indeed, let*

$$U_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} : \begin{bmatrix} \mathcal{H}_n \\ \mathcal{E}_n \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_{n+1} \\ \mathcal{E}_{*n} \end{bmatrix} \tag{7.9}$$

(where $\widetilde{\mathcal{X}}_n \subset \mathcal{H}_n$ *and* $\widetilde{\mathcal{X}}_{n+1} \subset \mathcal{H}_{n+1}$*) be any unitary-family extension of* V_n *(so* U_n *is unitary for each* n *and* $U_n|_{\mathcal{D}_{V_n}} = V_n$ *for all* n*). Then:*

(i) *Let* $\mathfrak{S}$ *be the* TVSS *system associated (as in Section* 5*) with a time-varying unitary colligation* $\{U_n\}_{n\in\mathbb{Z}}$ (7.9) *which extends the partial isometric colligation* $\{V_n\}$ (7.8), *let* $\mathfrak{S}^{\min}$ *(with scattering model space* $\mathcal{H}_n^{\min} \subset \mathcal{H}_n$*) be the minimal part of* $\mathfrak{S}$ *and define* $\mathbf{F}_n : \widetilde{\mathcal{X}}_n \to \mathcal{H}_n^{\min}$ *by*

$$\mathbf{F}_n = P^{\mathcal{H}_n}_{\mathcal{H}_n^{\min}}|_{\widetilde{\mathcal{X}}_n}.$$

Then $(\mathfrak{S}^{\min}, \{\mathbf{F}_n\})$ *solves the coordinate-free version of the* TVAIP, *and every solution of the coordinate-free* TVAIP *arises in this way.*

(ii) *Let* $S \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ *be the input-output operator of the time-varying unitary system*

$$\Sigma : \begin{cases} x(n+1) = A_n x_n + B_n u(n), \\ y(n) = C_n x_n + D_n u(n), \end{cases}$$

associated with the time-varying unitary colligation $\{U_n\}$ *from* (7.9) *extending* V_n *from* (7.8), *and define the map* $F_n : \widetilde{\mathcal{X}}_n \to \mathcal{H}_n^{\text{dBR}}(S)$ *by*

$$F_n : x_n \to \widehat{\Gamma}_n x_n = \begin{bmatrix} \Gamma_{*n} \\ \Gamma_n \end{bmatrix} x_n$$

(where Γ_n *is as in* (6.16) *and is given explicitly in Proposition* 6.4*). Then* $(S, \{F_n\})$ *is a solution of the de Branges-Rovnyak model* TVAIP, *and all solutions of the de Branges-Rovnyak model* TVAIP *arise in this way.*

Proof. The proof is a straightforward consequence of the correspondence between a TVSS $\mathfrak{S}$ with unitary scattering family $\{\mathcal{U}_n\}_{n\in\mathbb{Z}}$ and time-varying unitary colligations $\{U_n\}_{n\in\mathbb{Z}}$ according to the formula

$$U_n = P^{\mathcal{K}_{n+1}}_{\mathcal{H}_{n+1}\oplus\mathcal{E}_{*n}} \mathcal{U}_n^*|_{\mathcal{H}_n\oplus\mathcal{E}_n}.$$

The details for the time-invariant case can be found in [11]. ∎

There has recently been a lot of activity on extensions of tangential Nevanlinna-Pick interpolation and of the Nehari theorem on approximation of an L^∞ function by an H^∞ function in the infinity norm to the time-varying setting; for details we refer to the recent books [20], [19] and [15]. It is our contention that all these problems can be put into the framework of the TVAIP. Rather than attempting a formulation of the most general problem, for illustrative purposes we shall restrict ourselves to an informative special case, namely the *time-varying right* tangential Nevanlinna-Pick *interpolation problem* (TVRIP). The TVRIP is as follows. We are given families of Hilbert spaces

$$\mathcal{E} = \{\mathcal{E}_n\}_{n\in\mathbb{Z}}, \quad \mathcal{E}_* = \{\mathcal{E}_{*n}\}_{n\in\mathbb{Z}}$$

together with families of uniformly bounded, linear operators defined on $\mathbb{C}$

$$U_n : \mathbb{C} \to \mathcal{E}_n, \quad V_n : \mathbb{C} \to \mathcal{E}_{*n}, \quad w_n : \mathbb{C} \to \mathbb{C}. \tag{7.10}$$

