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SOLUTION OF A PROBLEM OF PELLER CONCERNING SIMILARITY

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ABSTRACT. We answer a question of Peller by showing that for any $c > 1$ there exists a power-bounded operator T on a Hilbert space with the property that any operator S similar to T satisfies $\sup\limits_n \|S^n\| > c$.

KEYWORDS: *Power bounded operators, similarities, multipliers, weights.*

MSC (2000): 47A65, 42A50.

1. INTRODUCTION

In this note we answer a question due to Peller ([13]) which has also recently been raised by Pisier ([14], p. 114). Peller's question is whether, for any $\varepsilon > 0$, every power-bounded operator T is similar to an operator S with $\sup\limits_n \|S^n\| < 1 + \varepsilon$.

It was shown by Foguel ([6]) in 1964 that there is a power-bounded operator T on a Hilbert space $\mathcal{H}$ which is not similar to a contraction. It was later shown by Lebow that this example is not polynomially bounded ([12]); for other examples see [2] and [14], Chapter 2. Recently, Pisier ([14]) answered a problem raised by Halmos by constructing an operator which is polynomially bounded and not similar to a contraction.

We shall construct a family of counter-examples to Peller's question. These counter-examples have a rather simple structure. Let w be an A_2-weight on the circle $\mathbb{T}$ and let $H^2(w)$ be the closed linear span of $\{\mathrm{e}^{\mathrm{i}n\theta} : n \geqslant 0\}$ in $L^2(w)$. We consider an operator

$$T\bigg(\sum_{n=0}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta}\bigg) = \sum_{n=0}^{\infty} \lambda_n a_n \mathrm{e}^{\mathrm{i}n\theta}$$

where $(\lambda_n)_{n=0}^{\infty}$ is a monotone increasing sequence of positive reals with $\lambda_n \uparrow 1$ and $\lambda_n < 1$ with

$$\lim_{n\to\infty} \frac{1-\lambda_{n+1}}{1-\lambda_n} = 0.$$

For such operators we can prove a rather precise result (Theorem 3.4):

$$\inf\{\sup_n \|(A^{-1}TA)^n\| : A \text{ invertible}\} = \sec\left(\frac{\pi}{2p}\right) \tag{1.1}$$

where $p = \sup\{a : w^a \in A_2\}$. By taking simple choices of A_2-weights where $p < \infty$ we can create a family of counter-examples.

The proof of Theorem 3.4 depends heavily on estimates for the norm of the Riesz projection in Section 2 particularly Theorem 2.6. These results can be obtained by a careful reading of the classical work of Helson and Szegö ([9]) on A_2-weights (cf. [7]). However, we present a self-contained argument, in which the reader will recognize many similarities with the Helson-Szegö theory.

We also show that our examples can only be polynomially bounded in the trivial situation when w is equivalent to the constant function and then T is similar to contraction. We also note that the case $p = \infty$ in (1.1) (when Peller's conjecture holds for T) corresponds to the case when $\log w$ is in the closure of $L^\infty(\mathbb{T})$ in $\mathrm{BMO}(\mathbb{T})$.

2. The norm of the Riesz projection on weighted L^2-spaces

We start by recalling an easy lemma concerning projections on a Hilbert space.

Lemma 2.1. *Let E and F be closed subspaces of a Hilbert space $\mathcal{H}$ so that $E + F$ is dense in $\mathcal{H}$. Suppose $0 \leqslant \varphi < \pi/2$. In order that there is a projection P of $\mathcal{H}$ onto E with $F = \ker P$ with $\|P\| \leqslant \sec\varphi$ it is necessary and sufficient that*

$$|(e, f)| \leqslant \sin\varphi \|e\|\,\|f\|, \quad e \in E,\ f \in F.$$

Remark 2.2. Note that a consequence of Lemma 2.1 is that if P is any non-trivial projection on a Hilbert space then $\|P\| = \|I - P\|$.

Now let $\mathbb{T}$ be the unit circle (which we identify with $(-\pi, \pi]$ in the usual way) equipped with the standard Haar measure $\mathrm{d}\theta/2\pi$. Let μ be any finite positive Borel measure on $\mathbb{T}$. We denote by $L^2(\mu) = L^2(\mathbb{T};\mu)$ the corresponding weighted L^2-space; if μ is absolutely continuous with respect to Haar measure so that $\mathrm{d}\mu = (2\pi)^{-1}w(\theta)\mathrm{d}\theta$ then we write $L^2(w)$. We refer to any nonnegative $w \in L^1(\mathbb{T})$ so that $w > 0$ on a set of positive measure as a weight.

