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# NON-SELF-ADJOINT HARMONIC OSCILLATOR, COMPACT SEMIGROUPS AND PSEUDOSPECTRA

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ABSTRACT. We provide new information concerning the pseudospectra of the complex harmonic oscillator. Our analysis illustrates two different techniques for getting resolvent norm estimates. The first uses the JWKB method and extends for this particular potential some results obtained recently by E.B. Davies. The second relies on the fact that the bounded holomorphic semigroup generated by the complex harmonic oscillator is of Hilbert-Schmidt type in a maximal angular region. In order to show this last property, we deduce a non-self-adjoint version of the classical Mehler's formula.

KEYWORDS: *Complex harmonic oscillator, non-self-adjoint, resolvent norm estimates, bounded holomorphic semigroups, pseudospectrum, JWKB method, Mehler's formula.*

MSC (2000): 34L40, 47A75, 47D06.

## 1. INTRODUCTION

The harmonic oscillator is known to be ubiquitous in many branches of quantum theory. Unlike most quantum mechanical problems, it can be solved explicitly in terms of elementary functions (see Section 2), so it provides an excellent illustration for the general principles of the quantum-theoretical formalism. It also plays a key role in quantum electrodynamics and quantum field theory, and has been a focus of great attention among the mathematical-physics community for many years.

Although it might seem very difficult to say something new about the harmonic oscillator, in a couple of recent papers ([5] and [6]) E.B. Davies provides a framework which opens the possibility of obtaining new information concerning the spectral theory of this and other Schrödinger operators. In the present paper we extend some results given in [5] and [6], and introduce some new ideas that will be useful in studying both spectral and stability properties of this operator when it has a complex coupling constant.

We define the complex harmonic oscillator to be the operator

$$H_c f(x) := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + cx^2 f(x)$$

acting on $L^2(\mathbb{R})$ with Dirichlet boundary conditions, where $c \in \mathbb{C}$ is such that $\mathrm{Re}(c) > 0$ and $\mathrm{Im}(c) > 0$. Our main interest is to investigate the resolvent norm of $H_c$ inside its numerical range. To this end, if we let the $\varepsilon$-pseudospectra of $H_c$ be

$$\mathrm{Spec}_\varepsilon(H_c) := \mathrm{Spec}(H_c) \cup \left\{ z \in \mathbb{C} : \|(H_c - z)^{-1}\| \geqslant \frac{1}{\varepsilon} \right\}$$

for all $\varepsilon > 0$, our interest is in obtaining information concerning the shape of these sets for small $\varepsilon$.

We adopt the notation and basic ideas about pseudospectra recently developed by E.B. Davies, L.N. Trefethen and others. For general results and more examples in the present spirit, we refer to [7], [5] and [12], and to the bibliography there.

The results we obtain here extend in two ways those given by E.B. Davies in [5] and [6] where he shows that for any positive constant $b$ independent of $\eta > 0$,

$$\|(H_c - \eta[b + c])^{-1}\| \to \infty$$

as $\eta \to \infty$. On the one hand in Section 3 (Theorems 3.3) we demonstrate that for all $b > 0$ and $1/3 < p < 3$ fixed,

$$\| (H_c - [b\eta + c\eta^p])^{-1} \| \to \infty$$

as $\eta \to \infty$. On the other hand in Section 5 we prove (as a consequence of Theorem 5.3) that for all $b > 0$ there exists a constant $M_b > 0$ such that

$$\lim_{\eta \to \infty} \|(H_c - [\eta + \mathrm{i}b])^{-1}\| \leqslant M_b$$

and

$$\lim_{\eta \to \infty} \|(H_c - [c\eta - \mathrm{i}b])^{-1}\| \leqslant M_b.$$

The study of pseudospectra provides important information about the stability of operator $H_c$. By virtue of the identity

$$\mathrm{Spec}_\varepsilon(H_c) = \bigcup \{\mathrm{Spec}(H_c + A) : \|A\| \leqslant \varepsilon\}$$

(for a proof see [11]), knowing the size of these sets for $\varepsilon$ close to zero, allow us to obtain very precise information about the stability under small perturbation of the eigenvalues of $H_c$ ([7]). It will turn out (Sections 3 and 5) that high energy eigenvalues are far more unstable than the first excited states (see also [1] and [5]).

The method we use in Section 3 is analogous to the techniques in [5] and [6], and involves the construction of a continuous family of approximate eigenstates for $H_c$ by means of JWKB analysis (see [9] for an introduction to this topic and some applications). We should mention that such a procedure is examined for more general potentials and from the numerical point of view in [2].

In Section 4 we deduce a non-self-adjoint version of the classical Mehler's formula, i.e. we construct an explicit formula for the heat kernel of $-H_c$ (Theorem 4.2). This formula allows us to show that the bounded holomorphic semigroup generated by $-H_c$ is compact in a maximal angular domain. The results of Section 4 provides a new approach to get resolvent norm estimates for $H_c$. Based on these estimates, in Section 5 we obtain a set that encloses $\mathrm{Spec}_\varepsilon(H_c)$ for small enough $\varepsilon$ (Corollary 5.2 and Theorem 5.3). This confirms the numerical evidence given in [5] about the shape of such sets.

## 2. DEFINITIONS AND NOTATION

We will suppose in this paper that the parameter $c \in \mathbb{C}$ satisfies $\mathrm{Re}(c) > 0$ and unless explicitly stated we do not impose conditions under $\mathrm{Im}(c)$.

We assume that $H_c$ acts in $L^2(\mathbb{R})$ with Dirichlet boundary conditions as follows. Take the closed $m$-sectorial quadratic form

$$Q_c(f,g) := \int_{\mathbb{R}} f'(x)\overline{g'(x)}\,\mathrm{d}x + c\int_{\mathbb{R}} x^2 f(x)\overline{g(x)}\,\mathrm{d}x$$

for all $f, g \in \mathrm{W}^{1,2}(\mathbb{R}) \cap \{f \in L^2(\mathbb{R}) : \int x^2|f(x)|^2\,\mathrm{d}x < \infty\}$. Then $H_c$ is defined to be the $m$-sectorial operator associated to $Q_c$ via the Friedrichs representation theorem (see [10]).

The subspaces $C_\mathrm{c}^\infty(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ (the Schwartz space) are form cores for $H_c$. Since

$$C_\mathrm{c}^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}) \subset \mathrm{W}^{1,2}(\mathbb{R}) \cap \left\{ f \in L^2(\mathbb{R}) : \int x^2|f(x)|^2\,\mathrm{d}x < \infty \right\},$$

it is enough to check the desired property for $C_\mathrm{c}^\infty(\mathbb{R})$. By Theorem VI-1.21 from [10] a subspace is a core for an $m$-sectorial form, if and only if it is a core for its real part. Notice that the real part of $Q_c$ is the non-negative quadratic form $\mathrm{Re}(Q_c) = Q_{\mathrm{Re}(c)}$, thus using for example Theorem 1.13 from [3] we can deduce that $C_\mathrm{c}^\infty(\mathbb{R})$ is a core for $\mathrm{Re}(Q_c)$ and therefore that it is also a core for $Q_c$ as needed.

