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ON A CLASS OF NON-SELF-ADJOINT QUADRATIC MATRIX OPERATOR PENCILS ARISING IN ELASTICITY THEORY

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Abstract. This paper deals with a class of non-self-adjoint quadratic pencils of block operator matrices. The main results concern the structure and location of the spectrum and theorems about the minimality, completeness and basis properties of the eigenvectors and associated vectors corresponding to certain parts of the spectrum. Finally, an application to the problem of vibrations of a rotating beam is given.

Keywords: *quadratic operator pencil, completeness of basis of eigenvectors.*
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1. INTRODUCTION

The spectral theory of quadratic operator pencils is a classical subject with many applications in elasticity theory. A fundamental contribution to the theory of self-adjoint operator pencils is due to Krein and Langer ([7]). Also, operators which have a certain block matrix representation occur frequently in mathematical physics. Recent contributions to this area may be found e.g. in [2], [1], [8], and [9].

In this paper we are going to study a class of damped non-self-adjoint quadratic operator pencils the coefficients of which are unbounded block operator matrices. Our aim is to investigate the spectrum of such pencils, to study the properties of the eigenvectors and associated vectors corresponding to certain parts of the spectrum, and to apply the results to the problem of vibrations of a rotating beam with inner and outer damping in a possibly inhomogeneous outer medium. A fundamental tool here are factorization theorems by Markus, Matsaev and Russu ([11], [12], [13], [10]).

Let $\mathcal{H}$ be a separable (infinite dimensional) Hilbert space. We consider a quadratic operator pencil $\mathcal{L}$ acting in the product space $\mathcal{H} \times \mathcal{H}$ and given by the matrix representation

$$(1.1)\quad \mathcal{L}(\lambda) = \lambda^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \lambda \begin{pmatrix} \alpha A + K_1 & 0 \\ 0 & \alpha A + K_2 \end{pmatrix} + \begin{pmatrix} A & \beta A \\ -\beta A & A \end{pmatrix}, \quad \lambda \in \mathbb{C}.$$

Here A is an unbounded self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$ having compact resolvent, $A \geqslant \delta I$ with some $\delta > 0$, α and β are positive constants, and K_1, K_2 are bounded operators, $0 \leqslant K_1, K_2 \leqslant \gamma I$ with some $\infty > \gamma \geqslant 0$. The domain of $\mathcal{L}(\lambda)$ is independent of λ and given by $\mathcal{D}(\mathcal{L}(\lambda)) = \mathcal{D}(A) \times \mathcal{D}(A)$.

An example for such a quadratic operator pencil arises in elasticity theory: The system of differential equations

$$EI\frac{\partial^4 u}{\partial z^4} + \omega\kappa EI\frac{\partial^4 v}{\partial z^4} + \kappa EI\frac{\partial^5 u}{\partial z^4 \partial t} + \varepsilon_1\frac{\partial u}{\partial t} + m\frac{\partial^2 u}{\partial t^2} = 0, \tag{1.2}$$

$$EI\frac{\partial^4 v}{\partial z^4} - \omega\kappa EI\frac{\partial^4 u}{\partial z^4} + \kappa EI\frac{\partial^5 v}{\partial z^4 \partial t} + \varepsilon_2\frac{\partial v}{\partial t} + m\frac{\partial^2 v}{\partial t^2} = 0 \tag{1.3}$$

on the finite interval $[0, l]$ describes the vibrations of a rotating beam of length l and mass density m per unit length. Here $EI > 0$ is the (constant) bending stiffness of the beam sections, $\omega > 0$ is the angular frequency of the rotation, $\kappa > 0$ is the coefficient of inner damping (Voigt material), and $\varepsilon_1, \varepsilon_2$ are nonnegative continuous functions on $[0, l]$ describing the outer viscous damping. In the general case when the outer medium is inhomogeneous, one has $\varepsilon_1 \not\equiv \varepsilon_2$ (see e.g. [3]).

The boundary conditions to be imposed e.g. in the case of hinged ends are

$$\begin{aligned} u(0,t) = u(l,t) = \frac{\partial^2 u}{\partial z^2}(0,t) = \frac{\partial^2 u}{\partial z^2}(l,t) = 0, \\ v(0,t) = v(l,t) = \frac{\partial^2 v}{\partial z^2}(0,t) = \frac{\partial^2 v}{\partial z^2}(l,t) = 0. \end{aligned} \tag{1.4}$$

For simplicity, we assume that $m \equiv 1$. Then separation of variables

$$(u(z,t), v(z,t))^{\mathrm{t}} = \mathrm{e}^{\lambda t}(y_1(z), y_2(z))^{\mathrm{t}}, \quad z \in [0, l],\ t \geqslant 0, \tag{1.5}$$

leads to a spectral problem of the form

$$\mathcal{L}(\lambda)y = 0, \quad \lambda \in \mathbb{C},$$

for $y = (y_1, y_2)^{\mathrm{t}}$ in the Hilbert space $L_2(0,l) \times L_2(0,l)$ where the operators A and K in $L_2(0,l)$ are given by

$$Ay := EI\, y^{(4)},\ \mathcal{D}(A) := \{y \in L_2(0,l) : y(0) = y(l) = y''(0) = y''(l) = 0\}, \tag{1.6}$$

$$K_i y := \varepsilon_i y, \quad \mathcal{D}(K_i) := L_2(0,l), \quad i = 1, 2, \tag{1.7}$$

and the constants α and β are given by $\alpha := \kappa$, $\beta := \omega\kappa$.

In Section 2 we first determine the structure of the spectrum of the quadratic operator pencil (1.1). We show that its essential spectrum consists of the points $-\frac{1+\mathrm{i}\beta}{\alpha}, -\frac{1-\mathrm{i}\beta}{\alpha}$, and that outside the essential spectrum $\mathcal{L}$ has 3 branches of eigenvalues accumulating at the points of the essential spectrum and at ∞. Secondly, we prove a criterion for the stability of the pencil $\mathcal{L}$, that is, a criterion guaranteeing that the spectrum of $\mathcal{L}$ lies in the open left half plane.

In Section 3 we consider the case $K_1 = K_2$ where $\mathcal{L}$ in fact decomposes into two quadratic pencils in $\mathcal{H}$. In this case the eigenvalue branch accumulating at ∞ splits again into two branches. We derive theorems about the minimality, completeness and basis properties of the eigenvectors and associated vectors corresponding to the 4 branches of eigenvalues of $\mathcal{L}$.

In Section 4 we consider the case $K_1 \neq K_2$. We prove a theorem about the minimality, completeness and basis properties of the eigenvectors and associated vectors corresponding to the branch of eigenvalues of $\mathcal{L}$ accumulating at ∞. Finally, in Section 5, we apply all results to the problem (1.2)–(1.4) of vibrations of a rotating beam.

2. THE SPECTRUM OF $\mathcal{L}$

We define the *resolvent set* $\rho(\mathcal{L})$ of the quadratic operator pencil $\mathcal{L}$ as

$$\rho(\mathcal{L}) := \{\lambda \in \mathbb{C} : \mathcal{L}(\lambda) : \mathcal{D}(A) \times \mathcal{D}(A) \to \mathcal{H} \times \mathcal{H} \text{ is bijective, } \mathcal{L}(\lambda)^{-1} \text{ is bounded}\}$$

and its *spectrum* $\sigma(\mathcal{L})$ as $\sigma(\mathcal{L}) := \mathbb{C} \setminus \rho(\mathcal{L})$. For $\lambda \in \mathbb{C}$, the operator $\mathcal{L}(\lambda)$ is called Fredholm if $\mathcal{L}(\lambda)$ is closed, its kernel is finite dimensional and its range is finite codimensional (see e.g. [6], Chapter IV, Section 5.1). A point $\lambda_0 \in \mathbb{C}$ is said to be an *eigenvalue* of $\mathcal{L}$ if $\mathcal{L}(\lambda)$ is not injective. An eigenvalue $\lambda_0 \in \mathbb{C}$ of $\mathcal{L}$ is called *normal* (or of finite type) if λ_0 is isolated and $\mathcal{L}(\lambda_0)$ is Fredholm. The *essential spectrum* of $\mathcal{L}$ is defined as

$$\sigma_{\mathrm{ess}}(\mathcal{L}) := \{\lambda \in \mathbb{C} : \mathcal{L}(\lambda) \text{ is not Fredholm}\}.$$

