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RELATIVE TENSOR PRODUCTS AND INFINITE C^*-ALGEBRAS

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Abstract. Let A, B, C be C^*-algebras. Given A-B and B-C normed bimodules V and W respectively, whose unit ball is convex with respect to the actions of the C^*-algebras, we study the reasonable seminorms on the relative tensor product $V \otimes_B W$, having the same convexity property. This kind of bimodule is often encountered and retains many features of the usual normed space. We show that the classical Grothendieck program extends nicely in this setting. Fixing B, we then establish that there exists an unique such seminorm on $V \otimes_B W$ for any V, W if and only if B is infinite in a weaker sense than proper infiniteness and stronger than the non existence of tracial states (the equivalence of these two latter notions still remaining open). Applying this result when B is a stable C^*-algebra, we show that the relative Haagerup tensor product of operator bimodules is both injective and projective.

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0. INTRODUCTION

Several fundamental problems in functional analysis involve normed spaces V endowed with contractive commuting left and right actions of C^*-algebras A and B respectively. Let us mention for instance the higher dimensional cohomology problem ([5], [19]) or the Morita theory of operator algebras ([3]). In view of the Gelfand representation theorem for C^*-algebras, it is natural to seek for concrete representation theorems for such A-B normed bimodules.

Assume that A and B are C^*-subalgebras of the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H and that V is a normed A-B submodule of $\mathcal{B}(H)$. Then each of the matrix spaces $M_n(V)$ can be provided with the relative operator norm $\|\cdot\|_n$ determined by the inclusion $M_n(V) \subset M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$. Equipped with this canonical matrix normed structure $(\|\cdot\|_n)_{n\geqslant 1}$, this space V is called a *concrete* A, B *operator bimodule* ([6]). These operator bimodules have been abstractly characterized in [5] (see also [6]) as the L^∞-matricially normed spaces (in the sense of [17] and [7]) for which the actions of A and B are completely contractive.

Another natural question is the following one: characterize those abstract (only) normed A-B bimodules $(V, \|\cdot\|)$ such that there exist faithful representations $(\pi, H), (\rho, K)$ of A and B respectively and an isometric map $J : V \to \mathcal{B}(K, H)$ with $J(avb) = \pi(a)J(v)\rho(b)$ for $a \in A$, $b \in B$, $v \in V$. This problem has been solved independently by B. Magajna in [13] and by the second author in [16]. These bimodules, distinguished by a very simple axiom, namely the A-B convexity of their unit ball (see Theorem 1.5), are called *representables.* Of course, when $A = B = \mathbb{C}$, every normed space is representable, being isometric to a subspace of the C^*-algebra of continuous functions on the unit ball of its dual.

As expected, there are remarkable parallels between normed spaces and representable bimodules. In this paper, we show that the elementary theory of tensor products of normed spaces carries over to the category of representables bimodules. Given a representable A-B bimodule V and a representable B-C bimodule W, we define in Section 2 the relative projective tensor product ${}_AV \otimes_B^\Gamma W_C$, and in Section 3 the relative injective tensor product ${}_AV \otimes_B^\Lambda W_C$. They are representable A-C bimodules and behave exactly as the corresponding classical tensor products $V \otimes^\gamma W$ and $V \otimes^\lambda W$ that we get back when A, B and C are reduced to the complex numbers. We remark in Section 4 that these relative tensor products can degenerate into the zero space even when V and W are not so. However, this cannot occur when B is any C^*-subalgebra of the algebra of all compact operators in a Hilbert space.

The classical H and H$'$ Hilbertian tensor products of Grothendieck have also their counterpart in our setting, as shown in Section 5.

A duality theory can also be developed, but at present we do not see how it can be used to extend the famous result of Grothendieck: a normed space V has the approximation property if and only if, for every normed space W, the natural map $\iota : V \otimes^\gamma W \to V \otimes^\lambda W$ is injective. Instead, we concentrate on B. We show that ${}_AV \otimes_B^\Gamma W_C = {}_AV \otimes_B^\Lambda W_C$ isometrically for all representable A-B bimodule V and all representable B-C bimodule W if and only if B is infinite in a certain sense that we call condition (I) (see Definition 6.1). This property is intermediate between the properties for $M(B)$ and B'' to be properly infinite, where $M(B)$ denotes the multiplier algebra of B and B'' its enveloping von Neumann algebra. Recall that a unital C^*-algebra is properly infinite if it contains two orthogonal projections equivalent to its unit. Of course, B'' is properly infinite if and only if B has no tracial states, but it is a very difficult outstanding problem to show whether this implies that $M(B)$ is properly infinite. Let us mention in this direction the remarkable result of Haagerup stating that for every unital exact C^*-algebra B without any tracial state there exists an integer n such that $M_n(B)$ is properly infinite ([10]).

As a consequence of our study, for every stable C^*-algebra B we have ${}_AV \otimes_B^\Gamma W_C = {}_AV \otimes_B^\Lambda W_C$ isometrically. In the last section, this result is applied in the following context. Let V be a right B operator module, and W a left B operator module. We denote by $V \otimes_B^{\mathrm{h}} W$ the relative Haagerup operator space tensor product (that is the operator space quotient of the usual Haagerup operator space tensor product $V \otimes^{\mathrm{h}} W$ by the closed subspace spanned by the tensors $vb \otimes w - v \otimes bw$). Let $\mathcal{K}$ be the C^*-algebra of all compact operators in a separable infinite dimensional Hilbert space. Then the spatial tensor products $\mathcal{K} \otimes_{\min} V$, $\mathcal{K} \otimes_{\min} W$ are respectively representable $\mathcal{K}$-$\mathcal{K} \otimes_{\min} B$ and $\mathcal{K} \otimes_{\min} B$-$\mathcal{K}$ bimodules, and the normed space $\mathcal{K} \otimes_{\min} (V \otimes_B^{\mathrm{h}} W)$ is naturally isometric to their unique tensor product in the category of representables bimodules. It follows immediately that the relative Haagerup tensor product is both injective and projective. This is well known when $B = \mathbb{C}$ but we have not found this result for a general C^*-algebra B in the very rich literature on the subject.

1. PRELIMINARIES AND NOTATIONS

For the reader's convenience, we recall in this section some definitions and results whose details are mostly contained in [15] (see also [16]). Given a C^*-algebra A, we denote by $\operatorname{Rep} A$ the class of its non degenerated representations (π, H). It is often convenient to write H instead of (π, H). We first introduce a class of representations which plays an important role in our study.

Definition A representation (π, H) of A is called *locally cyclic* if for any $h_1, h_2, \ldots, h_n \in H$ there exists $h \in H$ such that $h_i \in \overline{\pi(A)h}$ for $i = 1, 2, \ldots, n$.

Every cyclic representation is locally cyclic. The standard form of the enveloping von Neumann algebra A^{**} of A defines a locally cyclic representation of A, as a consequence of Lemma 2.3 from [20]. We call it *the standard representation of A*. Let us recall the following useful characterization.

Proposition ([16], [15]) *Every representation of A is locally cyclic if and only if A has no tracial states (or equivalently, A^{**} is properly infinite).*

This applies for instance to stable C^*-algebras. On the other hand, it is easily checked that the only locally cyclic representations of the C^*-algebra $\mathbb{M}_n(\mathbb{C})$ are those of the form p times the fundamental one, with $1 \leqslant p \leqslant n$. In particular, the one dimensional representation of $\mathbb{C}$ is its only locally cyclic representation.

Let us consider now two C^*-algebras A and B. A normed A-B bimodule is a normed space V equipped with structures of left A-module and right B-module, such that

$$a(vb) = (av)b, \quad \|avb\| \leqslant \|a\| \, \|v\| \, \|b\|$$

for all $v \in V$, $a \in A$, $b \in B$. In addition, we always assume that V is essential in the sense that $\overline{AV} = V = \overline{VB}$. Without loss of generality we can also assume that V is complete. We shall denote by V^* the dual of V.

Note that, given representations H_A and H_B of A and B respectively, the space $\mathcal{B}(H_B, H_A)$ of all bounded linear operators from H_B into H_A is in a natural way a normed A-B bimodule.

Definition Let V be a normed A-B bimodule. A bounded A-B-linear map from V into $\mathcal{B}(H_B, H_A)$ is called a *representation* of V. We say that the representation is *locally cyclic* (respectively *cyclic*) when H_A and H_B are so.

We denote by $\mathrm{Hom}_{A,B}(V, \mathcal{B}(H_B, H_A))$ the Banach space of all representations of V into $\mathcal{B}(H_B, H_A)$. The A-B bimodules admitting an isometric representation have a nice characterization as follows.

Definition Let V be an algebraic A-B bimodule. We say that a seminorm N on V has property (R) (or is a R-seminorm) if

$$\text{(R)}\quad N(a_1v_1b_1 + a_2v_2b_2) \leqslant \|a_1a_1^* + a_2a_2^*\|^{1/2} \max\{N(v_1), N(v_2)\} \|b_1^*b_1 + b_2^*b_2\|^{1/2}$$

for all $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $v_1, v_2 \in V$.

Observe that this condition says that the unit ball of V is A-B convex in an obvious sense.

Theorem ([16], [15] and [13]) *Let V be a normed A-B bimodule. The following conditions are equivalent:*

(i) *the norm of V has property* (R)*;*

(ii) *there exist faithful representations (π_A, H_A), (π_B, H_B) of A and B respectively and a linear isometric map $J : V \to \mathcal{B}(H_B, H_A)$ such that*

$$J(avb) = \pi_A(a)J(v)\pi_B(b), \quad \forall a \in A, b \in B, v \in V.$$

Obviously, we have (ii) $\Rightarrow$ (i). The converse is based on the following crucial lemma.