If $c$ is positive, $H_c$ is self-adjoint, but if $\mathrm{Im}(c) \neq 0$, $H_c$ is not even a normal operator. Moreover, (see [5]) there does not exists an invertible operator $U$ such that $UH_cU^{-1}$ is normal.

If we put $\lambda_n := c^{1/2}(2n+1)$, the spectrum of $H_c$ is

$$\mathrm{Spec}(H_c) = \{\lambda_n : n = 0, 1, \ldots\}.$$

It consists entirely of eigenvalues of multiplicity one and it is routine to check that if $H_n$ is the $n^{\mathrm{th}}$-Hermite polynomial, then

$$\Psi_n(x) := c^{1/8} H_n(c^{1/4}x)\mathrm{e}^{-(c^{1/2}x^2)/2}$$

are the eigenfunctions of $H_c$, so that

$$H_c\Psi_n = \lambda_n\Psi_n, \quad n = 0, 1, \ldots.$$

For $0 \leqslant \alpha,\ \beta \leqslant \pi/2$, we shall denote the angular sector

$$S(-\alpha,\beta) := \{z \in \mathbb{C} : -\alpha < \arg(z) < \beta\}.$$

We will also put $S(\alpha) := S(-\alpha,\alpha)$. It is clear from the definition that the numerical range of the operator $H_c$ is contained in the angular sector $S(0,\arg(c))$.

PROPOSITION *For all* $\operatorname{Re}(c) > 0$,

$$\operatorname{Num}(H_c) = \Big\{t_1 + ct_2 \in \mathbb{C} \,:\, t_1,\, t_2 \geqslant 0,\, t_1t_2 \geqslant \frac{1}{4}\Big\}.$$

*If* $\operatorname{Im}(c) \neq 0$, *then*

$$\operatorname{Spec}(H_c) \subset \operatorname{Int}(\operatorname{Num}(H_c)).$$

*Proof.* Since $\mathcal{S}(\mathbb{R})$ is a core for $Q_c$,

$$\overline{\operatorname{Num}(H_c)} = \overline{\{Q_c(f) : \|f\| = 1,\, f \in \mathcal{S}(\mathbb{R})\}}.$$

By the Heisenberg inequality, we know that for any $f \in \mathcal{S}(\mathbb{R})$

$$\Big(\int_{\mathbb{R}} |f'(x)|^2 \,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}} x^2|f(x)|^2 \,\mathrm{d}x\Big) \geqslant \frac{1}{4}\|f\|^4.$$

Therefore clearly

$$\operatorname{Num}(H_c) \subset W := \Big\{t_1 + ct_2 \in \mathbb{C} : t_1, t_2 > 0,\, t_1t_2 \geqslant \frac{1}{4}\Big\}.$$

Let us check the reverse inclusion. For this we need to find test functions $f \in \mathcal{S}(\mathbb{R})$ such that

$$Q_c(f) \in \partial(W) = \Big\{t + \frac{c}{4t} \in \mathbb{C} \,:\, t > 0\Big\}.$$

For $t > 0$, let $f_t(x) := \mathrm{e}^{-tx^2} \in \mathcal{S}(\mathbb{R})$. Using elementary properties of the Gamma function we can calculate

$$\int_{\mathbb{R}} |f_t'(x)|^2 \,\mathrm{d}x = 2t^2\pi^{1/2}(2t)^{-3/2},$$

$$\int_{\mathbb{R}} x^2|f_t(x)|^2 \,\mathrm{d}x = \frac{\pi^{1/2}}{2}(2t)^{-3/2},$$

$$\|f_t\|^2 = \pi^{1/2}(2t)^{-1/2}.$$

Combining this three equalities, we obtain for all $t > 0$

$$\Big(\int_{\mathbb{R}} |f_t'(x)|^2 \,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}} x^2|f_t(x)|^2 \,\mathrm{d}x\Big) = \frac{1}{4}\|f_t\|^4.$$

Therefore

$$Q_c\Big(\frac{f_t}{\|f_t\|}\Big) = t + \frac{c}{4t}, \quad t > 0,$$

and so every point in $\partial(W)$ is in $\operatorname{Num}(H_c)$. Since both sets are convex we have *a fortiori*

$$W \subset \operatorname{Num}(H_c).$$

For the second part, suppose that $\operatorname{Im}(c) \neq 0$. Since $\operatorname{Spec}(H_c) \subset \operatorname{Num}(H_c)$, it is enough to show that $\partial(W) \cap \operatorname{Spec}(H_c) = \emptyset$. The eigenvalues of $H_c$ lie on the line $c^{1/2}r$, $r > 0$. The boundary $\partial(W)$ only intersect this line at the point

$$z_0 := \frac{|c|^{1/2}}{2}\Big(1 + \frac{c}{|c|}\Big).$$

Clearly $|z_0| < |c^{1/2}|$, therefore the desired property follows from the fact that $z_0$ is never an eigenvalue of $H_c$. ∎

From the basic theory (see for instance [7] and [12]), we know that $\mathrm{Spec}_\varepsilon(H_c)$ contains the $\varepsilon$-neighbourhood of $\mathrm{Spec}(H_c)$ and it is contained in the $\varepsilon$-neighbourhood of $\mathrm{Num}(H_c)$. We will show later (Corollary 5.2) that if $\mathrm{Im}(c) > 0$, there exists an $E > 0$ depending on $c$ such that

$$\mathrm{Spec}_\varepsilon(H_c) \subset \mathrm{Num}(H_c)$$

for all $\varepsilon < E$. Observe that if $c$ is real, since $H_c$ is self-adjoint, $\mathrm{Spec}_\varepsilon(H_c)$ is actually equal to the $\varepsilon$-neighbourhood of $\mathrm{Spec}(H_c)$, we stress that this property is false in general for non-self-adjoint operators.

PROPOSITION *For* $\mathrm{Im}(c) > 0$ *fixed, the resolvent norm of* $H_c$ *is symmetric with respect to the axis* $c^{1/2}r$, $r \in \mathbb{R}$. *As a consequence for all* $\varepsilon > 0$ *the* $\varepsilon$-*pseudospectrum of* $H_c$ *is also symmetric with respect to this axis.*

*Proof.* For all $a > 0$ let

$$T_a f(x) := a^{1/2} f(ax), \quad x \in \mathbb{R},$$

the operator $T_a$ is isometric in $L^2(\mathbb{R})$ and such that $T_a^{-1} = T_{a^{-1}}$. Putting $\arg(c) := \vartheta$, since

$$T_{|c|^{1/4}} H_{\mathrm{e}^{\mathrm{i}\vartheta}} T_{|c|^{-1/4}} = |c|^{-1/2} H_c,$$

it becomes evident that we can assume without lost of generality $c = \mathrm{e}^{\mathrm{i}\vartheta}$ for $0 < \vartheta < \pi/2$.