In order to determine the essential spectrum of the operator pencil $\mathcal{L}$, we consider the transformed pencil $\mathcal{L}_{\mathrm{d}}$ given by $\mathcal{L}_{\mathrm{d}}(\lambda) := S^{-1}\mathcal{L}(\lambda)S$ on $\mathcal{D}(A) \times \mathcal{D}(A)$ for $\lambda \in \mathbb{C}$ where the operator matrix S in $\mathcal{H} \times \mathcal{H}$ is of the form

$$S := \begin{pmatrix} I & \mathrm{i}I \\ \mathrm{i}I & I \end{pmatrix}.$$

Then, for $\lambda \in \mathbb{C}$,

$$\mathcal{L}_{\mathrm{d}}(\lambda) := \lambda^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \lambda \begin{pmatrix} \alpha A + \frac{1}{2}(K_1 + K_2) & \frac{\mathrm{i}}{2}(K_1 - K_2) \\ -\frac{\mathrm{i}}{2}(K_1 - K_2) & \alpha A + \frac{1}{2}(K_1 + K_2) \end{pmatrix} + \begin{pmatrix} (1+\mathrm{i}\beta)A & 0 \\ 0 & (1-\mathrm{i}\beta)A \end{pmatrix},$$

and $\mathcal{L}_{\mathrm{d}}(\lambda)$ is closed (Fredholm) if and only if $\mathcal{L}(\lambda)$ is closed (Fredholm).

THEOREM 2.1. *The essential spectrum of $\mathcal{L}$ consists of the two points*

$$-\frac{1+\mathrm{i}\beta}{\alpha}, -\frac{1-\mathrm{i}\beta}{\alpha}.$$

The other points of the spectrum of $\mathcal{L}$ are normal eigenvalues which accumulate at most at the points $-\frac{1+\mathrm{i}\beta}{\alpha}$, $-\frac{1-\mathrm{i}\beta}{\alpha}$, and at ∞.

Proof. Let $\lambda \in \mathbb{C}$. If we write $\mathcal{L}_{\mathrm{d}}(\lambda)$ in the form

$$\begin{pmatrix} (\lambda\alpha+(1+\mathrm{i}\beta))A+\lambda^2 I+\lambda\frac{1}{2}(K_1+K_2) & \lambda\frac{\mathrm{i}}{2}(K_1-K_2) \\ -\lambda\frac{\mathrm{i}}{2}(K_1-K_2) & (\lambda\alpha+(1-\mathrm{i}\beta))A+\lambda^2 I+\lambda\frac{1}{2}(K_1+K_2) \end{pmatrix},$$

it is not difficult to see that $\mathcal{L}_{\mathrm{d}}(\lambda)$ (with domain $\mathcal{D}(A) \times \mathcal{D}(A)$) is closed if and only if $\lambda \neq -\frac{1\pm\mathrm{i}\beta}{\alpha}$. Now let $\lambda \neq -\frac{1\pm\mathrm{i}\beta}{\alpha}$. Since

$$\begin{aligned}
&\mathcal{L}_{\mathrm{d}}(\lambda)\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1}\end{pmatrix} \\
&= \lambda^2 \begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1}\end{pmatrix} + \lambda \begin{pmatrix} \alpha I + \frac{1}{2}(K_1+K_2)A^{-1} & \frac{\mathrm{i}}{2}(K_1-K_2)A^{-1} \\ -\frac{\mathrm{i}}{2}(K_1-K_2)A^{-1} & \alpha I + \frac{1}{2}(K_1+K_2)A^{-1}\end{pmatrix} \\
&\qquad + \begin{pmatrix} (1+\mathrm{i}\beta)I & 0 \\ 0 & (1-\mathrm{i}\beta)I\end{pmatrix} \\
&= \begin{pmatrix} (1+\mathrm{i}\beta+\lambda\alpha)I & 0 \\ 0 & (1-\mathrm{i}\beta+\lambda\alpha)I\end{pmatrix} + K(\lambda),
\end{aligned} \tag{2.1}$$

where $K(\lambda)$ is a compact operator in $\mathcal{H}\times\mathcal{H}$, the operator on the left hand side of (2.1) is Fredholm (see e.g. [4], Chapter XI, Theorem 4.2). Hence the same is true for $\mathcal{L}_{\mathrm{d}}(\lambda)$. On the other hand, since $\mathcal{H}$ is infinite dimensional, it follows from (2.1) that $\mathcal{L}_{\mathrm{d}}\big(-\frac{1\pm\mathrm{i}\beta}{\alpha}\big)$ is not Fredholm. Moreover, the operator in (2.1) is bijective for $\lambda = 0$. Now the theorem follows e.g. from a theorem about analytic Fredholm operator valued functions (see [4], Chapter XI, Corollary 8.4). ∎

REMARK 2.2. From the minimality and completeness results which will be proved in the next two sections more precise statements about the accumulation of the eigenvalues will follow if the resolvent of A belongs to a certain von Neumann–Schatten class: In the next section we are going to show that in the case $K_1 = K_2$ the normal eigenvalues of $\mathcal{L}$ split into 4 branches of eigenvalues, two branches accumulating at the points $-\frac{1+\mathrm{i}\beta}{\alpha}$, $-\frac{1-\mathrm{i}\beta}{\alpha}$, and two branches at ∞. In Section 4, for the case $K_1 \neq K_2$, it will turn out that again ∞ is an accumulation point of eigenvalues, but the branches of eigenvalues accumulating at ∞ can, in general, not be separated as in the case $K_1 = K_2$.

THEOREM 2.3. *Assume that there exists a* $\mu > 0$ *with* $K_1 \geqslant \mu I$, $K_2 \geqslant \mu I$, *and such that*

$$\frac{\mu}{\alpha} \geqslant \delta \quad \text{and} \quad \frac{\beta^2}{4\mu\alpha} < 1, \qquad \text{or} \qquad \frac{\mu}{\alpha} < \delta \quad \text{and} \quad \frac{\beta^2\delta}{(\alpha\delta+\mu)^2} < 1. \tag{2.2}$$

Then the spectrum of $\mathcal{L}$ *lies in the open left half plane.*

Proof. If $\lambda_0 \in \sigma_{\mathrm{ess}}(\mathcal{L})$, then obviously $\mathrm{Re}\,(\lambda_0) < 0$ by the above theorem. Otherwise, if $\lambda_0 \in \sigma(\mathcal{L}) \setminus \sigma_{\mathrm{ess}}(\mathcal{L})$, then λ_0 is an eigenvalue of $\mathcal{L}$. Let $Y = (y,z)^{\mathrm{t}} \in \mathcal{D}(A) \times \mathcal{D}(A)$, $\|y\|^2 + \|z\|^2 = 1$, be a corresponding eigenvector. Then $(\mathcal{L}(\lambda_0)Y, Y) = 0$ implies that

$$\begin{aligned}
&\mathrm{Re}\,(\lambda_0)^2 - \mathrm{Im}\,(\lambda_0)^2 + \mathrm{Re}\,(\lambda_0)(\alpha((Ay,y)+(Az,z)) + (K_1 y,y) \\
&\qquad + (K_2 z,z)) + (Ay,y) + (Az,z) = 0,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&2\,\mathrm{Re}\,(\lambda_0)\mathrm{Im}\,(\lambda_0) + \mathrm{Im}\,(\lambda_0)(\alpha((Ay,y)+(Az,z)) + (K_1 y,y) \\
&\qquad + (K_2 z,z)) + 2\beta\,\mathrm{Im}\,(Az,y) = 0.
\end{aligned} \tag{2.4}$$