Lemma *Assume that the norm of V has property* (R) *and let $F \in V^*$. Then there exist states φ and ψ of A and B respectively such that*

$$|F(avb)| \leqslant \varphi(aa^*)^{1/2}\psi(b^*b)^{1/2}\|v\|\,\|F\|, \quad \forall a \in A, b \in B, v \in V.$$

Given $F \in V^*$, this lemma ensures the existence of a cyclic representation $R : V \to \mathcal{B}(H_B, H_A)$ with $\|R\| = \|F\|$ and of unit vectors $\xi_A \in H_A$, $\xi_B \in H_B$ such that

$$F(v) = \langle R(v)\xi_B, \xi_A\rangle, \quad \forall v \in V.$$

As an immediate consequence we get

Proposition *Let V be a normed A-B bimodule with property* (R)*. Then for any $v \in V$ we have*

$$\|v\| = \sup \|R(v)\| = \max \|R(v)\|,$$

where R runs over all the contractive cyclic representations of V.

The desired isometric representation of V is obtained in an obvious way, by taking the direct sum of all cyclic contractive representations of V.

Definition An A-B bimodule V satisfying the equivalent conditions from Theorem 1.5 is called *representable*.

For the general study of these objects, we refer to [16]. By replacing the algebra of complex numbers with arbitrary C^*-algebras A and B, most of the basic facts from the theory of normed spaces extend to the category of representable A-B bimodules. In this framework the bounded linear forms have to be replaced by the locally cyclic representations. Indeed, the Hahn-Banach extension theorem becomes:

Theorem ([16] and [15]) *Let W be a representable A-B bimodule and $V \subset W$ a submodule. Then every locally cyclic representation $R : V \to \mathcal{B}(H_B, H_A)$ extends to a representation $\widetilde{R} : W \to \mathcal{B}(H_B, H_A)$ with $\|\widetilde{R}\| = \|R\|$.*

Note that since $\mathbb{C}$ has only one locally cyclic representation, Proposition 1.7 and Theorem 1.9 are basic results in classical functional analysis, when $A = B = \mathbb{C}$.

We now turn to operator bimodules in the sense of [6], which are important particular cases of representable bimodules.

Throughout this paper, given two integers p, q and a vector space V, we denote by $\mathbb{M}_{p,q}(V)$ the vector space of $p \times q$ matrices with entries in V. Also, we set $\mathbb{M}_p = \mathbb{M}_{p,p}$. If W is another linear space and $R : V \to W$ is a linear map, we denote as usual by $R_{p,q}$ the map $[v_{ij}] \mapsto [R(v_{ij})]$ from $\mathbb{M}_{p,q}(V)$ to $\mathbb{M}_{p,q}(W)$, and we write $R_p = R_{p,p}$. When $p < q$, we have a natural embedding $\mathbb{M}_p(V) \subset \mathbb{M}_q(V)$. Let us set $\mathbb{M}_\infty(V) = \bigcup_{n=1}^{\infty} \mathbb{M}_n(V)$.

Definition An *operator A-B bimodule* is an A-B bimodule together with a norm $\|\cdot\|_n$ on each matrix space $\mathbb{M}_n(V)$, $n \geqslant 1$, satisfying:

$$\|avb\|_n \leqslant \|a\|_n \|v\|_n \|b\|_n; \tag{R_1}$$

$$\left\| \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \right\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}, \tag{R_2}$$

for all $n, m \geqslant 1$, $a \in \mathbb{M}_n(A)$, $b \in \mathbb{M}_n(B)$, $v \in \mathbb{M}_n(V)$ and $w \in \mathbb{M}_m(W)$. When $A = \mathbb{C}$ (respectively $B = \mathbb{C}$), V is called a *right operator B-module* (respectively a *left operator A-module*).

If $A = B = \mathbb{C}$, then V is an operator space in the usual sense. For simplicity, given $v \in \mathbb{M}_n(V)$, we often write $\|v\|$ instead of $\|v\|_n$.

Recall that a linear map $R : V \to W$ between operator spaces is *completely bounded* if $\|R\|_{\mathrm{cb}} = \sup_{n \geqslant 1} \|R_n\| < \infty$. When $\|R\|_{\mathrm{cb}} \leqslant 1$ we say that R is *completely contractive*. If R_n is isometric for all $n \geqslant 1$ then R is *completely isometric*.

For any representations $H_A \in \mathrm{Rep} A$, $H_B \in \mathrm{Rep} B$, the linear space $\mathcal{B}(H_B, H_A)$ is canonically viewed as an operator A-B bimodule. Christensen, Effros and Sinclair have shown that every operator bimodule can be represented completely isometrically in some $\mathcal{B}(H_B, H_A)$.

THEOREM ([5]) *Let V be an operator A-B bimodule. There exist faithful representations (π_A, H_A), (π_B, H_B) of A and B respectively and a completely isometric map $J : V \to \mathcal{B}(H_B, H_A)$ such that*

$$J(avb) = \pi_A(a)J(v)\pi_B(b), \quad \forall a \in A, b \in B, v \in V.$$

In fact, this result can be obtained as a consequence of Theorem 1.5, as follows. Let us first introduce some notations: $\mathcal{K}$ will be the C^*-algebra of compact operators on a separable Hilbert space, and for any closed subspace F of $\mathcal{B}(H)$ we denote by $\mathcal{K} \otimes_{\min} F$ the spatial tensor product. Consider now an operator A-B bimodule V. By Axiom (R_2), there is a natural norm on $\mathbb{M}_\infty(V)$. The axioms (R_1) and (R_2) mean exactly that the completion W of $\mathbb{M}_\infty(V)$ is a representable $\mathcal{K}\otimes_{\min} A$-$\mathcal{K}\otimes_{\min} B$ bimodule. Then Theorem 1.5 suitably interpreted gives Theorem 1.11.

Let us recall the following useful result of R.R. Smith (Theorem 2.1 and Lemma 2.3 from [20]).

THEOREM *Let V be an operator A-B bimodule. Every locally cyclic representation $R : V \to \mathcal{B}(H_B, H_A)$ is completely bounded and $\|R\|_{\mathrm{cb}} = \|R\|$.*

DEFINITION Let $(V, \|\cdot\|)$ be a normed space. A system of matrix norms $\|\cdot\|_n$ on $\mathbb{M}_n(V)$ is said to be *compatible* with $(V, \|\cdot\|)$ if $\|\cdot\|_1 = \|\cdot\|$. We say that $(\|\cdot\|_n)_{n\geqslant 1}$ is *smaller* than another matrix normed structure $(\|\cdot\|'_n)_{n\geqslant 1}$ if for all $n \geqslant 1$

$$\|[v_{ij}]\| \leqslant \|[v_{ij}]\|', \quad \forall [v_{ij}] \in \mathbb{M}_n(V).$$

Let V be a representable A-B bimodule. Obviously on V there exist structures of operator A-B bimodule compatible with $(V, \|\cdot\|)$. It was noted in [16] and [13] that among such matrix normed structures there is a minimal one $\mathrm{MIN}_{A,B}$ and a maximal one $\mathrm{MAX}_{A,B}$. We recall now how these structures are characterized.

PROPOSITION ([16] and [15]) *Let V be a representable A-B bimodule. Given $\mathbf{v} = [v_{ij}] \in \mathbb{M}_n(V)$, we have the following expressions for the norm $\|\mathbf{v}\|_{\mathrm{MIN}}$ induced by $\mathrm{MIN}_{A,B}(V)$:*

(i) $\|\mathbf{v}\|_{\mathrm{MIN}} = \sup \|[R(v_{ij})]\|$ *where R runs over the cyclic contractive representations of V;*

(ii) $\|\mathbf{v}\|_{\mathrm{MIN}} = \sup \|[R(v_{ij})]\|$ *where R runs over the locally cyclic contractive representations of V;*

(iii) $\|\mathbf{v}\|_{\mathrm{MIN}} = \sup \Big\{ \Big\| \sum_{k=1}^{n} \sum_{l=1}^{n} a_k v_{kl} b_l \Big\|; \ \Big\| \sum_{k=1}^{n} a_k a_k^* \Big\| \leqslant 1, \Big\| \sum_{l=1}^{n} b_l^* b_l \Big\| \leqslant 1 \Big\}.$

PROPOSITION ([16] and [15]) *Let V be a representable A-B bimodule. Given $\mathbf{v} = [v_{ij}] \in \mathbb{M}_n(V)$, we have the following expressions for the norm $\|\mathbf{v}\|_{\mathrm{MAX}}$ induced by $\mathrm{MAX}_{A,B}(V)$:*

(i) $\|\mathbf{v}\|_{\mathrm{MAX}} = \sup \|[R(v_{ij})]\|$ *where R runs over all the contractive representations of V;*

(ii) $\|\mathbf{v}\|_{\mathrm{MAX}} = \inf\{\|a\| \max_k \|v_k\| \, \|b\|; \ \mathbf{v} = a \operatorname{diag}(v_k)\, b\}$ *with $a \in \mathbb{M}_{n,p}(A)$, $b \in \mathbb{M}_{p,n}(B)$ and where $\operatorname{diag}(v_k)$ denotes the diagonal matrix with diagonal entries $v_1, v_2, \ldots, v_p$.*

Finally, let us give the following consequence of Proposition 1.14, which is certainly known.

LEMMA *Let K be a Hilbert space and (π, H) be a locally cyclic representation of a C^*-algebra A. Then for $T_1, T_2, \ldots, T_n \in \mathcal{B}(K,H)$ we have*

$$\Big\| \sum T_i T_i^* \Big\|^{1/2} = \sup \Big\{ \Big\| \sum a_i T_i \Big\|;\ a_i \in A,\ \Big\| \sum a_i a_i^* \Big\| \leqslant 1 \Big\}.$$

In particular, for $\xi_1, \xi_2, \ldots, \xi_n \in H$,

$$\Big(\sum \|\xi_i\|^2 \Big)^{1/2} = \sup \Big\{ \Big\| \sum a_i \xi_i \Big\|_H,\ a_i \in A,\ \Big\| \sum a_i a_i^* \Big\| \leqslant 1 \Big\}.$$

Proof. By Proposition 1.14 (ii), we see that the natural operator space structure of $\mathcal{B}(K,H)$ is the minimal compatible one with respect to its A-$\mathcal{B}(K)$ bimodule structure.