For simplicity we rewrite the operator

$$H_c = P^2 + cQ^2$$

where $Pf(x) := \mathrm{i}f'(x)$ and $Qf(x) = xf(x)$ (respectively the quantum mechanical observables of momentum and position). Thus, applying Fourier transform we obtain

$$\begin{aligned} \|(P^2 + cQ^2 - z)^{-1}\| &= \|(Q^2 + cP^2 - z)^{-1}\| = \|(P^2 + \mathrm{e}^{-\mathrm{i}\vartheta}Q^2 - \mathrm{e}^{-\mathrm{i}\vartheta}z)^{-1}\| \\ &= \|[(P^2 + \mathrm{e}^{-\mathrm{i}\vartheta}Q^2 - \mathrm{e}^{-i\vartheta}z)^{-1}]^*\| = \|(P^2 + cQ^2 - \mathrm{e}^{\mathrm{i}\vartheta}\overline{z})^{-1}\| \end{aligned}$$

for all $z \notin \mathrm{Spec}(H_c)$, which is precisely our claim. ∎

## 3. HIGH ENERGY EIGENVALUES

In this section we show that if the coupling constant $c$ is such that $\mathrm{Im}(c) > 0$, and $z_\eta \in \mathrm{Num}(H_c)$ parameterized by $\eta > 0$ is such that

$$|z_\eta - (b\eta + c\eta^p)| \to 0, \quad \text{as } \eta \to \infty$$

for some $b > 0$ and $1/3 < p < 3$, then

$$\lim_{\eta\to\infty} \|(H_c - z_\eta)^{-1}\| = \infty.$$

Such a result will be a consequence of Theorem 3.3. Notice that by Proposition 2.2 it is enough to assume that $1/3 < p \leqslant 1$.

Our first aim is to obtain test functions $f_\eta \in C_c^\infty(\mathbb{R})$, parameterized by $\eta > 0$, such that if

$$z_\eta = \mathrm{i}c\eta^{1/2-\gamma/2} + \alpha^2\eta^\gamma + c\alpha^2\eta,$$

where $\alpha > 0$ and $1 \leqslant \gamma < 3$ are fixed constants independent of $\eta > 0$,

$$\lim_{\eta\to\infty} \frac{\|H_c f_\eta - z_\eta f_\eta\|}{\|f_\eta\|} = 0. \tag{3.1}$$

We follow a similar procedure as the one given originally in [5]. Let

$$\Phi(x) := \mathrm{e}^{-\Psi(x)}, \quad x \in \mathbb{R},$$

where the polynomial

$$\Psi(x_0 + s) := \Psi_1 s + \frac{\Psi_2 s^2}{2} + \frac{\Psi_3 s^3}{3}, \quad s \in \mathbb{R},$$

centered in $x_0 := \alpha\eta^{1/2}$, has coefficients

$$\begin{aligned}
\Psi_1 &:= \mathrm{i}\alpha\eta^{\gamma/2},\\
\Psi_2 &:= -\mathrm{i}c\eta^{1/2-\gamma/2},\\
\Psi_3 &:= -\frac{\mathrm{i}c}{2\alpha}\eta^{-\gamma/2}(1 + c\eta^{1-\gamma}).
\end{aligned}$$

Clearly

$$H_c\Phi(x_0 + s) = (p(s) + z_\eta)\Phi(x_0 + s)$$

where

$$p(s) = c_1 s + c_2 s^2 + c_3 s^3 + c_4 s^4, \quad s \in \mathbb{R} \tag{3.2}$$

has coefficients

$$\begin{aligned}
c_1 &= -\frac{\mathrm{i}c}{\alpha}\eta^{-\gamma/2}(1 + c\eta^{1-\gamma}),\\
c_2 &= 0,\\
c_3 &= \frac{c^2}{\alpha}\eta^{1/2-\gamma}(1 + c\eta^{1-\gamma}),\\
c_4 &= \frac{c^2}{4\alpha^2}\eta^{-\gamma}(1 + 2c\eta^{1-\gamma} + c^2\eta^{2-2\gamma}).
\end{aligned}$$

Let us establish some properties of the function $\Phi$ above defined. A straightforward calculation implies that

$$|\Phi(x_0+s)|^2 = \exp\left[-\beta_2(\eta)s^2 - \beta_3(\eta)s^3\right],$$

for

$$\beta_2(\eta) := \mathrm{Im}(c)\eta^{1/2-\gamma/2} > 0,$$
$$\beta_3(\eta) := \frac{\mathrm{Im}(c)(1+2\mathrm{Re}(c)\eta^{1-\gamma})}{3\alpha}\eta^{-\gamma/2} > 0.$$

Taking derivatives of $|\Phi(x_0+s)|^2$ with respect to $s$, allows us to conclude that this function has a local maximum at $s=0$, and a local minimum at

$$s = s_0 := -\frac{2\beta_2(\eta)}{3\beta_3(\eta)} = -\frac{2\alpha\eta^{1/2}}{1+2\mathrm{Re}(c)\eta^{1-\gamma}} \to -\infty$$

as $\eta \to \infty$.

The required $f_\eta$ can be defined truncating the function $\Phi$ as follows. It is routine to define a $C^\infty(\mathbb{R})$ compact support function

$$g(x) = \begin{cases} 1 & \text{if } |x-x_0| < \eta^{\delta_0}, \\ 0 & \text{if } |x-x_0| > 2\eta^{\delta_0}; \end{cases}$$

for $\delta_0 := \gamma/6$, such that there exist constants $q_1 > 0$ and $q_2 > 0$ independent of $x$, $\eta$, $\alpha$ or $\gamma$, with the property

$$|g'(x)| \leqslant q_1\eta^{-\delta_0},$$
$$|g''(x)| \leqslant q_2\eta^{-2\delta_0},$$

for any $x \in \mathbb{R}$. Then,

$$f_\eta(x) := g(x)\Phi(x), \quad x \in \mathbb{R}.$$

The next two lemmas point out some properties of $f_\eta$ which can be employed to demonstrate (3.1). The constants $a_k$ for $k = 1, 2, \ldots$ below are real and independent of $\eta$, but can possibly depend on $\alpha$ or $\gamma$.