Calculating $\mathrm{Im}\,(\lambda_0)$ from (2.4), using the estimate

$$\begin{aligned}|2\mathrm{Im}\,(Az,y)| &\leqslant 2|(Az,y)| = 2|(A^{1/2}z, A^{1/2}y)|\\ &\leqslant \|A^{1/2}z\|^2 + \|A^{1/2}y\|^2 = (Ay,y) + (Az,z)\end{aligned}$$

and substituting it into (2.3), we arrive at

$$\begin{aligned}&\mathrm{Re}\,(\lambda_0)^2 - \frac{\beta^2((Ay,y)+(Az,z))^2}{(2\,\mathrm{Re}\,(\lambda_0) + \alpha((Ay,y)+(Az,z)) + (K_1y,y) + (K_2y,y))^2}\\ &+\mathrm{Re}\,(\lambda_0)\,(\alpha((Ay,y)+(Az,z))+(K_1y,y)+(K_2z,z))+(Ay,y)+(Az,z)\leqslant 0.\end{aligned} \tag{2.5}$$

The left hand side is monotonically increasing for $\mathrm{Re}\,(\lambda_0) \in [0,\infty)$. But the condition (2.2) and $A \geqslant \delta > 0$ imply that

$$-\frac{\beta^2((Ay,y)+(Az,z))^2}{(\alpha((Ay,y)+(Az,z)) + (K_1y,y) + (K_2z,z))^2} + (Ay,y) + (Az,z) > 0,$$

a contradiction to (2.5). Hence $\mathrm{Re}\,(\lambda_0) < 0$. ∎

3. THE CASE $K_1 = K_2 = K$

In this case the operator pencil $\mathcal{L}_\mathrm{d}$ in $\mathcal{H}\times\mathcal{H}$ is the orthogonal sum of two quadratic operator pencils $\mathcal{L}_\pm$ in $\mathcal{H}$,

$$\mathcal{L}_\mathrm{d}(\lambda) = \begin{pmatrix} \mathcal{L}_+(\lambda) & 0\\ 0 & \mathcal{L}_-(\lambda)\end{pmatrix}, \quad \lambda \in \mathbb{C},$$

where

$$\mathcal{L}_\pm(\lambda) := \lambda^2 I + \lambda(\alpha A + K) + (1 \pm \mathrm{i}\beta)A, \quad \lambda \in \mathbb{C}.$$

In the following we are going to prove minimality, completeness and basis results for the eigenvectors and associated vectors corresponding to various branches of eigenvalues of $\mathcal{L}$.

With regard to the eigenvalues which will prove to accumulate at ∞, we introduce the auxiliary operator pencils $\mathcal{L}^1_\pm$ given by

$$\begin{aligned}\mathcal{L}^1_\pm(\lambda) &:= \frac{\lambda^2}{\alpha} A^{-1/2}\mathcal{L}_\pm\left(\frac{1}{\lambda}\right) A^{-1/2}\\ &= \lambda^2 \frac{1\pm \mathrm{i}\beta}{\alpha} I + \lambda\left(I + \frac{1}{\alpha}A^{-1/2}KA^{-1/2}\right) + \frac{1}{\alpha}A^{-1}, \quad \lambda \in \mathbb{C}.\end{aligned}$$

LEMMA 3.1. *The spectrum and the essential spectrum of $\mathcal{L}^1_\pm$ are given by:*
(i) $\sigma(\mathcal{L}^1_\pm) = \left\{\frac{1}{\lambda} : \lambda \in \sigma(\mathcal{L}_\pm)\right\} \cup \{0\}$;
(ii) $\sigma_{\mathrm{ess}}(\mathcal{L}^1_\pm) = \left\{0, -\frac{\alpha}{1\pm \mathrm{i}\beta}\right\}$.

Proof. The assertions are immediate from Theorem 2.1 and from the definition of $\mathcal{L}^1_\pm$. ∎

The *numerical range* (or root domain) of a quadratic operator polynomial $\mathcal{T}$ in $\mathcal{H}$ is the set of all roots of all possible polynomials $(\mathcal{T}(\,\cdot\,)y,y)$, $y \in \mathcal{H}$, $y \neq 0$ (see [10], Section 26.3). It consists of at most two components. If the numerical range of $\mathcal{T}$ consists of two disjoint components (called the *root zones* of $\mathcal{T}$), then, clearly, for any $y \in \mathcal{H}$, $y \neq 0$, the polynomial $(\mathcal{T}(\,\cdot\,)y,y)$ has exactly one root in each component.

LEMMA 3.2. *The numerical range of $\mathcal{L}_+^1$ consists of two components Δ_+^1, $\widetilde{\Delta}_+^1$ which are bounded and separated by the strip*

$$\left\{\lambda \in \mathbb{C} : 0 < \mathrm{Im}\,(\lambda) < \frac{\alpha\beta}{1+\beta^2}\right\}, \tag{3.1}$$

say Δ_+^1 below and $\widetilde{\Delta}_+^1$ above this strip. There exists a simple contour Γ_+^1 surrounding $\overline{\Delta_+^1} \cup \{0\}$ and separating $\overline{\Delta_+^1} \cup \{0\}$ from $\widetilde{\Delta}_+^1$, and such that

$$\inf_{\lambda\in\Gamma_+^1,\,\|y\|=1} \left|(\mathcal{L}_+^1(\lambda)y,y)\right| > 0.$$

Proof. The first assertion follows from the fact that $\mathcal{L}_+^1$ is a pencil of bounded operators. For the proof of the second statement, fix an element $y \in \mathcal{H}$, $\|y\| = 1$, consider the function

$$\varphi(\lambda,\eta) := \lambda^2\frac{1+\mathrm{i}\beta}{\alpha} + \lambda\left(1+\eta\frac{1}{\alpha}\big(KA^{-1/2}y, A^{-1/2}y\big)\right) + \eta\frac{1}{\alpha}\big(A^{-1/2}y, A^{-1/2}y\big)$$

for $\lambda \in \mathbb{C}$, $\eta \in [0,1]$, and assume that there exists a point λ_0 in the strip (3.1) and an $\eta \in [0,1]$ such that $\varphi(\lambda_0,\eta) = 0$. The point λ_0 has a representation $\lambda_0 = t + \mathrm{i}d$ with $t \in \mathbb{R}$ and $0 < d < \frac{\alpha\beta}{1+\beta^2}$. Hence

$$(t^2+2\mathrm{i}td-d^2)\frac{1+\mathrm{i}\beta}{\alpha} + \big(t+\mathrm{i}d\big)\left(1+\eta\frac{1}{\alpha}\big(KA^{-1/2}y, A^{-1/2}y\big)\right) + \eta\frac{1}{\alpha}\big\|A^{-1/2}y\big\|^2 = 0.$$

Taking the imaginary part and multiplying by $\frac{\alpha}{\beta}$ yields

$$t^2 + 2\frac{td}{\beta} - d^2 + d\frac{\alpha}{\beta}\left(1+\eta\frac{1}{\alpha}\big(KA^{-1/2}y, A^{-1/2}y\big)\right) = 0.$$

This is equivalent to

$$\left(t+\frac{d}{\beta}\right)^2 = \frac{d}{\beta^2}\left(d(1+\beta^2) - \alpha\beta\left(1+\eta\frac{1}{\alpha}\big(KA^{-1/2}y, A^{-1/2}y\big)\right)\right),$$

which implies, since $K \geqslant 0$, that

$$d \geqslant \frac{\alpha\beta}{1+\beta^2},$$

a contradiction. Thus $\varphi(\lambda,\eta) \neq 0$, $\eta \in [0,1]$, and in particular $(\mathcal{L}_+^1(\lambda)y,y) = \|y\|^2\varphi(\lambda,1) \neq 0$ for all λ in the strip (3.1).