The norm of ${}^{\mathrm{t}}[T_1 \cdots T_n] \in \mathbb{M}_{n,1}(\mathcal{B}(K,H))$ can be computed by the formula given in Proposition 1.14 (iii). We get easily

$$\begin{aligned}\Big\| \sum T_i^* T_i \Big\| &= \sup \Big\{ \Big\| \sum_{i=1}^n a_i T_i m \Big\|;\ \Big\| \sum a_i a_i^* \Big\| \leqslant 1, \|m\| \leqslant 1 \Big\} \\ &= \sup \Big\{ \Big\| \sum_{i=1}^n a_i T_i \Big\|;\ \Big\| \sum a_i a_i^* \Big\| \leqslant 1 \Big\}. \quad \blacksquare\end{aligned}$$

2. THE PROJECTIVE TENSOR PRODUCT

Let A, B, C be C^*-algebras, V a representable A-B bimodule and W a representable B-C bimodule. We denote by $V \otimes_B W$ the algebraic tensor product over B. By definition, it is the quotient of $V \otimes W$ by the vector space spanned by the elements of the form

$$vb \otimes w - v \otimes bw, \quad v \in V, w \in W, b \in B.$$

We write $v \otimes_B w$ for the equivalence class of $v \otimes w$ in $V \otimes_B W$.

The space $V \otimes_B W$ has a natural structure of A-C bimodule. Our aim is the study of the possible R-seminorms on $V \otimes_B W$ which are compatible with the norms of V and W as follows.

DEFINITION A seminorm N on $V \otimes_B W$ is called a *subcross seminorm* if $N(v \otimes_B w) \leqslant \|v\| \, \|w\|$ for all $v \in V$, $w \in W$.

For $u \in V \otimes_B W$ we set

$$\Gamma_{A,C}(u) = \inf_{u = a \,\mathrm{diag}(v_k \otimes_B w_k) b} \|a\| \Big(\max_{1 \leqslant k \leqslant n} \|v_k\| \Big) \Big(\max_{1 \leqslant k \leqslant n} \|w_k\| \Big) \|b\|$$

with $a \in \mathbb{M}_{1,n}(A)$, $b \in \mathbb{M}_{n,1}(B)$. When there is no risk of confusion we write $\Gamma(u)$ instead of $\Gamma_{A,C}(u)$.

Proposition *The functional* $\Gamma_{A,C}$ *is the largest subcross* R*-seminorm on* $V \otimes_B W$.

Proof. The only non completely obvious point is that Γ has property (R). Consider $a_1, \dots, a_n \in A$, $c_1, \dots, c_n \in B$ and $u_1, \dots, u_n \in V \otimes_B W$ with $\Gamma(u_1) < 1, \dots, \Gamma(u_n) < 1$. By definition of Γ, for $k = 1, \dots, n$, we can write

$$u_k = \sum_r a_{kr} v_{kr} \otimes_B w_{kr} c_{kr}$$

with $\Big\| \sum_r a_{kr} a_{kr}^* \Big\|, \Big\| \sum_r c_{kr}^* c_{kr} \Big\|, \max_r \|v_{kr}\|, \max_r \|w_{kr}\| < 1$. Then

$$u = \sum_{k,r} a_k a_{kr} v_{kr} \otimes_B w_{kr} c_{kr} c_k$$

and since

$$\sum_{k,r} a_k a_{kr} a_{kr}^* a_k^* \leqslant \sum_k a_k a_k^*,$$
$$\sum_{k,r} c_k^* c_{kr}^* c_{kr} c_k \leqslant \sum_k c_k^* c_k$$

we deduce from the definition of Γ that

$$\Gamma(u) < \Big\| \sum_k a_k a_k^* \Big\|^{1/2} \Big\| \sum_k c_k^* c_k \Big\|^{1/2}. \qquad \blacksquare$$

We denote by ${}_AV \otimes_B^\Gamma W_C$ the completion of the quotient of $V \otimes_B W$ by the null space of Γ. Let $[u]$ be the equivalence class of $u \in V \otimes_B W$ and set $\|[u]\|_\Gamma = \Gamma(u)$.

Recall that a map φ from a normed space X onto a normed space Y is a *quotient map* if the image of the open unit ball of X is the unit ball of Y.

The tensor product $\otimes_B^\Gamma$ is projective in the following sense:

Theorem *Let* V, V' *and* W, W' *be two representable* A-B *and* B-C *bimodules respectively. Let* $\Phi : V \to V'$ *be a* A-B *bimodule quotient map and* $\Psi : W \to W'$ *be a* B-C *bimodule quotient map. Then* $\Phi \otimes_B \Psi : V \otimes_B W \to V' \otimes_B W'$ *induces a* A-C *bimodule quotient map from* ${}_AV \otimes_B^\Gamma W_C$ *onto* ${}_AV' \otimes_B^\Gamma W'_C$.

Proof. For $u \in V \otimes_B W$ one obviously has

$$\Gamma(\Phi \otimes_B \Psi(u)) \leqslant \Gamma(u),$$

and therefore $\Phi \otimes_B \Psi$ induces a contractive A-C bimodule map from ${}_AV \otimes_B^\Gamma W_C$ into ${}_AV' \otimes_B^\Gamma W'_C$, that we still denote by $\Phi \otimes_B \Psi$. It remains to show that every $y \in {}_AV' \otimes_B^\Gamma W'_C$ with $\|y\| < 1$ belongs to the image of the open unit ball of ${}_AV \otimes_B^\Gamma W_C$.

Let us assume first that $y = [u']$ with $u' \in V' \otimes_B W'$. Then u' can be written as

$$u' = \sum_{k=1}^n a_k v'_k \otimes_B w'_k c_k$$

where $a_k \in A$, $c_k \in C$, $v'_k \in V'$ and $w'_k \in W'$ satisfy

$$\|v'_k\| < 1, \quad \|w'_k\| < 1, \quad \forall k = 1, \ldots, n$$
$$\Big\| \sum_{k=1}^n a_k a_k^* \Big\| \leqslant 1, \quad \Big\| \sum_{k=1}^n c_k^* c_k \Big\| \leqslant 1.$$

Since Φ and Ψ are quotient maps, for $k = 1, \ldots, n$ there exist $v_k \in V$, $w_k \in W$ with $\|v_k\| < 1$, $\|w_k\| < 1$ and $\Phi(v_k) = v'_k$, $\Psi(w_k) = w'_k$. By setting $u = \sum\limits_{k=1}^n a_k v_k \otimes_B w_k c_k$, we have obviously $\|[u]\|_\Gamma < 1$ and $(\Phi \otimes_B \Psi)[u] = [u']$.

In the general case, we choose η such that $\|y\| < 1-\eta$ and a sequence $(u'_n)_{n\geqslant 0}$ in $V' \otimes_B W'$ such that $\Gamma(u'_0) < 1 - \eta$, $\Gamma(u'_n) < \frac{\eta}{2^n}$, $\forall n \geqslant 1$ and $y = \sum\limits_{n=0}^\infty [u'_n]$. By the first part of the proof, we can find a sequence $(u_n)_{n\geqslant 0}$ in $V \otimes_B W$ with

$$\Gamma(u_0) < 1 - \eta, \quad \Gamma(u_n) < \frac{\eta}{2^n}, \quad \forall n \geqslant 1$$
$$(\Phi \otimes_B \Psi)([u_n]) = [u'_n], \quad \forall n \geqslant 0.$$

Letting $x = \sum\limits_{n=1}^\infty [u_n] \in {}_A V \otimes_B^\Gamma W_C$, we have $\|x\|_\Gamma < 1$ and $\Phi \otimes_B \Psi(x) = y$. ∎

Definition The tensor product $\otimes_B^\Gamma$ is called *the projective tensor product* of representable bimodules.

Given a representable A-C bimodule Z, denote by $\mathrm{Bil}_{A,C}^B(V, W; Z)$ the Banach space of all B-balanced, A-C linear bounded maps from $V \times W$ into Z. By definition, for $Q \in \mathrm{Bil}_{A,C}^B(V, W; Z)$ we have

$$Q(avb, wc) = aQ(v, bw)c, \quad \forall a \in A,\ b \in B,\ c \in C,\ v \in V,\ w \in W.$$

The projective tensor product $\otimes_B^\Gamma$ linearizes such maps.

Proposition (i) *For any $Q \in \mathrm{Bil}_{A,C}^B(V, W; Z)$ there exists a unique A-C bimodule morphism $\widetilde{Q} : {}_A V \otimes_B^\Gamma W_C \to Z$ such that for all $v \in V$, $w \in W$ we have*

$$\widetilde{Q}([v \otimes_B w]) = Q(v, w).$$

Moreover, we have $\|\widetilde{Q}\| = \|Q\|$.

(ii) *For $u = \sum v_k \otimes_B w_k \in V \otimes_B W$ we have*

$$\Gamma(u) = \sup_Q \Big\| \sum Q(v_k, w_k) \Big\|,$$

where the supremum is taken over all contractive bilinear maps $Q \in \mathrm{Bil}_{A,C}^B(V, W; \mathcal{B}(H_C, H_A))$ and H_A, H_C ranges over the set of cyclic (or locally cyclic, or all) representations of A and C respectively.

Proof. (i) For $u = \sum a_k v_k \otimes_B w_k c_k$ we have

$$\Big\| \sum_k a_k Q(v_k, w_k) c_k \Big\| \leqslant$$
$$\|Q\| \, \Big\| \sum_k a_k a_k^* \Big\|^{1/2} \max_k \|v_k\| \, \max_k \|w_k\| \, \Big\| \sum_k b_k^* b_k \Big\|^{1/2},$$

since Z is representable. Therefore Q induces an A-C linear bounded map $\widetilde{Q}$: ${}_AV \otimes_B^\Gamma W_C \to Z$ such that $\|\widetilde{Q}\| \leqslant \|Q\|$. The reverse inequality is obvious.