LEMMA *For $f_\eta$ as above and $1 \leqslant \gamma < 3$, there exist positive constants $a_1, a_2$ and $E_\gamma$, independent of $\eta$, such that*

$$a_1\eta^{(\gamma-1)/4} \leqslant \|f_\eta\|^2 \leqslant a_2\eta^{(\gamma-1)/4}$$

*for all $\eta > E_\gamma$.*

*Proof.* Since $\delta_0 < 1/2$,

$$\lim_{\eta\to\infty} \frac{2\beta_3(\eta)\eta^{3\delta_0}t^3}{\beta_2(\eta)\eta^{2\delta_0}t^2} = \lim_{\eta\to\infty} \frac{2(1+2\mathrm{Re}(c)\eta^{1-\gamma})\eta^{\delta_0}t}{3\alpha\eta^{1/2}} = 0,$$

uniformly for $0 \leqslant t \leqslant 2$. Therefore, there exists $E_\gamma > 0$ such that for any $\eta > E_\gamma$

$$0 \leqslant \beta_3(\eta)\eta^{3\delta_0}t^3 \leqslant \frac{\beta_2(\eta)\eta^{2\delta_0}t^2}{2}, \tag{3.3}$$

for any $0 \leqslant t \leqslant 2$.

For the first inequality: If $\eta > E_\gamma$, we have

$$\|f_\eta\|^2 = \int_{\mathbb{R}} |g(x_0+s)|^2 |\Phi(x_0+s)|^2 \,\mathrm{d}s \geqslant \int_0^{\eta^{\delta_0}} \exp[-\beta_2(\eta)s^2 - \beta_3(\eta)s^3]\,\mathrm{d}s$$

$$\geqslant \int_0^1 \exp[-\beta_2(\eta)\eta^{2\delta_0}t^2 - \beta_3(\eta)\eta^{3\delta_0}t^3]\eta^{\delta_0}\,\mathrm{d}t \geqslant \int_0^1 \exp[-2\beta_2(\eta)\eta^{2\delta_0}t^2]\eta^{\delta_0}\,\mathrm{d}t$$

$$\geqslant \int_0^{\eta^{\delta_0+(1-\gamma)/4}} \exp[-2\mathrm{Im}(c)u^2]\eta^{-(1-\gamma)/4}\,\mathrm{d}u \geqslant a_1\eta^{-(1-\gamma)/4}.$$

Observe that we are using the fact that $\delta_0 + (1-\gamma)/4 = 1/4 - \gamma/12 > 0$. The second inequality is similar. ∎

We now estimate the numerator in the left hand side of (3.1). By (3.2) we have

$$\|H_c f_\eta - z_\eta f_\eta\| \leqslant \|2g'(x_0+s)\Phi'(x_0+s)\| + \|g''(x_0+s)\Phi(x_0+s)\| + \sum_{k=1}^{4} \|c_k s^k \Phi(x_0+s) g(x_0+s)\|. \tag{3.4}$$

LEMMA *Let $E_\gamma > 0$ be as in Lemma* 3.1. *There exist positive constants $a_4, a_5$ and $a_6$, independent of $\eta > 0$, such that*

$$\|2g'(x_0+s)\Phi'(x_0+s)\|^2 \leqslant a_4\eta^{5\gamma/6}\exp\Big[-\frac{\mathrm{Im}(c)}{2}\eta^{(3-\gamma)/6}\Big]$$
$$\|g''(x_0+s)\Phi(x_0+s)\|^2 \leqslant a_5\eta^{-\gamma/2}\exp\Big[-\frac{\mathrm{Im}(c)}{2}\eta^{(3-\gamma)/6}\Big]$$
$$\|s^k\Phi(x_0+s)g(x_0+s)\|^2 \leqslant a_6\eta^{(\gamma-1)(2k+1)/4}$$

*for all $\eta > E_\gamma$ and $k = 1, 3$ or 4.*

*Proof.* We use similar arguments as the one provided in the proof of Lemma 3.1. Let $\Omega := \{s \in \mathbb{R} : \eta^{\delta_0} \leqslant |s| \leqslant 2\eta^{\delta_0}\}$. For all $s \in \Omega$, we have

$$|\Psi'(x_0+s)|^2 = |\Psi_1 + \Psi_2 s + \Psi_3 s^2|^2$$
$$\leqslant \eta^\gamma(\alpha + 2|c|\eta^{\delta_0+1/2-\gamma} + \frac{2|c|}{\alpha}\eta^{2\delta_0-\gamma}(1+|c|\eta^{1-\gamma}))^2 \leqslant \eta^\gamma a_3.$$

Then for all $\eta > E_\gamma$, by the conditions imposed on $g(x)$ above and by (3.3), we have

$$\begin{aligned}
\|2g'(x_0+s)\Phi'(x_0+s)\|^2 &= 4\int_{\mathbb{R}} |g'(x_0+s)|^2|\Psi'(x_0+s)|^2|\Phi(x_0+s)|^2\,\mathrm{d}s \\
&\leqslant \frac{a_4}{2}\eta^{\gamma-2\delta_0}\int_{\Omega}\exp[-\beta_2(\eta)s^2-\beta_3(\eta)s^3]\,\mathrm{d}s \\
&\leqslant a_4\eta^{\gamma-2\delta_0}\int_{\eta^{\delta_0}}^{2\eta^{\delta_0}}\exp[-\beta_2(\eta)s^2+\beta_3(\eta)s^3]\,\mathrm{d}s \\
&\leqslant a_4\eta^{\gamma-\delta_0}\int_1^2\exp\Big[-\frac{\beta_2(\eta)}{2}\eta^{2\delta_0}t^2\Big]\,\mathrm{d}t \\
&\leqslant a_4\eta^{5\gamma/6}\exp\Big[-\frac{\mathrm{Im}(c)}{2}\eta^{(3-\gamma)/6}\Big].
\end{aligned}$$

The second estimate is similar so that we can find $a_5$ without difficulty.

Finally for $\eta > E_\gamma$ and $k = 1, 3$ or $4$, using again (3.3),

$$\begin{aligned}
\|s^k\Phi(x_0+s)g(x_0+s)\|^2 &\leqslant \int_{-2\eta^{\delta_0}}^{2\eta^{\delta_0}} s^{2k}|\Phi(x_0+s)|^2\,\mathrm{d}s \\
&\leqslant 2\int_0^{2\eta^{\delta_0}} s^{2k}\exp[-\beta_2(\eta)s^2+\beta_3(\eta)s^3]\,\mathrm{d}s \\
&\leqslant 2\eta^{(2k+1)\delta_0}\int_0^2 t^{2k}\exp\Big[-\frac{\beta_2(\eta)}{2}\eta^{2\delta_0}t^2\Big]\,\mathrm{d}t \\
&\leqslant 2\eta^{(\gamma-1)(2k+1)/4}\int_0^\infty u^{2k}\exp\Big[-\frac{\mathrm{Im}(c)}{2}u^2\Big]\,\mathrm{d}u \\
&\leqslant a_6\eta^{(\gamma-1)(2k+1)/4}. \quad \blacksquare
\end{aligned}$$

Notice that (3.1) can be easily obtained from Lemma 3.1, Lemma 3.2 and equation (3.4). Using such result, we can achieve the following theorem.