Suppose w is a weight. We recall that $H^2(w)$ is the closed subspace of $L^2(w)$ generated by the functions $\{\mathrm{e}^{\mathrm{i}n\theta} : n \geqslant 0\}$. We recall that w is an *A_2-weight* if there is a bounded projection R of $L^2(w)$ onto $H^2(w)$ with $R(\mathrm{e}^{\mathrm{i}n\theta}) = 0$ if $n < 0$. In this case we always have that $w > 0$ a.e., w^{-1} is an A_2-weight and $L^2(w) \subset L^1$; the operator R must coincide with the Riesz projection $Rf \sim \sum_{n\geqslant 0} \widehat{f}(n)\mathrm{e}^{\mathrm{i}n\theta}$. Let us denote by $\|R\|_w$ the norm of the Riesz projection on $L^2(w)$. Note that for an A_2-weight $H^2(w) = H^1 \cap L^2(w)$. In particular we can define $f(z) = \sum_{n\geqslant 0} \widehat{f}(n)z^n$ for $|z| < 1$.

The following proposition can be derived from the classical work of Helson-Szegö [9] or [7]. However, we give a self-contained direct proof. We note that it is also close to some work of Cotlar-Sadosky, see e.g. [5].

PROPOSITION 2.3. *Let w be a weight function on $\mathbb{T}$. Assume $0 \leqslant \varphi < \frac{\pi}{2}$. The following conditions are equivalent:*

(i) *w is an A_2-weight and $\|R\|_w \leqslant \sec\varphi$;*

(ii) *there exists $h \in H^1$ so that $|w-h| \leqslant w\sin\varphi$ a.e.*

Proof. First note that by Lemma 2.1, (i) is equivalent to

$$\Big|\int_{-\pi}^{\pi} f(\theta)g(\theta)w(\theta)\frac{\mathrm{d}\theta}{2\pi}\Big| \leqslant \sin\varphi\Big(\int_{-\pi}^{\pi}|f(\theta)|^2 w(\theta)\frac{\mathrm{d}\theta}{2\pi}\Big)^{1/2}\Big(\int_{-\pi}^{\pi}|g(\theta)|^2 w(\theta)\frac{\mathrm{d}\theta}{2\pi}\Big)^{1/2}, \tag{2.1}$$

whenever $f, g \in H^2(w)$ with $g(0) = 0$.

To prove (i) implies (ii) we note that if w is an A_2-weight so that $\log w \in L^1$ we can find an outer function $F \in H^2$ so that $w = |F|^2$ a.e.. Then (2.1) gives

$$\Big|\int_{-\pi}^{\pi} fgwF^{-2}\,\frac{\mathrm{d}\theta}{2\pi}\Big| \leqslant \sin\varphi\Big(\int_{-\pi}^{\pi}|f|^2\frac{\mathrm{d}\theta}{2\pi}\Big)^{1/2}\Big(\int_{-\pi}^{\pi}|g|^2\frac{\mathrm{d}\theta}{2\pi}\Big)^{1/2},$$

for $f, g \in H^2$ with $g(0) = 0$. This in turn implies that

$$\Big|\int_{-\pi}^{\pi} fwF^{-2}\frac{\mathrm{d}\theta}{2\pi}\Big| \leqslant \sin\varphi\|f\|_1$$

for all $f \in H^1$, with $f(0) = 0$. By the Hahn-Banach Theorem this implies there exists $G \in H^\infty$ so that $\|wF^{-2} - G\|_\infty \leqslant \sin\varphi$ or $|w-h| \leqslant w\sin\varphi$ where $h = F^2G \in H^1$.

For the reverse direction just note that if $f, g \in H^2(w)$ with $g(0) = 0$ then

$$\int_{-\pi}^{\pi} fgw\frac{\mathrm{d}\theta}{2\pi} = \int_{-\pi}^{\pi} fg(w-h)\frac{\mathrm{d}\theta}{2\pi}$$

so that (2.1) follows from the Cauchy-Schwarz inequality. ∎

Let us isolate a simple special case of the above proposition.