(ii) follows immediately from (i) and Proposition 1.7. ∎

Remark that when $A = B = C = \mathbb{C}$, the tensor product $\otimes_B^\Gamma$ is just the usual projective tensor product $\otimes^\gamma$ in the category of normed spaces.

3. THE INJECTIVE TENSOR PRODUCT

Recall that when V and W are ordinary Banach spaces, the injective cross norm λ is the minimal cross norm α such that the functional α^* on $V^* \otimes W^*$ defined by

$$\alpha^*(f) = \sup_{\alpha(u)\leqslant 1} |f(u)|$$

is also a cross norm.

Assume now that V and W are representable bimodules as in Section 2. As already mentioned, the role of V^* and W^* is now played by $\mathrm{Hom}_{A,B}(V, \mathcal{B}(H_B, H_A))$ and $\mathrm{Hom}_{B,C}(W, \mathcal{B}(H_C, H_B))$ where H_A, H_B and H_C range over the locally cyclic representations of the corresponding C^*-algebras (or even only on their cyclic representations).

For $R \in \mathrm{Hom}_{A,B}(V, \mathcal{B}(H_B, H_A))$, $S \in \mathrm{Hom}_{B,C}(W, \mathcal{B}(H_C, H_B))$, we define $RS \in \mathrm{Bil}_{A,C}^B(V, W; \mathcal{B}(H_C, H_A))$ by

$$RS(v, w) = R(v)S(w).$$

We still denote by RS the corresponding A-C linear map from $V \otimes_B W$ into $\mathcal{B}(H_C, H_A)$.

LEMMA *For $u \in V \otimes_B W$ let us define:*

(i) $\alpha_1(u) = \sup \|RS(u)\|$, *the supremum being taken over all contractive, locally cyclic representations*

$$R \in \mathrm{Hom}_{A,B}(V, \mathcal{B}(H_A, H_B)) \quad \textit{and} \quad S \in \mathrm{Hom}_{B,C}(W, \mathcal{B}(H_C, H_B))$$

and H_A, H_B, H_C range over the locally cyclic representations of the corresponding C^*-algebras;

(ii) $\alpha_2(u) = \sup \|RS(u)\|$ *where we only consider cyclic representations;*

(iii) $\alpha_3(u) = \sup \|RS(u)\|$ *where we consider only the standard representations $H_{\mathrm{s}}(A), H_{\mathrm{s}}(B), H_{\mathrm{s}}(C)$ of the corresponding C^*-algebras.*

Then

$$\alpha_1(u) = \alpha_2(u) = \alpha_3(u).$$

Proof. Obviously we have $\alpha_3(u) \leqslant \alpha_1(u)$ and $\alpha_2(u) \leqslant \alpha_1(u)$.

Let us consider now contractive locally cyclic representations $R : V \to \mathcal{B}(H_B, H_A)$, $S : W \to \mathcal{B}(H_C, H_B)$. Put $u = \sum_{k=1}^{n} v_k \otimes_B w_k$. Given $\varepsilon > 0$, there exist unit vectors $\xi \in H_A$, $\zeta \in H_C$ such that

$$\Big\| \sum R(v_k)S(w_k) \Big\| < \Big| \Big\langle \sum R(v_k)S(w_k)\zeta, \xi \Big\rangle \Big| + \varepsilon.$$

Since (π_B, H_B) is locally cyclic, there exists $\eta \in H_B$ such that

$$S(w_k)\zeta, R(v_k)^*\xi \in \overline{\pi_B(B)\eta}, \quad \forall k = 1, \ldots, n.$$

Let us define

$$\begin{aligned} p &= [\pi_B(B)\eta] \in \pi_B(B)' \subset \mathcal{B}(H_B) \\ q &= [\pi_A(A)\xi] \in \pi_A(A)' \subset \mathcal{B}(H_A) \\ r &= [\pi_C(C)\zeta] \in \pi_C(C)' \subset \mathcal{B}(H_C) \end{aligned}$$

to be the cyclic projections associated with η, ξ and ζ. We have

$$\begin{aligned} \Big\langle \sum_{k=1}^n R(v_k)S(w_k)\zeta, \xi \Big\rangle &= \sum_{k=1}^n \langle S(w_k)\zeta, R(v_k)^*\xi\rangle \\ &= \sum_{k=1}^n \langle pS(w_k)r\zeta, pR(v_k)^*q\xi\rangle \\ &= \sum_{k=1}^n \langle qR(v_k)ppS(w_k)r\zeta, \xi\rangle. \end{aligned}$$

The representation π_A reduced to qH_A is cyclic and therefore unitarily equivalent to a subrepresentation of the standard representation of A. The same observation applies to π_B reduced to pH_B and to π_C reduced to rH_C. It follows that we may view the map $v \mapsto qR(v)p$ as a contractive representation with values into $\mathcal{B}(H_{\mathrm{s}}(B), H_{\mathrm{s}}(A))$ and $w \mapsto pS(w)r$ as a contractive representation with values in $\mathcal{B}(H_{\mathrm{s}}(C), H_{\mathrm{s}}(B))$. Therefore we have

$$\Big\| \sum R(v_k)S(w_k) \Big\| < \alpha_3(u) + \varepsilon$$

from which we can conclude that $\alpha_1(u) \leqslant \alpha_3(u)$. By the same argument we also get that $\alpha_1(u) \leqslant \alpha_2(u)$. ∎

We set $\Lambda_{A,C}(u) = \alpha_1(u)$ and for simplicity we often write $\Lambda(u)$ instead of $\Lambda_{A,C}(u)$.

Proposition $\Lambda_{A,C}$ *is a subcross seminorm having property* (R).

Proof. Clearly, given locally cyclic representations R, S, we have

$$\|R(v)S(w)\| \leqslant \|v\|\,\|w\|, \quad \forall v \in V, w \in W$$

and hence Λ is a subcross seminorm.

Moreover, for $u_i \in V \otimes_B W$, $a_i \in A$, $c_i \in C$, $i = 1, 2$, we have

$$RS\Big(\sum_{k=1}^2 a_k u_k c_k\Big) = \sum_{k=1}^2 a_k RS(u_k) c_k,$$

and therefore

$$\Big\|RS\Big(\sum_{k=1}^2 a_k u_k c_k\Big)\Big\| \leqslant \Big\|\sum_{k=1}^2 a_k a_k^*\Big\|^{1/2} \Big\|\sum_{k=1}^2 c_k^* c_k\Big\|^{1/2} \max\{\Lambda(u_1), \Lambda(u_2)\},$$

and we conclude that Λ has property (R). ∎

Let us show now that Λ is minimal in a reasonable sense.

PROPOSITION *Let α be a subcross* R*-seminorm on $V \otimes_B W$. Then the following conditions are equivalent:*

(i) $\Lambda_{A,C} \leqslant \alpha$;

(ii) *For every $u \in V \otimes_B W$ and every locally cyclic representations $R \in \mathrm{Hom}_{A,B}(V, \mathcal{B}(H_B, H_A))$, $S \in \mathrm{Hom}_{B,C}(W, \mathcal{B}(H_C, H_B))$, we have*

$$\|RS(u)\| \leqslant \|R\|\,\|S\|\,\alpha(u).$$

The proof is immediate.

It follows that the reasonable R-seminorms on $V \otimes_B W$ stay between $\Lambda_{A,C}$ and $\Gamma_{A,C}$.

Denote by ${}_AV \otimes_B^\Lambda W_C$ the completion of the quotient of $V \otimes_B W$ by the null space of $\Lambda_{A,C}$. We define $[u]$ to be the equivalence class of $u \in V \otimes_B W$ and set $\|[u]\|_\Lambda = \Lambda(u)$. We now show that the tensor product $\otimes_B^\Lambda$ is injective.

THEOREM *Let V, V' and W, W' be two representable A-B and B-C bimodules respectively. Let $\Phi : V \to V'$ be an isometric A-B bimodule map and $\Psi : W \to W'$ be an isometric B-C bimodule map. Then $\Phi \otimes_B \Psi : V \otimes_B W \to V' \otimes_B W'$ induces an isometric A-C bimodule map from ${}_AV \otimes_B^\Lambda W_C$ into ${}_AV' \otimes_B^\Lambda W'_C$.*

Proof. It is enough to show that for $u = \sum v_k \otimes_B w_k \in V \otimes_B W$, we have $\Lambda(\Phi \otimes_B \Psi(u)) = \Lambda(u)$. The inequality

$$\Lambda(\Phi \otimes_B \Psi(u)) \leqslant \Lambda(u)$$

is obvious. To show the reverse inequality, consider two contractive locally cyclic representations $R \in \mathrm{Hom}_{A,B}(V, \mathcal{B}(H_B, H_A))$ and $S \in \mathrm{Hom}_{B,C}(W, \mathcal{B}(H_C, H_B))$. Owing to Theorem 1.9, we can find contractive, locally cyclic representations R' and S' of V' and W' respectively, such that $R' \circ \Phi = R$ and $S' \circ \Psi = S$. It follows that

$$\|RS(u)\| = \|R'S'(\Phi \otimes_B \Psi(u))\| \leqslant \Lambda(\Phi \otimes_B \Psi(u)).$$

The theorem is proved. ∎

REMARK (a) In the definition

$$\Lambda_{A,C}(u) = \sup \|RS(u)\|$$

we may relax the conditions on the representations of A and C, and allow any representation instead of the locally cyclic ones. This comes immediately from the proof of Lemma 3.1. On the other hand, it is very important to impose the local cyclicity condition for the representations of B. Otherwise, we get the H$'$-tensor product to be introduced in Section 5.