THEOREM *Let* $H_c$ *be the complex harmonic oscillator such that* $\mathrm{Re}(c) > 0$ *and* $\mathrm{Im}(c) > 0$. *If*

$$z_\eta = b\eta + c\eta^p$$

*where* $b > 0$ *and* $1/3 < p < 3$ *are constants independent of* $\eta > 0$, *then*

$$\lim_{\eta\to\infty}\|(H_c - z_\eta)^{-1}\| = \infty.$$

*Proof.* Assume $1/3 < p \leqslant 1$. If we put the unitary operator $T_a$ for $a > 0$ as in the proof of Proposition 2.2, recall that for $\arg(c) := \vartheta$

$$T_{|c|^{1/4}} H_{\mathrm{e}^{\mathrm{i}\vartheta}} T_{|c|^{-1/4}} = |c|^{-1/2} H_c.$$

For all $r > 0$ and $\beta > 0$ let $w_{\eta,r} := \beta\eta + \beta r \mathrm{e}^{\mathrm{i}\vartheta}\eta^p$. Since

$$\|(H_{\mathrm{e}^{\mathrm{i}\vartheta}} - w_{\eta,|c|})^{-1}\| = \|(|c|^{-1/2} H_c - w_{\eta,|c|})^{-1}\| = |c|^{1/2} \|(H_c - |c|^{1/2} w_{\eta,|c|})^{-1}\|,$$

using (3.1) putting $\alpha^2 = |c|^{1/2}\beta$ and thinking of $|c| = r$, it is clear that

$$\lim_{\eta\to\infty} \|(H_{\mathrm{e}^{\mathrm{i}\vartheta}} - w_{\eta,r})^{-1}\| = \infty$$

for all $r > 0$ and $\beta > 0$.

Now if we put $\beta = |c|^{1/2} b$ and $r = |c|/b$, we obtain

$$\begin{aligned} \|(H_c - z_\eta)^{-1}\| &= \|(|c|^{1/2} H_{\mathrm{e}^{\mathrm{i}\vartheta}} - z_\eta)^{-1}\| \\ &= |c|^{-1/2} \|(H_{\mathrm{e}^{\mathrm{i}\vartheta}} - w_{\eta,|c|/b})^{-1}\| \to \infty \end{aligned}$$

as $\eta \to \infty$. The proof can be now completed by Proposition 2.2. ∎

This theorem has some consequences for the $\varepsilon$-pseudospectra of $H_c$. It implies that for $\varepsilon > 0$ fixed, any curve $z_\eta \in \mathrm{Num}(H_c)$ parameterized by $\eta > 0$ such that

$$|z_\eta - (b\eta + c\eta^p)| \to 0, \quad \text{as } \eta \to \infty$$

for some $b > 0$ and $1/3 < p < 3$, will eventually be inside $\mathrm{Spec}_\varepsilon(H_c)$ and it will stay there as $\eta \to \infty$. In particular this shows that high energy eigenvalues are increasingly unstable under small perturbations.

## 4. NON-SELF-ADJOINT MEHLER'S FORMULA

In this section we show that the bounded holomorphic semigroup of contractions generated by $-H_c$ is compact in a maximal angular sector. In order to deduce this property for the semigroup, we obtain an explicit formula for the heat kernel of $H_c$ as in [8] and show directly that this kernel is of Hilbert-Schmidt type.

We say that for $\alpha, \beta > 0$, a parameterized family of bounded operators $T_\tau$ in the Banach space $\mathcal{B}$, is a *bounded holomorphic semigroup of contractions* in the sector $S(-\alpha, \beta)$, if and only if:

(i) $T_{\tau_1+\tau_2} = T_{\tau_1} T_{\tau_2}$ for all $\tau_k \in S(-\alpha, \beta)$;
(ii) $\|T_\tau\| \leqslant 1$ for all $\tau \in S(-\alpha, \beta)$;
(iii) $T_\tau$ is a holomorphic family of operators in $\tau \in S(-\alpha, \beta)$;
(iv) for all $f \in \mathcal{B}$ and $\varepsilon > 0$,

$$\lim_{\tau\to 0} T_\tau f = f$$

for $\tau$ inside $S(-\alpha + \varepsilon, \beta - \varepsilon)$.

It follows directly from the definition that, for $-\alpha < \vartheta < \beta$ fixed, $T_{\mathrm{e}^{\mathrm{i}\vartheta} t}$ for $t > 0$ is a $C_0$ one-parameter semigroup in the standard sense ([4]). The *generator* of $T_\tau$ is, by definition, the infinitesimal generator of the one-parameter semigroup $T_t$ where $t > 0$.

For convenience, we shall put

$$S_c := \begin{cases} \overline{S(-\frac{\pi}{2}, \frac{\pi}{2} - \arg(c))} \setminus \{0\} & \text{if } \operatorname{Im}(c) > 0, \\ S(-\frac{\pi}{2}, \frac{\pi}{2}) & \text{if } \operatorname{Im}(c) = 0, \\ \overline{S(-\frac{\pi}{2} - \arg(c), \frac{\pi}{2})} \setminus \{0\} & \text{if } \operatorname{Im}(c) < 0. \end{cases}$$

THEOREM *Let $H_c$ be the complex harmonic oscillator for* $\operatorname{Re}(c) > 0$ *as defined above, then:*

(i) *for c fixed, $-H_c$ is the generator of a bounded holomorphic semigroup of contractions* $\mathrm{e}^{-H_c\tau}$*, with parameter $\tau$ in the open sector* $\operatorname{Int}(S_c)$*;*

(ii) *if* $\operatorname{Im}(c) > 0$*, then* $\mathrm{i}H_c$ *and* $-\mathrm{e}^{\mathrm{i}(\pi/2-\arg(c))}H_c$ *are also generators of a one-parameter semigroup such that* $\mathrm{e}^{-H_c\tau}$ *is strong continuous for all $\tau \in S_c$;*

(iii) *if* $\operatorname{Im}(c) < 0$*, then* $-\mathrm{i}H_c$ *and* $-\mathrm{e}^{-\mathrm{i}(\pi/2+\arg(c))}H_c$ *are also generators of a one-parameter semigroup such that* $\mathrm{e}^{-H_c\tau}$ *is strongly continuous for all $\tau \in S_c$;*

(iv) *for $\tau > 0$ fixed,* $\mathrm{e}^{-H_c\tau}$ *is a holomorphic family of bounded operators parameterized by c for all* $\operatorname{Re}(c) > 0$.

*Proof.* Properties (i) is deduced without difficulty from Theorem IX-1.24 of [10] and Theorem 2.24 of [4].

Property (ii) and (iii): if $\operatorname{Im}(c) > 0$, operator $\mathrm{i}H_c$ is maximal dissipative operators, therefore is the generator of a one-parameter semigroup (the same argument works for the other three cases). From analyticity, follows strong continuity for all $\tau \in \operatorname{Int}(S_c)$ and strong continuity in the edges can be checked using Corollary 3.18 from [4].

Observe that if $\operatorname{Im}(c) = 0$, $\operatorname{Spec}(\pm \mathrm{i}H_c)$ is purely imaginary, so we cannot apply the above argument.