(Thus U_n amounts to a vector in $\mathcal{E}_n$, V_n corresponds to a choice of vector in $\mathcal{E}_{*n}$ and w_n amounts to a complex number.) Associated with these families of operators are the block diagonal operators

$$\begin{aligned} U &= \operatorname{diag}\{U_n\}_{n\in\mathbb{Z}} : \ell^2(\mathbb{Z}, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathcal{E}) \\ V &= \operatorname{diag}\{V_n\}_{n\in\mathbb{Z}} : \ell^2(\mathbb{Z}, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathcal{E}) \\ W &= \operatorname{diag}\{w_n\}_{n\in\mathbb{Z}} : \ell^2(\mathbb{Z}, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathbb{C}). \end{aligned} \tag{7.11}$$

We assume that the operator $Z^{-1}W$ on $\ell^2(\mathbb{Z}, \mathbb{C})$ has spectral radius $r_\sigma(Z^{-1}A)$ less than 1. The TVRIP then is: *find (if possible) all contractive elements F of $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ such that*

$$(FU)^{\wedge R}(W) = V$$

where the right point evaluation $\mathbf{f} \to \mathbf{f}^{\wedge R}(A)$ is as in (2.18). The solution is: *Let $\Lambda = \operatorname{diag}\{\Lambda_n\}_{n\in\mathbb{Z}} \in \mathcal{D}(\mathbb{C}, \mathbb{C})$ be the unique solution of the time-varying Stein equation*

$$Z^{-1}\Lambda Z - A^*\Lambda A = U^*U - V^*V. \tag{7.12}$$

Then solutions to the TVRIP *exist if and only if $\Lambda \geqslant 0$ (i.e. $\Lambda_n \geqslant 0$ for each $n \in \mathbb{Z}$).* When solutions exist, there are procedures and formulas for constructing one solution or for parametrizing the set of all solutions under various hypotheses in various places in the literature (see e.g. [20], [15], [17] and [18]), but we shall not get into the details of this aspect; our purpose is to make the connection with the TVAIP.

To see how the TVAIP can be applied to the TVRIP, we must do two things: (1) *specify how to associate a* TVAIP *data set ω_{TVAIP} with an admissible* TVRIP *data set*

$$\omega_{\text{TVRIP}} = \{U = \{U_n\}, V = \{V_n\}, W = \{w_n\}\} \tag{7.13}$$

where U_n, V_n and w_n are as in (7.10), and (2) *indicate how a solution $(S, \{F_n\})$ of the* TVAIP *for ω_{TVAIP} generates a solution of the* TVRIP *for ω_{TVRIP}, and vice versa.*

As for (1), we assume that we are given a TVRIP data set (7.13) such that the associated numbers Λ_n are all nonnegative. We let $\mathcal{X}_n^0 = \mathbb{C}$, we let $\mathcal{X}_n$ equal $\mathbb{C}$ with the inner product induced by the positive number Λ_{n-1} if $\Lambda_{n-1} > 0$ and $\mathcal{X}_n = \{0\}$ if $\Lambda_{n-1} = 0$, and we take $\mathcal{E}_n$ and $\mathcal{E}_{*n}$ for ω_{TVAIP} the same as in ω_{TVRIP}. Then define operators

$$\begin{aligned} &T_{1,n} = [w_n] : \mathcal{X}_n^0 \to \mathcal{X}_n, \quad &&T_{2,n} = [I] : \mathcal{X}_n^0 \to \mathcal{X}_{n+1}, \\ &M_n = U_n : \mathcal{X}_n^0 \to \mathcal{E}_n, &&M_{*n} = V_n : \mathcal{X}_n^0 \to \mathcal{E}_{*n}. \end{aligned}$$

(Here the brackets in the definition of $T_{1,n}$ indicate that one takes $T_{1,n}$ to be zero in case $\mathcal{X}_n$ degenerates to the zero space, and similarly for $T_{2,n}$.) Note that these definitions give us a data set ω_{TVAIP} which satisfies the time-varying Potapov identity (7.1) since $\Lambda = \operatorname{diag}\{\Lambda_n\}_{n\in\mathbb{Z}}$ satisfies the time-varying Stein equation (7.12). In this way we have associated a TVAIP data set ω_{TVAIP} with any TVRIP data set ω_{TVRIP} for which the solution $\Lambda = \operatorname{diag}\{\Lambda_n\}_{n\in\mathbb{Z}}$ of (7.12) is positive semidefinite.