The zeros $\lambda_1(\eta), \lambda_2(\eta)$ of $\varphi(\,\cdot\,,\eta) = 0$ depend continuously on the parameter η, and $\lambda_1(0) = 0$, $\lambda_2(0) = -\frac{\alpha}{1+\mathrm{i}\beta}$ are separated by the strip (3.1). Then, according to what was proved above, so are $\lambda_1(\eta), \lambda_2(\eta)$ for all $\eta \in [0,1]$. In particular, the zeros of $(\mathcal{L}_+^1(\,\cdot\,)y,y) = \|y\|^2\varphi(\,\cdot\,,1)$ are separated by the strip (3.1). This proves the second statement. The remaining assertions are immediate. ∎

The subsequent corollary about the existence of a spectral root of $\mathcal{L}_+^1$ (or, equivalently, of a canonical factorization of $\lambda^{-1}\mathcal{L}_+^1(\lambda)$) follows from the above lemma by some general results of Markus and Matsaev ([11], [12], see also [10], Theorems 26.19 and 26.12).

Corollary 3.3. *The operator pencil $\mathcal{L}_+^1$ has a spectral root Z_+^1 such that*

$$\sigma(Z_+^1) = \sigma(\mathcal{L}_+^1) \cap \Delta_+^1.$$

In the following let $\mathcal{S}_p, 1 \leqslant p \leqslant \infty$, denote the von Neumann–Schatten classes of compact operators (see [5], Chapter III, Section 7, or [10], Section 2.4). Further, for an operator $T \in \mathcal{S}_\infty$ we denote by $n(\tau, T)$ the sum of the algebraic multiplicities of the eigenvalues of T in $\{\lambda \in \mathbb{C} : |\lambda| > \tau^{-1}\}$. For a quadratic operator pencil $\mathcal{T}$ the nonzero spectrum of which in some bounded domain G containing 0 consists of a sequence of eigenvalues of finite algebraic multiplicity converging to 0, we denote by $n(\tau, G, \mathcal{T})$ the sum of the algebraic multiplicities of the eigenvalues of $\mathcal{T}$ in $\{\lambda \in \mathbb{C} : |\lambda| > \tau^{-1}\} \cap G$ (see [10], Section 22.4). Using a theorem of Markus, Matsaev and Russu ([13]), we obtain:

Theorem 3.4. (i) *The set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_+^1$ in Δ_+^1 is minimal in $\mathcal{H}$.*

(ii) *If $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$, then the set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_+^1$ in Δ_+^1 is complete in $\mathcal{H}$. If, in addition, $n\left(\tau, \frac{1}{\alpha}A^{-1}\right) \sim c_1\tau^{c_2}$ as $\tau \to \infty$ with some $0 < c_1, c_2 < \infty$, then $n(\tau, G_+^1, \mathcal{L}_+^1) \sim c_1\tau^{c_2}$, where G_+^1 is the interior of the curve Γ_+^1.*

(iii) *If $n(\tau, A^{-1}) = \mathrm{O}(\tau^\gamma)$ for some $\gamma \in (0, \frac{1}{2}]$, then the set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_+^1$ in Δ_+^1 is a Riesz basis with parentheses in $\mathcal{H}$. If, in addition, $n\left(\tau, \frac{1}{\alpha}A^{-1}\right) = c_1\tau^{c_2} + \mathrm{O}(\tau^\beta)$ for some $0 < c_1, c_2 < \infty$, $0 \leqslant \beta < \alpha \leqslant \beta + \gamma$, then also $n(\tau, G_+^1, \mathcal{L}_+^1) = c_1\tau^{c_2} + \mathrm{O}(\tau^\beta)$.*

Proof. The theorem follows from a general result of Markus, Matsaev and Russu (see [13]) which is contained in [10], Theorem 22.13 and Corollary 26.20. To apply the statements therein, we choose $H = \frac{1}{\alpha}A^{-1}$, $T = 0$ for (ii) and $H = \frac{1}{\alpha}A^{-1}$, $D_0 = 0$, $D_1 = \frac{1}{\alpha}A^{-1/2}KA^{-1/2+\gamma}$ where $\gamma \in (0, \frac{1}{2}]$ for (iii). ∎

Remark 3.5. Analogous assertions hold for the pencil $\mathcal{L}_-^1$.

With regard to the branches of eigenvalues possibly accumulating at $-\frac{1 \pm \mathrm{i}\beta}{\alpha}$, we first consider the pencils $\mathcal{L}_\pm$ themselves.

Lemma 3.6. *The numerical range of $\mathcal{L}_+$ consists of two components $\Delta_+, \widetilde{\Delta}_+$ which are separated by the strip*

$$\{\lambda \in \mathbb{C} : -\delta_2 < \mathrm{Im}\,(\lambda) < \delta_2\}$$

where

$$\delta_2 := \beta\big(\big(\alpha^2 + (2\alpha\|K\| + 4)\,\delta^{-1} + \|K\|^2\delta^{-2}\big)^2 + 16\beta^2\delta^{-2}\big)^{-1/4}.$$

The component located in the half plane $\{\lambda \in \mathbb{C} : \operatorname{Im}(\lambda) \leqslant -\delta_2\}$*, say* Δ_+*, is bounded, and there exists a simple closed curve* Γ_+ *surrounding* Δ_+ *and separating it from* $\widetilde{\Delta}_+$*, and such that*

$$\inf_{\substack{\lambda\in\Gamma_+\\ y\in\mathcal{D}(A),\,\|y\|=1}} |(\mathcal{L}_+(\lambda)y,y)| > 0.$$

Proof. Let $y \in \mathcal{D}(A)$, $\|y\| = 1$. Then the solutions $\lambda_\pm(y)$ of $(\mathcal{L}_+(\lambda)y, y) = 0$ are given by

$$\lambda_\pm(y) = -\frac{\alpha(Ay,y)+(Ky,y)}{2} \pm \sqrt{\frac{(\alpha(Ay,y)+(Ky,y))^2}{4} - (1+\mathrm{i}\beta)(Ay,y)}. \tag{3.2}$$

Hence

$$\begin{aligned}
|\operatorname{Im}(\lambda_\pm(y))| &= \frac{1}{2}\left|\operatorname{Im}\left((\alpha(Ay,y)+(Ky,y))^2 - 4(1+\mathrm{i}\beta)(Ay,y)\right)^{1/2}\right| \\
&= \frac{1}{2\sqrt{2}}\Big(4(Ay,y) - (\alpha(Ay,y)+(Ky,y))^2 + \big(\big((\alpha(Ay,y)+(Ky,y))^2 \\
&\qquad - 4(Ay,y)\big)^2 + 16\beta^2(Ay,y)^2\big)^{1/2}\Big)^{1/2} \\
&\geqslant \beta\left(\left(\frac{(\alpha(Ay,y)+(Ky,y))^2 - 4(Ay,y)}{(Ay,y)^2}\right)^2 + \frac{16\beta^2}{(Ay,y)^2}\right)^{-1/4} \geqslant \delta_2.
\end{aligned}$$

This proves the first assertion. The second statement follows from the fact that the root $\lambda_+(y)$ lying in the half plane $\{\lambda \in \mathbb{C} : \operatorname{Im}(\lambda) \leqslant -\delta_2\}$ tends to $-\frac{1+\mathrm{i}\beta}{\alpha}$ when (Ay, y) tends to infinity. The remaining assertions are then immediate. ∎

In order to apply the results of Markus, Matsaev and Russu used before we need to consider pencils of bounded operators. Therefore we introduce

$$\begin{aligned}
\mathcal{L}^2_\pm(\lambda) &:= \frac{1}{\alpha}A^{-1/2}\mathcal{L}_\pm\left(\lambda - \frac{1\pm\mathrm{i}\beta}{\alpha}\right)A^{-1/2} \\
&= \frac{\lambda^2}{\alpha}A^{-1} + \lambda\left(I - \frac{2(1\pm\mathrm{i}\beta)}{\alpha^2}A^{-1} + \frac{1}{\alpha}A^{-1/2}KA^{-1/2}\right) + B_\pm
\end{aligned}$$

where

$$B_\pm = \frac{(1\pm\mathrm{i}\beta)^2}{\alpha^3}A^{-1} - \frac{1\pm\mathrm{i}\beta}{\alpha^2}A^{-1/2}KA^{-1/2}.$$