Property (iv): by the way in which we define its domain, the operator $H_c$ is a holomorphic family of type (B) for $\operatorname{Re}(c) > 0$. Using the fact that holomorphic families of this type are locally $m$-sectorial, this property can be demonstrated by analogy to Theorem 2.6 from [10]. ∎

For $\tau \in S_c$ let the coefficients:

$$\begin{aligned} \lambda &:= \exp[-2c^{1/2}\tau], \\ w_1 &:= c^{1/4}\lambda^{1/2}[\pi(1-\lambda^2)]^{-1/2}, \\ w_2 &:= \frac{c^{1/2}(1+\lambda^2)}{2(1-\lambda^2)}, \\ w_3 &:= \frac{2c^{1/2}\lambda}{(1-\lambda^2)}. \end{aligned}$$

For all $x, y \in \mathbb{R}$, put

$$K_c(\tau, x, y) := w_1 \exp\left[w_3 xy - w_2(x^2 + y^2)\right],$$

and define the integral operator

$$A_{c,\tau} f(x) := \int_{\mathbb{R}} K_c(\tau, x, y) f(y)\, \mathrm{d}y, \tag{4.1}$$

for all $f \in L^2(\mathbb{R})$ for which this formula makes sense.

Fixing $\tau > 0$, $K_c(\tau, x, y)$ is holomorphic for $\mathrm{Re}(c) > 0$, and fixing $c$ it is continuous for $\tau \in S_c$ and analytic in its interior. Observe that for $c$ real, $K_c(\tau, x, y)$ has poles if $\tau$ runs along the complex axis; this situation is avoided by the definition we have made of the angular region $S_c$.

As pointed out in [8], modifications of the classical argument (see [3] and [4]) show that

$$\mathrm{e}^{-H_c\tau} = A_{c,\tau} \tag{4.2}$$

for all $c > 0$ and $\tau > 0$. Using analytic continuation this equality can be extended to complex $c$ and $\tau$.

THEOREM (Non-self-adjoint Mehler's Formula) *Let $K_c(\tau, x, y)$ be defined as above. Then for all $\mathrm{Re}(c) > 0$ and $\tau \in S_c$,*

$$\mathrm{e}^{-H_c\tau} f(x) = \int_{\mathbb{R}} K_c(\tau, x, y) f(y)\,\mathrm{d}y, \quad f \in L^2(\mathbb{R}).$$

The rest of this section is devoted to proving this theorem. First we establish some local bounds for $|K_c(\tau, x, y)|$, this allows us to show that $A_{c,\tau}$ is a holomorphic family of Hilbert-Schmidt operators in both parameters and then we obtain the desired equality by analytic continuation.

LEMMA *For all $\tau_0 \in S_c$, there exists a neighbourhood $\tau_0 \in V_0 \subset \mathbb{C}$ and real constants $\alpha_1, \alpha_2$ and $\alpha_3$ such that for all $\tau \in V_0 \cap S_c$:*

$$|K_c(\tau, x, y)| \leqslant \alpha_1 \exp\left[\alpha_3 xy - \alpha_2(x^2 + y^2)\right], \quad \textit{all } x, y \in \mathbb{R}.$$

*The constants satisfy: $\alpha_1, \alpha_2 > 0$ and $2\alpha_2 \pm \alpha_3 > 0$, locally uniformly in $c$ and $\tau$.*

*Proof.* Clearly

$$|K_c(\tau, x, y)| = |w_1| \exp[\mathrm{Re}(w_3)xy - \mathrm{Re}(w_2)(x^2 + y^2)].$$

Since $w_i$ are continuous in $\tau$ and $c$, it is enough to show that $\mathrm{Re}(w_2) > 0$ and $\mathrm{Re}(2w_2 \pm w_3) > 0$, for all $\mathrm{Re}(c) > 0$ and $\tau \in S_c$.

Let $\vartheta_c := \arg(c^{1/2})$ and $\vartheta_\tau := \arg(\tau)$. Without loss of generality we will assume that $\mathrm{Im}(c) > 0$ and $|c^{1/2}| = 2$. Define the Moebius transforms by

$$M_\pm(z) := \frac{1 \pm z}{1 \mp z}, \quad z \in \mathbb{C}.$$

Then

$$w_2 = \frac{c^{1/2}}{2} M_+(\lambda^2) = \mathrm{e}^{\mathrm{i}\vartheta_c} M_+(\lambda^2)$$

and

$$2w_2 \pm w_3 = c^{1/2} \frac{1 \pm \lambda}{1 \mp \lambda} = 2\mathrm{e}^{\mathrm{i}\vartheta_c} M_\pm(\lambda).$$

Notice that $M_+$ maps the interior of the unitary disk into the open right half plane, and multiplying by $\mathrm{e}^{\mathrm{i}\vartheta_c}$ rotates about the origin by $\vartheta_c$. Every disk centered in the origin, of radius $r_1 < 1$, changes under $M_+$ into a disk $D_2$ of radius

$$r_2 := \frac{2r_1}{1 - r_1^2}$$

centered at

$$c_2 := \frac{1+r_1^2}{1-r_1^2} > 0.$$

If $r_1$ is close to 0, $r_2$ is small, $c_2$ is close to 1 and the rotation of $D_2$ about the origin by $\vartheta_c$ remains in the right half plane. If $r_1$ is close to 1, $r_2$ is big, and rotating by $\vartheta_c$ can send some points of $D_2$ to the left half plane. In spite of this possibility, notice that points in $D_2$ which are to the right of $c_2$ do not cross to the left half plane when rotated. It is elementary to find the maximum radius $r_1$ which allows $e^{i\vartheta_c} D_2$ stay in the right half plane. We call, the disk with center the origin having this radius, the *critical disk*.

Now if we consider $\vartheta_c$ and $\vartheta_\tau$ fixed, $\lambda^2$ describes a spiral with radius decreasing exponentially as $|\tau|$ increases, starting in 1 and coiling about the origin. We leave to the reader, to check that all these spirals cross the critical disk with real part large enough to guarantee that after the mapping by $M_+$ and the rotation by $\vartheta_c$, the resulting curve stays in the right half plane. This gives $\mathrm{Re}(w_2) > 0$ and a similar analysis gives $\mathrm{Re}(2w_2 \pm w_3) > 0$. ∎

LEMMA *For all* $\mathrm{Re}(c) > 0$ *and* $\tau \in S_c$, *the operator* $A_{c,\tau}$ *as defined in* (4.1) *is of Hilbert-Schmidt type and*

(i) *for* $c$ *fixed,* $A_{c,\tau}$ *is norm continuous for* $\tau \in S_c$ *and holomorphic for* $\tau \in \mathrm{Int}(S_c)$*;*

(ii) *for* $\tau > 0$ *fixed,* $A_{c,\tau}$ *is a holomorphic family of bounded operators for* $\mathrm{Re}(c) > 0$.