PROPOSITION 2.4. *Let $0 \neq f \in H^1$ be such that $\|\arg f(\theta)\| \leqslant \varphi < \pi/2$ almost everywhere. If f is not identically zero then $w = \operatorname{Re} f$ is an A_2-weight for which $\|R\|_w \leqslant \sec\varphi$.*

Proof. In this case $w = \operatorname{Re} f \geqslant 0$ a.e. and $|\operatorname{Im} f| \leqslant w\tan\varphi$ a.e. Furthermore:

$$|w - \cos^2\varphi f|^2 \leqslant (\sin^4\varphi + \cos^4\varphi\tan^2\varphi)w^2 \leqslant w^2\sin^2\varphi$$

a.e., so that we obain the result from Proposition 2.3. ∎

REMARK 2.5. Suppose $0 < \alpha < 1$ and $f \in H^1$ is given by

$$f(z) = \Big(\frac{z-1}{z+1}\Big)^\alpha$$

(taking the usual branch of $\zeta \mapsto \zeta^\alpha$). Then

$$w = \operatorname{Re} f = \cos\frac{\alpha\pi}{2}\Big|\tan\frac{\theta}{2}\Big|^\alpha.$$

It follows that

$$\|R\|_{|\tan(\theta/2)|^\alpha} \leqslant \sec\frac{\alpha\pi}{2}. \tag{2.2}$$

In fact (2.2) is well-known (see [11], for example). We are grateful to Igor Verbitsky for bringing this reference to our attention.

We will say that two weights v, w are equivalent ($v \sim w$) if $v/w, w/v \in L^\infty$.

THEOREM 2.6. *Suppose w is an A_2-weight on $\mathbb{T}$. Then*

$$\inf\{\|R\|_v : v \sim w\} = \sec\Big(\frac{\pi}{2p}\Big)$$

where

$$p = \sup\{a > 0 : w^a \in A_2\}.$$

Proof. First suppose $v \sim w$ and $\|R\|_v = \sec\psi$ where $0 \leqslant \psi < \pi/2$. Then there exists $h \in H^1$ with $|v - h| \leqslant v \sin\psi$ a.e. In particular, $|\arg h| \leqslant \psi$ a.e. and so h maps $\mathbb{D}$ into the same sector. It follows that we can define $h^r \in H^{1/r}$ for all $r > 0$. Choose r so that $r\psi < \pi/2$, and let $g = h^r$. Then $\operatorname{Re} g \geqslant 0$ and $|\operatorname{Im} g| \leqslant \tan(r\psi)\operatorname{Re} g$ so that $g \in H^1$. Now by Proposition 2.4 we have that $\operatorname{Re} g$ is an A_2-weight. However $\operatorname{Re} g \sim |h|^r \sim w^r$ so that $r \leqslant p$. We deduce that $\psi \geqslant \pi/(2p)$.

For the converse direction assume that w^r is an A_2-weight. Then there exists $h \in H^1$ so that $|w^r - h| \leqslant w^r \sin\psi$ where $0 \leqslant \psi < \pi/2$. Arguing as above we have $g = h^{1/r} \in H^1$ and $\operatorname{Re} g$ is an A_2-weight with $\|R\|_{\operatorname{Re} g} \leqslant \sec(\psi/r)$. Note that $\operatorname{Re} g \sim w$, and this establishes the other direction. ∎

REMARK 2.7. If we now let $w(\theta) = |\tan\theta/2|^\alpha$ where $0 < \alpha < 1$ then we can apply (2.2) to deduce that, for this particular weight the infimum is attained, i.e.

$$\inf\{\|R\|_v : v \sim w\} = \|R\|_{|\tan(\theta/2)|^\alpha} = \sec\Big(\frac{\alpha\pi}{2}\Big). \tag{2.3}$$