COROLLARY 3.7. *The numerical range of* $\mathcal{L}^2_+$ *consists of two components* Δ^2_+, $\widetilde{\Delta}^2_+$ *which are bounded and separated by the strip*

$$\left\{\lambda \in \mathbb{C} : \frac{\beta}{\alpha} - \delta_2 < \operatorname{Im}(\lambda) < \frac{\beta}{\alpha} + \delta_2\right\},$$

say Δ^2_+ *below and* $\widetilde{\Delta}^2_+$ *above this strip. There exists a simple closed curve* Γ^2_+ *surrounding* $\Delta^2_+ \cup \{0\}$ *and separating it from* $\widetilde{\Delta}^2_+$*, and such that*

$$\inf_{\lambda\in\Gamma^2_+,\,\|y\|=1} \left|(\mathcal{L}^2_+(\lambda)y,y)\right| > 0.$$

Proof. All assertions follow immediately from Lemma 3.6 by definition of $\mathcal{L}^2_+$ (note that $\delta_2 < \frac{\beta}{\alpha}$). ∎

THEOREM 3.8. (i) *The set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_+^2$ in Δ_+^2 is minimal in $\mathcal{H}$.*

(ii) *If $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$, then the set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_+^2$ in Δ_+^2 is complete in $\mathcal{H}$. If, in addition, $n(\tau, B_+) \sim c_1\tau^{c_2}$ as $\tau \to \infty$ with some $0 < c_1, c_2 < \infty$, then $n(\tau, G_+^2, \mathcal{L}_+^2) \sim c_1\tau^{c_2}$, where G_+^2 is the interior of the curve Γ_+^2.*

Proof. As in the proof of Theorem 3.4, we use [10], Theorem 22.13 and Corollary 26.20. To apply the statements therein, we now choose $H = B_+$, $T = 0$ for (ii), and we note that if $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$, then also $B_+ \in \mathcal{S}_p$ (see e.g. [5]). ∎

REMARK 3.9. Analogous assertions hold for the pencil $\mathcal{L}_-^2$.

In order to formulate statements for the original pencil $\mathcal{L}$, we first note that

$$\sigma(\mathcal{L}) = \sigma(\mathcal{L}_{\mathrm{d}}) = \sigma(\mathcal{L}_+) \cup \sigma(\mathcal{L}_-).$$

We denote by λ_k^1 and λ_k^2, $k = 1, 2, \ldots$, the eigenvalues of the pencil $\mathcal{L}_+$ located in the upper and lower half plane, respectively (counted according to their algebraic multiplicities). It is not difficult to see that then the complex conjugates $\overline{\lambda_k^1}$ and $\overline{\lambda_k^2}$, $k = 1, 2, \ldots$, are the eigenvalues of the pencil $\mathcal{L}_-$ located in the lower and upper half plane, respectively. We denote the corresponding eigenvectors and associated vectors of $\mathcal{L}_+$ by y_k^1 and y_k^2, and those of $\mathcal{L}_-$ by $\overline{y_k^1}$ and $\overline{y_k^2}$.

Then the eigenvalues of $\mathcal{L}_{\mathrm{d}}$ (and hence those of $\mathcal{L}$) can be separated into the 4 branches $\{\lambda_k^1\}$, $\{\lambda_k^2\}$ and $\{\overline{\lambda_k^1}\}$, $\{\overline{\lambda_k^2}\}$, which lie symmetrically to the real axis. The corresponding eigenvectors and associated vectors of $\mathcal{L}_{\mathrm{d}}$ are of the form

$$(y_k^1, 0)^{\mathrm{t}},\ (y_k^2, 0)^{\mathrm{t}},\ (0, \overline{y_k^1})^{\mathrm{t}},\ (0, \overline{y_k^2})^{\mathrm{t}}.$$

The respective eigenvectors of $\mathcal{L}$ are given by

$$(y_k^1, \mathrm{i}y_k^1)^{\mathrm{t}},\ (y_k^2, \mathrm{i}y_k^2)^{\mathrm{t}},\ (\mathrm{i}\overline{y}_k^1, \overline{y}_k^1)^{\mathrm{t}},\ (\mathrm{i}\overline{y_k^2}, \overline{y_k^2})^{\mathrm{t}},$$

and there are analogous formulas for the associated vectors of $\mathcal{L}$.

THEOREM 3.10. (i) *The set of eigenvectors and associated vectors of $\mathcal{L}$ corresponding to the eigenvalues λ_k^1 and $\overline{\lambda_k^1}$, $k = 1, 2, \ldots$, is minimal in the space $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ where $\mathcal{H}_{A^{-1}} = (\mathcal{H}, (A^{-1}\cdot, A^{-1}\cdot))$, complete in $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ if $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$, and a Riesz basis with parentheses in $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ if $n(\tau, A^{-1}) = \mathrm{O}(\tau^\gamma)$ for some $\gamma \in (0, \frac{1}{2}]$.*

(ii) *The set of eigenvectors and associated vectors of $\mathcal{L}$ corresponding to the eigenvalues λ_k^2 and $\overline{\lambda_k^2}$, $k = 1, 2, \ldots$, is minimal in the space $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ and complete in $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ if $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$.*

In particular, λ_k^1, $\overline{\lambda_k^1} \to \infty$, $\lambda_k^2 \to -\frac{1+\mathrm{i}\beta}{\alpha}$, $\overline{\lambda_k^2} \to -\frac{1-\mathrm{i}\beta}{\alpha}$ for $k \to \infty$ if $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$.

Proof. The assertions in (i) and (ii) follow from Theorems 3.4 and 3.8. The statement about the accumulation at the points of the essential spectrum follows from Theorem 2.1 together with the minimality and completeness from (i) and (ii). ∎

4. THE GENERAL CASE

Now we consider the general case of a quadratic block operator matrix pencil (1.1) with possibly different K_1, K_2. In this case $\mathcal{L}$ cannot be written as the orthogonal sum of two pencils in $\mathcal{H}$, and hence there exists no decomposition of the spectrum as it was used in the previous section.

In order to guarantee a certain subdivision of the spectrum of $\mathcal{L}$ also here and to obtain minimality, completeness and basis results for the eigenvectors and associated vectors, we have to assume in addition that

$$\delta > \frac{4}{\alpha^2}. \tag{4.1}$$

We choose $\rho > 0$ such that

$$\rho > \frac{\beta\alpha\delta}{\alpha^2\delta - 4}. \tag{4.2}$$

Lemma 4.1. *On the segment* $\Gamma_1 := \{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) = -\frac{2}{\alpha},\ |\mathrm{Im}\,(\lambda)| \leqslant \rho\}$ *we have the estimate*

$$\inf_{\substack{\lambda\in\Gamma_1 \\ Y\in\mathcal{D}(A)\times\mathcal{D}(A),\,\|Y\|=1}} |(\mathcal{L}(\lambda)Y,Y)| > 0.$$

Proof. Let $\lambda = -\frac{2}{\alpha} + \mathrm{i}\tau$ with $\tau \in \mathbb{R}$, $Y = (y,z)^{\mathrm{t}}$ with $y, z \in \mathcal{D}(A)$ and $\|y\|^2 + \|z\|^2 = 1$. Then

$$\begin{aligned}
&|(\mathcal{L}(\lambda)Y,Y)| \\
&= \bigg|\Big(-\frac{2}{\alpha} + \mathrm{i}\tau\Big)^2 + \Big(-\frac{2}{\alpha} + \mathrm{i}\tau\Big)\big(\alpha(Ay,y) + \alpha(Az,z) + (K_1y,y) + (K_2z,z)\big) \\
&\qquad + (Ay,y) + (Az,z) + \beta(Az,y) - \beta(Ay,z)\bigg| \\
&\geqslant \bigg|\frac{4}{\alpha^2} - \tau^2 - (Ay,y) - (Az,z) - \frac{2}{\alpha}(K_1y,y) - \frac{2}{\alpha}(K_2z,z)\bigg| \geqslant \delta - \frac{4}{\alpha^2} > 0
\end{aligned}$$

by assumption (4.1). ∎

Lemma 4.2. *On the rays* $\Gamma_2^{\pm} := \{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) \geqslant -\frac{2}{\alpha},\ \mathrm{Im}\,(\lambda) = \pm\rho\}$ *we have the estimate*

$$\inf_{\substack{\lambda\in\Gamma_2^{\pm} \\ Y\in\mathcal{D}(A)\times\mathcal{D}(A),\,\|Y\|=1}} |(\mathcal{L}(\lambda)Y,Y)| > 0.$$