(b) Let A_1 be a C^*-subalgebra of A with $\overline{A_1A} = A$, and C_1 be a C^*-subalgebra of C with $\overline{C_1C} = C$. Every representable A-B bimodule V is a representable $A_1 - B$ bimodule, and the same remark applies for W when replacing C by C_1. It is obvious that on $V \otimes_B W$ we have

$$\Lambda_{A_1,C_1} \leqslant \Lambda_{A,C} \leqslant \Gamma_{A,C} \leqslant \Gamma_{A_1,C_1}.$$

(c) When $A = B = C = \mathbb{C}$, $V \otimes_B^\Lambda W$ is just the usual injective Banach space tensor product $V \otimes^\lambda W$, since $\mathbb{C}$ has only one locally cyclic representation.

Let us emphasize that in this paper we use the original notations λ and γ of Schatten for the injective and projective Banach tensor space products respectively ([18]). Their analogues for representable bimodules are consequently denoted by Λ and Γ. We warn the reader that the symbol Λ should not be confused with the symbol $\frown$, which is Grothendieck's notation for γ.

4. ARE Λ AND Γ NORMS?

The following example, due to E. Blanchard, shows that in general the seminorm Γ (and a fortiori the other subcross seminorms) on $V \otimes_B W$ is not a norm. Specifically, let us consider two countable disjoint and dense subsets $\{a_n, n \in \mathbb{N}\}$, $\{b_n, n \in \mathbb{N}\}$ in $[0,1]$, and define

$$V = W = c_0$$

to be the space of sequences converging to zero. We take $A = C = \mathbb{C}$ and $B = C[0,1]$, the C^*-algebra of continuous functions on $[0,1]$. Let B act on V and W by

$$(vf)_n = f(a_n)v_n, \quad (fw)_n = f(b_n)w_n, \quad \forall f \in C[0,1], v \in V, w \in W.$$

In [2], E. Blanchard shows that if $\alpha \in V$ and $\beta \in W$ have all their coefficients non zero, then $\alpha \otimes_B \beta \neq 0$. However, we prove now that $\mathrm{Bil}^B(V, W; \mathbb{C}) = 0$. This will imply, as a consequence of Proposition 2.5, that $\Gamma(u) = 0$ for all $u \in V \otimes_B W$.

Let $\alpha \in V$, $\beta \in W$ and $Q \in \mathrm{Bil}^B(V, W; \mathbb{C})$. Given $\varepsilon > 0$, we fix integers N and M such that

$$|\alpha_n| \leqslant \varepsilon, \quad |\beta_m| \leqslant \varepsilon, \quad \forall n \geqslant N, m \geqslant M,$$

and we define $\alpha' \in V$, $\beta' \in W$ by

$$\alpha'_n = \begin{cases} \alpha_n, & n \leqslant N, \\ 0, & n > N; \end{cases} \qquad \beta'_m = \begin{cases} \beta_m, & m \leqslant M, \\ 0, & m > M. \end{cases}$$

Setting $\alpha'' = \alpha - \alpha'$, $\beta'' = \beta - \beta'$, we have obviously $\|\alpha''\| \leqslant \varepsilon$, $\|\beta''\| \leqslant \varepsilon$.

On the other hand, there exists $f \in C[0,1]$ such that $f(a_n) = 1$ for $n \leqslant N$, $f(b_m) = 0$ for $m \leqslant M$. It follows that $\alpha' = \alpha' f$ and $0 = \beta' f$, and therefore

$$Q(\alpha', \beta') = 0.$$

Finally, we have

$$\begin{aligned} |Q(\alpha, \beta)| &= |Q(\alpha' + \alpha'', \beta' + \beta'')| \\ &= |Q(\alpha', \beta') + Q(\alpha', \beta'') + Q(\alpha'', \beta') + Q(\alpha'', \beta'')| \\ &\leqslant \|Q\|(\varepsilon\|\alpha\| + \varepsilon\|\beta\| + \varepsilon^2). \end{aligned}$$

Since ε is arbitrary, $Q(\alpha, \beta) = 0$. ∎

However, when B is finite dimensional, or more generally when B is a C^*-subalgebra of the C^*-algebra $\mathcal{K}(H)$ of all compact operators in a Hilbert space H, we will see that Λ is a norm on $V \otimes_B W$. Observe first that this property of B is equivalent to the fact that B is a restricted product of C^*-algebras $\mathcal{K}(H_i)$, or to the property of B of being an ideal in its enveloping von Neumann algebra B^{**} (see [11]).

PROPOSITION *Let A, B, C be C^*-algebras and assume that B is an ideal in B^{**}. Let V be a representable A-B bimodule and W be a representable B-C bimodule. Then $\Lambda_{A,C}$ is a norm on $V \otimes_B W$.*

Proof. Following Remark 3.5 it suffices to consider the case $A = C = \mathbb{C}$. Denote by $H_{\mathrm{s}}(B)$ the standard (locally cyclic) representation of B. We identify $\mathrm{Hom}_{B,\mathbb{C}}(W, \mathcal{B}(\mathbb{C}, H_{\mathrm{s}}(B)))$ with the space $\mathrm{Hom}_B(W, H_{\mathrm{s}}(B))$ of bounded linear maps $S : W \to H_{\mathrm{s}}(B)$ such that $S(bw) = bS(w)$ for $b \in B$ and $w \in W$. Similarly we identify $\mathrm{Hom}_{\mathbb{C},B}(V, \mathcal{B}(H_{\mathrm{s}}(B), \mathbb{C}))$ with the space $\mathrm{Hom}_B(V, H_{\mathrm{s}}(B)^*)$ of all bounded linear maps $R : V \to H_{\mathrm{s}}(B)^*$ such that $R(vb) = R(v)b$, where the dual of $H_{\mathrm{s}}(B)$ is endowed with its usual structure of right B-module. Denoting by j the antilinear isomorphism from $H_{\mathrm{s}}(B)^*$ onto $H_{\mathrm{s}}(B)$ we have $j(\varphi b) = b^* j(\varphi)$ for all $\varphi \in H_{\mathrm{s}}(B)^*$ and $b \in B$.

Let $u = \sum_{k=1}^{n} v_k \otimes_B w_k \in V \otimes_B W$ such that $\Lambda(u) = 0$. Then for $R \in \mathrm{Hom}_B(V, H_{\mathrm{s}}(B)^*)$ and $S \in \mathrm{Hom}_B(W, H_{\mathrm{s}}(B))$ we have

$$\sum \langle S(w_k), jR(v_k)\rangle = 0. \tag{4.1}$$

Let $p \in \mathcal{B}(\mathbb{C}^n \otimes H_{\mathrm{s}}(B))$ be the projection on the closed span of

$$\{(jR(v_k))_{1\leqslant k\leqslant n}; R \in \mathrm{Hom}_B(V, H_{\mathrm{s}}(B)^*)\}.$$

Similarly, $q \in \mathcal{B}(\mathbb{C}^n \otimes H_{\mathrm{s}}(B))$ is defined as the projection onto the closed span of

$$\{(S(w_k))_{1\leqslant k\leqslant n}; S \in \mathrm{Hom}_B(W, H_{\mathrm{s}}(B))\}.$$

Using (4.1), we get $pq = 0$.

Denote by B' the commutant of B in $\mathcal{B}(H_{\mathrm{s}}(B))$. Obviously, for any $b' \in B'$ we have $Rb' \in \mathrm{Hom}_B(V, H_{\mathrm{s}}(B)^*)$. It follows that p commutes with $\mathbf{1}_n \otimes B'$, that is $p \in \mathbb{M}_n(B'')$. In the same way we get $q \in \mathbb{M}_n(B'')$.

Let $p = [p_{ij}]$ and $q = [q_{ij}]$. Then

$$R(v_k) = \sum_i R(v_i)p_{ik}, \quad S(w_k) = \sum_j q_{kj} S(w_j)$$

for $k = 1, \dots, n$. On the other hand, by Cohen's factorization theorem, there exist $b_k \in B$ and $v_k' \in V$ such that $v_k = v_k' b_k$ and since B is an ideal in B'' we get

$$R(v_k) = \sum_i R(v_i')b_i p_{ik} = R\Big(\sum_i v_i'(b_i p_{ik})\Big)$$

for all $R \in \mathrm{Hom}_B(V, H_{\mathrm{s}}(B)^*)$. As every cyclic representation is equivalent to a subrepresentation of its standard one, Proposition 1.7 yields $v_k = \sum_i v_i'(b_i p_{ik})$. Finally, let us define $w_k' = \sum_i (b_k p_{ki}) w_i$, $k = 1, \dots, n$. We have $\sum v_k \otimes_B w_k = \sum v_k' \otimes_B w_k'$. But, for any $S \in \mathrm{Hom}_B(W, H_{\mathrm{s}}(B))$, observe that

$$\begin{aligned} S(w_k') &= \sum_i (b_k p_{ki}) S(w_i) = \sum_i \sum_j b_k p_{ki} q_{ij} S(w_j) \\ &= \sum_j b_k \Big(\sum_i p_{ki} q_{ij}\Big) S(w_j) = 0. \end{aligned}$$

Using again Proposition 1.7, we find $w_k' = 0$ for $k = 1, \dots, n$, thus

$$u = \sum v_k' \otimes_B w_k' = 0. \quad \blacksquare$$

5. THE HAAGERUP AND GROTHENDIECK TENSOR PRODUCTS

In the beginning of this section, we consider a right operator B-module W and a left operator B-module V. The relative Haagerup tensor product $V \otimes_B^{\mathrm{h}} W$ has been considered by several authors ([4], [12] and [3]) in relation with the study of spaces of completely bounded multilinear maps. Let us recall first a few basic facts. For more details we refer to [3].

DEFINITION Let Z be an operator space. A B-balanced bilinear map $Q : V \times W \to Z$ is said to be *completely bounded* if there exists a constant K such that

$$\Big\| \Big[\sum_{k=1}^{n} Q(v_{ik}, w_{kj}) \Big] \Big\| \leqslant K \, \|[v_{ij}]\| \, \|[w_{ij}]\|$$

for all $[v_{ij}] \in \mathbb{M}_n(V)$, $[w_{ij}] \in \mathbb{M}_n(W)$, $n \geqslant 1$.