*Proof.* Lemma 4.3 implies that for all $\tau_0 \in S_c$, there exist a neighbourhood $\tau_0 \in V_0 \subset \mathbb{C}$ and a constant $M < \infty$ such that:

$$\iint |K_c(\tau, x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y < M,$$

for all $\tau \in V_0 \cap S_c$, where the constant $M$ depends locally uniformly on $c$. Therefore $A_{c,\tau}$ is Hilbert-Schmidt operator and its norm depends locally uniformly on $c$ and $\tau$.

By the dominated convergence theorem, it is elementary to check that

$$\|K_c(\tau, \cdot, \cdot) - K_c(\tau_0, \cdot, \cdot)\|_2 \to 0$$

as $\tau \to \tau_0$. This provides norm continuity of $A_{c,\tau}$ in $\tau \in S_c$ for $c$ fixed. Finally the analyticity of the family of operator in both variables, can be deduced without difficulty by differentiating under the integral sign. ∎

By Theorem 4.1 and Lemma 4.4, we know that the families of bounded operators $e^{-H_c\tau}$ and $A_{c,\tau}$ are both holomorphic in each parameter. They coincide when those parameters are real. If we fix $\tau > 0$, by analytic continuation in $c$, equation (4.2) is also true for all $\mathrm{Re}(c) > 0$. Now fixing $c$ with $\mathrm{Re}(c) > 0$, by analytic continuation in $\tau$, the same equation is true for all $\tau \in \mathrm{Int}(S_c)$. Finally, since both families are strongly continuous in $\tau$ at the edges of $S_c$, we have proved Theorem 4.2 as requested.

Lemma 4.4 is one of the crucial points for the analysis of pseudospectra that we intend to carry out in the next section. In particular, we use strongly the fact that $e^{-H_c\tau}$ is a compact operator for all $\tau \in S_c$.

## 5. PSEUDOSPECTRA

Our aim in this section is to employ the compactness of the bounded holomorphic semigroup generated by $-H_c$ (see previous section) to obtain estimates on the resolvent norm of $H_c$ inside $S(0,\arg(c))$. The technique is based on the spectral radius formula for the semigroup and a convenient Jordan decomposition of the operators involved. For simplicity we will assume from now on that $\mathrm{Im}(c)>0$, but with a few corrections, the same results and techniques apply to the case $\mathrm{Im}(c)<0$.

We establish first a formula for the spectral radius of one-parameter semigroups; a proof of this fact can be found in Theorem 1.22 from [4]. If $\mathrm{e}^{-Tt}$ is a $C_0$ one-parameter semigroup, the limit

$$a:=\lim_{t\to\infty}t^{-1}\log\|\mathrm{e}^{-Tt}\|$$

always exists with $-\infty\leqslant a<\infty$, and the spectral radius

$$\mathrm{rad}(\mathrm{e}^{-Tt}):=\max\left\{|\lambda|:\lambda\in\mathrm{Spec}(\mathrm{e}^{-Tt})\right\}=\mathrm{e}^{at}$$

for all $t>0$. This implies in particular that for all $\alpha<-a$

$$\lim_{t\to\infty}\mathrm{e}^{\alpha t}\|\mathrm{e}^{-Tt}\|=0. \tag{5.1}$$

Applying this estimate to $\mathrm{e}^{-H_c\tau}$ for $\tau$ on the edges of $S_c$, we can show that the resolvent norm of $H_c$ is uniformly bounded in lines parallel and close enough to the edges of $S(0,\arg(c))$.

THEOREM *For fixed* $\mathrm{Im}(c)>0$, *consider the complex parameters* $z_{\mathrm{low}}=\eta+\mathrm{i}\varepsilon$ *and* $z_{\mathrm{upp}}=c(\eta-\mathrm{i}\varepsilon)/|c|$, *with* $\eta>0$ *and* $\varepsilon>0$. *Then, for* $0<d<\mathrm{Im}(\lambda_0)$ *there exists a constant* $M_{c,d}<\infty$, *independent of* $\eta$ *and* $\varepsilon$, *such that*

$$\|(H_c-z_{\mathrm{low}})^{-1}\|\leqslant M_{c,d}$$

*and*

$$\|(H_c-z_{\mathrm{upp}})^{-1}\|\leqslant M_{c,d},$$

*for all* $\eta>0$ *and* $0\leqslant\varepsilon\leqslant d$.

*Proof.* By Theorem 4.2 and Lemma 4.4, the bounded holomorphic semigroup $\mathrm{e}^{-H_c\tau}$ is compact for all $\tau$ in the maximal sector $S_c$. Therefore ([4]),

$$\mathrm{Spec}(\mathrm{e}^{-H_c\tau})=\{0\}\cup\{\mathrm{e}^{-\lambda_n\tau}:n=0,1,\ldots\}.$$

By Proposition 2.2 we can just concentrate on the lower edge. Putting $\tau=-\mathrm{i}t$ for $t>0$, we obtain $\mathrm{rad}(\mathrm{e}^{\mathrm{i}H_ct})=\mathrm{e}^{-\mathrm{Im}(\lambda_0)t}$. Fix $0<d<\mathrm{Im}(\lambda_0)$; by formula (5.1) with $T=-\mathrm{i}H_c$, $a=-\mathrm{Im}(\lambda_0)$ and $\alpha=(d-a)/2$, there exists $t_\alpha>0$ such that $\|\mathrm{e}^{\mathrm{i}H_ct}\|\leqslant\mathrm{e}^{-\alpha t}$ for all $t>t_\alpha$. Then

$$\begin{aligned}\|(H_c-(\eta+\mathrm{i}\varepsilon))^{-1}\|&=\|(\mathrm{i}\eta-\varepsilon-\mathrm{i}H_c)^{-1}\|\leqslant\int_0^\infty\mathrm{e}^{\varepsilon s}\|\mathrm{e}^{\mathrm{i}H_cs}\|\,\mathrm{d}s\\&\leqslant\int_0^{\tau_\alpha}\mathrm{e}^{\varepsilon s}\,\mathrm{d}s+\int_{\tau_\alpha}^\infty\mathrm{e}^{(\varepsilon-\alpha)s}\,\mathrm{d}s\leqslant\int_0^{\tau_\alpha}\mathrm{e}^{\varepsilon s}\,\mathrm{d}s+\int_{\tau_\alpha}^\infty\mathrm{e}^{(d-\alpha)s}\,\mathrm{d}s\\&\leqslant\int_0^{\tau_\alpha}\mathrm{e}^{\varepsilon s}\,\mathrm{d}s+\int_0^\infty\mathrm{e}^{-(\mathrm{Im}(\lambda_0)-d)s/2}\,\mathrm{d}s\leqslant M_{c,d}.\end{aligned}$$

∎

This result provides information about the shape of the $\varepsilon$-pseudospectra of operator $H_c$. The result below will be extended in Theorem 5.3.