As for (2), suppose next that $(S, \{F_n\}_{n\in\mathbb{Z}})$ solves the TVAIP for ω_{TVAIP}, where we use the augmented de Branges-Rovnyak model formulation. Thus we write

$$F = [\cdots \quad F_{-1} \quad F_0 \quad F_1 \cdots] : \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathbb{C}) \to \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{E}_* \oplus \mathcal{E})$$

and $(S, \{F_n\})$ being a solution of TVAIP problem for data set ω_{TVAIP} means that condition (7.7) holds. For our situation this means that

$$F : \mathcal{D}_{\text{HS}}(\mathbb{C}, \mathbb{C}) \to \mathcal{H}^{\text{dBR,aug}}(S)$$

and

$$FR_W x = FR_Z x - \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} -V \\ U \end{bmatrix} x \quad \text{for all } x \in \mathcal{D}_{\text{HS}}(\mathbb{C}, \mathbb{C}).$$

Since by assumption $r_\sigma(Z^{-1}W) < 1$, it follows that $Z - A$ is invertible on ℓ^2 and we can solve uniquely for F in terms of S:

$$F = \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} -V \\ U \end{bmatrix} R_{(Z-A)^{-1}}. \tag{7.14}$$

Thus S uniquely determines F if (S, F) solves the augmented de Branges-Rovnyak model TVAIP for this case, and furthermore, for any such S it is always the case that the corresponding F satisfies

$$F : \mathcal{D}_{\text{HS}}(\mathbb{C}, \mathbb{C}) \to \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{E}_*) \\ \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{E}) \end{bmatrix}.$$

The only issue then is whether the image of $\mathcal{D}_{\text{HS}}(\mathbb{C}, \mathbb{C})$ under F is also contained in $\mathcal{L}_{\text{HS}}(\mathbb{C}, \mathcal{E}_*) \oplus \mathcal{U}_{-,\text{HS}}(\mathbb{C}, \mathcal{E})$. Let us write $Fx = \begin{bmatrix} F^{(1)} \\ F^{(2)} \end{bmatrix} x \subset \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{E}_*) \oplus \mathcal{X}_{\text{HS}}(\mathbb{C}, \mathcal{E})$. Then from (7.14) we see that

$$F^{(1)}x = (SU - V)x(Z - A)^{-1}, \quad F^{(2)}x = (U - S^*V)x(Z - A)^{-1}. \tag{7.15}$$

The condition that (S, F) solve the TVAIP is simply that $F^{(1)}x \in \mathcal{L}_{\text{HS}}(\mathbb{C}, \mathcal{E}_*)$ and that $F^{(2)}x \in \mathcal{U}_{-,\text{HS}}(\mathbb{C}, \mathcal{E})$ for each $x \in \mathcal{D}_{\text{HS}}(\mathbb{C}, \mathbb{C})$. As U, V and x are diagonal, S^* is upper triangular and $(Z-A)^{-1} = (I-Z^{-1}A)^{-1}Z^{-1}$ is strictly upper triangular, it is clear that $F^{(2)}x$ is strictly upper triangular for any such S (no interpolation conditions required). On the other hand, it is well known (see e.g. [2]) that the value of the right point evaluation $(SUx)^{\wedge R}(W)$ is characterized as that diagonal operator D such that

$$[SUx - D](Z - W)^{-1} \in \mathcal{L}_{\text{HS}}(\mathbb{C}, \mathcal{E}_*).$$

We thus see from (7.15) that $F^{(1)}x \in \mathcal{L}_{\text{HS}}(\mathbb{C}, \mathcal{E}_*)$ exactly when $Vx = (SUx)^{\wedge R}(W)$. Another easy property concerning right point evaluations is that

$$(SUx)^{\wedge R}(W) = (SU)^{\wedge R}(W)x \quad \text{for } x \text{ diagonal.}$$

We conclude that (S, F) solves the TVAIP if and only if $(SU)^{\wedge R}(W) = V$, i.e., if and only S solves the TVRIP, as expected.

REMARK 7.2. In the time-invariant case, it is possible to see the Nehari problem as an instance of the AIP (see [11] and the references there). Recent work of Kheifets ([26]) formulates a more general abstract interpolation problem whereby axiom (3) of a scattering system (the orthogonality between the outgoing and incoming spaces) is removed; this gives a more natural framework into which to fit the Nehari problem. It should be possible to pursue this idea also for the time-varying setting.

REMARK 7.3. The time-invariant version of the Abstract Interpolation Problem has applications to many other types of interpolation problems, such as 2-block and 4-block interpolation, boundary interpolation and the Hamburger moment problem (where one must use a linear-fractional change of variable to convert the original continuous-time setting to a discrete-time setting, see [24]). A possible line of future research is to understand a time-varying analogue of boundary interpolation by using the TVAIP formalism.

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Received August 23, 1999; revised November 6, 2000.