3. MULTIPLIERS

Suppose $(e_n)_{n=0}^\infty$ be any Schauder basis of a Hilbert space $\mathcal{H}$; note that we do not assume (e_n) to be orthonormal or even unconditional. Let (P_n) be the associated partial sum operators $P_n\Big(\sum\limits_{k=0}^\infty a_k e_k\Big) = \sum\limits_{k=0}^n a_k e_k$. Let $Q_n = I - P_n$ and note that $\|Q_n\| = \|P_n\|$ for all $n \geqslant 0$. Since (e_n) is a basis we have that $\sup\limits_n \|P_n\| = b < \infty$ where b is the *basis constant.* We call an operator $T : \mathcal{H} \to \mathcal{H}$ a *monotone multiplier* (with respect to the given basis) if there is an increasing sequence $(\lambda_k)_{k=0}^\infty$ in $\mathbb{R}$ so that $0 \leqslant \lambda_k \leqslant 1$ so that

$$T\Big(\sum_{k=0}^\infty a_k e_k\Big) = \sum_{k=0}^\infty \lambda_k a_k e_k.$$

Lemma 3.1. *If T is defined as above then T is (well-defined and) bounded and* $\sup_n \|T^n\| \leqslant b$.

Proof. It is enough to show T is bounded and $\|T\| \leqslant b$ since T^n is also a monotone multiplier. To see this note that if $(a_k)_{k=0}^{\infty}$ is finitely nonzero and $x = \sum_{k=0}^{\infty} a_k e_k$, then

$$Tx = \lambda_0 x + \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k-1}) Q_{k-1} x$$

so that $\|Tx\| \leqslant \sup_n \|Q_n\| = b$. ∎

We shall say that T is a *fast monotone multiplier* if in addition, $\lambda_k < 1$ for all k and

$$\lim_{k\to\infty} \frac{1-\lambda_k}{1-\lambda_{k-1}} = 0. \tag{3.1}$$

Lemma 3.2. *Suppose T is a fast monotone multiplier. Then there is an increasing sequence of integers $(N_n)_{n=0}^{\infty}$ so that* $\lim_{n\to\infty} \|T^{N_n} - Q_n\| = 0$.

Proof. Note that if $x = \sum_{k=0}^{\infty} a_k e_k$ then

$$T^{N_n} x - Q_n x = \sum_{k=0}^{n} \lambda_k^{N_n} a_k e_k - (1 - \lambda_{n+1}^{N_n}) Q_n x + \sum_{k=n+1}^{\infty} (\lambda_k^{N_n} - \lambda_{n+1}^{N_n}) a_k e_k$$

whence a calculation as in Lemma 3.1 gives

$$\|T^{N_n} x - Q_n x\| \leqslant b \lambda_n^{N_n} \|P_n x\| + (b+1)(1 - \lambda_{n+1}^{N_n}) \|Q_n x\|.$$

It follows that

$$\|T^{N_n} - Q_n\| \leqslant b\big(b\lambda_n^{N_n} + (b+1)(1 - \lambda_{n+1}^{N_n})\big).$$

It remains therefore only to select N_n so that $\lim_{n\to\infty} \lambda_n^{N_n} = 0$ and $\lim_{n\to\infty} \lambda_{n+1}^{N_n} = 1$.

For convenience we write $\lambda_n = e^{-\nu_n}$ where $\nu_n/\nu_{n+1} = \kappa_n^2$ and $\kappa_n \to \infty$. For any $n \geqslant 0$, pick N_n to be the greatest integer so that $N_n \nu_n^{1/2} \nu_{n+1}^{1/2} \leqslant 1$. Then

$$N_n \nu_{n+1}^{1/2} \nu_n^{1/2} \geqslant \frac{N_n}{N_n + 1}$$

and $\lim N_n = \infty$.

Now

$$N_n \nu_n \geqslant \frac{N_n \kappa_n}{N_n + 1} \quad \text{and} \quad N_n \nu_{n+1} \leqslant \kappa_n^{-1}.$$

This yields the desired result. ∎

We now turn to the case when $\mathcal{H} = H^2(w)$ where w is an A_2-weight and $e_k(\theta) = \mathrm{e}^{\mathrm{i}k\theta}$ for $k \geqslant 0$.

LEMMA 3.3. *The basis constant of* $(e_k)_{k=0}^{\infty}$ *in* $H^2(w)$ *is given by* $b = \|R\|_w$.