The best constant K will be denoted by $\|Q\|_{\mathrm{cb}}$ and $\mathrm{Cb}^B(V, W; Z)$ will be the Banach space of all the completely bounded B-balanced bilinear maps.

The operator space $V \otimes_B^{\mathrm{h}} W$ is designed to linearize these bilinear maps. It is the quotient of the Haagerup tensor product by its closed subspace spanned by the elementary tensors $vb \otimes w - v \otimes bw$. For $u \in V \otimes_B W$ we denote by $[u]$ its equivalence class in $V \otimes_B^{\mathrm{h}} W$. We denote by $\| \cdot \|_{\mathrm{h}}$ the Haagerup norm and we set $\mathrm{h}(u) = \|[u]\|_{\mathrm{h}}$. The operator space structure of $V \otimes_B^{\mathrm{h}} W$ is defined globally on $\mathcal{K} \otimes_{\min} (V \otimes_B^{\mathrm{h}} W)$ by

$$\Big\| \sum k_i \otimes [v_i \otimes_B w_i] \Big\| = \sup \Big\| \sum k_i \otimes Q(v_i, w_i) \Big\| \tag{5.1}$$

where Q ranges over the completely contractive elements of $\mathrm{Cb}^B(V, W; \mathcal{B}(H))$ and H runs over all possible choices of Hilbert spaces.

Recall also (see Remark 2.7 from [3]) that for $u \in V \otimes_B W$ we have $\mathrm{h}(u) < 1$ if and only if u can be expressed as

$$u = \sum_{k=1}^{n} v_k \otimes_B w_k \tag{5.2}$$

with $\|[v_1 \cdots v_n]\| < 1$ and $\|{}^{\mathrm{t}}[w_1 \cdots w_n]\| < 1$.

If, in addition, V carries a structure of operator A-B bimodule and W a structure of operator B-C bimodule, then $V \otimes_B^{\mathrm{h}} W$ is an operator A-C bimodule (see Lemma 2.4 from [3]). In particular, $V \otimes_B^{\mathrm{h}} W$ is a representable A-C bimodule. Therefore, the canonical morphism $V \otimes_B W \to V \otimes_B^{\mathrm{h}} W$ yields an R-seminorm h on $V \otimes_B W$. Clearly, given locally cyclic contractive representations $R : V \to \mathcal{B}(H_B, H_A)$, $S : W \to \mathcal{B}(H_C, H_B)$, then $RS \in \mathrm{Cb}^B(V, W; \mathcal{B}(H_C, H_A))$ with $\|RS\|_{\mathrm{cb}} \leqslant 1$, since R and S are automatically completely bounded, with $\|R\| = \|R\|_{\mathrm{cb}}$, $\|S\| = \|S\|_{\mathrm{cb}}$. Then, it follows from Propositions 2.2 and 3.3 that

$$\Lambda_{A,C} \leqslant \mathrm{h} \leqslant \Gamma_{A,C}.$$

The following representation theorem, due to Paulsen and Smith, gives a very useful description of the B-balanced A-C bilinear completely contractive maps on $V \times W$.

Theorem ([14]) *Let (π_A, H_A) and (π_C, H_C) be two representations of A and C. If $Q \in \mathrm{Bil}^B_{A,C}(V, W, \mathcal{B}(H_C, H_A))$ is completely contractive, then there exist a representation (π_B, H_B) of B and two completely contractive representations $R \in \mathrm{Hom}_{A,B}(V, \mathcal{B}(H_B, H_A))$, $S \in \mathrm{Hom}_{B,C}(W, \mathcal{B}(H_C, H_B))$ such that $Q = RS$. Moreover, R and S can be chosen such that $\|Q\|_{\mathrm{cb}} = \|R\|_{\mathrm{cb}}\|S\|_{\mathrm{cb}}$.*

Proof. Knowing that V and W can be isometrically concretely represented, as bimodules, this theorem follows easily from Theorem 2.9 of [14], as already pointed out in Remark 2.13 from [14]. ∎

Corollary *Let V and W be two operator A-B and B-C bimodules respectively. Then for $u = \sum v_k \otimes_B w_k \in V \otimes_B W$ we have*

$$\mathrm{h}(u) = \sup \Big\| \sum R(v_k)S(w_k) \Big\|$$

where the supremum runs over all the completely contractive representations $R \in \mathrm{Hom}_{A,B}(V, \mathcal{B}(H_B, H_A))$, $S \in \mathrm{Hom}_{B,C}(W, \mathcal{B}(H_C, H_B))$ and H_A, H_B and H_C range over the representations of the corresponding algebras.

Proof. Obviously we have $\sup \| \sum R(v_k)S(w_k)\| \leqslant \mathrm{h}(u)$. Conversely, by Proposition 1.7 there exists a cyclic contractive representation $T : V \otimes^{\mathrm{h}}_B W \to \mathcal{B}(H_C, H_A)$ such that $\mathrm{h}(u) = \|[u]\|_{\mathrm{h}} = \|T[u]\|$. Then we conclude by observing that T comes from a completely contractive element in $\mathrm{Bil}^B_{A,C}(V, W; \mathcal{B}(H_C, H_A))$ and by applying Theorem 5.2. ∎

In the rest of this section we assume that V and W are only representable bimodules. Then to every possible choice of compatible operator bimodule structure on V and W, the associated Haagerup tensor product gives a reasonable subcross seminorm on $V \otimes_B W$. We denote by $\mathrm{H}_{A,C}$ and $\mathrm{H}'_{A,C}$ the seminorms defined by choosing as compatible structures MIN and MAX respectively. Note that we have

$$\Lambda_{A,C} \leqslant \mathrm{H}_{A,C} \leqslant \mathrm{H}'_{A,C} \leqslant \Gamma_{A,C}.$$

Theorem *Let V and W be two representable A-B and B-C operator bimodules respectively. Given $u \in V \otimes_B W$, we have:*

(i) $\mathrm{H}'_{A,C}(u) = \sup \|RS(u)\|$, *where the supremum is taken over all contractive representations $R \in \mathrm{Hom}_{A,B}(V, \mathcal{B}(H_B, H_A))$ and $S \in \mathrm{Hom}_{B,C}(W, \mathcal{B}(H_C, H_B))$.*

(ii) $\mathrm{H}'_{A,C}(u) = \inf \|a\|\,\|b\|\,\|c\| \max_k \|v'_k\| \max_k \|w'_k\|$, *where the infimum is taken over all possible expressions of u as*

$$u = \sum_{k,l} a_k v'_k \otimes_B b_{kl} w'_l c_l$$

with $v'_k \in V$, $w'_k \in W$, $a = [a_1 \cdots a_p] \in \mathbb{M}_{1,p}(A)$, $b = [b_{ij}] \in \mathbb{M}_{p,q}(B)$, $c = {}^{\mathrm{t}}[c_1 \cdots c_q] \in \mathbb{M}_{q,1}(C)$.

Proof. Each representation $R : \mathrm{MAX}_{A,B}(V) \to \mathcal{B}(H_B, H_A)$ is completely bounded and $\|R\| = \|R\|_{\mathrm{cb}}$. The same argument applies to $\mathrm{MAX}_{B,C}(W)$. Then (i) is a consequence of Corollary 5.3.

Let us now prove (ii). Assuming that $\mathrm{H}'(u) < 1$, it follows from (5.2) that u can be written as

$$u = \sum v_k \otimes_B w_k$$

with $\|[v_1 \cdots v_n]\|_{\mathrm{MAX}} < 1$, $\|{}^t[w_1 \cdots w_n]\|_{\mathrm{MAX}} < 1$. Then by Proposition 1.15, there exist an integer $p \geqslant 1$, $a \in \mathbb{M}_{1,p}(A)$, $b' \in \mathbb{M}_{p,n}(B)$, $v'_1, \ldots, v'_n \in V$, all with norm < 1, such that

$$v_i = \sum_{k=1}^{p} a_k v'_k b'_{ki}, \quad \forall i = 1, \ldots, n.$$

Similarly, there exist an integer $q \geqslant 1$, $b'' \in \mathbb{M}_{n,q}$, $c \in \mathbb{M}_{q,1}$ and $w'_1, \ldots, w'_q \in W$, all with norm < 1, such that

$$w_i = \sum_{l=1}^{q} b''_{il} w'_l c_l, \quad \forall i = 1, \ldots, n.$$

Set $b = b'b''$. We have

$$u = \sum_i v_i \otimes_B w_i = \sum_{i,k,l} a_k v'_k b'_{ki} \otimes_B b''_{il} w'_l c_l = \sum_{k,l} a_k v'_k \otimes_B b_{kl} w'_l c_l.$$

Since $\|b\| < 1$, the infimum in the statement is strictly less than 1.

Conversely, assume that u has an expression of the form

$$u = \sum_{k,l} a_k v'_k \otimes_B b_{kl} w'_l c_l$$

with a, b, c, v'_k and w'_l as in (ii). Then, given two contractive representations $R : V \to \mathcal{B}(H_B, H_A)$, $S : W \to \mathcal{B}(H_C, H_B)$, we have

$$\begin{aligned} \|RS(u)\| &= \Big\| \sum_{k,l} R(a_k v'_k) S(b_{kl} w'_l c_l) \Big\| \\ &= \|\pi_{1,p}(a) \operatorname{diag}\{R(v'_k)\}\, \rho_{p,q}(b) \operatorname{diag}\{S(w'_k)\}\, \sigma_{q,1}(c)\| < 1, \end{aligned}$$

where $(\pi, H_A), (\rho, H_B)$ and (σ, H_C) denote here the C^*-algebras representations. ∎

Theorem *Let V and W be two representable bimodules as above. For $u \in V \otimes_B W$ we have*

$$\mathrm{H}_{A,C}(u) = \inf \sup_{R,S} \Big\| \sum R(v_k) R(v_k)^* \Big\|^{1/2} \Big\| \sum S(w_k)^* S(w_k) \Big\|^{1/2}$$

where R and S range over all cyclic (or locally cyclic) representations of V and W respectively, and the infimum is taken over all possible expressions of u in the form $u = \sum v_k \otimes_B w_k$.