Proof. Let $\lambda = t + \mathrm{i}\rho$ with $t \geqslant -\frac{2}{\alpha}$, $Y = (y,z)^{\mathrm{t}}$ with $y, z \in \mathcal{D}(A)$ and $\|y\|^2 + \|z\|^2 = 1$. Then

$$
\begin{aligned}
&\big|\big(\mathcal{L}(\lambda)Y,Y\big)\big| \geqslant \big|\mathrm{Im}\,\big(\mathcal{L}(\lambda)Y,Y\big)\big| \\
&= \big|2t\rho + \rho\big(\alpha(Ay,y)+\alpha(Az,z)+(K_1y,y)+(K_2z,z)\big) + 2\beta\,\mathrm{Im}\,(Az,y)\big| \\
&= \big|2t + \big(\alpha(Ay,y)+\alpha(Az,z)+(K_1y,y)+(K_2z,z)\big)\big| \\
&\qquad \cdot \left|\rho + \frac{2\beta\,\mathrm{Im}\,(Az,y)}{2t+\big(\alpha(Ay,y)+\alpha(Az,z)+(K_1y,y)+(K_2z,z)\big)}\right| \\
&\geqslant \left(-\frac{4}{\alpha}+\alpha\delta\right)\left|\rho - \frac{2\beta\,|\mathrm{Im}\,(Az,y)|}{\big|2t+\big(\alpha(Ay,y)+\alpha(Az,z)+(K_1y,y)+(K_2z,z)\big)\big|}\right| \\
&\geqslant \alpha\left(\delta-\frac{4}{\alpha^2}\right)\left|\rho - \frac{2\alpha\beta\,|(Az,y)|}{\alpha^2\delta-4}\right| \\
&\geqslant \alpha\left(\delta-\frac{4}{\alpha^2}\right)\left|\rho - \frac{2\alpha\beta\,\|A^{1/2}z\|\,\|A^{1/2}y\|}{\alpha^2\delta-4}\right| \\
&\geqslant \alpha\left(\delta-\frac{4}{\alpha^2}\right)\left|\rho - \frac{\alpha\beta\big(\|A^{1/2}z\|^2+\|A^{1/2}y\|^2\big)}{\alpha^2\delta-4}\right| \\
&\geqslant \alpha\left(\delta-\frac{4}{\alpha^2}\right)\left|\rho - \frac{\alpha\beta\delta}{\alpha^2\delta-4}\right| > 0
\end{aligned}
$$

by assumption (4.1) and the choice of ρ according to (4.2). ∎

Lemma 4.3. *On the segment $\Gamma_3 := \{\lambda\in\mathbb{C} : \mathrm{Re}\,(\lambda)=\rho_1,\ |\mathrm{Im}\,(\lambda)|\leqslant\rho\}$ with $\rho_1>\rho$ we have*

$$\inf_{\substack{\lambda\in\Gamma_3\\ Y\in\mathcal{D}(A)\times\mathcal{D}(A),\ \|Y\|=1}} |(\mathcal{L}(\lambda)Y,Y)| > 0.$$

Proof. Let $\lambda=\rho_1+\mathrm{i}\tau$ with $|\tau|\leqslant\rho$, $Y=(y,z)^{\mathrm{t}}$ with $y,z\in\mathcal{D}(A)$ and $\|y\|^2+\|z\|^2=1$. Then

$$
\begin{aligned}
&|(\mathcal{L}(\lambda)Y,Y)| \geqslant \big|\mathrm{Re}\,\big(\mathcal{L}(\lambda)Y,Y\big)\big| \\
&= \big|\rho_1^2-\tau^2+\rho_1\big(\alpha(Ay,y)+\alpha(Az,z)+(K_1y,y)+(K_2z,z)\big)+(Ay,y)+(Az,z)\big| \\
&> 2\,(\alpha\delta\rho_1+\delta) > 0
\end{aligned}
$$

since $\rho_1>\rho$. ∎

By Γ we now denote the rectangle the sides of which are Γ_1,Γ_3 and parts of the rays Γ_2^+,Γ_2^-. Further, we denote by G^- the interior and by G^+ the exterior of Γ (without the boundary of Γ).

Lemma 4.4. *For any fixed $Y\in\mathcal{D}(A)\times\mathcal{D}(A)$, the polynomial $(\mathcal{L}(\lambda)Y,Y)$ has one root in G^- and one root in G^+.*

Proof. Consider the auxiliary pencil

$$\mathcal{L}_1(\lambda)=\mathcal{L}\left(\lambda-\frac{2}{\alpha}\right),\quad \lambda\in\mathbb{C},$$

and the corresponding polynomial $(\mathcal{L}_1(\lambda)Y,Y)$ for fixed $Y=(y,z)^{\mathrm{t}}$ with $y\in\mathcal{D}(A)$, $z\in\mathcal{D}(A)$, $\|y\|^2+\|z\|^2=1$. The roots of $(\mathcal{L}_1(\lambda)Y,Y)=0$ are the roots of the equation

$$(4.3)\qquad \begin{aligned}&\lambda^2+\lambda\left(-\frac{4}{\alpha}+\alpha(Ay,y)+\alpha(Az,z)+(K_1y,y)+(K_2z,z)\right)\\&\quad+\frac{4}{\alpha^2}-(Ay,y)-(Az,z)-\frac{2}{\alpha}(K_1y,y)-\frac{2}{\alpha}(K_2z,z)+2\mathrm{i}\beta\,\mathrm{Im}\,(Az,y)=0.\end{aligned}$$

Due to assumption (4.1) the quadratic equation

$$(4.4)\qquad \lambda^2+\lambda\left(\alpha\delta-\frac{4}{\alpha}\right)+\frac{4}{\alpha^2}-\delta=0$$

possesses exactly one solution on the positive half axis and exactly one on the negative half axis (excluding 0). Now we consider A and K_i, $i=1,2$, as perturbations of δI and 0, respectively, i.e., we consider

$$A(\eta):=\eta(A-\delta I)+\delta I,\quad K_i(\eta)=\eta K_i,\quad i=1,2,$$

for $\eta\in[0,1]$ and the pencil $\mathcal{L}_1(\eta,\lambda)$ which arises if we substitute A and K_i, $i=1,2$, in $\mathcal{L}_1(\lambda)$ by $A(\eta)$ and $K_i(\eta)$, $i=1,2$, i.e.,

$$\begin{aligned}\mathcal{L}_1(\eta,\lambda):=\lambda^2\begin{pmatrix}I&0\\0&I\end{pmatrix}&+\lambda\begin{pmatrix}\alpha A(\eta)+K_1(\eta)-\frac{4}{\alpha}&0\\0&\alpha A(\eta)+K_2(\eta)-\frac{4}{\alpha}\end{pmatrix}\\&+\begin{pmatrix}-A(\eta)-\frac{2}{\alpha}K_1(\eta)+\frac{4}{\alpha^2}&\beta A\\-\beta A&-A(\eta)-\frac{2}{\alpha}K_2(\eta)+\frac{4}{\alpha^2}\end{pmatrix},\end{aligned}$$