COROLLARY *For all* $0 < \delta < 1$ *there exists an* $\varepsilon > 0$ *such that*

$$\mathrm{Spec}_\varepsilon(H_c) \subset S(0, \arg(c)) + \delta c^{1/2}.$$

As a consequence of this corollary together with Proposition 2.1, we obtain

$$\mathrm{Spec}_\varepsilon(H_c) \subset \mathrm{Num}(H_c)$$

for $\varepsilon$ small enough.

The situation for the other eigenvalues, requires more care and the argument involves a Jordan decomposition of the problem. Let $Q_n$ be the spectral projector of operator $H_c$ associated to the eigenvalue $\lambda_n$, i.e.

$$Q_n f := \frac{1}{2\pi \mathrm{i}} \int_{\gamma_n} (z - H_c)^{-1} f \,\mathrm{d}z$$

where $\gamma_n$ is a smooth curve whose interior just contains eigenvalue $\lambda_n$. Observe that $Q_n$ are not orthogonal projections in $L^2(\mathbb{R})$.

For $m = 0, 1, \ldots$ put

$$P_m := \sum_{n=0}^{m} Q_n.$$

It is clear that $P_m$ is the spectral projector with rank $m$ associated to eigenvalues $\lambda_0, \ldots, \lambda_m$. Notice that $L^2(\mathbb{R})$ can be decomposed as the direct sum of the closed subspaces $\mathrm{Ran}(Q_n)$ for $n \leqslant m$ and $\mathrm{Ran}(I - P_m)$, in the sense that the subspaces are linearly independent and

$$L^2(\mathbb{R}) = \mathrm{Ran}(Q_0) + \cdots + \mathrm{Ran}(Q_m) + \mathrm{Ran}(I - P_m).$$

It is easy to show that each of the $m + 2$ subspaces above is invariant under the semigroup $\mathrm{e}^{-H_c \tau}$ for all $\tau \in S_c$. Even more, the generator of the bounded holomorphic semigroups of contractions

$$\mathrm{e}^{-H_c \tau} | \mathrm{Ran}(Q_n)$$

and

$$\mathrm{e}^{-H_c \tau} | \mathrm{Ran}(I - P_m),$$

are respectively $-H_c|(\mathrm{Ran}(Q_n))$ and $-H_c|(\mathrm{Ran}(I - P_m))$. Notice as well that, by compactness of the restriction of the semigroup, we have

$$\mathrm{Spec}(\mathrm{e}^{-H_c \tau} | \mathrm{Ran}(I - P_m)) = \{0\} \cup \{\mathrm{e}^{-\lambda_n \tau}\}_{n=m+1}^{\infty}. \tag{5.2}$$

In order to extend Corollary 5.2 beyond the first eigenvalue, observe that for all $z \notin \mathrm{Spec}(H_c)$

$$(H_c - z)^{-1} = \sum_{n=0}^{m} (H_c - z)^{-1} Q_n + (H_c - z)^{-1}(I - P_m).$$

Then if we call $\kappa_m := 1 + \sum_{n=0}^{m} \|Q_n\|$, a straightforward calculation allows us to obtain the following estimate: for all $z \notin \mathrm{Spec}(H_c)$

$$\begin{aligned} &\|(H_c - z)^{-1}\| \\ &\leqslant \kappa_m \Big( \sum_{n=0}^{m} \|(H_c|(\mathrm{Ran}(Q_n)) - z)^{-1}\| + \|(H_c|(\mathrm{Ran}(I - P_m)) - z)^{-1}\| \Big). \end{aligned} \tag{5.3}$$

With the help of this inequality, we can achieve the theorem below.

THEOREM *Let $H_c$ the complex harmonic oscillator such that* $\mathrm{Re}(c) > 0$ *and* $\mathrm{Im}(c) > 0$. *For all* $0 < \delta < 1$ *and* $m = 0, 1, \ldots$ *there exists an* $\varepsilon > 0$ *such that*

$$\mathrm{Spec}_\varepsilon(H_c) \subset [S(0, \arg(c)) + (\lambda_{m+1} - \delta)] \bigcup \bigcup_{n=0}^{m} \{z \in \mathbb{C} : |z - \lambda_n| < \delta\}.$$

*Proof.* To estimate the first sum of norms in (5.3), since $\mathrm{Ran}(Q_n)$ is the one dimensional subspace generated by the $n^{th}$ eigenvector $\Psi_n$, the operator $H_c|(\mathrm{Ran}(Q_n))$ acts on this subspace as the operator of multiplication by $\lambda_n$. Then for all $z \neq \lambda_n$,

$$\|(H_c|(\mathrm{Ran}(Q_n)) - z)^{-1}\| = \frac{1}{|\lambda_n - z|}.$$

To estimate the last resolvent norm in (5.3), we use equation (5.2), applying an analogous of Theorem 5.1 to $H_c|(\mathrm{Ran}(I - P_m))$ instead of $H_c$. Notice that now the first eigenvalue is $\lambda_{m+1}$. ∎

In the notation of Aslanyan and Davies ([1] and [8]),

$$\kappa_m = 1 + \sum_{n=1}^{m} \kappa(\lambda_m),$$

where $\kappa(\lambda_n)$ is the index of instability of the eigenvalue $\lambda_n$. Based on the results in [8], as $n$ increases, the indices of instability $\kappa(\lambda_n)$ grow faster than any power of $n$. This is reflected in the above theorem in the fact that as $m$ gets bigger the $\varepsilon$ we must choose gets exponentially smaller.

This theorem confirms the numerical calculations made by Davies in [5], where he uses a computer package to find level curves for the resolvent norm of a discretization of the operator $H_c$.

Going beyond the scope of Theorem 5.3, our conjecture is as follows. Let $0 < p < 1/3$, $m = 0, 1, \ldots$ and $0 < \delta < 1$ be fixed. Let $b_{m,p} > 0$ such that there exists $E > 0$ (possibly depending on $m$ or $p$) verifying $b_{k,p}E + cE^p = \lambda_m$. Put

$$z_\eta := b_{m,p}\eta + c\eta^p, \quad \eta > 0$$

and

$$\Omega_{m,p} := \left\{ |z_\eta| \mathrm{e}^{\mathrm{i}\vartheta} \in \mathbb{C} : \eta \geqslant E,\, \arg(z_\eta) \leqslant \vartheta \leqslant \arg(c\overline{z_\eta}/|c|) \right\}.$$

CONJECTURE There exists an $\varepsilon > 0$ such that

$$\mathrm{Spec}_\varepsilon(H_c) \subset \Omega_{m,p} \bigcup \bigcup_{n=0}^{m} \{z \in \mathbb{C} : |z - \lambda_n| < \delta\}.$$

This would be a substantial improvement of the results provided in this paper. The case $p = 0$ is precisely Theorem 5.3. Because of Theorem 3.3 the statement is false for $1/3 < p \leqslant 1$ so in this sense the constraint $0 < p < 1/3$ is optimal.

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