Proof. In fact $Q_{n-1}f = e_n R(e_{-n}f)$ so it is clear that $\|Q_{n-1}\| \leqslant \|R\|_w$. For the other direction suppose f is a trigonometric polynomial in $L^2(w)$. Then for large enough n we have $e_n f \in H^2(w)$ and then $Rf = e_{-n}Q_{n-1}(e_n f)$. This quickly yields $\|R\|_w \leqslant b$. ∎

THEOREM 3.4. *Let* w *be an* A_2*-weight on* $\mathbb{T}$ *and let* $T : H^2(w) \to H^2(w)$ *be a fast monotone multiplier corresponding to the sequence* (λ_n). *Then*

$$\inf\{\sup_n \|(A^{-1}TA)^n\| : A \text{ invertible}\} = \sec\Big(\frac{\pi}{2p}\Big) \tag{3.2}$$

where

$$p = \sup\{a > 0 : w^a \in A_2\}.$$

Proof. We shall prove that if $\sigma \geqslant 1$ then the existence of an invertible A so that $\sup_n \|(A^{-1}TA)^n\| \leqslant \sigma$ is equivalent to the existence of a weight v equivalent to w so that $\|R\|_v \leqslant \sigma$. Once this is done, the result follows from Theorem 2.6.

In one direction this is easy. Assume v equivalent to w and $\|R\|_v \leqslant \sigma$. This means that there is an equivalent inner-product norm on $H^2(w)$ in which the basis constant of $(e_k)_{k=0}^{\infty}$ is bounded by σ. It follows from Lemma 3.1 that in this equivalent norm we have $\sup_n \|T^n\|_v \leqslant \sigma$. Hence T is similar to an operator $A^{-1}TA$ such that $\sup_n \|(A^{-1}TA)^n\| \leqslant \sigma$.

We now consider the converse. Let $S : H^2(w) \to H^2(w)$ be the operator $Sf = e_1 f$. Suppose A is an invertible operator such that $\|(A^{-1}TA)^n\| \leqslant \sigma$. We will define a new inner-product on $H^2(w)$ by

$$\langle f, g\rangle = \mathrm{LIM}(A^{-1}S^n f, A^{-1}S^n g)$$

where LIM denotes any Banach limit (see e.g. [4], p. 85). Since S is an isometry on $H^2(w)$ and A is invertible this defines an equivalent inner-product $|\cdot|$ norm on $H^2(w)$. Now for any $f \in H^2(w)$ and fixed $m \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \|A^{-1}Q_{m+n}S^n f - A^{-1}T^{N_{m+n}}S^n f\| = 0$$

where (N_n) is given in Lemma 3.2. Hence

$$\limsup_{n\to\infty}(\|A^{-1}Q_{m+n}S^n f\|^2 - \sigma^2\|A^{-1}S^n f\|^2) \leqslant 0.$$

Now

$$|Q_m f|^2 = \mathrm{LIM}\|A^{-1}S^n Q_m f\|^2 = \mathrm{LIM}\|A^{-1}Q_{m+n}S^n f\|^2 \leqslant \sigma^2|f|^2.$$

Thus with respect to the new norm $|\cdot|$ the basis constant is at most σ.

Now let $c_k = \langle e_0, e_k\rangle$ for $k \geqslant 0$ and let $c_k = \overline{c}_{-k}$ when $k < 0$. Then it follows easily that $\langle e_k, e_l\rangle = c_{l-k}$ for all k, l and that for all finitely nonzero sequences (a_k) of complex numbers we have that

$$\sum_{k,l} a_k \overline{a}_l c_{k-l} \geqslant 0.$$

This implies (see [10], p. 38) that there is a finite positive measure μ on $\mathbb{T}$ so that

$$\int \mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\mu(\theta) = c_k.$$

Thus

$$\langle f, g\rangle = \int f\overline{g}\,\mathrm{d}\mu.$$

However this norm is equivalent to the original norm so that μ is absolutely continuous with respect to Lebesgue measure and of the form $(2\pi)^{-1}v(\theta)\mathrm{d}\theta$ where $v \sim w$.