$\eta\in[0,1]$, $\lambda\in\mathbb{C}$. Then $\mathcal{L}_1(1,\lambda)=\mathcal{L}_1(\lambda)$ and the equation $(\mathcal{L}_1(0,\lambda)Y,Y)=0$ is just the equation (4.4). The quadratic form $(\mathcal{L}_1(\eta,\lambda)Y,Y)$ is analytic with respect to η and λ. The roots $\lambda_1(\eta)$ and $\lambda_2(\eta)$ of $(\mathcal{L}_1(\eta,\lambda)Y,Y)=0$ are piecewise analytic, they may fail to be analytic in $[0,1]$ only if for some $\eta\in[0,1]$, we have $\lambda_1(\eta)=\lambda_2(\eta)$. Since $\lambda_1(0)$ and $\lambda_2(0)$ are the roots of (4.4), one of them, say $\lambda_1(0)$, lies in the open left half plane and the other in the open right half plane. According to Lemmas 4.1, 4.2 and 4.3, we have

$$\inf_{\substack{\lambda\in\widehat{\Gamma}\\Y\in\mathcal{D}(A)\times\mathcal{D}(A),\,\|Y\|=1}}|\mathcal{L}_1(\lambda)Y,Y)|>0,$$

where $\widehat{\Gamma}\subset\{\lambda\in\mathbb{C}:\mathrm{Re}\,(\lambda)>0\}$ is the rectangle $\widehat{\Gamma}:=\{\lambda+\frac{2}{\alpha}:\lambda\in\Gamma\}$. Now we choose $\rho_1>\rho$ such that $\rho_1>\lambda_2(0)$. Then $\lambda_2(\eta)$ remains inside $\widehat{\Gamma}$ and $\lambda_1(\eta)$ outside $\widehat{\Gamma}$ for all $\eta\in[0,1]$. The roots of $(\mathcal{L}(\lambda)Y,Y)$ are given by $\lambda_1(1)-\frac{2}{\alpha}$ and $\lambda_2(1)-\frac{2}{\alpha}$. Hence the first one lies outside Γ and the second one inside Γ. ∎

In order to prove results about the eigenvectors and associated vectors corresponding to the branches of eigenvalues possibly accumulating at ∞, we introduce

$$\widetilde{A}^{-1}:=\frac{1}{\alpha}\begin{pmatrix}A^{-1}&0\\0&A^{-1}\end{pmatrix}$$

and consider the pencil

$$
\begin{aligned}
\mathcal{L}_2(\lambda) &:= \frac{\lambda^2}{\alpha}\widetilde{A}^{-1/2}\mathcal{L}\left(\frac{1}{\lambda}\right)\widetilde{A}^{-1/2} \\
&= \lambda^2\frac{1}{\alpha}\begin{pmatrix} I & \beta \\ -\beta & I \end{pmatrix} + \lambda\begin{pmatrix} I+\frac{1}{\alpha}A^{-1/2}K_1A^{-1/2} & 0 \\ 0 & I+\frac{1}{\alpha}A^{-1/2}K_2A^{-1/2} \end{pmatrix} \\
&\qquad + \frac{1}{\alpha}\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix}
\end{aligned}
$$

for $\lambda \in \mathbb{C}$.

By $\widetilde{\Gamma}$ we denote the closed simple curve obtained from the rectangle Γ after the transformation $\lambda \mapsto \frac{1}{\lambda}$. Let $\widetilde{G}^+$ ($\widetilde{G}^-$) denote the interior (exterior) of $\widetilde{\Gamma}$.

Theorem 4.5. (i) *The set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_2$ in $\widetilde{G}^+$ is minimal in $\mathcal{H}\times\mathcal{H}$.*

(ii) *If $A^{-1}\in\mathcal{S}_p$ for some $p<\infty$, then the set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_2$ in $\widetilde{G}^+$ is complete in $\mathcal{H}\times\mathcal{H}$. If, in addition, $n(\tau,\frac{1}{\alpha}A^{-1}) \sim c_1\tau^{c_2}$ as $\tau\to\infty$ with some $0<c_1,c_2<\infty$, then $n(\tau,\widetilde{G}^+,\mathcal{L}_2)\sim 2c_1\tau^{c_2}$.*

(iii) *If $n(\tau,A^{-1}) = \mathrm{O}(\tau^{\gamma})$ for some $\gamma\in(0,\frac{1}{2}]$, then the set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_2$ in $\widetilde{G}^+$ is a Riesz basis with parentheses in $\mathcal{H}\times\mathcal{H}$. If, in addition, $n(\tau,\frac{1}{\alpha}A^{-1}) = c_1\tau^{c_2}+\mathrm{O}(\tau^{\beta})$ for some $0<c_1,c_2<\infty$, $0\leqslant\beta<\alpha\leqslant\beta+\gamma$, then also $n(\tau,\widetilde{G}^+,\mathcal{L}_2) = 2c_1\tau^{c_2}+\mathrm{O}(\tau^{\beta})$.*

Proof. Again we invoke the results contained in [10], Theorem 22.13 and Corollary 26.20, and apply them with

$$
H = \frac{1}{\alpha}\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad T = 0
$$

for (ii) and

$$
\begin{aligned}
&H = \frac{1}{\alpha}\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix}, \\
&D_0 = 0, \quad D_1 = \frac{1}{\alpha}\begin{pmatrix} A^{-1/2}K_1A^{-1/2+\gamma} & 0 \\ 0 & A^{-1/2}K_2A^{-1/2+\gamma} \end{pmatrix}
\end{aligned}
$$

where $\gamma\in(0,\frac{1}{2}]$ for (iii). ∎

Theorem 4.6. *The set of eigenvectors and associated vectors of $\mathcal{L}$ corresponding to the eigenvalues in the half plane $\{\lambda\in\mathbb{C} : \mathrm{Re}\,(\lambda) < -\frac{2}{\alpha}\})$ is minimal in the space $\mathcal{H}_{A^{-1}}\times\mathcal{H}_{A^{-1}}$ where $\mathcal{H}_{A^{-1}} = (\mathcal{H},(A^{-1}\cdot,A^{-1}\cdot))$. It is complete in $\mathcal{H}_{A^{-1}}\times\mathcal{H}_{A^{-1}}$ if $A^{-1}\in\mathcal{S}_p$ for some $p<\infty$, and a Riesz basis with parentheses in $\mathcal{H}_{A^{-1}}\times\mathcal{H}_{A^{-1}}$ if $n(\tau,A^{-1}) = \mathrm{O}(\tau^{\gamma})$ for some $\gamma\in(0,\frac{1}{2}]$. In particular, the eigenvalues in $\{\lambda\in\mathbb{C} : \mathrm{Re}\,(\lambda) < -\frac{2}{\alpha}\})$ accumulate at ∞ if $A^{-1}\in\mathcal{S}_p$ for some $p<\infty$.*

Proof. The statements of the theorem follow from Theorem 4.5 and Theorem 2.1. ∎

5. APPLICATION TO THE PROBLEM OF VIBRATIONS OF A ROTATING BEAM

In this section we are going to apply the results of the previous sections to the system (1.2), (1.3) of partial differential equations with boundary conditions (1.4). After the separation of variables (1.5) (and assuming $m \equiv 1$ for simplicity), it takes the form

$$EI\, y_1^{(4)} + \omega\kappa EI\, y_2^{(4)} + \lambda\bigl(\kappa EI\, y_1^{(4)} + \varepsilon_1\, y_1\bigr) + \lambda^2\, y_1 = 0, \tag{5.1}$$

$$EI\, y_2^{(4)} - \omega\kappa EI\, y_1^{(4)} + \lambda\bigl(\kappa EI\, y_2^{(4)} + \varepsilon_2\, y_2\bigr) + \lambda^2\, y_2 = 0 \tag{5.2}$$

with boundary conditions

$$y_1(0) = y_1(l) = y_1''(0) = y_1''(l) = 0, \quad y_2(0) = y_2(l) = y_2''(0) = y_2''(l) = 0. \tag{5.3}$$

Here the operators A and K_1, K_2 are determined by (1.6) and (1.7), and the constants α, β by $\alpha = \kappa$ (the coefficient of inner damping) and $\beta = \omega\kappa$ (where ω is the angular frequency of the rotation of the beam).

Obviously, the operator A has compact resolvent, and it is not difficult to see that the eigenvalues λ_k, $k = 1, 2, \dots$, of A are all simple and given by

$$\lambda_k = EI\left(\frac{k\pi}{l}\right)^4, \quad k = 1, 2, \dots.$$

Hence the lower bound δ of A is its least eigenvalue,

$$\delta = EI\left(\frac{\pi}{l}\right)^4.$$

It is also easy to see that the number $n(\tau, A^{-1})$ of eigenvalues of A^{-1} greater than τ^{-1}, i.e., the number of eigenvalues of A less than τ satisfies

$$n(\tau, A^{-1}) \sim \frac{l}{\pi}\left(\frac{1}{EI}\right)^{1/4} \tau^{1/4}. \tag{5.4}$$

An immediate consequence of Theorem 2.1 is the following statement.