Proof. By definition,

$$\mathrm{H}(u) = \inf \Big\{ \|v\|_{\mathrm{MIN}} \|w\|_{\mathrm{MIN}};\ u = \sum v_k \otimes_B w_k \Big\},$$

where $v = [v_1 \cdots v_n]$ runs over $\mathbb{M}_{1,n}(V)$ and $w = {}^{\mathrm{t}}[w_1 \cdots w_n]$ runs over $\mathbb{M}_{n,1}(W)$ for $n \geqslant 1$. Now we conclude by using Proposition 1.14. ∎

Remark We define ${}_AV \otimes_B^{\mathrm{H}'} W_C$ to be the completion of the quotient of $V \otimes_B W$ by the null space of $\mathrm{H}'_{A,C}$, and ${}_AV \otimes_B^{\mathrm{H}} W_C$ is defined similarly.

Using the expression of $\mathrm{H}'_{A,C}$ given in Theorem 5.4 (ii), it is easily checked that the tensor product $\otimes_B^{\mathrm{H}'}$ is projective. An application of Theorem 1.9 shows that $\otimes_B^{\mathrm{H}}$ is injective.

When $A = B = C = \mathbb{C}$ we recover the Hilbertian norms $\otimes^{\mathrm{H}}$ and $\otimes^{H'}$ introduced by Grothendieck in [8]. Recall that in this case, the dual of $V \otimes^{H'} W$ is naturally identified with the space of all bilinear hilbertian forms on $V \times W$. They are precisely those bilinear forms Q on $V \times W$ for which there exist a Hilbert space H and bounded linear operators $R : V \to H$, $S : W \to H^*$ such that

$$Q(v,w) = \langle R(v), S(v) \rangle, \quad \forall\, v \in V, w \in W. \tag{5.3}$$

Moreover $\|Q\| = \inf \|R\|\,\|S\|$, the infimum being taken over all Hilbert spaces H and factorizations (5.3).

For representable bimodules, given two representations (π_A, H_A) and (π_C, H_C) of A and C respectively, we have

$$\mathrm{Hom}_{A,C}({}_AV \otimes_B^{\Gamma} W_C, \mathcal{B}(H_C, H_A)) = \mathrm{Bil}_{A,C}^{B}(V, W; \mathcal{B}(H_C, H_A)).$$

The space of representations $T : {}_AV \otimes_B^{\mathrm{H}'} W_C \to \mathcal{B}(H_C, H_A)$ is contained in $\mathrm{Bil}_{A,C}^{B}(V, W; \mathcal{B}(H_C, H_A))$ and has the following characterization, similar to the scalar case:

Proposition *Let $Q \in \mathrm{Bil}_{A,C}^{B}(V, W; \mathcal{B}(H_C, H_A))$. The following conditions are equivalent:*

(i) *There exists $c > 0$ such that $\|Q(u)\| \leqslant c\|[u]\|_{\mathrm{H}'}$ for all $u \in V \otimes_B W$;*

(ii) *There exist a representation (π, H) of B and representations $R : V \to \mathcal{B}(H, H_A)$, $S : W \to \mathcal{B}(H_C, H)$ such that*

$$Q(v,w) = R(v)S(w), \quad \forall\, v \in V, w \in W.$$

and $\|R\|\,\|S\| \leqslant c$.

Proof. This follows from Theorem 5.2 and from the automatic complete boundedness of representations on a MAX operator bimodule. ∎

6. THE CASE $\Lambda = \Gamma$

DEFINITION Let B be a C^*-algebra and $n \geqslant 2$. We say that B has property (I_n) if every matrix $[b_{ij}] \in \mathbb{M}_n(B)$ with norm < 1 has a decomposition

$$[b_{ij}] = \sum_{k=1}^{p} c^k r^k$$

where $c^k \in \mathbb{M}_{n,1}(B)$, $r^k \in \mathbb{M}_{1,n}(B)$ for $k = 1, \dots, n$ and

$$\sum_{k=1}^{p} \|c^k\| \, \|r^k\| < 1.$$

When the decomposition can be achieved with $p = 1$ we say that B has property (I'_n).

We say that B has property (I) if it has property (I_n) for every $n \geqslant 2$. Property (I') is defined similarly.

These conditions are related to the non-existence of tracial states on B. Their equivalence is an open problem. Let us give the following relations between these various notions of infiniteness.

PROPOSITION *Let us consider the following conditions:*

(i) *the unit of the multiplier algebra $M(B)$ of B is a properly infinite projection;*

(ii) *(respectively* (ii$'$)*) B has property* (I) *(respectively* (I')*);*

(iii) *(respectively* (iii$'$)*) B has property* (I_2) *(respectively* (I'_2)*);*

(iv) *B has no tracial states.*

Then:

(i) $\Rightarrow$ (ii$'$) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv)*;*

(ii$'$) $\Rightarrow$ (iii$'$)*;*

(iii$'$) $\Rightarrow$ (i) *when B is unital.*

Proof. Assume (i) and let $n \geqslant 2$. There exist n isometries $s_1, \dots, s_n$ in $M(B)$ such that $s_i^* s_j = 0$ for $i \neq j$. Consider $b = [b_{ij}] \in \mathbb{M}_n(B)$ with norm < 1. We write $b = \alpha\beta$ with $\alpha, \beta \in \mathbb{M}_n(B)$ of norm < 1 and we set for $i = 1, \dots, n$

$$c_i = \sum_{k=1}^{n} \alpha_{ik} s_k^* \in B, \quad r_i = \sum_{k=1}^{n} s_k \beta_{ki} \in B, \quad r = [r_1 \cdots r_n], \quad c = {}^{\mathrm{t}}[c_1 \cdots c_n].$$

Then we have $b = cr$ with

$$\|c\|^2 = \left\| \sum c_i^* c_i \right\| = \|[s_1 \cdots s_n] \alpha^* \alpha \, {}^{\mathrm{t}}[s_1^* \cdots s_n^*]\| < \left\| \sum s_i s_i^* \right\| \leqslant 1$$

and similarly $\|r\| < 1$. Therefore (i) $\Rightarrow$ (ii$'$).

The implications (ii$'$)$\Rightarrow$ (ii) $\Rightarrow$(iii) and (ii$'$)$\Rightarrow$(iii$'$) are obvious. Let us show that (iii)$\Rightarrow$(iv). Assume that B has a tracial state τ. Given $\varepsilon > 0$ we choose $b \in B$ such that $\|b\| < 1$ and $|\tau(b)| \geqslant 1/2$. If the condition (iii) is fulfilled, we have a decomposition

$$\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \sum_{k=1}^{p} c^k r^k$$

with $c^k = {}^{\mathrm{t}}[\,c_1^k \quad c_2^k\,]$, $r^k = [r_1^k \; r_2^k]$ and

$$\sum_{k=1}^{p} \|c^k\| \, \|r^k\| < 1.$$

Using the Cauchy-Schwarz inequality and the fact that τ is a tracial state, we get

$$\begin{aligned} 2|\tau(b)| &= \Big|\sum_k \tau(c_1^k r_1^k + c_2^k r_2^k)\Big| \\ &\leqslant \sum_k \tau(c_1^k c_1^{k*} + c_2^k c_2^{k*})^{1/2} \tau(r_1^{k*} r_1^k + r_2^{k*} r_2^k)^{1/2} \\ &= \sum_k \tau(c_1^{k*} c_1^k + c_2^{k*} c_2^k)^{1/2} \tau(r_1^k r_1^{k*} + r_2^k r_2^{k*})^{1/2} \\ &\leqslant \sum_k \|c^k\| \, \|r^k\| < 1, \end{aligned}$$

which is a contradiction.

For (iii′)⇒(i) when B is unital, let us assume that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} [r_1 \; r_2].$$

We construct two orthogonal self-adjoint idempotents in B, equivalent to 1, as follows (see Proposition 4.6.2 from [1]).

Denote by p_1 the idempotent r_1c_1 and put $z_1 = 1 + (p_1 - p_1^*)(p_1^* - p_1)$. Then z_1 commutes with p_1 and $q_1 = p_1 p_1^* z_1^{-1}$ is a self-adjoint idempotent such that $p_1 q_1 = q_1$ and $q_1 p_1 = p_1$.

Similarly, we define $p_2 = r_2 c_2$ and z_2, but we set $q_2 = p_2^* p_2 z_2^{-1}$. Now we have $p_2 q_2 = p_2$ and $q_2 p_2 = q_2$ and therefore

$$q_2 q_1 = q_2 p_2 p_1 q_1 = q_2 r_2 c_2 r_1 c_1 q_1 = 0.$$

Since q_1 and q_2 are both equivalent to 1, we obtain that 1 is properly infinite. ∎

THEOREM *Let B be a C^*-algebra. Then the following conditions are equivalent:*

(i) $\Lambda = \Gamma$ *on $V \otimes_B W$ for every representable A-B bimodule V and every representable B-C bimodule W;*

(ii) *B has property* (I).

Proof. We show first that (ii)⇒(i). Since by Proposition 1.2 every nondegenerate representation of B is locally cyclic when B has no tracial states, it follows from the definition of Λ and Theorem 5.4 that $\Lambda = \mathrm{H}'$. It remains to show that $\Gamma \leqslant \mathrm{H}'$.