It follows that in $H^2(v)$ the basis constant of the exponential basis is at most σ and so by Lemma 3.3 we have $\|R\|_v \leqslant \sigma$ and the proof is complete. ∎

We can now give explicit examples by taking the weights $w(\theta) = |\theta|^\alpha$ where $0 < \alpha < 1$. It is clear that in Theorem 3.4 we have $p = \alpha^{-1}$ and so for any fast monotone multiplier we have

$$\inf\{\sup_n \|(A^{-1}TA)^n\| : A \text{ invertible}\} = \sec\Big(\frac{\pi\alpha}{2}\Big) > 1.$$

Note that we are essentially using here the original example of a conditional basis for Hilbert space due to Babenko ([1]). We can also utilize (2.3) to show that for this example the infimum in (3.2) is actually attained. In general the infimum in (3.2) need not be attained; this it will be seen easily from Theorem 3.6 below.

Theorem 3.5. *Let w be an A_2-weight and suppose $T : H^2(w) \to H^2(w)$ is a fast monotone multiplier, corresponding to the sequence (λ_n). Then the following are equivalent:*

(i) *T is similar to a contraction;*
(ii) *T is polynomially bounded;*
(iii) *$w \sim 1$.*

Proof. That (i) implies (ii) is a consequence of von Neumann's inequality (see [14]). Similarly, (iii) implies (i) is trivial. It therefore remains to prove that (ii) implies (iii). We shall treat the case when the λ_k are distinct; small modifications are necessary in the other cases. We shall also suppose the measure $\mathrm{d}\mu = (2\pi)^{-1}w(\theta)\mathrm{d}\theta$ is a probability measure so that $\|e_k\| = 1$ for all k.

First note that if $f \in H^\infty(\mathbb{D})$ then for any $r < 1$, then $f_r(T)$ is well-defined where $f_r(z) = f(rz)$ and if T is polynomially bounded we have an estimate

$$\|f_r(T)\| \leqslant C\|f\|_{H^\infty(\mathbb{D})},$$

or equivalently

$$\Big\|\sum_{k=0}^{\infty} f(r\lambda_k)a_k e_k\Big\| \leqslant C\|f\|_{H^\infty(\mathbb{D})}\Big\|\sum_{k=0}^{\infty} a_k e_k\Big\|$$

whenever (a_k) is finitely non-zero. Letting $r \to 1$ we obtain

$$\Big\|\sum_{k=0}^{\infty} f(\lambda_k)a_k e_k\Big\| \leqslant C\|f\|_{H^\infty(\mathbb{D})}\Big\|\sum_{k=0}^{\infty} a_k e_k\Big\|.$$

Recall that by Carleson's theorem ([3]) the sequence (λ_n) is *interpolating* (cf. [7], p. 287–288) so that there is a constant B such that for any sequence $\varepsilon_k = \pm 1$ there exists $f \in H^\infty(\mathbb{D})$ with $\|f\|_{H^\infty(\mathbb{D})} \leqslant B$ and $f(\lambda_k) = \varepsilon_k$ for all $k \geqslant 0$. Hence

$$\Big\|\sum_{k=0}^{\infty} \varepsilon_k a_k e_k\Big\| \leqslant BC \Big\|\sum_{k=0}^{\infty} a_k e_k\Big\|$$

for all finitely non-zero sequences (a_k). Hence by the parallelogram law we have

$$(BC)^{-1}\Big(\sum_{k=0}^{\infty} |a_k|^2\Big)^{1/2} \leqslant \Big\|\sum_{k=0}^{\infty} a_k e_k\Big\| \leqslant BC\Big(\sum_{k=0}^{\infty} |a_k|^2\Big)^{1/2}$$

from which it follows that $w \sim 1$. ∎

We conclude by considering the cases when

$$\inf\{\sup_n \|(A^{-1}TA)^n\| : A \text{ invertible}\} = 1.$$

THEOREM 3.6. *Let w be an A_2-weight and suppose $T : H^2(w) \to H^2(w)$ is a fast monotone multiplier, corresponding to the sequence (λ_n). Then the following are equivalent:*

(i) *for any $\varepsilon > 0$, T is similar to an operator S with $\sup_n \|S^n\| < 1 + \varepsilon$;*

(ii) *$\log w$ is in the closure of L^∞ in* BMO*;*

(iii) *$w^a \in A_2$ for every $a > 0$.*

Proof. The equivalence of (i) and (iii) is proved in Theorem 3.4. The equivalence of (ii) and (iii) is due to Garnett and Jones ([8]); see also [7], Corollary 6.6 and its proof (p. 258–9). ∎

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