THEOREM 5.1. *The essential spectrum of the problem* (5.1)–(5.3) *consists of the two points*

$$-\frac{1}{\kappa} - \mathrm{i}\omega, \; -\frac{1}{\kappa} + \mathrm{i}\omega.$$

The other points of the spectrum of the problem (5.1)–(5.3) *are normal eigenvalues which accumulate at most at the points* $-\frac{1}{\kappa} - \mathrm{i}\omega$, $-\frac{1}{\kappa} + \mathrm{i}\omega$, *and at* ∞.

From Theorem 2.3 we immediately get the following stability result.

THEOREM 5.2. *Set* $\mu := \min_{i=1,2} \min_{x\in[0,l]} \{\varepsilon_i(x)\}$. *Then the spectrum of the problem* (5.1)–(5.3) *lies in the open left half plane if*

$$\mu \geqslant \kappa EI\left(\frac{\pi}{l}\right)^4 \quad \text{and} \quad \mu > \frac{\omega^2\kappa}{4},$$

or if

$$\mu < \kappa EI\left(\frac{\pi}{l}\right)^4 \quad \text{and} \quad \left(\kappa EI\left(\frac{\pi}{l}\right)^4 + \mu\right)^2 > \omega^2\kappa^2 EI\left(\frac{\pi}{l}\right)^4.$$

Concerning results about the minimality, completeness and basis properties of the eigenvectors of the problem (5.1)–(5.3) corresponding to certain branches of eigenvalues, we have to distinguish the case when the outer medium is homogeneous, i.e., $\varepsilon_1 \equiv \varepsilon_2 \equiv: \varepsilon$ and hence $K_1 = K_2$, and the case when the outer medium is inhomogeneous, i.e., $\varepsilon_1 \not\equiv \varepsilon_2$ and hence $K_1 \neq K_2$.

In the case of a homogeneous outer medium, according to Section 3, the eigenvalues of the given problem (5.1)–(5.3) split into 4 branches $\{\lambda_k^1\} \cup \{\lambda_k^2\} \cup \{\overline{\lambda_k^1}\} \cup \{\overline{\lambda_k^2}\}$ where λ_k^1 and λ_k^2, $k = 1, 2, \ldots$, are the eigenvalues of the problem

$$(5.5)\qquad \begin{aligned} &(1 + \mathrm{i}\omega\kappa)EIy^{(4)} + \lambda(\kappa EIy^{(4)} + \varepsilon y) + \lambda^2 y = 0,\\ &y(0) = y(l) = y''(0) = y''(l) = 0, \end{aligned}$$

located in the upper and lower half plane, respectively (counted according to their algebraic multiplicities). The respective eigenfunctions and associated functions of the problem of (5.1)–(5.3) can be obtained from the eigenfunctions and associated functions y_k^1 and y_k^2 of the problem (5.5) and from the eigenfunctions and associated functions $\overline{y_k^1}$ and $\overline{y_k^2}$ of the problem

$$(5.6)\qquad \begin{aligned} &(1 - \mathrm{i}\omega\kappa)EIy^{(4)} + \lambda(\kappa EIy^{(4)} + \varepsilon y) + \lambda^2 y = 0,\\ &y(0) = y(l) = y''(0) = y''(l) = 0. \end{aligned}$$

For instance, the eigenfunctions of (5.1)–(5.3) are given by the formulas

$$(y_k^1, \mathrm{i}y_k^1)^{\mathrm{t}},\ (y_k^2, \mathrm{i}y_k^2)^{\mathrm{t}},\ (\mathrm{i}\overline{y_k^1}, \overline{y_k^1})^{\mathrm{t}},\ (\mathrm{i}\overline{y_k^2}, \overline{y_k^2})^{\mathrm{t}}.$$

In the following we denote by $W_2^4(0,l)$ the Sobolev space of order 4 associated with $L_2(0,l)$.

Theorem 5.3. (i) *The set of eigenfunctions and associated functions of problem* (5.1)–(5.3) *corresponding to the eigenvalues λ_k^1 and $\overline{\lambda_k^1}$, $k = 1, 2, \ldots$, forms a Riesz basis with parentheses in the space $W_2^4(0,l) \times W_2^4(0,l)$.*

(ii) *The eigenvalues λ_k^1 (and hence $\overline{\lambda_k^1}$) accumulate at ∞; if enumerated such that $|\lambda_k^1| \leqslant |\lambda_{k+1}^1|$, they satisfy the asymptotics*

$$\lambda_k^1 = -\kappa EI\left(\frac{k\pi}{l}\right)^4 + \mathrm{i}\omega + \xi_k^1 + \mathrm{i}\eta_k^1, \quad k \to \infty,$$

where ξ_k^1, η_k^1 are real and $\xi_k^1 = \mathrm{o}(k^4)$, $\eta_k^1 = \mathrm{o}(1)$.

Proof. The assertion in (i) and the first assertion in (ii) follow from Theorem 3.10 which we can apply due to (5.4) with $\gamma = \frac{1}{4}$. From Theorem 3.10 we also obtain that

$$(5.7)\qquad |\lambda_k^1| \sim \kappa EI\left(\frac{k\pi}{l}\right)^4, \quad k \to \infty.$$

Now let y_k^1, $\|y_k^1\| = 1$, be an eigenfunction of (5.5) (i.e., of $\mathcal{L}_+$) at λ_k^1. Then

$$(5.8)\qquad \left(\lambda_k^1\right)^2 + \lambda_k^1\left(\alpha(Ay_k^1, y_k^1) + (Ky_k^1, y_k^1)\right) + (1 + \mathrm{i}\beta)(Ay_k^1, y_k^1) = 0.$$

From this it follows that $|(Ay_k^1, y_k^1)| \to \infty$, $k \to \infty$, because otherwise

$$\left|\lambda_k^1\left(\lambda_k^1 + \alpha(Ay_k^1, y_k^1) + (Ky_k^1, y_k^1)\right)\right|$$

would also be bounded, a contradiction to (5.7). Then the assertion follows from the formula for the solutions of the quadratic equation (5.8) (see (3.2)) and from (5.7). ∎

Theorem 5.4. *The set of eigenfunctions and associated functions of problem* (5.1)–(5.3) *corresponding to the eigenvalues* λ_k^2 *and* $\overline{\lambda_k^2}$, $k = 1, 2, \dots$, *is minimal and complete in the space* $W_2^4(0,l) \times W_2^4(0,l)$. *In particular,* $\lambda_k^2 \to -\frac{1}{\kappa} - \mathrm{i}\omega$, $\overline{\lambda_k^2} \to -\frac{1}{\kappa} + \mathrm{i}\omega$ *for* $k \to \infty$.

Proof. The statements are immediate from Theorem 3.10. ∎

In the case of an inhomogeneous outer medium we obtain:

Theorem 5.5. *Assume that* $EI\big(\frac{\pi}{l}\big)^4 > \frac{4}{\kappa^2}$. *Then:*

(i) *The set of eigenfunctions and associated functions of problem* (5.1)–(5.3) *corresponding to the eigenvalues located in the half plane* $\{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) < -\frac{2}{\kappa}\}$ *forms a Riesz basis with parentheses in the space* $W_2^4(0,l) \times W_2^4(0,l)$.

(ii) *The eigenvalues* λ_k *of problem* (5.1)–(5.3) *in* $\{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) < -\frac{2}{\kappa}\}$ *accumulate at* ∞; *if enumerated such that* $|\lambda_k| \leqslant |\lambda_{k+1}|$, *they satisfy the asymptotics*

$$\lambda_k = -\kappa EI \left(\frac{k\pi}{2l}\right)^4 + \mathrm{o}(k^4), \quad k \to \infty.$$

Proof. The first assertion follows immediately from Theorem 4.6. The proof of the second statement is similar to the proof of Theorem 5.3 (ii). ∎

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