Let $u \in V \otimes_B W$ with $\mathrm{H}'(u) < 1$. By Theorem 5.4, u may be written

$$u = \sum_{k,l} \alpha_k v_k \otimes_B b_{kl} w_l \gamma_l$$

where $\alpha = [\alpha_1 \cdots \alpha_n] \in \mathbb{M}_{1,n}(A)$, $b = [b_{ij}] \in \mathbb{M}_n(B)$, $\gamma = {}^{\mathrm{t}}[\gamma_1 \cdots \gamma_n] \in \mathbb{M}_{n,1}(C)$, $v_k \in V$ and $w_k \in W$ are of norm < 1. Decomposing b as in Definition 6.1 we get

$$u = \sum_{j=1}^{p} \Big(\sum_{k=1}^{n} \alpha_k v_k c_k^j \Big) \otimes_B \Big(\sum_{l=1}^{n} r_l^j w_l \gamma_l \Big)$$

and therefore

$$\begin{aligned}\Gamma(u) &\leqslant \sum_{j=1}^{p} \Big\| \sum_{k=1}^{n} \alpha_k v_k c_k^j \Big\| \, \Big\| \sum_{l=1}^{n} r_l^j w_l \gamma_l \Big\| \\ &\leqslant \sum_{j=1}^{p} \Big\| \sum_{k} \alpha_k \alpha_k^* \Big\|^{1/2} \max_k \|v_k\| \, \Big\| \sum_k c_k^{j*} c_k^j \Big\|^{1/2} \\ &\qquad \cdot \Big\| \sum_l r_l^j r_l^{j*} \Big\|^{1/2} \max_l \|w_l\| \, \Big\| \sum_l \gamma_l^* \gamma_l \Big\|^{1/2} \\ &\leqslant \sum_{j=1}^{p} \|c^j\| \, \|r^j\| < 1.\end{aligned}$$

Let us show now that (i)⇒(ii). Fix $n > 1$ and let V be the representable $\mathbb{C}$-B bimodule $\mathbb{M}_{n,1}(B)$. By definition we have

$${}^{\mathrm{t}}[b_1 \cdots b_n] b = {}^{\mathrm{t}}[b_1 b \cdots b_n b], \quad \|{}^{\mathrm{t}}[b_1 \cdots b_n]\| = \Big\| \sum b_i^* b_i \Big\|^{1/2}.$$

Similarly, let W the representable B-$\mathbb{C}$ bimodule $\mathbb{M}_{1,n}(B)$ with its natural structure. The map

$$({}^{\mathrm{t}}[b_1 \cdots b_n], [b'_1 \cdots b'_n]) \mapsto [b_i b'_j] \in \mathbb{M}_n(B)$$

defines a $\mathbb{C}$-linear isomorphism from $V \otimes_B W$ onto $\mathbb{M}_n(B)$, by which we identify the two spaces. On $\mathbb{M}_n(B)$ the norm Λ is the usual C^*-algebra norm (see Lemma 6.4 below). Assume that $\Lambda = \Gamma$ on $V \otimes_B W$ and consider $b \in \mathbb{M}_n(B)$ with $\|b\| < 1$. By definition of Γ we can write b as

$$b = \sum_{k=1}^{p} \lambda_k v_k \otimes_B w_k \mu_k$$

with $\sum |\lambda_k|^2 \sum |\mu_k|^2 < 1$ and $\|v_k\| \leqslant 1$, $\|w_k\| \leqslant 1$ for all k. To conclude, it suffices to set $c^k = \lambda_k v_k$ and $r^k = \mu_k w_k$, since $b = \sum c^k r^k$ and $\sum \|c^k\| \, \|r^k\| \leqslant \sum |\lambda_k| \, |\mu_k| < 1$. ∎

LEMMA *Let V be the representable $\mathbb{C}$-B bimodule $\mathbb{M}_{n,1}(B)$ and W be the representable B-$\mathbb{C}$ bimodule $\mathbb{M}_{1,n}(B)$. Then ${}_AV \otimes_B^{\Lambda} W_C$ is canonically isometric with $\mathbb{M}_n(B)$ with its C^*-algebra norm.*

Proof. By Lemma 3.1, we have, for $u \in V \otimes_B W$,

$$\Lambda(u) = \sup \|RS(u)\|,$$

the supremum being taken over the contractive representations of the form $R \in \mathrm{Hom}_B(V, \mathcal{B}(H_{\mathrm{s}}(B), \mathbb{C}))$ and $S \in \mathrm{Hom}_B(W, \mathcal{B}(\mathbb{C}, H_{\mathrm{s}}(B)))$, where $H_{\mathrm{s}}(B)$ is the standard representation of B.

For every $S \in \mathrm{Hom}_B(W, \mathcal{B}(\mathbb{C}, H_{\mathrm{s}}(B)))$ there exists a unique vector $(\xi_1, \dots, \xi_n) \in H_{\mathrm{s}}(B)^n$ such that

$$S([b_1 \cdots b_n]) = (b_1\xi_1, \dots, b_n\xi_n), \quad \forall [b_1 \cdots b_n] \in \mathbb{M}_{1,n}(B).$$

Using Lemma 1.16 we see that

$$\|S\| = \sup\Big\{\|S([b_1 \cdots b_n])\|;\ \Big\|\sum b_k b_k^*\Big\| \leqslant 1\Big\} = \Big(\sum \|\xi_k\|^2\Big)^{1/2}.$$

Therefore, $\mathrm{Hom}_B(W, H_{\mathrm{s}}(B))$ is canonically isometric to the Hilbert space $H_{\mathrm{s}}(B)^n$.

Similarly, the space $\mathrm{Hom}_B(V, H_{\mathrm{s}}(B)^*)$ is canonically isometric to $\overline{H_{\mathrm{s}}(B)}^n$, where $\overline{H_{\mathrm{s}}(B)}$ is the conjugate Hilbert space of $H_{\mathrm{s}}(B)$.

Let us compute now the Λ-seminorm of the matrix $[b_{ij}] \in \mathbb{M}_n(B)$, viewed as an element of $V \otimes_B W$. We have

$$\|[b_{ij}]\| = \sup \|RS([b_{ij}])\| = \sup\Big\{\Big|\sum_{i,j}\langle b_{ij}\xi_j, \mu_i\rangle\Big|;\ \sum \|\xi_i\|^2 \leqslant 1, \sum \|\mu_i\|^2 \leqslant 1\Big\}$$

from the above observation, and this concludes the proof. ∎

7. APPLICATION TO THE RELATIVE HAAGERUP TENSOR PRODUCT

In this last section we consider a C^*-algebra B, a right operator B-module V and a left operator B-module W. Then $V_1 = \mathcal{K} \otimes_{\min} V$ is a representable $\mathcal{K}$-$\mathcal{K} \otimes_{\min} B$ bimodule and $W_1 = \mathcal{K} \otimes_{\min} W$ is a representable $\mathcal{K} \otimes_{\min} B$-$\mathcal{K}$ bimodule. Note that $\mathcal{K} \otimes_{\min} B$ satisfies the condition (I) and therefore the results of the preceding section apply.

Theorem *The map $(k \otimes v, k' \otimes w) \mapsto kk' \otimes v \otimes_B w$ induces in a natural way an isometric $\mathcal{K}$-$\mathcal{K}$ bimodule map φ from $V_1 \otimes^{\Gamma}_{\mathcal{K}\otimes_{\min} B} W_1$ onto $\mathcal{K} \otimes_{\min} (V \otimes^{\mathrm{h}}_B W)$.*

Proof. Recall that V_1 and W_1 are defined as completions of $\mathbb{M}_\infty(V)$ and $\mathbb{M}_\infty(W)$ respectively. We first define

$$\varphi : \mathbb{M}_\infty(V) \otimes \mathbb{M}_\infty(W) \to \mathbb{M}_\infty(V \otimes^{\mathrm{h}}_B W)$$

by

$$\varphi([v_{ij}], [w_{ij}]) = \Big[\sum_k v_{ik} \otimes_B w_{kj}\Big].$$

Clearly we have

$$\varphi(avb, wc) = a\varphi(v, bw)c$$

for any $a, c \in \mathbb{M}_\infty(\mathbb{C}), b \in \mathbb{M}_\infty(B), v \in \mathbb{M}_\infty(V), w \in \mathbb{M}_\infty(W)$. Also

$$\|\varphi(v, w)\|_{\mathrm{h}} \leqslant \|v\|\,\|w\|, \quad \forall v \in \mathbb{M}_\infty(V), w \in \mathbb{M}_\infty(W).$$

In particular, φ extends to a bounded bilinear map

$$\varphi : V_1 \times W_1 \to \mathcal{K} \otimes_{\min} (V \otimes^{\mathrm{h}}_B W).$$

Let us show that φ is $\mathcal{K} \otimes_{\min} B$-balanced. It is sufficient to check that

$$\varphi(v(k \otimes b), w) = \varphi(v, (k \otimes b)w), \quad \forall k \in \mathcal{K}, v \in V_1, w \in W_1.$$

Consider $(v,w) \in V_1 \times W_1$ and choose an approximate unit p_n of $\mathcal{K}$ made of finite rank operators. Then

$$\varphi(v(k\otimes b),w) = \lim_n \varphi(v(kp_n\otimes b),w) = \lim_n \varphi(v,(kp_n\otimes b)w) = \varphi(v,(k\otimes b)w).$$

The same argument shows that φ is $\mathcal{K}$-$\mathcal{K}$-linear.

The norm on $\mathcal{K}\otimes_{\min}(V\otimes^{\mathrm{h}}_B W)$ induces, via the map φ, a reasonable R-seminorm on $V_1\otimes^{\Gamma}_{\mathcal{K}\otimes_{\min}B} W_1$. By the preceding section, the latter seminorm is unique, and the result follows. ∎

Corollary *The relative Haagerup tensor product is both injective and projective. In other words, given two right operator B-modules V and V', two left operator B-modules W and W', a completely isometric (respectively quotient) B-linear map $\varphi : V \to V'$ and a completely isometric (respectively quotient) B-linear map $\psi : W \to W'$, then $\varphi\otimes_B\psi : V\otimes_B W \to V'\otimes_B W'$ induces a completely isometric (respectively quotient) map from $V\otimes^{\mathrm{h}}_B W$ into (respectively onto) $V'\otimes^{\mathrm{h}}_B W'$.*

The injectivity of the relative Haagerup tensor product in some particular cases has been proved by Magajna in Theorem 2.2 from [12].

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